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λ-Aluthge transforms and Schatten ideals

Jorge Antezana *, Pedro Massey, Demetrio Stojanoff [1]

Depto. de Matemática, FCE-UNLP and IAM-CONICET, La Plata, Argentina

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Dedicated to the memory of Jorge Samur

Abstract

Let $T \in L(\mathcal{H})$, and let $T = U|T| = |T^*|U$ be the polar decomposition of T. Then, for every $\lambda \in [0, 1]$ the λ-Aluthge transform is defined by $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$. We show that several properties which are known for the usual Aluthge transform (i.e. the case $\lambda = 1/2$) also hold for λ-Aluthge transforms with $\lambda \in (0, 1)$. Moreover, we get several results which are new, even for the usual Aluthge transform.

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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and let $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Given $T \in L(\mathcal{H})$, consider its (left) polar decomposition $T = U|T|$. In order to study the relationship among p-hyponormal operators, Aluthge introduced in [1] the transformation $\Delta_{1/2}(\cdot) : L(\mathcal{H}) \to L(\mathcal{H})$ defined by

* Corresponding author.
E-mail addresses: antezana@mate.unlp.edu.ar (J. Antezana), massey@mate.unlp.edu.ar (P. Massey), demetrio@mate.unlp.edu.ar (D. Stojanoff).
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$$\Delta_{1/2}(T) = |T|^{1/2}U|T|^{1/2}.$$

Later on, this transformation, now called Aluthge transform, was also studied in other contexts by several authors, such as Jung, Ko and Pearcy [15] and [16], Foias, Jung, Ko and Pearcy [12], Ando [2], Ando and Yamazaki [3], Yamazaki [23], Okubo [17], Wang [21] and Wu [22] among others.

In this paper, given $\lambda \in [0, 1]$ and $T \in L(\mathscr{H})$, we study the so-called λ-Aluthge transform of T defined by

$$\Delta_\lambda(T) = |T|^\lambda U |T|^{1-\lambda}.$$

This notion has already been considered by Okubo in [17]. For $\lambda = 0$, $|T|^\lambda$ will be considered as the orthogonal projection onto the closure of $R(|T|)$. For $\lambda = 1$, $\Delta_\lambda(T) = |T|U$, which is known as Duggal's transform of T [12], or *hinge* of T [19].

The main tool we use to study the λ-Aluthge transforms is Young's inequality (see, [4,14] or Section 2). Some results of this paper are devoted to the generalization of well known properties of Aluthge transform to λ-Aluthge transforms. For $\lambda \in (0, 1)$, we prove that the map $T \mapsto \Delta_\lambda(T)$ is continuous at every closed range operator T (see [15] for the case $\lambda = 1/2$). For every analytic function f defined in an open neighborhood of $\sigma(T)$, we show that

$$\|f(\Delta_\lambda(T))\| \leqslant \|f(\Delta_1(T))\|^\lambda \, \|f(\Delta_0(T))\|^{1-\lambda} \leqslant \|f(T)\|,$$

(see [12,17]). When, $\dim \mathscr{H} = n < \infty$, we prove that the limit points of the sequence $\{\Delta_\lambda^m(T)\}$ are normal matrices, from which we deduce Yamazaki's spectral radius formula $\rho(T) = \lim_{n\to\infty} \|\Delta_\lambda^m(T)\|$ (only in the finite dimensional case), where $\rho(T)$ denotes the spectral radius of T.

On the other hand, we show several results which are new even for the usual Aluthge transform. Given $1 \leqslant p < \infty$, we prove that the Schatten p-norms of the λ-Aluthge transforms decrease with respect to the Schatten p-norms of the original operator. Moreover, if $\|\Delta_\lambda(T)\|_p = \|T\|_p < \infty$ (for any fixed $1 \leqslant p < \infty$), then T must be normal. This was proved for $\lambda = 1/2$ and $p = 2$ in [12]. In this case, we show the following estimation: if T is a Hilbert Schmidt operator, $\lambda \in (0, 1)$, and $\alpha = \min\{\lambda, 1-\lambda\}$, then

$$\alpha^2 \| |T| - |T^*| \|_2^2 \leqslant \|T\|_2^2 - \|\Delta_\lambda(T)\|_2^2.$$

When $\dim \mathscr{H} = 2$, Ando and Yamazaki proved that the sequence of iterated Aluthge transforms $\{\Delta_{1/2}^m(T)\}$ converges (see [3]). Motivated by their ideas, we show that the sequence $\{\Delta_\lambda^m(T)\}$ converges for every $\lambda \in (0, 1)$ and every 2×2 matrix T. Moreover, if $\Delta_\lambda^\infty(T) = \lim_{m\to\infty} \Delta_\lambda^m(T)$, we prove that the map $T \mapsto \Delta_\lambda^\infty(T)$ is jointly continuous in both parameters, $\lambda \in (0, 1)$ and $T \in \mathscr{M}_2(\mathbb{C})$.

Finally, we study some properties of the Jordan structure of the iterated Aluthge transforms. Given $T \in \mathscr{M}_n(\mathbb{C})$ and $\mu \in \sigma(T)$, let $\mathscr{H}_{\mu,T}$ denote the spectral subspace of T associated to the eigenvalue μ (see Definition 4.18 for a precise definition). We prove that given two different eigenvalues of T, γ and μ, the angle between $\mathscr{H}_{\mu,\Delta_\lambda^m(T)}$ and $\mathscr{H}_{\gamma,\Delta_\lambda^m(T)}$ converges to $\pi/2$, for every $\lambda \in (0, 1)$. In other words

$$P_{\mathscr{H}_{\mu,\Delta_\lambda^m(T)}} P_{\mathscr{H}_{\gamma,\Delta_\lambda^m(T)}} \xrightarrow[m\to\infty]{} 0,$$

where, for any subspace $\mathscr{S} \subseteq \mathscr{H}$, $P_{\mathscr{S}}$ denotes the orthogonal projection onto $\mathscr{S}$. Concerning the conjecture of the convergence of the sequence $\{\Delta_\lambda^m(T)\}$ for $T \in \mathscr{M}_n(\mathbb{C})$, we show a reduction to the invertible case.

The paper is organized as follows: Section 2 contains preliminary results on Riesz's functional calculus, Schatten ideals, and a list of known inequalities which we use in the paper. Section 3 deals with the properties of λ-Aluthge transform in the infinite dimensional setting. In Section 4 we study the finite dimensional case.

2. Preliminaries

In this paper $\mathscr{H}$ denotes a complex Hilbert space, $L(\mathscr{H})$ the algebra of bounded linear operators on $\mathscr{H}$, $GL(\mathscr{H})$ the group of all invertible elements of $L(\mathscr{H})$, $\mathscr{U}(\mathscr{H})$ the group of unitary operators, $L(\mathscr{H})^+$ the cone of all positive operators and $L_0(\mathscr{H})$ the ideal of compact operators. When $\dim \mathscr{H} = n < \infty$ the elements of $L(\mathscr{H})$ are identified with $n \times n$ matrices, and we write $\mathscr{M}_n(\mathbb{C})$ instead of $L(\mathscr{H})$. Given $T \in L(\mathscr{H})$, $R(T)$ denotes the range or image of T, $N(T)$ the null space of T, $\sigma(T)$ the spectrum of T, $\rho(T)$ the spectral radius of T, T^* the adjoint of T, and $\|T\|$ the usual norm of T (also called spectral norm, we sometimes write $\|T\|_{sp}$); a norm $|||\cdot|||$ in $\mathscr{M}_n(\mathbb{C})$ (or defined in some adequate ideal of compact operators) is called unitarily invariant if $|||UTV||| = |||T|||$ for unitary U, V. If $R(T)$ is closed, $T^\dagger$ denotes the Moore–Penrose pseudoinverse of T. Given a closed subspace $\mathscr{S} \subseteq \mathscr{H}$, $P_{\mathscr{S}} \in L(\mathscr{H})$ denotes the orthogonal projection onto $\mathscr{S}$.

Given $T \in L(\mathscr{H})$, $\operatorname{Hol}(\sigma(T))$ denotes the set of all complex analytic functions defined in an open neighborhood of $\sigma(T)$. In this set, we identify two functions if they agree in an open neighborhood of $\sigma(T)$. If $T \in L(\mathscr{H})$ and $f \in \operatorname{Hol}(\sigma(T))$, $f(T)$ indicates the evaluation of f at T, by using the Riesz functional calculus. The reader is referred to Brown and Pearcy's book [8] (see also [9]) for general properties of this calculus, and a proof of the following statement.

Proposition 2.1. *Given $T_0 \in L(\mathscr{H})$ such that $\sigma(T_0)$ is contained in an open set $U \subseteq \mathbb{C}$, let $\{f_n\}$ be a sequence of locally analytic functions on U converging to a limit f_0 uniformly on compact subsets of U, and likewise let $\{T_n\}$ be a sequence in $L(\mathscr{H})$, converging to T_0 (in norm). Then, $f_n(T_n)$ is defined for all sufficiently large n and $f_n(T_n) \xrightarrow[n\to\infty]{\|\cdot\|} f_0(T_0)$.*

Given $A \in L_0(\mathscr{H})$, $s_k(A)$, $k \in \mathbb{N}$ denote the singular values of A, arranged in non-increasing order. If we denote by *tr* the canonical semifinite trace in $L(\mathscr{H})$ then the Schatten p-ideals ($1 \leqslant p < \infty$) are defined in the following way:

$$L^p(\mathscr{H}) = \left\{T \in L_0(\mathscr{H}) : \operatorname{tr}(|T|^p) < \infty\right\}.$$

Each $L^p(\mathscr{H})$, endowed with the norm

$$\|T\|_p = (\operatorname{tr}(|T|^p))^{1/p} = \left(\sum_{k\in\mathbb{N}} s_k(T)^p\right)^{1/p}.$$

is a Banach space. If $p > 1$, then $L^p(\mathscr{H})^* \cong L^q(\mathscr{H})$, where $1/p + 1/q = 1$. In this rest of this section, we list some inequalities which will be useful in the sequel. We begin with the following two versions of Young's inequality.

