Abstract

Let \( T \in L(\mathcal{H}) \), and let \( T = U |T| = |T^*|U \) be the polar decomposition of \( T \). Then, for every \( \lambda \in [0, 1] \) the \( \lambda \)-Aluthge transform is defined by \( \Delta_{\lambda} (T) = \frac{|T|^\lambda U |T|^{1-\lambda}}{\|T\|} \). We show that several properties which are known for the usual Aluthge transform (i.e. the case \( \lambda = 1/2 \)) also hold for \( \lambda \)-Aluthge transforms with \( \lambda \in (0, 1) \). Moreover, we get several results which are new, even for the usual Aluthge transform.

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AMS classification: 47A30; 15A60; 47B10

Keywords: Aluthge transform; Schatten norms; Riesz calculus; Polar decomposition

1. Introduction

Let \( \mathcal{H} \) be a complex Hilbert space, and let \( L(\mathcal{H}) \) be the algebra of bounded linear operators on \( \mathcal{H} \). Given \( T \in L(\mathcal{H}) \), consider its (left) polar decomposition \( T = U |T| \). In order to study the relationship among p-hyponormal operators, Aluthge introduced in [1] the transformation \( \Delta_{1/2} (\cdot) : L(\mathcal{H}) \to L(\mathcal{H}) \) defined by

\[ \Delta_{1/2} (T) = \frac{|T|^2}{\|T\|} \]
\[ \Delta_{1/2}(T) = |T|^{1/2} U |T|^{1/2}. \]

Later on, this transformation, now called Aluthge transform, was also studied in other contexts by several authors, such as Jung, Ko and Pearcy [15] and [16], Foias, Jung, Ko and Pearcy [12], Ando [2], Ando and Yamazaki [3], Yamazaki [23], Okubo [17], Wang [21] and Wu [22] among others.

In this paper, given \( \lambda \in [0, 1] \) and \( T \in L(H) \), we study the so-called \( \lambda \)-Aluthge transform of \( T \) defined by
\[ \Delta_{\lambda}(T) = |T|^{1/2} U |T|^{1/2}. \]

This notion has already been considered by Okubo in [17]. For \( \lambda = 0 \), \( |T|^1 \) will be considered as the orthogonal projection onto the closure of \( R(|T|) \). For \( \lambda = 1 \), \( \Delta_1(T) = |T|U \), which is known as Duggal’s transform of \( T \) [12], or hinge of \( T \) [19].

The main tool we use to study the \( \lambda \)-Aluthge transforms is Young’s inequality (see, [4,14] or Section 2). Some results of this paper are devoted to the generalization of well known properties of Aluthge transform to \( \lambda \)-Aluthge transforms. For \( \lambda \in (0, 1) \), we prove that the map \( T \mapsto \Delta_{\alpha}(T) \) is continuous at every closed rank operator \( T \) (see [15] for the case \( \lambda = 1/2 \)). For every analytic function \( f \) defined in an open neighborhood of \( \sigma(T) \), we show that
\[ \|f(\Delta_{\alpha}(T))\| \leq \|f(\Delta_1(T))\|^0 \|f(\Delta_0(T))\|^{1-\lambda} \leq \|f(T)\|, \]
(see [12,17]). When \( \dim H = n < \infty \), we prove that the limit points of the sequence \( \{\Delta_{\alpha}^m(T)\} \) are normal matrices, from which we deduce Yamazaki’s spectral radius formula \( \rho(T) = \lim_{m\to\infty} \|\Delta_{\alpha}^m(T)\| \) (only in the finite dimensional case), where \( \rho(T) \) denotes the spectral radius of \( T \).

On the other hand, we show several results which are new even for the usual Aluthge transform. Given \( 1 \leq p < \infty \), we prove that the Schatten \( p \)-norms of the \( \lambda \)-Aluthge transforms decrease with respect to the Schatten \( p \)-norms of the original operator. Moreover, if \( \|\Delta_{\lambda}(T)\|_p = |T| \) for any fixed \( 1 \leq p < \infty \), then \( T \) must be normal. This was proved for \( \lambda = 1/2 \) and \( p = 2 \) in [12]. In this case, we show the following estimation: if \( T \) is a Hilbert Schmidt operator, \( \lambda \in (0, 1) \), and \( \alpha = \min \{\lambda, 1 - \lambda\} \), then
\[
\alpha^2 ||T| - |T^*||_2^2 \leq ||T\|_2^2 - ||\Delta_{\lambda}(T)\|_2^2.
\]

When \( \dim H = 2 \), Ando and Yamazaki proved that the sequence of iterated Aluthge transforms \( \{\Delta_{\alpha}^m(T)\} \) converges (see [3]). Motivated by their ideas, we show that the sequence \( \{\Delta_{\alpha}^m(T)\} \) converges for every \( \lambda \in (0, 1) \) and every \( 2 \times 2 \) matrix \( T \). Moreover, if \( \Delta_{\alpha}^\infty(T) = \lim_{m\to\infty} \Delta_{\alpha}^m(T) \), we prove that the map \( T \mapsto \Delta_{\alpha}^\infty(T) \) is jointly continuous in both parameters, \( \lambda \in (0, 1) \) and \( T \in \mathcal{H}_2(C) \).

Finally, we study some properties of the Jordan structure of the iterated Aluthge transforms. Given \( T \in \mathcal{H}_n(C) \) and \( \mu \in \sigma(T) \), let \( \mathcal{H}_{\mu,T} \) denote the spectral subspace of \( T \) associated to the eigenvalue \( \mu \) (see Definition 4.18 for a precise definition). We prove that given two different eigenvalues of \( T \), \( \gamma \) and \( \mu \), the angle between \( \mathcal{H}_{\gamma,T} \) and \( \mathcal{H}_{\mu,T} \) converges to \( \pi/2 \), for every \( \lambda \in (0, 1) \). In other words
Concerning the conjecture of the convergence of the sequence $\{\Delta^n_m(T)\}$ for $T \in \mathcal{H}_n(\mathbb{C})$, we show a reduction to the invertible case.

The paper is organized as follows: Section 2 contains preliminary results on Riesz’s functional calculus, Schatten ideals, and a list of known inequalities which we use in the paper. Section 3 deals with the properties of $\lambda$-Aluthge transform in the infinite dimensional setting. In Section 4 we study the finite dimensional case.

2. Preliminaries

In this paper $\mathcal{H}$ denotes a complex Hilbert space, $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$, $GL(\mathcal{H})$ the group of all invertible elements of $L(\mathcal{H})$, $\mathcal{U}(\mathcal{H})$ the group of unitary operators, $L(\mathcal{H})^+$ the cone of all positive operators and $L_0(\mathcal{H})$ the ideal of compact operators. When $\dim \mathcal{H} = n < \infty$ the elements of $L(\mathcal{H})$ are identified with $n \times n$ matrices, and we write $\mathcal{H}_n(\mathbb{C})$ instead of $L(\mathcal{H})$. Given $T \in L(\mathcal{H})$, $R(T)$ denotes the range or image of $T$, $N(T)$ the null space of $T$, $\sigma(T)$ the spectrum of $T$, $\rho(T)$ the spectral radius of $T$, $T^*$ the adjoint of $T$, and $\|T\|$ the usual norm of $T$ (also called spectral norm, we sometimes write $\|T\|_{sp}$); a norm $\|\cdot\|$ in $\mathcal{H}_n(\mathbb{C})$ (or defined in some adequate ideal of compact operators) is called unitarily invariant if $\|UTV\| = \|T\|$ for unitary $U, V$. If $R(T)$ is closed, $T^\perp$ denotes the Moore–Penrose pseudoinverse of $T$. Given a closed subspace $\mathcal{F} \subseteq \mathcal{H}$, $P_\mathcal{F} \in L(\mathcal{H})$ denotes the orthogonal projection onto $\mathcal{F}$.

Given $T \in L(\mathcal{H})$, $\text{Hol}(\sigma(T))$ denotes the set of all complex analytic functions defined in an open neighborhood of $\sigma(T)$. In this set, we identify two functions if they agree in an open neighborhood of $\sigma(T)$. If $T \in L(\mathcal{H})$ and $f \in \text{Hol}(\sigma(T))$, $f(T)$ indicates the evaluation of $f$ at $T$, by using the Riesz functional calculus. The reader is referred to Brown and Pearcy’s book [8] (see also [9]) for general properties of this calculus, and a proof of the following statement.

Proposition 2.1. Given $T_0 \in L(\mathcal{H})$ such that $\sigma(T_0)$ is contained in an open set $U \subseteq \mathbb{C}$, let $\{f_n\}$ be a sequence of locally analytic functions on $U$ converging to a limit $f_0$ uniformly on compact subsets of $U$, and likewise let $\{T_n\}$ be a sequence in $L(\mathcal{H})$, converging to $T_0$ (in norm). Then, $f_n(T_n)$ is defined for all sufficiently large $n$ and $f_n(T_n) \xrightarrow[n \to \infty]{} f_0(T_0)$.