Proposition 2.2 (Argerami–Farenick [4]). *Let $A \in L^p(\mathscr{H})$ and $B \in L^q(\mathscr{H})$ be positive operators and $1/p + 1/q = 1$. Then, $AB \in L^1(\mathscr{H})$ and*

$$\operatorname{tr}(|AB|) \leqslant \frac{\operatorname{tr}(A^p)}{p} + \frac{\operatorname{tr}(B^q)}{q}.$$

Moreover, equality holds if and only if $A^p = B^q$.

Proposition 2.3 (Hirzallah–Kittaneh [14]). *Let $A, B \in L(\mathscr{H})^+$, and let $p, q > 1$ with $1/p + 1/q = 1$. Suppose that $A^p, B^q \in L^2(\mathscr{H})$. Then $AB \in L^2(\mathscr{H})$, and*

$$\|AB\|_2^2 + \frac{1}{r^2}\,\|A^p - B^q\|_2^2 \leqslant \left\|\frac{A^p}{p} + \frac{B^q}{q}\right\|_2^2,$$

where $r = \max\{p, q\}$.

Now, we state a version of the well known Corde's inequality [10], for unitarily invariant norms. In the proof we use standard techniques and properties of the *k*th antisymmetric tensor powers $\bigwedge^k A$, $A \in L(\mathscr{H})$ and majorization, which can be found in B. Simon's book [20] or Bhatia's book [6].

Proposition 2.4. *Let A and B be positive compact operators. If $p \geqslant 1$, then*

$$\sum_{i=1}^{k} s_i\left(|AB|^p\right) \leqslant \sum_{i=1}^{k} s_i\left(A^p B^p\right), \quad k \in \mathbb{N}. \tag{1}$$

Proof. Fix $k \in \mathbb{N}$. Since $\|\bigwedge^k A\| = \prod_{i=1}^k s_i(A)$, Cordes' inequality

$$\|CD\|^p \leqslant \|C^p D^p\|, \quad C, D \in L(\mathscr{H})^+,$$

implies that

$$\begin{aligned}\|\textstyle\bigwedge^k A^p B^p\| &= \|(\textstyle\bigwedge^k A)^p(\textstyle\bigwedge^k B)^p\| \geqslant \|\textstyle\bigwedge^k A \textstyle\bigwedge^k B\|^p\\ &= \|\textstyle\bigwedge^k AB\|^p = \|\textstyle\bigwedge^k |AB|^p\|.\end{aligned}$$

Then, $\prod_{i=1}^k s_i(|AB|^p) \leqslant \prod_{i=1}^k s_i(A^p B^p)$, $k \in \mathbb{N}$, which implies inequality (1). □

Finally, we include the next inequality, proved by Bhatia and Kittaneh [7]:

Proposition 2.5. *Let* $A, B \in \mathcal{M}_n(\mathbb{C})^+$, *and* $r \in [0, 1]$. *Then*

$$|||A^r - B^r||| \leqslant |||I|||^{1-r}|||A - B|||^r$$

for every unitarily invariant norm $|||\cdot|||$.

3. λ-Aluthge transforms

Definition 3.1. Let $T \in L(\mathcal{H})$, and suppose that $T = U|T| = |T^*|U$ is the polar decomposition of T. Then, for every $\lambda \in [0, 1]$ we define the λ-Aluthge transform of T in the following way:

$$\Delta_\lambda(T) = |T|^\lambda U |T|^{1-\lambda}.$$

When $\lambda = 0$, $|T|^\lambda$ will be considered as the orthogonal projection onto $\overline{R(|T|)}$.

Remark 3.2. Let $T \in L(\mathcal{H})$ and let $T = W|T|$ be an arbitrary polar decomposition of T. It was shown in [17] that $\Delta_\lambda(T) = |T|^\lambda W|T|^{1-\lambda}$ for every $\lambda \in [0, 1)$ i.e., the λ-Aluthge transform does not depend on the partial isometry for $\lambda \in [0, 1)$. We shall use this fact repeatedly in the sequel. On the other hand, for $\lambda = 1$, it is necessary to fix the unique partial isometry U such that $T = U|T|$ and $N(U) = N(T)$. For example, if $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $U = T$ and $|T| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, but the unitary matrix $W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ also satisfies $T = W|T|$, while $\Delta_1(T) = |T|U = 0 \neq |T|W = T^*$.

In the next proposition, we describe some properties which follow easily from the definitions.

Proposition 3.3. *Let* $T \in L(\mathcal{H})$ *and* $\lambda \in [0, 1]$. *Then*:

1. $\Delta_\lambda(VTV^*) = V\Delta_\lambda(T)V^*$ *for every* $V \in \mathcal{U}(\mathcal{H})$.
2. $\|\Delta_\lambda(T)\| \leqslant \|T\|$.
3. $\sigma(\Delta_\lambda(T)) = \sigma(T)$.
4. *If* $\dim \mathcal{H} < \infty$, *then* T *and* $\Delta_\lambda(T)$ *have the same characteristic polynomial.*

Proposition 3.4. *Let* $T \in L(\mathcal{H})$, $\lambda \in [0, 1]$ *and let* f *be a function, which is locally analytic in a neighborhood of* $\sigma(T)$. *If* $T = U|T|$ *is the polar decomposition of* T *then*,

1. $f(T)U = Uf(\Delta_1(T))$.
2. $|T|^\lambda f(T) = f(\Delta_\lambda(T))|T|^\lambda$.

Proof. A simple induction argument proves the statement for $f(t) = t^n$. This can be extended to every polynomial by linearity. This can be applied to show the statement for rational functions (with poles outside $\sigma(T)$). Finally, using Runge's theorem (see, for example, Conway's book [9]), the result generalizes to analytic functions. □

In [15], Jung, Ko and Pearcy proved that the Aluthge transformation is continuous at every closed range operator, with respect to the norm topology, for $\lambda = 1/2$. In order to generalize this property for $\lambda \in (0, 1)$, we need the following result. Recall that, if $B \in L(\mathscr{H})$ has closed range, there exists a unique pseudo-inverse $B^\dagger$ of B such that $BB^\dagger$ and $B^\dagger B$ are selfadjoint projections. $B^\dagger$ is called the Moore–Penrose pseudo-inverse of B (see, for example, [5]).

Lemma 3.5. *Let $B \in L(\mathscr{H})$, selfadjoint with closed range, and let $\{B_n\}$ be a sequence of closed range selfadjoint operators such that $B_n \xrightarrow[n\to\infty]{} B$ in norm. If $P_{R(B_n)} \xrightarrow[n\to\infty]{} P_{R(B)}$ in norm, then also $B_n^\dagger \xrightarrow[n\to\infty]{} B^\dagger$ in norm.*

Proof. Denote by $P_n = P_{R(B_n)}$ and $P = P_{R(B)}$. If $P_n \xrightarrow[n\to\infty]{} P$ then there exists a sequence $\{U_n\}$ of unitary operators such that $U_n \xrightarrow[n\to\infty]{} 1$ and $U_n^* P U_n = P_n$, $n \in \mathbb{N}$. Indeed, we can take U_n as the unitary part in the polar decomposition of $PP_n + (1 - P)(1 - P_n)$, which is invertible for large n. Note that, if $S_n = U_n B_n U_n^*$, then $S_n \xrightarrow[n\to\infty]{} B$ in norm, $R(S_n) = R(B)$ and $S_n^\dagger = U_n B_n^\dagger U_n^*$, $n \in \mathbb{N}$. Hence, it suffices to prove that $S_n^\dagger \xrightarrow[n\to\infty]{} B^\dagger$. But this is clear by continuity of the map $A \mapsto A^{-1}$ (on the fixed subspace $R(B) = R(S_n)$, $n \in \mathbb{N}$). □

Theorem 3.6. *Let T be an operator with closed range. Then, for every $\lambda \in (0, 1)$, the λ-Aluthge transform $\Delta_\lambda(\cdot)$ is continuous at T.*