Given $A \in L_0(\mathcal{H})$, $s_k(A)$, $k \in \mathbb{N}$ denote the singular values of $A$, arranged in non-increasing order. If we denote by $tr$ the canonical semifinite trace in $L(\mathcal{H})$ then the Schatten $p$-ideals ($1 \leq p < \infty$) are defined in the following way:

$$L^p(\mathcal{H}) = \{ T \in L_0(\mathcal{H}) : tr(|T|^p) < \infty \}.$$
Each $L^p(\mathcal{H})$, endowed with the norm
\[
\|T\|_p = (\text{tr}(|T|^p))^{1/p} = \left(\sum_{k \in \mathbb{N}} s_k(T)^p\right)^{1/p},
\]
is a Banach space. If $p > 1$, then $L^p(\mathcal{H})^* \cong L^q(\mathcal{H})$, where $1/p + 1/q = 1$. In this rest of this section, we list some inequalities which will be useful in the sequel. We begin with the following two versions of Young’s inequality.

**Proposition 2.2** (Argerami–Farenick [4]). Let $A \in L^p(\mathcal{H})$ and $B \in L^q(\mathcal{H})$ be positive operators and $1/p + 1/q = 1$. Then, $AB \in L^1(\mathcal{H})$ and
\[
\text{tr}(AB) = \text{tr}(A) \text{tr}(B).
\]
Moreover, equality holds if and only if $A^p = B^q$.

**Proposition 2.3** (Hirzallah–Kittaneh [14]). Let $A, B \in L(\mathcal{H})^+$, and let $p, q > 1$ with $1/p + 1/q = 1$. Suppose that $A^p, B^q \in L^2(\mathcal{H})$. Then $AB \in L^2(\mathcal{H})$, and
\[
\|AB\|_2^2 + \frac{1}{r^2} \|A^p - B^q\|_2^2 \leq \left\|\frac{A^p}{p} + \frac{B^q}{q}\right\|_2^2,
\]
where $r = \max\{p, q\}$.

Now, we state a version of the well known Corde’s inequality [10], for unitarily invariant norms. In the proof we use standard techniques and properties of the $k$th antisymmetric tensor powers $\wedge^k A$, $A \in L(\mathcal{H})$ and majorization, which can be found in B. Simon’s book [20] or Bhatia’s book [6].

**Proposition 2.4.** Let $A$ and $B$ be positive compact operators. If $p \geq 1$, then
\[
\sum_{i=1}^k s_i(|AB|^p) \leq \sum_{i=1}^k s_i(A^p B^p), \quad k \in \mathbb{N}.
\]

**Proof.** Fix $k \in \mathbb{N}$. Since $\|\wedge^k A\| = \prod_{i=1}^k s_i(A)$, Corde’s inequality
\[
\|CD\|^p \leq \|C^p D^p\|, \quad C, D \in L(\mathcal{H})^+, \quad p \geq 1,
\]
implies that
\[
\|\wedge^k A^p B^p\| = \|\wedge^k A^p (\wedge^k B)^p\| \geq \|\wedge^k A^p \wedge^k B^p\| = \|\wedge^k |AB|^p\|.
\]
Then, $\prod_{i=1}^k s_i(|AB|^p) \leq \prod_{i=1}^k s_i(A^p B^p), \quad k \in \mathbb{N}$, which implies inequality (1). \(\square\)
Finally, we include the next inequality, proved by Bhatia and Kittaneh [7]:

**Proposition 2.5.** Let $A, B \in \mathcal{M}_n(C)$, and $r \in [0, 1]$. Then

$$\|A^r - B^r\| \leq \|I^{1-r}\|A - B\|_r$$

for every unitarily invariant norm $\|\cdot\|$.

### 3. $\lambda$-Aluthge transforms

**Definition 3.1.** Let $T \in L(\mathcal{H})$, and suppose that $T = U|T| = |T^*|U$ is the polar decomposition of $T$. Then, for every $\lambda \in [0, 1]$ we define the $\lambda$-Aluthge transform of $T$ in the following way:

$$\Delta_\lambda(T) = |T|^{\lambda}U|T|^{1-\lambda}.$$  

When $\lambda = 0$, $|T|^\lambda$ will be considered as the orthogonal projection onto $R(\|T\|)$.

**Remark 3.2.** Let $T \in L(\mathcal{H})$ and let $T = W|T|$ be an arbitrary polar decomposition of $T$. It was shown in [17] that $\Delta_\lambda(T) = |T|^\lambda W|T|^{1-\lambda}$ for every $\lambda \in [0, 1]$ i.e., the $\lambda$-Aluthge transform does not depend on the partial isometry for $\lambda \in [0, 1)$. We shall use this fact repeatedly in the sequel. On the other hand, for $\lambda = 1$, it is necessary to fix the unique partial isometry $U$ such that $T = U|T|$ and $N(U) = N(T)$. For example, if $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $U = T$ and $|T| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, but the unitary matrix $W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ also satisfies $T = W|T|$, while $\Delta_1(T) = |T|U = 0 \neq |T|W = T^*$.

In the next proposition, we describe some properties which follow easily from the definitions.

**Proposition 3.3.** Let $T \in L(\mathcal{H})$ and $\lambda \in [0, 1]$. Then:

1. $\Delta_\lambda(VTV^*) = V\Delta_\lambda(T)V^*$ for every $V \in \mathcal{U}(\mathcal{H})$.
2. $\|\Delta_\lambda(T)\| \leq \|T\|$.
3. $\sigma(\Delta_\lambda(T)) = \sigma(T)$.
4. If $\dim \mathcal{H} < \infty$, then $T$ and $\Delta_\lambda(T)$ have the same characteristic polynomial.

**Proposition 3.4.** Let $T \in L(\mathcal{H})$, $\lambda \in [0, 1]$ and let $f$ be a function, which is locally analytic in a neighborhood of $\sigma(T)$. If $T = U|T|$ is the polar decomposition of $T$ then:

1. $f(T)U = Uf(\Delta_1(T))$.
2. $|T|^\lambda f(T) = f(\Delta_\lambda(T))|T|^\lambda$. 
Proof. A simple induction argument proves the statement for \( f(t) = t^n \). This can be extended to every polynomial by linearity. This can be applied to show the statement for rational functions (with poles outside \( \sigma(T) \)). Finally, using Runge’s theorem (see, for example, Conway’s book [9]), the result generalizes to analytic functions. \( \Box \)

In [15], Jung, Ko and Pearcy proved that the Aluthge transformation is continuous at every closed range operator, with respect to the norm topology, for \( \lambda = 1/2 \). In order to generalize this property for \( \lambda \in (0, 1) \), we need the following result. Recall that, if \( B \in L(H) \) has closed range, there exists a unique pseudo-inverse \( B^\dagger \) of \( B \) such that \( BB^\dagger \) and \( B^\dagger B \) are selfadjoint projections. \( B^\dagger \) is called the Moore–Penrose pseudo-inverse of \( B \) (see, for example, [5]).

**Lemma 3.5.** Let \( B \in L(H) \), selfadjoint with closed range, and let \( \{B_n\} \) be a sequence of closed range selfadjoint operators such that \( B_n \xrightarrow{n \to \infty} B \) in norm. If \( P_{R(B_n)} \xrightarrow{n \to \infty} P_{R(B)} \) in norm, then also \( B_n^\dagger \xrightarrow{n \to \infty} B^\dagger \) in norm.

**Proof.** Denote by \( P_n = P_{R(B_n)} \) and \( P = P_{R(B)} \). If \( P_n \xrightarrow{n \to \infty} P \) then there exists a sequence \( \{U_n\} \) of unitary operators such that \( U_n \xrightarrow{n \to \infty} 1 \) and \( U_n^* PU_n = P_n \), \( n \in \mathbb{N} \).

Indeed, we can take \( U_n \) as the unitary part in the polar decomposition of \( PP_n + (1 - P)(1 - P_n) \), which is invertible for large \( n \). Note that, if \( S_n = U_n B_n U_n^* \), then \( S_n \xrightarrow{n \to \infty} B \) in norm, \( R(S_n) = R(B) \) and \( S_n^\dagger = U_n B_n^\dagger U_n^* \), \( n \in \mathbb{N} \). Hence, it suffices to prove that \( S_n^\dagger \xrightarrow{n \to \infty} B^\dagger \). But this is clear by continuity of the map \( A \mapsto A^{-1} \) (on the fixed subspace \( R(B) = R(S_n) \), \( n \in \mathbb{N} \)). \( \Box \)

**Theorem 3.6.** Let \( T \) be an operator with closed range. Then, for every \( \lambda \in (0, 1) \), the \( \lambda \)-Aluthge transform \( \Delta_\lambda(\cdot) \) is continuous at \( T \).