Proof. Let $\{T_n\}$ be a sequence of operators such that $\|T_n - T\| \to 0$. For each $n \in \mathbb{N}$, let $T_n = U_n|T_n|$ be a polar decomposition of T_n. On the other hand, take $\varepsilon > 0$ such that $\sigma(|T|) \subseteq \{0\} \cup (2\varepsilon, +\infty)$ and suppose, without loss of generality, that $\sigma(|T_n|) \subseteq (-\varepsilon, \varepsilon) \cup (2\varepsilon, +\infty)$ for all n. Define, for $n \in \mathbb{N}$,

$$P_n = |T_n| E_{|T_n|}(-\varepsilon, \varepsilon) \quad \text{and} \quad A_n = U_n P_n, \tag{2}$$

$$Q_n = |T_n| E_{|T_n|}(2\varepsilon, +\infty) \quad \text{and} \quad B_n = U_n Q_n, \tag{3}$$

where $E_{|T_n|}(I)$ denotes the spectral projection of $|T_n|$ corresponding to the interval $I \subseteq \mathbb{R}$. Note that $A_n + B_n = T_n$, and (2) and (3) are polar decompositions of A_n and B_n, respectively. Therefore

$$\begin{aligned} \|\Delta_\lambda(T) - \Delta_\lambda(T_n)\| \leqslant {} & \|\Delta_\lambda(A_n)\| + \|P_n^\lambda U_n Q_n^{1-\lambda}\| \\ & + \|Q_n^\lambda U_n P_n^{1-\lambda}\| + \|\Delta_\lambda(T) - \Delta_\lambda(B_n)\|. \end{aligned}$$

By Proposition 2.1, $P_n = |T_n|E_{|T_n|}(-\varepsilon, \varepsilon) \xrightarrow[n\to\infty]{\|\cdot\|} |T|E_{|T|}(-\varepsilon, \varepsilon) = 0$. Then

$$\|\Delta_\lambda(A_n)\| + \left\|P_n^\lambda U_n Q_n^{1-\lambda}\right\| + \left\|Q_n^\lambda U_n P_n^{1-\lambda}\right\| \xrightarrow[n\to\infty]{} 0.$$

On the other hand, $|B_n| = Q_n$ which have closed ranges. Since the maps $\chi_{(-\varepsilon,\varepsilon)}$ and $\chi_{(2\varepsilon,+\infty)}$ admit complex analytic extensions to the set $\{z \in \mathbb{C} : \mathrm{Re}(z) \in (-\varepsilon, \varepsilon) \cup (2\varepsilon, +\infty)\}$, we can apply Proposition 2.1, and obtain that

$$P_{R(Q_n)} = E_{|T_n|}(2\varepsilon, +\infty) \xrightarrow[n\to\infty]{\|\cdot\|} E_{|T|}(2\varepsilon, +\infty) = P_{R(|T|)}.$$

Hence, $|B_n| \xrightarrow[n\to\infty]{} |T|$ and $P_{R(|B_n|)} \xrightarrow[n\to\infty]{} P_{R(|T|)}$, both in the norm topology. By Lemma 3.5, we conclude that $|B_n|^\dagger \xrightarrow[n\to\infty]{} |T|^\dagger$ in norm. Therefore

$$\|\Delta_\lambda(T) - \Delta_\lambda(B_n)\| = \left\| |T|^\lambda T(|T|^\dagger)^\lambda - |B_n|^\lambda B_n(B_n^\dagger)^\lambda \right\| \xrightarrow[n\to\infty]{} 0,$$

which completes the proof. □

Remark 3.7. Theorem 3.6 fails for $\lambda = 0$ and $\lambda = 1$, even in the finite dimensional case. Indeed, take $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T_n = \begin{pmatrix} 0 & 1 \\ 1/n & 0 \end{pmatrix}$, $n \in \mathbb{N}$. It is easy to check that $\Delta_0(T_n) = T_n$ and $\Delta_1(T_n) = T_n^*$, which do not converge to $0 = \Delta_0(T) = \Delta_1(T)$. Compare with Remark 3.2.

3.1. Schatten norms and ideals

In this subsection we characterize those operators in $L^p(\mathcal{H})$ which satisfy $\|\Delta_\lambda(T)\|_p = \|T\|_p$. Naturally, the equality holds if T is normal, because $T = \Delta_\lambda(T)$. It was proved in [16] that, for the Frobenius norm and for $\lambda = 1/2$, the equality holds if and only if T is normal. In the following proposition we estimate from below the difference between the Frobenius norms of T and $\Delta_\lambda(T)$.

Proposition 3.8. *Let $T \in L^2(\mathcal{H})$ and $\lambda \in (0, 1)$. If $\alpha = \min\{\lambda, 1-\lambda\}$, then*

$$\alpha^2\||T| - |T^*|\|_2^2 \leqslant \|T\|_2^2 - \|\Delta_\lambda(T)\|_2^2. \tag{4}$$

Proof. Note that, if $T = U|T|$ is the polar decomposition of T, then $|T^*|^r = U|T|^r U^*$, for every $r > 0$. Then

$$\begin{aligned}\|\Delta_\lambda(T)\|_2^2 &= \mathrm{tr}\big(\Delta_\lambda(T)\Delta_\lambda(T)^*\big) = \mathrm{tr}\big(|T|^\lambda U|T|^{2(1-\lambda)}U^*|T|^\lambda\big)\\ &= \mathrm{tr}\big(|T|^\lambda|T^*|^{2(1-\lambda)}|T|^\lambda\big) = \||T|^\lambda|T^*|^{(1-\lambda)}\|_2^2.\end{aligned}$$

Using Hirzallah–Kittaneh's inequality (Proposition 2.3) with $A = |T|^\lambda$, $B = |T^*|^{1-\lambda}$, $p = \lambda^{-1}$, $q = (1-\lambda)^{-1}$ and $\alpha = \min\{\lambda, 1-\lambda\} = \max\{\lambda^{-1}, (1-\lambda)^{-1}\}^{-1}$, we get

$$\|\Delta_\lambda(T)\|_2^2 + \alpha^2 \||T| - |T^*|\|_2^2 \leqslant \|\lambda|T| + (1-\lambda)|T^*|\|_2^2 \leqslant \|T\|_2^2,$$

where the last inequality follows from the triangle inequality. □

Now, we prove that equality in other Schatten norms also implies that T is normal.

Theorem 3.9. *Let $\lambda \in (0,1)$, $1 \leqslant p < \infty$ and $T \in L^p(\mathscr{H})$. Then, $\Delta_\lambda(T) \in L^p(\mathscr{H})$ and*

$$\|\Delta_\lambda(T)\|_p \leqslant \|T\|_p.$$

Moreover, the equality holds if and only if T is normal.

In order to prove this result, we need the following lemma.

Lemma 3.10. *Let $A, B \in L(\mathscr{H})$ and let $B = U|B|$ be the polar decomposition of B. Then, for every $p > 0$,*

$$|AB^*|^p = U\,\big||A|\,|B|\big|^p U^*.$$

Proof. Let $P = \big||A||B|\big|^2$. Then, for every continuous function f defined on $[0,+\infty)$ such that $f(0) = 0$,

$$f(UPU^*) = Uf(P)U^*. \tag{5}$$

In fact, since $R(P) \subseteq R(|B|)$, and U^*U is the orthogonal projection onto $\overline{R(|B|)}$, then $(UPU^*)^n = UP^nU^*$, for every $n \geqslant 1$. Therefore, by linearity, formula (5) holds for every polynomial f such that $f(0) = 0$. On the other hand, given a continuous function f defined in $[0,+\infty)$ such that $f(0) = 0$, there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ of polynomials such that $p_n(0) = 0$, $n \in \mathbb{N}$, and $p_n \xrightarrow[n\to\infty]{} f$ uniformly on $\sigma(P) \cup \{0\} = \sigma(UPU^*) \cup \{0\}$. So, standard limit arguments prove formula (5). Now, the result follows from the equality

$$|AB^*|^2 = BA^*AB^* = U|B||A|^2|B|U^* = U\big||A||B|\big|^2 U^*,$$

by applying the function $f(x) = x^{p/2}$ to both sides. □

Proof of Theorem 3.9. Let $T = U|T|$ be the polar decomposition of T. Fix $1 \leqslant p < \infty$. Then, using Lemma 3.10 with $A = |T|^\lambda$ and $B^* = U|T|^{1-\lambda}$, we get

$$\operatorname{tr}|\Delta_\lambda(T)|^p = \operatorname{tr}\big||T|^\lambda\,|T^*|^{1-\lambda}\big|^p.$$

Using Proposition 2.4 with $A = |T|^\lambda$ and $B = |T^*|^{1-\lambda}$, we get

$$\operatorname{tr}\big||T|^\lambda\,|T^*|^{1-\lambda}\big|^p \leqslant \operatorname{tr}\big|\,|T|^{p\lambda}\,|T^*|^{p(1-\lambda)}\big|.$$

Then, by Proposition 2.2, for the conjugate numbers λ^{-1} and $(1-\lambda)^{-1}$,

$$\begin{aligned}\operatorname{tr}|\Delta_\lambda(T)|^p &\leqslant \operatorname{tr}\big|\,|T|^{p\lambda}\,|T^*|^{p(1-\lambda)}\big| \\ &\leqslant \lambda\operatorname{tr}|T|^p + (1-\lambda)\operatorname{tr}|T^*|^p = \operatorname{tr}|T|^p.\end{aligned}$$

Therefore, if $\|\Delta_\lambda(T)\|_p = \|T\|_p$, then equality holds in Young's inequality, and by Proposition 2.2, we conclude that $|T|^p = |T^*|^p$. Hence T is normal. □

Remark 3.11. Theorem 3.9 fails for $\lambda = 1$. Take, for example, $T \in L^2(\mathscr{H})$ with polar decomposition $T = U|T|$, with $U \in \mathscr{U}(\mathscr{H})$. In this case, $\|\Delta_1(T)\|_2 = \|T\|_2$. The following example shows that Theorem 3.9 may be false for other unitarily invariant norms. In particular, for the spectral norm.