**Proof.** Let \( \{T_n\} \) be a sequence of operators such that \( \|T_n - T\| \to 0 \). For each \( n \in \mathbb{N} \), let \( T_n = U_n |T_n| \) be a polar decomposition of \( T_n \). On the other hand, take \( \varepsilon > 0 \) such that \( \sigma(|T|) \subseteq [0, 2\varepsilon) \cup (2\varepsilon, +\infty) \) and suppose, without loss of generality, that \( \sigma(|T_n|) \subseteq \{\lambda : |\lambda| < \varepsilon\} \cup (2\varepsilon, +\infty) \) for all \( n \). Define, for \( n \in \mathbb{N} \),

\[
P_n = |T_n| E_{[T_n]}(-\varepsilon, \varepsilon) \quad \text{and} \quad A_n = U_n P_n, \quad (2)
\]

\[
Q_n = |T_n| E_{[T_n]}(2\varepsilon, +\infty) \quad \text{and} \quad B_n = U_n Q_n, \quad (3)
\]

where \( E_{[T_n]}(I) \) denotes the spectral projection of \( |T_n| \) corresponding to the interval \( I \subseteq \mathbb{R} \). Note that \( A_n + B_n = T_n \), and (2) and (3) are polar decompositions of \( A_n \) and \( B_n \), respectively. Therefore

\[
\| \Delta_\lambda(T) - \Delta_\lambda(T_n) \| \leq \| \Delta_\lambda(A_n) \| + \| P_n^\lambda U_n Q_n^{1-\lambda} \|
\]

\[
+ \| Q_n^\lambda U_n P_n^{1-\lambda} \| + \| \Delta_\lambda(T) - \Delta_\lambda(B_n) \|.
\]
By Proposition 2.1, \( P_n = \| T_n \| E_n \| (-\varepsilon, \varepsilon) \) \( \| T \| E_n \| (-\varepsilon, \varepsilon) = 0 \). Then
\[
\| \Delta_\lambda(A_n) \| + \| P_n^* U_n P_n^{-\lambda} \| + \| Q_n U_n P_n^{-\lambda} \| \to 0.
\]

On the other hand, \( |B_n| = Q_n \) which have closed ranges. Since the maps \( \chi((-\varepsilon, \varepsilon)) \) and \( \chi((\varepsilon, +\infty)) \) admit complex analytic extensions to the set \( \{ z \in \mathbb{C} : \text{Re}(z) \in (-\varepsilon, \varepsilon) \cup (\varepsilon, +\infty) \} \), we can apply Proposition 2.1, and obtain that
\[
P_R(Q_n) = E_{[T_n]}(2\varepsilon, +\infty) \to E_T(2\varepsilon, +\infty) = P_R(T).
\]
Hence, \( |B_n| \to |T| \) and \( P_R(|B_n|) \to P_R(|T|) \), both in the norm topology. By Lemma 3.5, we conclude that \( |B_n| \to |T| \) in norm. Therefore
\[
\| \Delta_\lambda(T) - \Delta_\lambda(B_n) \| = \| |T|^{\lambda} T |T|^\lambda - |B_n|^{\lambda} B_n(B_n)^{\lambda} \| \to 0,
\]
which completes the proof. \( \square \)

**Remark 3.7.** Theorem 3.6 fails for \( \lambda = 0 \) and \( \lambda = 1 \), even in the finite dimensional case. Indeed, take \( T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( T_n = \begin{pmatrix} 0 & 1/n \\ 1/n & 0 \end{pmatrix} \), \( n \in \mathbb{N} \). It is easy to check that \( \Delta_0(T_n) = T_n \) and \( \Delta_1(T_n) = T_n^2 \), which do not converge to 0 = \( \Delta_0(T) = \Delta_1(T) \). Compare with Remark 3.2.

### 3.1. Schatten norms and ideals

In this subsection we characterize those operators in \( L^p(\mathcal{H}) \) which satisfy \( \| \Delta_\lambda(T) \|_p = \| T \|_p \). Naturally, the equality holds if \( T \) is normal, because \( T = \Delta_\lambda(T) \). It was proved in [16] that, for the Frobenius norm and for \( \lambda = 1/2 \), the equality holds if and only if \( T \) is normal. In the following proposition we estimate from below the difference between the Frobenius norms of \( T \) and \( \Delta_\lambda(T) \).

**Proposition 3.8.** Let \( T \in L^2(\mathcal{H}) \) and \( \lambda \in (0, 1) \). If \( \alpha = \min \{ \lambda, 1 - \lambda \} \), then
\[
\alpha^2 \| |T| - |T^\lambda| \|_2^2 \leq \| T \|_2^2 - \| \Delta_\lambda(T) \|_2^2.
\]

**Proof.** Note that, if \( T = U|T| \) is the polar decomposition of \( T \), then \( |T^\lambda| = U|T^\lambda|U^* \), for every \( r > 0 \). Then
\[
\| \Delta_\lambda(T) \|_2^2 = \text{tr}(\Delta_\lambda(T) \Delta_\lambda(T)^*) = \text{tr}(|T^\lambda|U|T|^{2(1-\lambda)}U^*|T^\lambda|)
\]
\[
= \text{tr}(|T^\lambda|T^{2(1-\lambda)}|T^\lambda|) = \| |T^\lambda|T^{2(1-\lambda)}|T^\lambda| \|_2^2.
\]

Using Hirzallah–Kittaneh’s inequality (Proposition 2.3) with \( A = |T^\lambda|, \ B = |T^\lambda|^{1-\lambda}, \ p = \lambda^{-1}, \ q = (1-\lambda)^{-1} \) and \( \alpha = \min \{ \lambda, 1 - \lambda \} = \max \{ \lambda^{-1}, (1-\lambda)^{-1} \}^{-1} \), we get
\[ \| \Delta_k(T) \|_2^2 + \alpha^2 \| T^* \|_2^2 \leq \| \lambda |T| + (1 - \lambda) |T^*| \|_2^2 \leq \| T \|_2^2, \]

where the last inequality follows from the triangle inequality. \( \square \)

Now, we prove that equality in other Schatten norms also implies that \( T \) is normal.

**Theorem 3.9.** Let \( \lambda \in (0, 1), \) \( 1 \leq p < \infty \) and \( T \in L^p(\mathcal{H}) \). Then, \( \Delta_k(T) \in L^p(\mathcal{H}) \) and

\[ \| \Delta_k(T) \|_p \leq \| T \|_p. \]

Moreover, the equality holds if and only if \( T \) is normal.

In order to prove this result, we need the following lemma.

**Lemma 3.10.** Let \( A, B \in L(\mathcal{H}) \) and let \( B = U|B| \) be the polar decomposition of \( B \). Then, for every \( p > 0 \),

\[ |AB^*|^p = U \| A \| |B| |^p U^*. \]

**Proof.** Let \( P = \| A \| |B| |^2 \). Then, for every continuous function \( f \) defined on \([0, +\infty)\) such that \( f(0) = 0 \),

\[ f(U PU^*) = U f(P) U^*. \] (5)

In fact, since \( R(P) \subseteq R(|B|) \), and \( U^*U \) is the orthogonal projection onto \( R(|B|) \), then \( (U PU^*)^n = U P^n U^* \), for every \( n \geq 1 \). Therefore, by linearity, formula (5) holds for every polynomial \( f \) such that \( f(0) = 0 \). On the other hand, given a continuous function \( f \) defined in \([0, +\infty)\) such that \( f(0) = 0 \), there exists a sequence \( \{p_n\}_{n \in \mathbb{N}} \) of polynomials such that \( p_n(0) = 0, \) \( n \in \mathbb{N} \), and \( p_n \rightarrow f \) uniformly on \( \sigma(P) \cup \Omega = \sigma(U PU^*) \cup \Omega \). So, standard limit arguments prove formula (5).

Now, the result follows from the equality

\[ |AB^*|^2 = B A^* A B^* = U |B| |A|^2 |B| U^* = U |A| |B| |^2 U^*, \]

by applying the function \( f(x) = x^{p/2} \) to both sides. \( \square \)

**Proof of Theorem 3.9.** Let \( T = U |T| \) be the polar decomposition of \( T \). Fix \( 1 \leq p < \infty \). Then, using Lemma 3.10 with \( A = |T|^p \) and \( B^* = U |T|^{1 - \lambda} \), we get

\[ \text{tr} |\Delta_k(T)|^p \leq \text{tr} \| |T|^p \| |T^{1 - \lambda} \|^p. \]

Using Proposition 2.4 with \( A = |T|^p \) and \( B = |T^*|^{1 - \lambda} \), we get

\[ \text{tr} \| |T|^p \| |T^{1 - \lambda} \|^p \leq \text{tr} \| |T|^p |T^*|^{p(1 - \lambda)} \|. \]

Then, by Proposition 2.2, for the conjugate numbers \( \lambda^{-1} \) and \( (1 - \lambda)^{-1} \),

\[ \text{tr} |\Delta_k(T)|^p \leq \text{tr} |T|^{p\lambda} |T^*|^{p(1 - \lambda)} \]

\[ \leq \lambda \text{tr} |T|^p + (1 - \lambda) \text{tr} |T^*|^p = \text{tr} |T|^p. \]
Therefore, if \( \| \Delta_{\lambda}(T) \|_p = \| T \|_p \), then equality holds in Young’s inequality, and by Proposition 2.2, we conclude that \( |T|^p = |T^*|^p \). Hence \( T \) is normal. □

**Remark 3.11.** Theorem 3.9 fails for \( \lambda = 1 \). Take, for example, \( T \in L^2(\mathcal{H}) \) with polar decomposition \( T = U|T| \), with \( U \in \mathcal{U}(\mathcal{H}) \). In this case, \( \| \Delta_{\lambda}(T) \|_2 = \| T \|_2 \).