Let

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then,

$$\Delta_\lambda(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for every } \lambda \in (0, 1),$$

and therefore

$$1 = \|\Delta_\lambda(T)\|_p < \|T\|_p = 2^{1/p} \quad \text{but} \quad \|\Delta_\lambda(T)\| = \|T\| = 1.$$

The reader interested in the equality for the spectral norm is referred to [24]. In that work, Yamazaki proves that $\|\Delta_\lambda(T)\| = \|T\|$ if an only if T is normaloid, i.e., if $\rho(T) = \|T\|$.

Remark 3.12. Using standard techniques of alternate tensor powers, it can be proved that given $T \in L_0(\mathscr{H})$ and $\lambda \in [0, 1]$, then

$$\prod_{i=1}^{k} s_i(\Delta_\lambda(T)) \leqslant \prod_{i=1}^{k} s_i(T), \quad k \in \mathbb{N}.$$

This inequality says that the singular values of $\Delta_\lambda(T)$ are log-majorized by the singular values of T. Hence, we can deduce that for every unitarily invariant norm $|||\cdot|||$, we have that $|||\Delta_\lambda(T)||| \leqslant |||T|||$.

3.2. Riesz's functional calculus

An interesting result proved by Foias et al. [12] relates the Aluthge transform with completely contractive maps by using Riesz's functional calculus. Following similar ideas, in this subsection we study the relationship between Riesz's functional calculus and λ-Aluthge transforms. We begin with the following technical lemma.

Lemma 3.13. *Let $X \in L(\mathscr{H})$, $A \in GL(\mathscr{H})^+$ and $\lambda \in [0, 1]$. Then, given $n \in \mathbb{N}$, and $f_{11}, \ldots, f_{nn}$ analytic functions defined in a neighborhood of $\sigma(XA)$, we have*

$$\left\|\left(f_{ij}(A^\lambda X A^{1-\lambda})\right)_{ij}\right\| \leqslant \left\|\left(f_{ij}(AX)\right)_{ij}\right\|^\lambda \cdot \left\|\left(f_{ij}(XA)\right)_{ij}\right\|^{1-\lambda}.$$

Proof. Let $\Omega_{0,1}$ denote the open subset of the complex plane defined by

$$\Omega_{0,1} = \{z \in \mathbb{C} : \mathrm{Re}(z) \in (0,1)\}.$$

Given two unitary vectors $x = (x_1, \dots, x_n)$, and $y = (y_1, \dots, y_n)$ belonging to $\mathscr{H}^n$, define $\varphi_{x,y} : \overline{\Omega_{0,1}} \to \mathbb{C}$ in the following way:

$$\varphi_{xy}(z) = \left\langle \left(f_{ij}(A^z X A^{1-z})\right)_{ij} x,\ y \right\rangle.$$

If I_n denotes the identity operator on $\mathbb{C}^n$, then

$$\left(f_{ij}(A^z X A^{1-z})\right)_{ij} = \left(A^z f_{ij}(XA) A^{-z}\right)_{ij} = (A^z \otimes I_n)\left(f_{ij}(XA)\right)_{ij}(A^{-z} \otimes I_n).$$

Hence, it is easy to see that $\varphi_{x,y}$ is analytic in $\Omega_{0,1}$ and continuous in $\overline{\Omega_{0,1}}$. On the other hand, since A^{it} is unitary for every $t \in \mathbb{R}$,

$$\begin{aligned} |\varphi_{x,y}(it)| &= \left|\left\langle \left(f_{ij}(A^{it} X A^{1-it})\right)_{ij} x,\ y \right\rangle\right| \\ &= \left|\left\langle \left((A^{it} \otimes I_n)\left(f_{ij}(XA)\right)_{ij}(A^{-it} \otimes I_n)\right) x,\ y \right\rangle\right| \\ &\leqslant \left\|\left(f_{ij}(XA)\right)_{ij}\right\|. \end{aligned}$$

Analogously

$$\begin{aligned} |\varphi_{x,y}(1+it)| &= \left|\left\langle \left(f_{ij}(A^{1+it} X A^{-it})\right)_{ij} x,\ y \right\rangle\right| \\ &= \left|\left\langle \left((A^{it} \otimes I_n)\left(f_{ij}(AX)\right)_{ij}(A^{-it} \otimes I_n)\right) x,\ y \right\rangle\right| \\ &\leqslant \left\|\left(f_{ij}(AX)\right)_{ij}\right\|. \end{aligned}$$

Therefore, by the three lines theorem (see, for example, [18]), if $\lambda = \mathrm{Re}(z)$,

$$\left|\left\langle \left(f_{ij}(A^z X A^{1-z})\right)_{ij} x,\ y \right\rangle\right| \leqslant \left\|\left(f_{ij}(AX)\right)_{ij}\right\|^{\lambda} \cdot \left\|\left(f_{ij}(XA)\right)_{ij}\right\|^{1-\lambda}.$$

Taking supremum over all $x, y \in \mathscr{H}^n$, we get the desired inequality. □

Lemma 3.13 allows us to give an alternative proof of Jung, Ko and Pearcy's result, which also generalizes it for $\lambda \in (0,1)$.

Proposition 3.14. *Let* $T \in L(\mathscr{H})$, $\lambda \in (0,1)$ *and* $f \in \mathrm{Hol}(\sigma(T))$. *Then*

1. $\|f(\Delta_0(T))\| \leqslant \|f(T)\|$ *and* $\|f(\Delta_1(T))\| \leqslant \|f(T)\|$.
2. $\|f(\Delta_\lambda(T))\| \leqslant \|f(\Delta_1(T))\|^{\lambda}\, \|f(\Delta_0(T))\|^{1-\lambda} \leqslant \|f(T)\|$.

Proof. The inequality $\|f(\Delta_1(T))\| \leqslant \|f(T)\|$ was proved by Foias, Jung, Ko and Pearcy in [12], using Proposition 3.4. The inequality for $\Delta_0(T)$ can be proved by following similar ideas.

In order to prove the inequality of item 2, Let $T = U|T|$ be the polar decomposition of T and E the orthogonal projection onto $\overline{R(|T|)}$. Note that $(|T| + n^{-1})^{\lambda} \xrightarrow[n\to\infty]{\|\cdot\|} |T|^{\lambda}$, because the sequence of functions $f_n(x) = (x + n^{-1})^{\lambda}$ ($n \in \mathbb{N}$) converges uniformly to $f(x) = x^{\lambda}$ on compact subsets. So, given $f \in \mathrm{Hol}\,(\sigma(T))$, by Proposition 2.1 we have that

$$f\big((|T| + n^{-1})^{\lambda} E\, U(|T| + n^{-1})^{1-\lambda}\big).$$

$f(EU(|T| + n^{-1}))$ and $f((|T| + n^{-1})EU)$ are defined for all sufficiently large n. Moreover,

$$f\big(U(|T| + n^{-1})\big) \xrightarrow[n\to\infty]{\|\cdot\|} f\big(EU|T|\big),$$

$$f\big((|T| + n^{-1})EU\big) \xrightarrow[n\to\infty]{\|\cdot\|} f\big(|T|E\,U\big) = f\big(|T|U\big),$$

$$f\big((|T| + n^{-1})^{\lambda}EU(|T| + n^{-1})^{1-\lambda}\big) \xrightarrow[n\to\infty]{\|\cdot\|} f\big(|T|^{\lambda}U|T|^{1-\lambda}\big).$$

Using Lemma 3.13 and standard limit arguments, we get inequality 2. □

Remark 3.15. Using Lemma 3.13, it can be proved that given $n \in \mathbb{N}$, and $f_{11}, \ldots, f_{nn} \in \mathrm{Hol}\,(\sigma(T))$,

$$\big\|\big(f_{ij}(\Delta_{\lambda}(T))\big)_{ij}\big\| \leqslant \big\|\big(f_{ij}(\Delta_1(T))\big)_{ij}\big\|^{\lambda}\big\|\big(f_{ij}(\Delta_0(T))\big)_{ij}\big\|^{1-\lambda}.$$

It should be mentioned that $\|(f_{ij}(\Delta_0(T)))_{ij}\| \leqslant \|(f_{ij}(T))_{ij}\|$.

For $T \in L(\mathscr{H})$, we denote $W(T) = \{\langle Tx, x\rangle : x \in \mathscr{H}, \|x\| = 1\}$, its numerical range. As a corollary of Proposition 3.14, we obtain the next result about numerical ranges.

Corollary 3.16. *Let $T \in L(\mathscr{H})$ and $\lambda \in [0, 1]$. Then, for every complex analytic function f defined in a neighborhood of $\sigma(T)$,*

$$\overline{W\big(f(\Delta_{\lambda}(T))\big)} \subseteq \overline{W\big(f(T)\big)}.$$

Proof. Indeed, by Proposition 3.14 (item 1), for every $\mu \in \mathbb{C}$ it holds that $\|f(\Delta_{\lambda}(T)) - \mu I\| \leqslant \|f(T) - \mu I\|$. So, if $B(r, \zeta) = \{z \in \mathbb{C} : |z - \zeta| \leqslant r\}$, using the well known formula

$$\overline{W(T)} = \bigcap_{\lambda\in\mathbb{C}} B\big(\|T - \lambda I\|, \lambda\big),$$

we have that

$$\overline{W(f(\Delta_\lambda(T)))} = \bigcap_{\mu\in\mathbb{C}} B(\|f(\Delta_\lambda(T)) - \mu I\|, \lambda)$$
$$\subseteq \bigcap_{\mu\in\mathbb{C}} B(\|f(T) - \mu I\|, \lambda) = \overline{W(f(T))}. \qquad \square$$

Remark 3.17. The above Corollary, was proved in [12], for $\lambda = 1/2$, using that $\overline{W(T)}$ is the intersection of all half-planes H containing $W(T)$, which are spectral sets for T. In [17], Okubo obtains the same result for a polynomial function f, for every $\lambda \in (0, 1)$.

4. The finite dimensional case

In this section, we study the λ-Aluthge transformation in finite dimensional spaces. Given $T \in \mathcal{M}_n(\mathbb{C})$ and $\lambda \in (0, 1)$, we denote by $\Delta^n_\lambda(T)$ the n-times iterated λ-Aluthge transform of T, i.e.,

$$\Delta^0_\lambda(T) = T \quad \text{and} \quad \Delta^n_\lambda(T) = \Delta_\lambda\left(\Delta^{n-1}_\lambda(T)\right), \quad n \in \mathbb{N}.$$

The following proposition was proved, for $\lambda = 1/2$, by Ando in [2], and by Jung, Ko and Pearcy in [16].

Proposition 4.1. *Let $T \in \mathcal{M}_n(\mathbb{C})$. Then, the limit points of the sequence $\{\Delta^n_\lambda(T)\}_{n\in\mathbb{N}}$ are normal. Moreover, if L is a limit point, then $\sigma(L) = \sigma(T)$ with the same algebraic multiplicity.*

Proof. Let $\{\Delta^{n_k}_\lambda(T)\}_{k\in\mathbb{N}}$ be a subsequence which converge in norm to a limit point L. By the continuity of Aluthge transforms, $\Delta^{n_k+1}_\lambda(T) \underset{k\to\infty}{\longrightarrow} \Delta_\lambda(L)$. Then

$$\|\Delta_\lambda(L)\|_2 = \lim_{k\to\infty} \|\Delta^{n_k+1}_\lambda(T)\|_2 = \lim_{n\to\infty} \|\Delta^n_\lambda(T)\|_2$$
$$= \lim_{k\to\infty} \|\Delta^{n_k}_\lambda(T)\|_2 = \|L\|_2.$$

Hence, by Theorem 3.9 L is normal. It only remains to prove that $\sigma(L) = \sigma(T)$ with the same algebraic multiplicity, or equivalently, that $\operatorname{tr}(T^m) = \operatorname{tr}(L^m)$ for every $m \in \mathbb{N}$. Indeed,

$$\operatorname{tr} L^m = \lim_{k\to\infty} \operatorname{tr} \Delta^{n_k}_\lambda(T)^m = \operatorname{tr} T^m, \quad m \in \mathbb{N},$$

because, for each $k \in \mathbb{N}$, $\sigma\left(\Delta^{n_k}_\lambda(T)\right) = \sigma(T)$ (with algebraic multiplicity), and therefore $\operatorname{tr} \Delta^{n_k}_\lambda(T)^m = \operatorname{tr} T^m$. $\square$

As a consequence of this result, we obtain Yamazaki's spectral radius formula, for every $\lambda \in (0, 1)$. It should be mentioned that Yamazaki's formula holds for operators in Hilbert spaces (with $\lambda = 1/2$), but we can only prove the general case ($\lambda \neq 1/2$) in the finite dimensional case.

Corollary 4.2. *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\lambda \in (0, 1)$. Then,*

$$\rho(T) = \lim_{n \to \infty} \|\Delta_\lambda^n(T)\|.$$

Proof. Take a subsequence $\{\Delta_\lambda^{n_k}(T)\}$ that converges to a limit point L. Since L is normal and $\sigma(L) = \sigma(T)$, it holds that $\|L\| = \rho(L) = \rho(T)$. Hence

$$\lim_{k \to \infty} \|\Delta_\lambda^{n_k}(T)\| = \|L\| = \rho(L) = \rho(T).$$

Finally, since the whole sequence $\{\|\Delta_\lambda^n(T)\|\}$ converges because it is non-increasing, we obtain the desired result. □

Analogously we can deduce the following result, proved by Ando in [2] for $\lambda = 1/2$. We use the notation $\mathrm{co}(X)$ for the convex hull of the set X.

Corollary 4.3. *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\lambda \in (0, 1)$. Then,*

$$\mathrm{co}(\sigma(T)) = \bigcap_{n=1}^{\infty} W(\Delta_\lambda^n(T)).$$

Now we state the following result, which is a direct consequence of Theorem 3.6 and the fact that the map $T \to |T|^r$ is norm-continuous in $\mathcal{M}_n(\mathbb{C})$.

Proposition 4.4. *The map $(\lambda, T) \to \Delta_\lambda(T)$ from $(0, 1) \times \mathcal{M}_n(\mathbb{C})$ into $\mathcal{M}_n(\mathbb{C})$ is continuous when $\mathcal{M}_n(\mathbb{C})$ is endowed with the norm-topology and the interval $(0, 1)$ with the usual one.*

Proof. It follows by a standard $\frac{\varepsilon}{2}$-argument. □

4.1. The iterated Aluthge transforms in $\mathcal{M}_2(\mathbb{C})$

In this subsection we study the convergence of the sequence $\{\Delta_\lambda^n(T)\}$ when T is a 2×2 matrix. The convergence of this sequence for $n \times n$ matrices and $\lambda = 1/2$ was conjectured by Jung, Ko, and Pearcy in [15]. Although this conjecture is still open, there exists a result, due to T. Ando and T. Yamazaki [3], which answers the conjecture affirmatively for 2×2 matrices and $\lambda = 1/2$. We generalize this result for arbitrary $\lambda \in (0, 1)$ and we also prove that the map which assigns to each pair (λ, T) the limit of the sequence $\{\Delta^n_\lambda(T)\}$ is continuous in both variables T and λ.

Lemma 4.5. *Let $T \in \mathcal{M}_2(\mathbb{C})$ and $\lambda \in (0, 1)$. Suppose that $\sigma(T) = \{\mu_1, \mu_2\}$ with $\mu_1 \neq \mu_2$. Then, there exists $\gamma(T, \lambda) \in (0, 1)$ such that, for all $n \in \mathbb{N}$,*

$$\left\| \Delta^n_\lambda(T)^* \Delta^n_\lambda(T) - \Delta^n_\lambda(T) \Delta^n_\lambda(T)^* \right\|_2 \leqslant \gamma(T, \lambda)^n \, \|T^*T - TT^*\|_2.$$

Moreover, if $\alpha = \min\{\lambda, 1 - \lambda\}$, then we can take

$$\gamma(T, \lambda) = \left(1 - \frac{2\alpha^2 \, |\mu_1 - \mu_2|^2}{2\,|\mu_1\mu_2| + \|T\|_2^2}\right)^{1/2}.$$

Proof. Denote $T_n = \Delta^n_\lambda(T)$, $n \in \mathbb{N}$. In some orthonormal basis, which may be different for each $n \in \mathbb{N}$, T_n has the form

$$T_n = \begin{pmatrix} \mu_1 & a_n \\ 0 & \mu_2 \end{pmatrix}, \quad \text{with } a_n = \left(\|T_n\|_2^2 - [\,|\mu_1|^2 + |\mu_2|^2]\right)^{1/2} \geqslant 0.$$

Hence $a_{n+1} \leqslant a_n$, $n \in \mathbb{N}$, by Theorem 3.9. Easy computations show that, if $M = |\mu_1 - \mu_2|^2$ then

$$\left\|T_n^*T_n - T_nT_n^*\right\|_2^2 = 2\,a_n^2\left(M + a_n^2\right), \quad n \in \mathbb{N}. \tag{6}$$

Therefore, for all $n \in \mathbb{N}$,

$$\frac{\|T_{n+1}^*T_{n+1} - T_{n+1}T_{n+1}^*\|_2^2}{\|T_n^*T_n - T_nT_n^*\|_2^2} = \frac{a_{n+1}^2}{a_n^2}\,\frac{(M + a_{n+1}^2)}{(M + a_n^2)} \leqslant \frac{a_{n+1}^2}{a_n^2}. \tag{7}$$

Since $a_n^2 - a_{n+1}^2 = \|T_n\|_2^2 - \|T_{n+1}\|_2^2$, by Proposition 3.8 the following inequality holds for all $n \in \mathbb{N}$,

$$\frac{a_{n+1}^2}{a_n^2} = 1 - \frac{\|T_n\|_2^2 - \|T_{n+1}\|_2^2}{a_n^2} \leqslant 1 - \frac{\alpha^2\|\,|T_n| - |T_n^*|\,\|_2^2}{a_n^2}.$$

On the other hand, if $X \in \mathcal{M}_2(\mathbb{C})^+$ and $d = \det(X)^{1/2}$, then it is known that

$$X^{1/2} = \frac{X + dI}{\sqrt{2d + \operatorname{tr}(X)}}.$$

Hence, if we denote $d = \det(T_n^*T_n)^{1/2} = \det(T_nT_n^*)^{1/2} = |\det T| = |\mu_1\mu_2|$, we have that

$$\|\,|T_n| - |T_n^*|\,\|_2^2 = \frac{\|T_n^*T_n - T_nT_n^*\|_2^2}{2d + \|T_n\|_2^2}, \quad n \in \mathbb{N}.$$