The following example shows that Theorem 3.9 may be false for other unitarily invariant norms. In particular, for the spectral norm.

Let

\[
T = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]

Then,

\[
\Delta_{\lambda}(T) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

for every \( \lambda \in (0, 1) \),

and therefore

\[
1 = \| \Delta_{\lambda}(T) \|_p < \| T \|_p = 2^{1/p} \quad \text{but} \quad \| \Delta_{\lambda}(T) \| = \| T \| = 1.
\]

The reader interested in the equality for the spectral norm is referred to [24]. In that work, Yamazaki proves that \( \| \Delta_{\lambda}(T) \| = \| T \| \) if and only if \( T \) is normaloid, i.e., if \( \rho(T) = \| T \| \).

**Remark 3.12.** Using standard techniques of alternate tensor powers, it can be proved that given \( T \in L_0(\mathcal{H}) \) and \( \lambda \in [0, 1] \), then

\[
\prod_{i=1}^k x_i(\Delta_{\lambda}(T)) \leq \prod_{i=1}^k s_i(T), \quad k \in \mathbb{N}.
\]

This inequality says that the singular values of \( \Delta_{\lambda}(T) \) are log-majorized by the singular values of \( T \). Hence, we can deduce that for every unitarily invariant norm \( \| \cdot \| \), we have that \( \| \Delta_{\lambda}(T) \| \leq \| T \| \).

### 3.2. Riesz’s functional calculus

An interesting result proved by Foias et al. [12] relates the Aluthge transform with completely contractive maps by using Riesz’s functional calculus. Following similar ideas, in this subsection we study the relationship between Riesz’s functional calculus and \( \lambda \)-Aluthge transforms. We begin with the following technical lemma.

**Lemma 3.13.** Let \( X \in L(\mathcal{H}) \), \( A \in GL(\mathcal{H})^+ \) and \( \lambda \in [0, 1] \). Then, given \( n \in \mathbb{N} \), and \( f_{11}, \ldots, f_{nn} \) analytic functions defined in a neighborhood of \( \sigma (AX) \), we have

\[
\left\| \left((f_{ij}(A^k X A^{1-k}))_{ij}\right)_{ij} \right\| \leq \left\| (f_{ij}(AX))_{ij} \right\|^\lambda \cdot \left\| (f_{ij}(XA))_{ij} \right\|^{1-\lambda}.
\]
Proof. Let $\Omega_{0.1}$ denote the open subset of the complex plane defined by

$$\Omega_{0.1} = \{ z \in \mathbb{C} : \text{Re}(z) \in (0, 1) \}.$$

Given two unitary vectors $x = (x_1, \ldots, x_n)$, and $y = (y_1, \ldots, y_n)$ belonging to $\mathbb{R}^n$, define $\varphi_{x,y} : \Omega_{0.1} \to \mathbb{C}$ in the following way:

$$\varphi_{x,y}(z) = \{(f_{ij}(A^2 X A^{1-z}))_{ij}, x, y\}.$$

If $I_n$ denotes the identity operator on $\mathbb{C}^n$, then

$$\{f_{ij}(A^2 X A^{1-z})\}_{ij} = (A^2 f_{ij}(X A) A^{-z})_{ij} = (A^2 \otimes I_n)(f_{ij}(X A))_{ij}(A^{-z} \otimes I_n).$$

Hence, it is easy to see that $\varphi_{x,y}$ is analytic in $\Omega_{0.1}$ and continuous in $\Omega_{0.1}$. On the other hand, since $A^t$ is unitary for every $t \in \mathbb{R}$,

$$|\varphi_{x,y}(i t)| = \left\| (f_{ij}(A^t X A^{1-it}))_{ij}, x, y \right\|$$

$$= \left\| (A^t \otimes I_n)(f_{ij}(X A))_{ij}(A^{-it} \otimes I_n)x, y \right\|$$

$$\leq \left\| (f_{ij}(X A))_{ij} \right\|_2.$$

Analogously

$$|\varphi_{x,y}(1 + i t)| = \left\| (f_{ij}(A^{1+it} X A^{-it}))_{ij}, x, y \right\|$$

$$= \left\| (A^t \otimes I_n)(f_{ij}(X A))_{ij}(A^{-it} \otimes I_n)x, y \right\|$$

$$\leq \left\| (f_{ij}(X A))_{ij} \right\|_2.$$

Therefore, by the three lines theorem (see, for example, [18]), if $\lambda = \text{Re}(z)$,

$$\left\| (f_{ij}(A^2 X A^{1-z}))_{ij}, x, y \right\| \leq \left\| (f_{ij}(X A))_{ij} \right\|^{\lambda} \cdot \left\| (f_{ij}(X A))_{ij} \right\|^{1-\lambda}.$$

Taking supremum over all $x, y \in \mathbb{R}^n$, we get the desired inequality. □

Lemma 3.13 allows us to give an alternative proof of Jung, Ko and Pearcy's result, which also generalizes it for $\lambda \in (0, 1)$.

Proposition 3.14. Let $T \in L(\mathcal{H})$, $\lambda \in (0, 1)$ and $f \in \text{Hol}(\sigma(T))$. Then

1. $\|f(\Delta_0(T))\| \leq \|f(T)\|$ and $\|f(\Delta_1(T))\| \leq \|f(T)\|$.
2. $\|f(\Delta_\lambda(T))\| \leq \|f(\Delta_1(T))\|^\lambda \cdot \|f(\Delta_0(T))\|^{1-\lambda} \leq \|f(T)\|$.

Proof. The inequality $\|f(\Delta_1(T))\| \leq \|f(T)\|$ was proved by Foias, Jung, Ko and Pearcy in [12], using Proposition 3.4. The inequality for $\Delta_0(T)$ can be proved by following similar ideas.
In order to prove the inequality of item 2, Let $T = U |T|$ be the polar decomposition of $T$ and $E$ the orthogonal projection onto $R(|T|)$. Note that $(|T| + n^{-1})^{1/2}$ converges uniformly to $f(x) = x^{1/2}$ on compact subsets. So, given $f \in Hol (\sigma (T))$, by Proposition 2.1 we have that

$$f((|T| + n^{-1})^{1/2} E U (|T| + n^{-1})^{1-1/2}).$$

$f(EU(|T| + n^{-1}))$ and $f((|T| + n^{-1})EU)$ are defined for all sufficiently large $n$. Moreover,

$$f(U(|T| + n^{-1})) \xrightarrow{n \to \infty} f(EU|T|),$$

$$f((|T| + n^{-1})EU) \xrightarrow{n \to \infty} f(|T|E U) = f(|T|U),$$

$$f((|T| + n^{-1})^{1/2} EU (|T| + n^{-1})^{1-1/2}) \xrightarrow{n \to \infty} f(|T|^{1/2}U|T|^{1-1/2}).$$

Using Lemma 3.13 and standard limit arguments, we get inequality 2. □

Remark 3.15. Using Lemma 3.13, it can be proved that given $n \in \mathbb{N}$, and $f_{11}, \ldots, f_{nn} \in Hol (\sigma (T))$,

$$\| (f_{ij}(\Delta_{\lambda}(T)))_{ij} \| \leq \| (f_{ij}(\Delta_{1}(T)))_{ij} \|^2 \| (f_{ij}(\Delta_{0}(T)))_{ij} \|^{1-\lambda}.$$

It should be mentioned that $\| (f_{ij}(\Delta_{0}(T)))_{ij} \| \leq \| (f_{ij}(T))_{ij} \|$.

For $T \in L(H)$, we denote $W(T) = \{(TX, x) : x \in H, \|x\| = 1\}$, its numerical range. As a corollary of Proposition 3.14, we obtain the next result about numerical ranges.

Corollary 3.16. Let $T \in L(H)$ and $\lambda \in [0, 1]$. Then, for every complex analytic function $f$ defined in a neighborhood of $\sigma (T)$,

$$W(f(\Delta_{\lambda}(T))) \subseteq W(f(T)).$$

Proof. Indeed, by Proposition 3.14 (item 1), for every $\mu \in \mathbb{C}$ it holds that $\| f(\Delta_{\lambda}(T)) - \mu I \| \leq \| f(T) - \mu I \|$. So, if $B(r, \zeta) = \{ z \in \mathbb{C} : |z - \zeta| \leq r \}$, using the well known formula

$$W(T) = \bigcap_{\lambda \in \mathbb{C}} B(\|T - \lambda I\|, \lambda),$$

we can write

$$W(f(\Delta_{\lambda}(T))) \subseteq \bigcap_{\lambda \in \mathbb{C}} B(\|f(T) - \lambda I\|, \lambda).$$

This completes the proof.
we have that
$W(f(\Delta_\lambda(T))) = \bigcap_{\mu \in \mathbb{C}} B(\|f(\Delta_\lambda(T)) - \mu I\|, \lambda)$
\[
\leq \bigcap_{\mu \in \mathbb{C}} B(\|f(T) - \mu I\|, \lambda) = W(f(T)).
\]

**Remark 3.17.** The above Corollary, was proved in [12], for $\lambda = 1/2$, using that $W(T)$ is the intersection of all half-planes $H$ containing $W(T)$, which are spectral sets for $T$. In [17], Okubo obtains the same result for a polynomial function $f$, for every $\lambda \in (0, 1)$.

### 4. The finite dimensional case

In this section, we study the $\lambda$-Aluthge transformation in finite dimensional spaces. Given $T \in \mathcal{B}_n(\mathbb{C})$ and $\lambda \in (0, 1)$, we denote by $\Delta_\lambda^n(T)$ the $n$-times iterated $\lambda$-Aluthge transform of $T$, i.e.,
\[
\Delta_\lambda^0(T) = T \quad \text{and} \quad \Delta_\lambda^n(T) = \Delta_\lambda \left( \Delta_\lambda^{n-1}(T) \right), \quad n \in \mathbb{N}.
\]

The following proposition was proved, for $\lambda = 1/2$, by Ando in [2], and by Jung, Ko and Pearcy in [16].

**Proposition 4.1.** Let $T \in \mathcal{B}_n(\mathbb{C})$. Then, the limit points of the sequence $\{\Delta_\lambda^n(T)\}_{n \in \mathbb{N}}$ are normal. Moreover, if $L$ is a limit point, then $\sigma(L) = \sigma(T)$ with the same algebraic multiplicity.

**Proof.** Let $\{\Delta_\lambda^{n_k}(T)\}_{k \in \mathbb{N}}$ be a subsequence which converge in norm to a limit point $L$. By the continuity of Aluthge transforms, $\Delta_\lambda^{n_k+1}(T) \rightarrow \Delta_\lambda(L)$. Then
\[
\|\Delta_\lambda(L)\|_2 = \lim_{k \rightarrow \infty} \|\Delta_\lambda^{n_k+1}(T)\|_2 = \lim_{n \rightarrow \infty} \|\Delta_\lambda^n(T)\|_2
\]
\[
= \lim_{k \rightarrow \infty} \|\Delta_\lambda^{n_k}(T)\|_2 = \|L\|_2.
\]

Hence, by Theorem 3.9 $L$ is normal. It only remains to prove that $\sigma(L) = \sigma(T)$ with the same algebraic multiplicity, or equivalently, that $\text{tr}(T^m) = \text{tr}(L^m)$ for every $m \in \mathbb{N}$. Indeed,
\[
\text{tr} L^m = \lim_{k \rightarrow \infty} \text{tr} \Delta_\lambda^{n_k}(T)^m = \text{tr} T^m, \quad m \in \mathbb{N},
\]

because, for each $k \in \mathbb{N}$, $\sigma(\Delta_\lambda^{n_k}(T)) = \sigma(T)$ (with algebraic multiplicity), and therefore $\text{tr} \Delta_\lambda^{n_k}(T)^m = \text{tr} T^m$. $\square$
As a consequence of this result, we obtain Yamazaki’s spectral radius formula, for every \( \lambda \in (0, 1) \). It should be mentioned that Yamazaki’s formula holds for operators in Hilbert spaces (with \( \lambda = 1/2 \)), but we can only prove the general case (\( \lambda \neq 1/2 \)) in the finite dimensional case.

**Corollary 4.2.** Let \( T \in \mathcal{M}_n(\mathbb{C}) \) and \( \lambda \in (0, 1) \). Then,

\[
\rho(T) = \lim_{k \to \infty} \|\Delta_k^\lambda(T)\|
\]

**Proof.** Take a subsequence \( \{\Delta_k^\lambda(T)\} \) that converges to a limit point \( L \). Since \( L \) is normal and \( \sigma(L) = \sigma(T) \), it holds that \( \|L\| = \rho(L) = \rho(T) \). Hence

\[
\lim_{k \to \infty} \|\Delta_k^\lambda(T)\| = \|L\| = \rho(L) = \rho(T).
\]

Finally, since the whole sequence \( \{\|\Delta_k^\lambda(T)\|\} \) converges because it is non-increasing, we obtain the desired result. \( \square \)

Analogously we can deduce the following result, proved by Ando in [2] for \( \lambda = 1/2 \). We use the notation \( \text{co}(X) \) for the convex hull of the set \( X \).

**Corollary 4.3.** Let \( T \in \mathcal{M}_n(\mathbb{C}) \) and \( \lambda \in (0, 1) \). Then,

\[
\text{co}(\sigma(T)) = \bigcap_{n=1}^\infty W(\Delta_n^\lambda(T)).
\]

Now we state the following result, which is a direct consequence of Theorem 3.6 and the fact that the map \( T \to \|T\| \) is norm-continuous in \( \mathcal{M}_n(\mathbb{C}) \).

**Proposition 4.4.** The map \( (\lambda, T) \to \Delta_k^\lambda(T) \) from \( (0, 1) \times \mathcal{M}_n(\mathbb{C}) \) into \( \mathcal{M}_n(\mathbb{C}) \) is continuous when \( \mathcal{M}_n(\mathbb{C}) \) is endowed with the norm-topology and the interval \( (0, 1) \) with the usual one.

**Proof.** It follows by a standard \( \xi \)-argument. \( \square \)

### 4.1. The iterated Aluthge transforms in \( \mathcal{M}_2(\mathbb{C}) \)

In this subsection we study the convergence of the sequence \( \{\Delta_k^\lambda(T)\} \) when \( T \) is a 2 \times 2 matrix. The convergence of this sequence for \( n \times n \) matrices and \( \lambda = 1/2 \) was conjectured by Jung, Ko, and Pearcy in [15]. Although this conjecture is still open, there exists a result, due to T. Ando and T. Yamazaki [3], which answers the conjecture affirmatively for 2 \times 2 matrices and \( \lambda = 1/2 \). We generalize this result.
for arbitrary \( \lambda \in (0, 1) \) and we also prove that the map which assigns to each pair \((\lambda, T)\) the limit of the sequence \(\{\Delta^n(\lambda)\}\) is continuous in both variables \(T\) and \(\lambda\).

**Lemma 4.5.** Let \( T \in \mathcal{H}_2(\mathbb{C}) \) and \( \lambda \in (0, 1) \). Suppose that \( \sigma(T) = [\mu_1, \mu_2] \) with \( \mu_1 \neq \mu_2 \). Then, there exists \( \gamma(T, \lambda) \in (0, 1) \) such that, for all \( n \in \mathbb{N} \),

\[
\| \Delta^n_\lambda(T) - \Delta^n_\lambda(T)^* \| \leq \gamma(T, \lambda) \| T^* T - T T^* \|_2.
\]

Moreover, if \( \sigma = \min(\lambda, 1 - \lambda) \), then we can take

\[
\gamma(T, \lambda) = \left(1 - \frac{2 \sigma^2 | \mu_1 - \mu_2 |^2}{2 |\mu_1 \mu_2| + \| T \|^2_2} \right)^{1/2}.
\]

**Proof.** Denote \( T_n = \Delta^n_\lambda(T), n \in \mathbb{N} \). In some orthonormal basis, which may be different for each \( n \in \mathbb{N} \), \( T_n \) has the form

\[
T_n = \begin{pmatrix} \mu_1 & a_n \\ 0 & \mu_2 \end{pmatrix}, \quad \text{with } a_n = (\| T_n \|^2_2 - [| \mu_1 |^2 + | \mu_2 |^2])^{1/2} \geq 0.
\]

Hence \( a_{n+1} \leq a_n, n \in \mathbb{N} \), by Theorem 3.9. Easy computations show that, if \( M = | \mu_1 - \mu_2 |^2 \) then

\[
\| T_{n+1}^* T_n - T_n T_{n+1}^* \|^2_2 = 2 a_n^2 (M + a_n^2), \quad n \in \mathbb{N}.
\]

Therefore, for all \( n \in \mathbb{N} \),

\[
\frac{\| T_{n+1}^* T_n - T_{n+1}^* T_{n+1} \|^2_2}{\| T_n^* T_n - T_{n+1}^* T_{n+1} \|^2_2} = \frac{a_{n+1}^2 (M + a_{n+1}^2)}{a_n^2} \leq \frac{a_{n+1}^2}{a_n^2}.
\]

Since \( a_{n+1}^2 - a_n^2 = \| T_n \|^2_2 - \| T_{n+1} \|^2_2 \), by Proposition 3.8 the following inequality holds for all \( n \in \mathbb{N} \),

\[
a_{n+1}^2 - a_n^2 = 1 - \frac{\| T_n \|^2_2 - \| T_{n+1} \|^2_2}{a_n^2} \leq 1 - \frac{\| T_n \|^2_2}{a_n^2} \leq 1 - \frac{\| T_n \|^2_2}{a_n^2}.
\]

On the other hand, if \( X \in \mathcal{H}_2(\mathbb{C})^+ \) and \( d = \det(X)^{1/2} \), then it is known that

\[
X^{1/2} = \frac{X + dI}{\sqrt{2d + \text{tr}(X)}}.
\]

Hence, if we denote \( d = \det(T_n^* T_n)^{1/2} = \det(T_n T_n^*)^{1/2} = | \det T | = | \mu_1 \mu_2 | \), we have that

\[
\| |T_n| - |T_n^*| \|^2_2 = \frac{\| T_n^* T_n - T_n T_n^* \|^2_2}{2d + \| T_n \|^2_2}, \quad n \in \mathbb{N}.
\]