Therefore, by Eq. (6), for all $n \in \mathbb{N}$,

$$\begin{aligned} \frac{a_{n+1}^2}{a_n^2} &\leqslant 1 - \frac{\alpha^2\,\|T_n^*T_n - T_nT_n^*\|_2^2}{a_n^2(2d + \|T_n\|_2^2)} \\ &= 1 - \frac{2\alpha^2\,(M + a_n^2)}{2d + \|T_n\|_2^2} \leqslant 1 - \frac{2\alpha^2\,M}{2d + \|T\|_2^2}. \end{aligned} \tag{8}$$

Finally, taking

$$\gamma(T,\lambda) = \left(1 - \frac{2\alpha^2 M}{2d + \|T\|_2^2}\right)^{1/2},$$

by Eqs. (7) and (8), we get

$$\left\| T_{n+1}^* T_{n+1} - T_{n+1} T_{n+1}^* \right\|_2 \leqslant \gamma(T,\lambda) \left\| T_n^* T_n - T_n T_n^* \right\|_2, \quad n \in \mathbb{N},$$

and the result is proved by iterating this inequality. Note that $0 < \alpha^2 \leqslant 1/4$ and

$$0 < M = |\,\mu_1 - \mu_2\,|^2 \leqslant 2\,|\mu_1\mu_2| + |\mu_1|^2 + |\mu_2|^2 \leqslant 2d + \|T\|_2^2\,.$$

Then $0 < \gamma(T,\lambda) < 1$. □

Theorem 4.6. *Let $T \in \mathscr{M}_2(\mathbb{C})$ and $\lambda \in (0,1)$. Then, the sequence $\{\Delta_\lambda^n(T)\}$ converges.*

Proof. Suppose that $\sigma(T) = \{\mu_1, \mu_2\}$. Since we have proved (see Proposition 4.1) that the limit points of the sequence $\{\Delta_\lambda^n(T)\}$ are normal, if $\mu_1 = \mu_2 = c$, then $\Delta_\lambda^n(T) \underset{n\to\infty}{\longrightarrow} cI$. Thus, from now on we only consider the case in which $\mu_1 \neq \mu_2$. As in the Lemma 4.5, we denote $T_n = \Delta_\lambda^n(T)$.

Fix $n \geqslant 0$. If $T_n = U_n|T_n|$ is the polar decomposition of T_n, then $|T_n^*|^s = U_n|T_n|^s U_n^*$, for every $s > 0$. Therefore we obtain

$$\begin{aligned}(T_{n+1} - T_n)U_n^* &= |T_n|^\lambda U_n |T_n|^{1-\lambda} U_n^* - U_n|T_n|U_n^* \\ &= |T_n|^\lambda |T_n^*|^{1-\lambda} - |T_n^*| = \big(|T_n|^\lambda - |T_n^*|^\lambda\big)|T_n^*|^{1-\lambda}.\end{aligned}$$

Since $\|AB\|_2 \leqslant \|A\|_2\|B\|$, we can deduce that

$$\begin{aligned}\|T_{n+1} - T_n\|_2 &\leqslant \||T_n|^\lambda - |T_n^*|^\lambda\|_2 \cdot \||T_n^*|^{1-\lambda}\| \\ &\leqslant \||T_n|^\lambda - |T_n^*|^\lambda\|_2 \cdot \|T\|^{1-\lambda}.\end{aligned}$$

Using Proposition 2.5 with $A = T_n^* T_n$, $B = T_n T_n^*$ and $r = \lambda/2$, we get

$$\begin{aligned}\|T_{n+1} - T_n\|_2 &\leqslant \||T_n|^\lambda - |T_n^*|^\lambda\|_2 \cdot \|T\|^{1-\lambda} \\ &\leqslant (2\|T\|^{1-\lambda})\|T_n^* T_n - T_n T_n^*\|_2^{\lambda/2},\end{aligned}$$

because $\|I_2\|_2^{1-\lambda/2} \leqslant 2$. Let $a = \gamma(T,\lambda)^{\lambda/2} < 1$, where $\gamma(T,\lambda) \in (0,1)$ is the constant of Lemma 4.5. Then

$$\begin{aligned}\|T_{n+1} - T_n\|_2 &\leqslant (2\|T\|^{1-\lambda})\|T_n^* T_n - T_n T_n^*\|_2^{\lambda/2} \\ &\leqslant a^n\big(2\|T\|^{1-\lambda}\|T^*T - TT^*\|_2^{\lambda/2}\big).\end{aligned}$$

Denote $N(T,\lambda) = 2\|T\|^{1-\lambda}\|T^*T - TT^*\|_2^{\lambda/2}$. Then, if $n, m \in \mathbb{N}$, with $n < m$,

$$\|T_m - T_n\|_2 \leqslant \sum_{k=n}^{m-1} \|T_{k+1} - T_k\|_2$$
$$\leqslant N(T,\lambda) \sum_{k=n}^{m-1} a^k \underset{n,m\to\infty}{\longrightarrow} 0, \tag{9}$$

which shows that the $\lim_{n\to\infty} T_n = \lim_{n\to\infty} \Delta_\lambda^n(T)$ exists. □

In order to state precisely the next results, we need the following notations:

Definition 4.7

1. Given $T \in \mathcal{M}_2(\mathbb{C})$ and $\lambda \in (0,1)$, denote $\Delta_\lambda^\infty(T) = \lim_{n\to\infty} \Delta_\lambda^n(T)$.
2. Consider the map $\Gamma : (0,1) \times \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ defined by

$$\Gamma(\lambda, T) = \Delta_\lambda^\infty(T), \quad (\lambda, T) \in (0,1) \times \mathcal{M}_2(\mathbb{C}).$$

Theorem 4.8. *Let $\lambda \in (0,1)$ be fixed. Then the map $\Gamma(\lambda,\cdot) : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$, given by*

$$\mathcal{M}_2(\mathbb{C}) \ni T \mapsto \Delta_\lambda^\infty(T)$$

is continuous. Therefore $\Delta_\lambda^\infty(\cdot)$ is a continuous retraction from $\mathcal{M}_2(\mathbb{C})$ onto the space of normal matrices in $\mathcal{M}_2(\mathbb{C})$.

Proof. Take $T \in \mathcal{M}_2(\mathbb{C})$ and $\lambda \in (0,1)$. We shall consider two cases:
Case 1. Suppose that $\sigma(T) = \{\mu\}$. Let $S \in \mathcal{M}_2(\mathbb{C})$ with $\sigma(S) = \{\eta_1, \eta_2\}$. Since $\Delta_\lambda^\infty(T) = \mu I$ and $\Delta_\lambda^\infty(S)$ is a normal operator with the same spectrum as S, then

$$\left\|\Delta_\lambda^\infty(T) - \Delta_\lambda^\infty(S)\right\|_2^2 = |\mu - \eta_1|^2 + |\mu - \eta_2|^2.$$

Clearly, this implies that $\Delta_\lambda^\infty(\cdot)$ is continuous at T.
Case 2. Suppose that $\sigma(T) = \{\mu_1, \mu_2\}$ with $\mu_1 \neq \mu_2$ and let $\varepsilon > 0$. Take $\delta_1 > 0$ such that for every matrix S satisfying $\|T - S\|_2 \leqslant \delta_1$, the constant $\gamma(S,\lambda)$ of Lemma 4.5 applied to S satisfies $\gamma(S,\lambda) \leqslant r$, for some $r < 1$. Indeed, note that the formula for $\gamma(S,\lambda)$ given in Lemma 4.5 depends continuously on S (and its spectrum). Note that the constant $N(S,\lambda) = 4\|S\|^{1-\lambda}\|S^*S - SS^*\|_2^{\lambda/2}$ is bounded on the set $\mathcal{U} = \{S \in \mathcal{M}_2(\mathbb{C}) : \|T - S\|_2 \leqslant \delta_1\}$. Then, by formula 9, we can deduce that there exists $n \in N$, such that

$$\left\|\Delta_\lambda^\infty(S) - \Delta_\lambda^n(S)\right\|_2 \leqslant N(S,\lambda) \sum_{k=n}^{\infty} r^{k\lambda/2} \leqslant \frac{\varepsilon}{3},$$

for every $S \in \mathcal{U}$. Finally, since the map $\Delta_\lambda^n(\cdot)$ is continuous on $\mathcal{M}_2(\mathbb{C})$, we can take $0 < \delta_2 < \delta_1$ such that, if $\|T - S\|_2 \leqslant \delta_2$, then

$$\left\|\Delta_\lambda^n(T) - \Delta_\lambda^n(S)\right\|_2 \leqslant \frac{\varepsilon}{3}.$$

So, if $\|T - S\|_2 \leqslant \delta_2$, then

$$\begin{aligned}\left\|\Delta_\lambda^\infty(T) - \Delta_\lambda^\infty(S)\right\|_2 &\leqslant \left\|\Delta_\lambda^\infty(T) - \Delta_\lambda^n(T)\right\|_2 + \left\|\Delta_\lambda^n(T) - \Delta_\lambda^n(S)\right\|_2 \\ &\quad + \left\|\Delta_\lambda^n(S) - \Delta_\lambda^\infty(S)\right\|_2 \leqslant \varepsilon,\end{aligned}$$

which completes the proof. □

Theorem 4.9. *Let $T \in \mathcal{M}_2(\mathbb{C})$ be fixed. Then the map $\Gamma(\cdot, T) : (0,1) \to \mathcal{M}_2(\mathbb{C})$, given by*

$$(0,1) \ni \lambda \mapsto \Delta_\lambda^\infty(T)$$

is continuous. Moreover, if $\sigma(T) = \{\mu_1, \mu_2\}$ with $|\mu_1| = |\mu_2|$, then the map is constant.