Therefore, by Eq. (6), for all \( n \in \mathbb{N} \),

\[
\frac{a_{n+1}^2}{a_n^2} \leq 1 - \frac{\| T_n^* T_n - T_n T_n^* \|^2_2}{a_n^2 (2d + \| T_n \|^2_2)}
\]

\[
= 1 - \frac{2 \sigma^2 (M + a_n^2)}{2d + \| T_n \|^2_2} \leq 1 - \frac{2 \sigma^2 M}{2d + \| T \|^2_2}.
\]
Finally, taking
\[
\gamma(T, \lambda) = \left(1 - \frac{2\alpha^2 M}{2d + \|T\|^2}\right)^{1/2},
\]
by Eqs. (7) and (8), we get
\[
\|T_{n+1}^a - T_n^a\|_2 \leq \gamma(T, \lambda)\|T_n^a - T_n^a\|_2, \quad n \in \mathbb{N},
\]
and the result is proved by iterating this inequality. Note that \(0 < \alpha^2 \leq 1/4\) and
\[
0 < M = |\mu_1 - \mu_2|^2 \leq 2|\mu_1\mu_2| + |\mu_1|^2 + |\mu_2|^2 \leq 2d + \|T\|^2.
\]
Then \(0 < \gamma(T, \lambda) < 1\). □

**Theorem 4.6.** Let \(T \in \mathcal{H}_2(\mathbb{C})\) and \(\lambda \in (0, 1)\). Then, the sequence \(\{\Delta_n^\lambda(T)\}\) converges.

**Proof.** Suppose that \(\sigma(T) = \{\mu_1, \mu_2\}\). Since we have proved (see Proposition 4.1) that the limit points of the sequence \(\{\Delta_n^\lambda(T)\}\) are normal, if \(\mu_1 = \mu_2 = c\), then \(\Delta_n^\lambda(T) \to cI\). Thus, from now on we only consider the case in which \(\mu_1 \neq \mu_2\).

As in the Lemma 4.5, we denote \(T_n = \Delta_n^\lambda(T)\).

Fix \(n \geq 0\). If \(T_n = U_n|T_n|\) is the polar decomposition of \(T_n\), then \(|T_n^a|^2 = U_n|T_n|^2 U_n^*\), for every \(n > 0\). Therefore we obtain
\[
(T_{n+1}^a - T_n^a)U_n^* = |T_n|^2 U_n|T_n|^{1-\lambda} U_n^* - U_n|T_n|^2 U_n^* = |T_n|^{1-\lambda} - |T_n^a| = (|T_n|^{1-\lambda} - |T_n^a|)|T_n^a|^{1-\lambda}.
\]
Since \(\|AB\|_2 \leq \|A\|_2 \|B\|_2\), we can deduce that
\[
\|T_{n+1} - T_n\|_2 \leq \|T_n|^{1-\lambda} - |T_n^a|^{1-\lambda}\|_2 \cdot \|T_n^a|^{1-\lambda}\|_2.
\]
Using Proposition 2.5 with \(A = T_n^a, B = T_n^a\) and \(r = \lambda/2\), we get
\[
\|T_{n+1} - T_n\|_2 \leq \|T_n|^{1-\lambda} - |T_n^a|^{1-\lambda}\|_2 \cdot \|T\|^{1-\lambda}
\leq (2\|T\|^{1-\lambda})\|T_n^a - T_n^a\|_2^{\lambda/2},
\]
because \(\|T_n\|^{1-\lambda/2} \leq 2\). Let \(a = \gamma(T, \lambda)^{\lambda/2} < 1\), where \(\gamma(T, \lambda) \in (0, 1)\) is the constant of Lemma 4.5. Then
\[
\|T_{n+1} - T_n\|_2 \leq \gamma^{\lambda/2} \|T_n^a - T_n^a\|_2^{\lambda/2} \leq a^\lambda \gamma^{\lambda/2} \|T^a - TT^a\|_2^{\lambda/2}.
\]
Denote $N(T, \lambda) = 2\|T\|^{1-\lambda}\|T^*T - TT^*\|^{\lambda/2}/2$. Then, if $n, m \in \mathbb{N}$, with $n < m$,

\[
\|T_m - T_n\|_2 \leq \sum_{k=n}^{m-1} \|T_{k+1} - T_k\|_2 \\
\leq N(T, \lambda) \sum_{k=n}^{m-1} \alpha^k n, m \to \infty, (9)
\]

which shows that the \(\lim_{n \to \infty} T_n = \lim_{n \to \infty} \Delta^a_\lambda(T)\) exists. \(\square\)

In order to state precisely the next results, we need the following notations:

**Definition 4.7**

1. Given \(A \in \mathbb{H}_2(C)\) and \(\lambda \in (0, 1)\), denote \(\Delta^a_\lambda(A) = \lim_{n \to \infty} \Delta^a_\lambda(T)\).
2. Consider the map \(I : (0, 1) \times \mathbb{H}_2(C) \to \mathbb{H}_2(C)\) defined by \(I(\lambda, T) = \Delta^a_\lambda(T), (\lambda, T) \in (0, 1) \times \mathbb{H}_2(C)\).

**Theorem 4.8.** Let \(\lambda \in (0, 1)\) be fixed. Then the map \(I(\lambda, \cdot) : \mathbb{H}_2(C) \to \mathbb{H}_2(C)\), given by

\[\mathbb{H}_2(C) \ni T \mapsto \Delta^a_\lambda(T)\]

is continuous. Therefore \(\Delta^a_\lambda(\cdot)\) is a continuous retraction from \(\mathbb{H}_2(C)\) onto the space of normal matrices in \(\mathbb{H}_2(C)\).

**Proof.** Take \(T \in \mathbb{H}_2(C)\) and \(\lambda \in (0, 1)\). We shall consider two cases:

**Case 1.** Suppose that \(\sigma(T) = \{\mu\}\). Let \(S \in \mathbb{H}_2(C)\) with \(\sigma(S) = \{\eta_1, \eta_2\}\). Since \(\Delta^a_\lambda(T) = \mu I\) and \(\Delta^a_\lambda(S)\) is a normal operator with the same spectrum as \(S\), then

\[
\|\Delta^a_\lambda(T) - \Delta^a_\lambda(S)\|_2 = |\mu - \eta_1|^2 + |\mu - \eta_2|^2.
\]

Clearly, this implies that \(\Delta^a_\lambda(\cdot)\) is continuous at \(T\).

**Case 2.** Suppose that \(\sigma(T) = \{\mu_1, \mu_2\}\) with \(\mu_1 \neq \mu_2\) and let \(\ell > 0\). Take \(\delta_1 > 0\) such that for every matrix \(S\) satisfying \(\|T - S\|_2 \leq \delta_1\), the constant \(\gamma(S, \lambda)\) of Lemma 4.5 applied to \(S\) satisfies \(\gamma(S, \lambda) \leq r\), for some \(r < 1\). Indeed, note that the formula for \(\gamma(S, \lambda)\) given in Lemma 4.5 depends continuously on \(S\) (and its spectrum). Note that the constant \(N(S, \lambda) = 4\|S\|^{1-\lambda}\|S^*S - SS^*\|^{\lambda/2}/2\) is bounded on the set \(\mathcal{M} = \{S \in \mathbb{H}_2(C) : \|T - S\|_2 \leq \delta_1\}\). Then, by formula 9, we can deduce that there exists \(n \in \mathbb{N}\), such that

\[
\|\Delta^a_\lambda(S) - \Delta^a_\lambda(S)\|_2 \leq N(S, \lambda) \sum_{k=n}^{\infty} r^{k/2} \leq \frac{\ell}{\delta_1},
\]
for every $S \in \mathcal{H}$. Finally, since the map $\Delta^n(\cdot)$ is continuous on $\mathcal{H}(\mathbb{C})$, we can take $0 < \delta_2 < \delta_1$ such that, if $\|T - S\|_2 \leq \delta_2$, then

$$\|\Delta^n(T) - \Delta^n(S)\|_2 \leq \varepsilon.$$ 

So, if $\|T - S\|_2 \leq \delta_2$, then

$$\|\Delta^n(\infty)(T) - \Delta^n(\infty)(S)\|_2 \leq \|\Delta^n(\infty)(T) - \Delta^n(\infty)(S)\|_2 + \|\Delta^n(T) - \Delta^n(S)\|_2 \leq \varepsilon,$$

which completes the proof. □

Theorem 4.9. Let $T \in \mathcal{H}(\mathbb{C})$ be fixed. Then the map $\Gamma(\cdot, T) : (0, 1) \to \mathcal{H}(\mathbb{C})$, given by

$$(0, 1) \ni \lambda \mapsto \Delta^n(\infty)(T)$$

is continuous. Moreover, if $\sigma(T) = [\mu_1, \mu_2]$ with $|\mu_1| = |\mu_2|$, then the map is constant.

Proof. The proof of the continuity is similar to the proof of the previous theorem (see also Remark 4.10). Note that the constants $\gamma(T, \lambda)$ and $N(T, \lambda)$ depend continuously on both variables, in particular on $\lambda$. Also, by Proposition 4.4, the map $\lambda \mapsto \Delta^n(\lambda)(T)$ is continuous, for every $n \in \mathbb{N}$. Let $T \in \mathcal{H}(\mathbb{C})$ such that $|\mu_1| = |\mu_2|$. As Ando and Yamazaki pointed out in [3], without loss of generality we can assume that $T = \begin{pmatrix} a & b \\ -b & d \end{pmatrix} \in \mathcal{H}(\mathbb{R})$, with $b > 0$, and $\sigma(T) = \{u + iv, u - iv\}$ with $u^2 + v^2 = 1$ and $v > 0$. Then,

$$\Gamma(\lambda, T) = \begin{pmatrix} u \\ -v \\ u \end{pmatrix}, \quad \lambda \in (0, 1).$$

Indeed, if $\Delta^n(\lambda)(T) = \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix}$, by Theorem 4.6 and some simple computations, we get

$$\Delta^n(\lambda)(T)^T \Delta^n(\lambda)(T) - \Delta^n(\lambda)(T) \Delta^n(\lambda)(T)^T = (b_n - c_n) \begin{pmatrix} -b_n + c_n \\ a_n - d_n \\ b_n + c_n \end{pmatrix},$$

so, the sequences $a_n$ and $d_n$ converge to $\text{tr}(T)/2 = u$. On the other hand, following essentially the same lines as in Ando-Yamazaki’s proof, we get $0 < m = \inf_n (b_n - c_n)^2 = \lim_{n \to \infty} (b_n - c_n)^2$. Hence, $b_n - c_n$ must converge to $m^{1/2}$ or $-m^{1/2}$. Moreover, since $b_n + c_n \to 0$ by formula 10, then $m^{1/2} = 2v$, for each $\lambda \in (0, 1)$.

Therefore

$$\Gamma(\lambda, T) = \begin{pmatrix} u \\ 0 \\ v \end{pmatrix} = \Gamma(1/2, T) \quad \text{or} \quad \Gamma(\lambda, T) = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}.$$ 

But $\Gamma$ is continuous on $\lambda$, so $\Gamma(\lambda, T) = \Gamma(1/2, T)$ for every $\lambda \in (0, 1)$. □
Remark 4.10. With similar arguments to those used in the proofs of the previous two theorems, it can be proved that the map \( \Gamma \) is jointly continuous.

Example 4.11. If \( T \in \mathbb{M}_2(\mathbb{C}) \) has eigenvalues with different moduli, then the map \( \lambda \mapsto \Delta_{\lambda}^\infty(T) \) does not seem to be constant, in general. For example, if \( T = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix} \), numerical computations show that

\[
\Delta_{0.5}^\infty(T) \approx \begin{pmatrix} 2.22738 & 0.973807 \\ 0.973807 & 1.77262 \end{pmatrix},
\]

while

\[
\Delta_{0.7}^\infty(T) \approx \begin{pmatrix} 1.37162 & -0.777907 \\ -0.777907 & 2.62838 \end{pmatrix}.
\]

Nevertheless, for many other matrices \( T \) with different modulus eigenvalues, the map \( \lambda \mapsto \Delta_{\lambda}^\infty(T) \) seems to be constant.

4.2. The Jordan structure of Aluthge transforms

In this subsection, we study some properties of the Jordan structure of the iterated Aluthge transforms. We show a reduction of the conjecture on the convergence of the sequence \( \{\Delta_n^\infty(T)\} \) for \( T \in \mathbb{M}_n(\mathbb{C}) \), to the invertible case. We also study the behavior of the angles between the spectral subspaces of iterates of the Aluthge transform for \( T \in \mathbb{M}_n(\mathbb{C}) \).

The following result states a simple relation between the null spaces of polynomials in \( T \) and in \( \Delta_\lambda(T) \). This relation has some consequences regarding multiplicity and Jordan structure of eigenvalues of \( T \) and \( \Delta_\lambda(T) \). We denote by \( \mathbb{C}[x] \) the set of complex polynomials.

Lemma 4.12. Let \( T \in \mathbb{M}_n(\mathbb{C}) \) and \( \lambda \in (0, 1) \).

1. Given \( p \in \mathbb{C}[x] \), then \( \dim N(p(T)) \leq \dim N(p(\Delta_\lambda(T))) \).
2. For \( n \in \mathbb{N} \), \( n \geq 2 \), \( \dim N(T^n) = \dim N(\Delta_\lambda(T)^n) \).

Proof. Assume first that \( p(0) \neq 0 \). In this case \( N(T) \cap N(p(T)) = \{0\} \). Hence

\[
\dim |T|^\lambda(N(p(T))) = \dim N(p(T)),
\]

because \( N(T) = N(\|T\|) = N(\|T^\lambda\|) \). Using Proposition 3.4, we know that \( p(\Delta_\lambda(T)) |T|^\lambda = |T|^\lambda p(T) \), so that

\[
|T|^\lambda(N(p(T))) \subseteq N(p(\Delta_\lambda(T))).
\]

If \( p(0) = 0 \), Note that \( N(T) \subseteq N(p(T)) \) and also \( N(T) \subseteq N(p(\Delta_\lambda(T))) \). Denote by \( \mathcal{S} = N(p(T)) \) and \( N(T) \subseteq N(p(\Delta_\lambda(T))) \). Denote by \( \mathcal{S} = N(p(T)) \cap N(T) \). Then \( \dim |T|^\lambda(\mathcal{S}) = \dim \mathcal{S} \) and \( |T|^\lambda(\mathcal{S}) \subseteq N(T)^\perp \).

On the other hand, we get that \( |T|^\lambda(\mathcal{S}) \subseteq N(p(\Delta_\lambda(T))) \) as before. Then
\[ \dim N(p(T)) = \dim N(T) + \dim \mathcal{S} \]
\[ = \dim N(T) + \dim |T|^\lambda(\mathcal{S}) \]
\[ = \dim \left[ N(T) \oplus |T|^\lambda(\mathcal{S}) \right] \leq \dim N(p(\Delta_\lambda(T))). \]

Finally, note that if \( n \geq 2 \) we have
\[ N(\Delta_\lambda(T)^{n-1} |T|^\lambda) = N(|T|^\lambda T^{n-1}) = N(T^n). \]

Let \( \mathcal{S} = N(\Delta_\lambda(T)^{n-1} \oplus N(T) \). Since \(|T|^\lambda\) operates bijectively on \( N(T)^\lambda \), there is a subspace \( \mathcal{M} \subseteq N(\mathcal{T}) \) such that \( \dim \mathcal{M} = \dim \mathcal{S} \) and \( |T|^\lambda(\mathcal{M}) = \mathcal{S} \). Hence
\[ N(\Delta_\lambda(T)^{n-1} |T|^\lambda) = \{ x \in \mathbb{C}^n : |T|^\lambda(x) \in N(\Delta_\lambda(T)^{n-1}) \} = N(T) \oplus \mathcal{M}. \]

So that \( \dim N(\Delta_\lambda(T)^{n-1}) = \dim N(\Delta_\lambda(T)^{n-1} |T|^\lambda) = \dim N(T^n). \)

**Definition 4.13.** Let \( T \in \mathcal{M}_n(\mathbb{C}) \) and \( \mu \in \sigma(T) \). We denote
1. \( m(T, \mu) \) the algebraic multiplicity of the eigenvalue \( \mu \) for \( T \).
2. \( m_0(T, \mu) = \dim N(T - \mu I) \), the geometric multiplicity of the eigenvalue \( \mu \) for \( T \).
3. \( r(T, \mu) = \min\{ k \in \mathbb{N} : \dim N(T - \mu I)^k = m(T, \mu) \} \), usually called the index of \( \mu \). Note that \( r(T, \mu) \) is the size of the biggest Jordan block of \( T \) associated to \( \mu \).

We say that the Jordan structure of \( T \) for the eigenvalue \( \mu \) is trivial if \( m(T, \mu) = m_0(T, \mu) \), or equivalently, if \( r(T, \mu) = 1 \).

**Proposition 4.14.** Let \( T \in \mathcal{M}_n(\mathbb{C}) \) and \( \lambda \in (0, 1) \).

1. Suppose that \( 0 \in \sigma(T) \). Then
\[ m(T, 0) = m_0(\Delta_\lambda^{r(T, 0)-1} (T), 0) = \dim N(\Delta_\lambda^{r(T, 0)-1} (T)). \]

Therefore, after \( r(T, 0) - 1 \) iterations of the Aluthge transform, we get a matrix whose Jordan structure for the eigenvalue 0 is trivial.

2. If \( \mu \in \sigma(T)/\{0\} \), then
\[ m_0(T, \mu) \leq m_0(\Delta_\lambda(T), \mu) \] and \( r(T, \mu) \geq r(\Delta_\lambda(T), \mu) \).