Proof. The proof of the continuity is similar to the proof of the previous theorem (see also Remark 4.10). Note that the constants $\gamma(T, \lambda)$ and $N(T, \lambda)$ depend continuously on both variables, in particular on λ. Also, by Proposition 4.4, the map $\lambda \mapsto \Delta_\lambda^n(T)$ is continuous, for every $n \in \mathbb{N}$. Let $T \in \mathcal{M}_2(\mathbb{C})$ such that $|\mu_1| = |\mu_2|$. As Ando and Yamazaki pointed out in [3], without loss of generality we can assume that $T = \begin{pmatrix} a & b \\ -b & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$, with $b > 0$, and $\sigma(T) = \{u + \mathrm{i}v, u - \mathrm{i}v\}$ with $u^2 + v^2 = 1$ and $v > 0$. Then,

$$\Gamma(\lambda, T) = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}, \qquad \lambda \in (0,1).$$

Indeed, if $\Delta_\lambda^n(T) = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, by Theorem 4.6 and some simple computations, we get

$$\begin{aligned}&\Delta_\lambda^n(T)^* \Delta_\lambda^n(T) - \Delta_\lambda^n(T)\Delta_\lambda^n(T)^* \\ &\quad = (b_n - c_n)\begin{pmatrix} -(b_n + c_n) & a_n - d_n \\ a_n - d_n & b_n + c_n \end{pmatrix} \underset{n\to\infty}{\longrightarrow} 0.\end{aligned} \tag{10}$$

So, the sequences a_n and d_n converge to $\operatorname{tr}(T)/2 = u$. On the other hand, following essentially the same lines as in Ando-Yamazaki's proof, we get $0 < m = \inf_n (b_n - c_n)^2 = \lim_{n\to\infty}(b_n - c_n)^2$. Hence, $b_n - c_n$ must converge to $m^{1/2}$ or $-m^{1/2}$. Moreover, since $b_n + c_n \underset{n\to\infty}{\longrightarrow} 0$ by formula 10, then $m^{1/2} = 2v$, for each $\lambda \in (0,1)$. Therefore

$$\Gamma(\lambda, T) = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} = \Gamma(1/2, T) \quad \text{or} \quad \Gamma(\lambda, T) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}.$$

But Γ is continuous on λ, so $\Gamma(\lambda, T) = \Gamma(1/2, T)$ for every $\lambda \in (0,1)$. □

Remark 4.10. With similar arguments to those used in the proofs of the previous two theorems, it can be proved that the map Γ is jointly continuous.

Example 4.11. If $T \in \mathscr{M}_2(\mathbb{C})$ has eigenvalues with different moduli, then the map $\lambda \mapsto \Delta_\lambda^\infty(T)$ does not seem to be constant, in general. For example, if $T = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}$, numerical computations show that

$$\Delta_{0.3}^\infty(T) \cong \begin{pmatrix} 2.22738 & 0.973807 \\ 0.973807 & 1.77262 \end{pmatrix} \quad \text{while}$$

$$\Delta_{0.7}^\infty(T) \cong \begin{pmatrix} 1.37162 & -0.777907 \\ -0.777907 & 2.62838 \end{pmatrix}.$$

Nevertheless, for many other matrices T with different modulus eigenvalues, the map $\lambda \mapsto \Delta_\lambda^\infty(T)$ seems to be constant.

4.2. The Jordan structure of Aluthge transforms

In this subsection, we study some properties of the Jordan structure of the iterated Aluthge transforms. We show a reduction of the conjecture on the convergence of the sequence $\{\Delta_\lambda^m(T)\}$ for $T \in \mathscr{M}_n(\mathbb{C})$, to the invertible case. We also study the behavior of the angles between the spectral subspaces of iterates of the Aluthge transform for $T \in \mathscr{M}_n(\mathbb{C})$.

The following result states a simple relation between the null spaces of polynomials in T and in $\Delta_\lambda(T)$. This relation has some consequences regarding multiplicity and Jordan structure of eigenvalues of T and $\Delta_\lambda(T)$. We denote by $\mathbb{C}[x]$ the set of complex polynomials.

Lemma 4.12. *Let $T \in \mathscr{M}_n(\mathbb{C})$ and $\lambda \in (0, 1)$.*

1. *Given $p \in \mathbb{C}[x]$, then* $\dim N(p(T)) \leqslant \dim N(p(\Delta_\lambda(T)))$.
2. *For $n \in \mathbb{N}$, $n \geqslant 2$,* $\dim N(T^n) = \dim N(\Delta_\lambda(T)^{n-1})$.

Proof. Assume first that $p(0) \neq 0$. In this case $N(T) \cap N(p(T)) = \{0\}$. Hence

$$\dim |T|^\lambda(N(p(T))) = \dim N(p(T)),$$

because $N(T) = N(|T|) = N(|T|^\lambda)$. Using Proposition 3.4, we know that $p(\Delta_\lambda(T))|T|^\lambda = |T|^\lambda p(T)$, so that

$$|T|^\lambda(N(p(T)) \subseteq N(p(\Delta_\lambda(T)).$$

If $p(0) = 0$, Note that $N(T) \subseteq N(p(T))$ and also $N(T) \subseteq N(p(\Delta_\lambda(T)))$. Denote by $\mathscr{S} = N(p(T)) \ominus N(T)$. Then $\dim |T|^\lambda(\mathscr{S}) = \dim \mathscr{S}$ and $|T|^\lambda(\mathscr{S}) \subseteq N(T)^\perp$. On the other hand, we get that $|T|^\lambda(\mathscr{S}) \subseteq N(p(\Delta_\lambda(T)))$ as before. Then

$$\begin{aligned}\dim N(p(T)) &= \dim N(T) + \dim \mathscr{S} \\ &= \dim N(T) + \dim |T|^{\lambda}(\mathscr{S}) \\ &= \dim \left[N(T) \oplus |T|^{\lambda}(\mathscr{S})\right] \leqslant \dim N(p(\Delta_\lambda(T))).\end{aligned}$$

Finally, note that if $n \geqslant 2$ we have

$$N(\Delta_\lambda(T)^{n-1}|T|^{\lambda}) = N(|T|^{\lambda}T^{n-1}) = N(T^n).$$

Let $\mathscr{S} = N(\Delta_\lambda(T)^{n-1}) \ominus N(T)$. Since $|T|^{\lambda}$ operates bijectively on $N(T)^{\perp}$, there is a subspace $\mathscr{M} \subseteq N(T)^{\perp}$ such that $\dim \mathscr{M} = \dim \mathscr{S}$ and $|T|^{\lambda}(\mathscr{M}) = \mathscr{S}$. Hence

$$N(\Delta_\lambda(T)^{n-1}|T|^{\lambda}) = \left\{x \in \mathbb{C}^n : |T|^{\lambda}(x) \in N(\Delta_\lambda(T)^{n-1})\right\} = N(T) \oplus \mathscr{M}.$$

So that $\dim N(\Delta_\lambda(T)^{n-1}) = \dim N(\Delta_\lambda(T)^{n-1}|T|^{\lambda}) = \dim N(T^n)$. $\square$

Definition 4.13. Let $T \in \mathscr{M}_n(\mathbb{C})$ and $\mu \in \sigma(T)$. We denote

1. $m(T, \mu)$ the *algebraic multiplicity* of the eigenvalue μ for T.
2. $m_0(T, \mu) = \dim N(T - \mu I)$, the *geometric multiplicity* of the eigenvalue μ for T.
3. $r(T, \mu) = \min\{k \in \mathbb{N} : \dim N(T - \mu I)^k = m(T, \mu)\}$, usually called the *index* of μ. Note that $r(T, \mu)$ is the size of the biggest Jordan block of T associated to μ.

We say that the Jordan structure of T for the eigenvalue μ is *trivial* if $m(T, \mu) = m_0(T, \mu)$, or equivalently, if $r(T, \mu) = 1$.

Proposition 4.14. *Let $T \in \mathscr{M}_n(\mathbb{C})$ and $\lambda \in (0, 1)$.*

1. *Suppose that $0 \in \sigma(T)$. Then*

$$m(T, 0) = m_0(\Delta_\lambda^{r(T,0)-1}(T), 0) = \dim N(\Delta_\lambda^{r(T,0)-1}(T)).$$

   Therefore, after $r(T, 0) - 1$ iterations of the Aluthge transform, we get a matrix whose Jordan structure for the eigenvalue 0 is trivial.
2. *If $\mu \in \sigma(T)/\{0\}$, then*

$$m_0(T, \mu) \leqslant m_0(\Delta_\lambda(T), \mu) \quad \textit{and} \quad r(T, \mu) \geqslant r(\Delta_\lambda(T), \mu).$$

Proof

1. Denote $r(T, 0) = r$. If $r \geqslant 2$, by Lemma 4.12,

$$\begin{aligned}m(T, 0) &= \dim N(T^r) = \dim N(\Delta_\lambda(T)^{r-1}) = \dim N(\Delta_\lambda^2(T)^{r-2}) \\ &= \cdots = \dim N(\Delta_\lambda^{r-2}(T)^2) = \dim N(\Delta_\lambda^{r-1}(T)).\end{aligned}$$

If $r = 1$, then $\Delta_\lambda^{r-1}(T) = \Delta_\lambda^0(T) = T$ by definition, and

$$m(T, 0) = m_0(T, 0) = \dim(\Delta_\lambda^{r-1}(T)).$$

2. Consider $P_m(x) = (x - \mu)^m$, $m \in \mathbb{N}$. Taking $m = 1$, by Lemma 4.12,

$$m_0(T, \mu) = \dim N(T - \mu I) \leqslant \dim N(\Delta_\lambda(T) - \mu I) = m_0(\Delta_\lambda(T), \mu).$$

Taking $m = r(T, \mu)$, again by Lemma 4.12, we have that

$$\begin{aligned} m(T, \mu) &= \dim N((T - \mu I)^{r(T,\mu)}) \\ &\leqslant \dim N((\Delta_\lambda(T) - \mu I)^{r(T,\mu)}) \leqslant m(\Delta_\lambda(T), \mu). \end{aligned}$$

Since $m(\Delta_\lambda(T), \mu) = m(T, \mu)$, we get that $r(T, \mu) \geqslant r(\Delta_\lambda(T), \mu)$. □

Remark 4.15. In particular, Proposition 4.14 shows that if T is nilpotent of order n then $\Delta_\lambda^{n-1}(T) = 0$. This result was proved by Jung, Ko and Pearcy in [16].

Corollary 4.16. *Let $\lambda \in (0, 1)$. If the sequence $\{\Delta_\lambda^m(S)\}$ converges for every* invertible *matrix $S \in \mathcal{M}_n(\mathbb{C})$ and every $n \in \mathbb{N}$, then the sequence $\{\Delta_\lambda^m(T)\}$ converges for all $T \in \mathcal{M}_n(\mathbb{C})$ and every $n \in \mathbb{N}$.*

Proof. Let $T \in \mathcal{M}_n(\mathbb{C})$. By Lemma 4.14, we can assume that $m(T, 0) = m_0(T, 0)$. Note that, in this case, $N(\Delta_\lambda(T)) = N(T)$, because $N(T) \subseteq N(\Delta_\lambda(T))$ and $m_0(\Delta_\lambda(T), 0) = m(T, 0)$. On the other hand, $R(\Delta_\lambda(T)) \subseteq R(|T|)$ so that $R(\Delta_\lambda(T))$ and $N(\Delta_\lambda(T))$ are orthogonal subspaces. Thus, there exists a unitary matrix U such that

$$U\Delta_\lambda(T)U^* = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},$$

where $S \in M_s(\mathbb{C})$ is invertible ($s = n - m(T, 0)$). Since for every $m \geqslant 2$

$$\Delta_\lambda^m(T) = U^* \begin{pmatrix} \Delta_\lambda^{m-1}(S) & 0 \\ 0 & 0 \end{pmatrix} U,$$

the sequence $\{\Delta_\lambda^m(T)\}$ converges, because the sequence $\{\Delta_\lambda^{m-1}(S)\}$ converges by hypothesis. □

Remark 4.17. If $T \in \mathcal{M}_n(\mathbb{C})$ is invertible, then $|T|^\lambda$ is invertible for every $\lambda \in (0, 1)$, and

$$\Delta_\lambda(T) = |T|^\lambda T |T|^{-\lambda}. \tag{11}$$

Therefore, T and $\Delta_\lambda^m(T)$ are similar matrices, for every $m \in \mathbb{N}$. That is, $\Delta_\lambda^m(T)$ and T have the same Jordan structure. This shows that the geometric multiplicity of non-zero eigenvalues does not increase in general. On the other hand, Proposition 4.14 implies that for non-invertible operators T, $\Delta_\lambda(T)$ and T may be not similar. In particular, the Jordan structure of T and $\Delta_\lambda(T)$ may be different.

Numerical experiences show that the rate of convergence of the sequence $\{\Delta_\lambda^m(T)\}$ is smaller for non-diagonabilizable *T*, than for diagonabilizable examples.

Definition 4.18. Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\mu \in \sigma(T)$.

1. Denote $\mathcal{H}_{\mu,T} = N((T-\mu I)^{r(T,\mu)})$. Note that $\mathbb{C}^n = \bigoplus_{\gamma\in\sigma(T)} \mathcal{H}_{\gamma,T}$.
2. Denote $Q_{\mu,T} \in \mathcal{M}_n(\mathbb{C})$ the oblique projection with

$$R(Q_{\mu,T}) = \mathcal{H}_{\mu,T} \quad \text{and} \quad N(Q_{\mu,T}) = \bigoplus_{\gamma\neq\mu} \mathcal{H}_{\gamma,T}.$$

Proposition 4.19. *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\lambda \in (0,1)$. Then, for every $\mu \in \sigma(T)$,*

$$\|Q_{\mu,\Delta_\lambda^m(T)}\| \xrightarrow[n\to\infty]{\|\cdot\|} 1.$$

Proof. Let $f_\mu \in \mathrm{Hol}(T)$ be an analytic map which takes the value 1 in a neighborhood of μ, and the value 0 in a neighborhood of $\sigma(T)\setminus\{\mu\}$. Then it is known that $f_\mu(T) = Q_{\mu,T}$. Moreover, since $\sigma\left(\Delta_\lambda^m(T)\right) = \sigma(T)$, we have that $Q_{\mu,\Delta_\lambda^m(T)} = f_\mu(\Delta_\lambda^m(T))$, $m\in\mathbb{N}$, $\mu\in\sigma(T)$. Then, by Proposition 3.14,

$$\|Q_{\mu,\Delta_\lambda^m(T)}\| \geqslant \|Q_{\mu,\Delta_\lambda^{m+1}(T)}\|, \quad m\in\mathbb{N},\ \mu\in\sigma(T).$$

On the other hand, there exists a subsequence $\Delta_\lambda^{m_k}(T) \xrightarrow[k\to\infty]{} L$ for some normal matrix $L\in\mathcal{M}_n(\mathbb{C})$, with $\sigma(L) = \sigma(T)$. Then, by Proposition 2.1,

$$\left\|Q_{\mu,\Delta_\lambda^{m_k}(T)}\right\| = \left\|f_\mu(\Delta_\lambda^{m_k}(T))\right\| \xrightarrow[k\to\infty]{} \|f_\mu(L)\| = \|Q_{\mu,L}\| = 1.$$

because the spectral projections of normal operators are selfadjoint (i.e., orthogonal). □

Remark 4.20. Given two subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathbb{C}^n$ such that $\mathcal{M}\cap\mathcal{N} = \{0\}$, the **angle** between $\mathcal{M}$ and $\mathcal{N}$ is the angle in $[0,\pi/2]$ whose cosine is defined by

$$\begin{aligned} c[\mathcal{M},\mathcal{N}] &= \sup\left\{ |\langle x,y\rangle| : x\in\mathcal{M},\ y\in\mathcal{N} \text{ and } \|x\| = \|y\| = 1\right\} \\ &= \|P_{\mathcal{M}}P_{\mathcal{N}}\|, \end{aligned} \tag{12}$$

where $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$. The *sine* of this angle is $s[\mathcal{M},\mathcal{N}] = (1-c[\mathcal{M},\mathcal{N}]^2)^{1/2}$. If $\mathcal{M}\oplus\mathcal{N} = \mathbb{C}^n$ and *Q* is the oblique projection with range $\mathcal{M}$ and null space $\mathcal{N}$, it is known that

$$\begin{aligned} \|Q\| &= \left(1-\|P_{\mathcal{M}}P_{\mathcal{N}}\|^2\right)^{-1/2} = \left(1-c[\mathcal{M},\mathcal{N}]^2\right)^{-1/2} \\ &= s[\mathcal{M},\mathcal{N}]^{-1}. \end{aligned}$$

For proofs of these results, the reader is referred to Gohberg and Krein [13], Deutsch [11], or Ben-Israel and Greville [5].

Now we can see that Proposition 4.19 is equivalent to the following statement: given $\mu \in \sigma(T)$, the angle between the spectral subspaces $\mathscr{H}_{\mu,\Delta_\lambda^m(T)}$ and $\mathscr{N}_\mu = \bigoplus_{\gamma\neq\mu} \mathscr{H}_{\gamma,\Delta_\lambda^m(T)}$ converges to $\pi/2$. Given $\mu \neq \gamma \in \sigma(T)$, since $\mathscr{H}_{\gamma,\Delta_\lambda^m(T)} \subseteq \mathscr{N}_\mu$, it is easy to see that

$$c\left[\mathscr{H}_{\mu,\Delta_\lambda^m(T)}, \mathscr{H}_{\gamma,\Delta_\lambda^m(T)}\right] \leqslant c\left[\mathscr{H}_{\mu,\Delta_\lambda^m(T)}, \mathscr{N}_\mu\right] \xrightarrow[n\to\infty]{\|\cdot\|} 0.$$

Therefore, also the angle between $\mathscr{H}_{\mu,\Delta_\lambda^m(T)}$ and $\mathscr{H}_{\gamma,\Delta_\lambda^m(T)}$ converges to $\pi/2$. Another description of this fact is that

$$P_{\mathscr{H}_{\mu,\Delta_\lambda^m(T)}} P_{\mathscr{H}_{\gamma,\Delta_\lambda^m(T)}} \xrightarrow[n\to\infty]{\|\cdot\|} 0.$$

This also follows from Eq. (12).

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