**Proof**
1. Denote \( r(T, 0) = r \). If \( r \geq 2 \), by Lemma 4.12,
\[ m(T, 0) = \dim N(T^r) = \dim N(\Delta_\lambda(T)^{r-1}) = \dim N(\Delta_\lambda^2(T)^{r-2}) = \cdots = \dim N(\Delta_\lambda^{r-2}(T)^2) = \dim N(\Delta_\lambda^{r-1}(T)). \]
If \( r = 1 \), then \( \Delta_{\lambda}^{r-1} (T) = \Delta_{\lambda}^0 (T) = T \) by definition, and
\[
m(T, 0) = m_0 (T, 0) = \dim (\Delta_{\lambda}^{r-1} (T)).
\]

2. Consider \( P_m (x) = (x - \mu)^m, m \in \mathbb{N} \). Taking \( m = 1 \), by Lemma 4.12,
\[
m_0 (T, \mu) = \dim N (T - \mu I) \leq \dim N (\Delta_{\lambda} (T) - \mu I) = m_0 (\Delta_{\lambda} (T), \mu).
\]
Taking \( m = r (T, \mu) \), again by Lemma 4.12, we have that
\[
m(T, \mu) = \dim ((T - \mu I)^r (T, \mu)) \leq \dim ((\Delta_{\lambda} (T) - \mu I)^r (T, \mu)) = m (\Delta_{\lambda} (T), \mu).
\]
Since \( m (\Delta_{\lambda} (T), \mu) = m (T, \mu) \), we get that \( r (T, \mu) \geq r (\Delta_{\lambda} (T), \mu) \). \( \square \)

**Remark 4.15.** In particular, Proposition 4.14 shows that if \( T \) is nilpotent of order \( n \) then \( \Delta_{\lambda}^{n-1} (T) = 0 \). This result was proved by Jung, Ko and Pearcy in [16].

**Corollary 4.16.** Let \( \lambda \in (0, 1) \). If the sequence \( \{ \Delta_{\lambda}^m (S) \} \) converges for every invertible matrix \( S \in \mathcal{M}_n (\mathbb{C}) \) and every \( n \in \mathbb{N} \), then the sequence \( \{ \Delta_{\lambda}^m (T) \} \) converges for all \( T \in \mathcal{M}_n (\mathbb{C}) \) and every \( n \in \mathbb{N} \).

**Proof.** Let \( T \in \mathcal{M}_n (\mathbb{C}) \). By Lemma 4.14, we can assume that \( m (T, 0) = m_0 (T, 0) \).
Note that, in this case, \( N (\Delta_{\lambda} (T)) = N (T) \), because \( N (T) \leq N (\Delta_{\lambda} (T)) \) and \( m_0 (\Delta_{\lambda} (T), 0) = m (T, 0) \). On the other hand, \( R (\Delta_{\lambda} (T)) \leq R (|T|) \) so that \( R (\Delta_{\lambda} (T)) \) and \( N (\Delta_{\lambda} (T)) \) are orthogonal subspaces. Thus, there exists a unitary matrix \( U \) such that
\[
U \Delta_{\lambda} (T) U^* = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},
\]
where \( S \in \mathcal{M}_k (\mathbb{C}) \) is invertible \( (s = n - m (T, 0)) \). Since for every \( m \geq 2 \)
\[
\Delta_{\lambda}^m (T) = U^* \begin{pmatrix} \Delta_{\lambda}^{m-1} (S) & 0 \\ 0 & 0 \end{pmatrix} U,
\]
the sequence \( \{ \Delta_{\lambda}^m (T) \} \) converges, because the sequence \( \{ \Delta_{\lambda}^{m-1} (S) \} \) converges by hypothesis. \( \square \)

**Remark 4.17.** If \( T \in \mathcal{M}_n (\mathbb{C}) \) is invertible, then \( |T|^k \) is invertible for every \( \lambda \in (0, 1) \), and
\[
\Delta_{\lambda} (T) = |T|^k T |T|^{-\lambda}.
\]
Therefore, \( T \) and \( \Delta_{\lambda}^m (T) \) are similar matrices, for every \( m \in \mathbb{N} \). That is, \( \Delta_{\lambda}^m (T) \) and \( T \) have the same Jordan structure. This shows that the geometric multiplicity of non-zero eigenvalues does not increase in general. On the other hand, Proposition 4.14 implies that for non-invertible operators \( T, \Delta_{\lambda} (T) \) and \( T \) may be not similar. In particular, the Jordan structure of \( T \) and \( \Delta_{\lambda} (T) \) may be different.
Numerical experiences show that the rate of convergence of the sequence \( \{ \Delta^m_\lambda (T) \} \) is smaller for non-diagonizable \( T \), than for diagonizable examples.

**Definition 4.18.** Let \( T \in \mathcal{B}(\mathbb{C}) \) and \( \mu \in \sigma (T) \).

1. Denote \( \mathcal{H}_{\mu, T} = N((T - \mu I)^2(T, \mu)) \). Note that \( \mathbb{C} = \bigoplus_{\mu \in \sigma (T)} \mathcal{H}_{\mu, T} \).
2. Denote \( Q_{\mu, T} \in \mathcal{B}(\mathbb{C}) \) the oblique projection with \( R(Q_{\mu, T}) = \mathcal{H}_{\mu, T} \) and \( N(Q_{\mu, T}) = \bigoplus_{\mu \neq \mu} \mathcal{H}_{\mu, T} \).

**Proposition 4.19.** Let \( T \in \mathcal{B}(\mathbb{C}) \) and \( \lambda \in (0, 1) \). Then, for every \( \mu \in \sigma (T) \),

\[
\| Q_{\mu, \Delta^m_\lambda (T)} \| \geq \| Q_{\mu, \Delta^{m+1}_\lambda (T)} \|, \quad m \in \mathbb{N}, \mu \in \sigma (T).
\]

On the other hand, there exists a subsequence \( \Delta^{m_k}_\lambda (T) \rightarrow L \) for some normal matrix \( L \in \mathcal{B}(\mathbb{C}) \), with \( \sigma (L) = \sigma (T) \). Then, by Proposition 3.14,

\[
\| Q_{\mu, \Delta^{m_k}_\lambda (T)} \| = \| f_\mu (\Delta^{m_k}_\lambda (T)) \| \rightarrow \| f_\mu (L) \| = \| Q_{\mu, L} \| = 1.
\]

because the spectral projections of normal operators are selfadjoint (i.e., orthogonal).

**Remark 4.20.** Given two subspaces \( \mathcal{M} \) and \( \mathcal{N} \) of \( \mathbb{C}^n \), such that \( \mathcal{M} \cap \mathcal{N} = \{0\} \), the angle between \( \mathcal{M} \) and \( \mathcal{N} \) is the angle in \([0, \pi/2]\) whose cosine is defined by

\[
c [\mathcal{M}, \mathcal{N}] = \sup \left\{ |(x, y)| : x \in \mathcal{M}, y \in \mathcal{N} \text{ and } \|x\| = \|y\| = 1 \right\} 
\]

\[
= \| P_{\mathcal{M}} \|, \quad (12)
\]

where \( P_{\mathcal{M}} \) denotes the orthogonal projection onto \( \mathcal{M} \). The sine of this angle is

\[
s [\mathcal{M}, \mathcal{N}] = (1 - c [\mathcal{M}, \mathcal{N}])^{1/2}.
\]

If \( \mathcal{M} \oplus \mathcal{N} = \mathbb{C}^n \) and \( Q \) is the oblique projection with range \( \mathcal{M} \) and null space \( \mathcal{N} \), it is known that

\[
\| Q \| = \left( 1 - \| P_{\mathcal{M}} \| \right)^{-1/2} = \left( 1 - c [\mathcal{M}, \mathcal{N}] \right)^{-1/2}
\]

\[
= s [\mathcal{M}, \mathcal{N}]^{-1}.
\]

For proofs of these results, the reader is referred to Golberg and Krein [13], Deutsch [11], or Ben-Israel and Greville [5].
Now we can see that Proposition 4.19 is equivalent to the following statement: given \( \mu \in \sigma(T) \), the angle between the spectral subspaces \( \mathcal{H}_{\mu, \Delta^p_{\mu}(T)} \) and \( \mathcal{N}_{\mu} = \bigoplus_{\gamma \neq \mu} \mathcal{H}_{\gamma, \Delta^p_{\gamma}(T)} \) converges to \( \pi/2 \). Given \( \mu \neq \gamma \in \sigma(T) \), since \( \mathcal{H}_{\gamma, \Delta^p_{\gamma}(T)} \subseteq \mathcal{N}_{\mu} \), it is easy to see that

\[
\cos \left( \mathcal{H}_{\mu, \Delta^p_{\mu}(T)} : \mathcal{H}_{\gamma, \Delta^p_{\gamma}(T)} \right) \leq \cos \left( \mathcal{H}_{\mu, \Delta^p_{\mu}(T)} : \mathcal{N}_{\mu} \right) \xrightarrow{n \to \infty} 0.
\]

Therefore, also the angle between \( \mathcal{H}_{\mu, \Delta^p_{\mu}(T)} \) and \( \mathcal{H}_{\gamma, \Delta^p_{\gamma}(T)} \) converges to \( \pi/2 \).

Another description of this fact is that

\[
P_{\mathcal{H}_{\mu, \Delta^p_{\mu}(T)}} P_{\mathcal{H}_{\gamma, \Delta^p_{\gamma}(T)}} \xrightarrow{n \to \infty} 0.
\]

This also follows from Eq. (12).

Acknowledgement

We wish to acknowledge Prof. G. Corach who told us about the Aluthge transform, and shared with us fruitful discussions concerning these matters.

References