Algebraic Theory for the Clique Operator*

Marisa Gutierrez
Departamento de Matemática
Universidad Nacional de La Plata
C. C. 172, (1900) La Plata, Argentina
marisa@mate.unlp.edu.ar

João Meidanis
Instituto de Computação
Universidade Estadual de Campinas
Cx. Postal 6176, 13084-971 Campinas–SP, Brazil
meidanis@ic.unicamp.br

Abstract

In this text we attempt to unify many results about the K operator based on a new theory involving graphs, families and operators. We are able to build an "operator algebra" that helps to unify and automate arguments. In addition, we relate well-known properties, such as the Helly property, to the families and the operators.

As a result, we deduce many classic results in clique graph theory from the basic fact that $CS = I$ for conformal, reduced families. This includes Hamelink's construction, Roberts and Spencer theorem, and Bandelt and Prisner's partial characterization of clique-fixed classes [2]. Furthermore, we show the power of our approach proving general results that lead to polynomial recognition of certain graph classes.
Keywords: Helly graphs, intersection graphs

1 Introduction

In this text graphs are finite, undirected, and simple. In addition, we will not be concerned with the particular representation of graphs, so isomorphic graphs will be the same graph for us.

The clique operator (denoted by K) takes a graph G and returns the intersection graph of the maximal cliques of G. Besides being among the best studied graph operators (such as the line graph, the block graph and the power graph operators, for instance — see Prisner's book *Graph Dynamics* for a review [14]), K is far beyond the most interesting one.

Many questions about K are still open. Probably the most important of them is the one related to the computational complexity of recognizing clique graphs (i.e., graphs that are $K(G)$ for some G). Even though a characterization of clique graphs due to Roberts and Spencer [16] implies that the problem is in NP, still nobody knows whether it is polynomially solvable or NP-complete.

A good portion of the research on the clique operator has focused on *graph classes*. Images under K of chordal graphs, clique-Helly graphs, trees, and many others have been characterized. Many properties of these images have also been discovered. For instance, Escalante [3] has shown that the class of clique-Helly graphs is *clique-fixed*.

In the last few years [8, 9, 10] we have been trying to unify the results about the K operator based on the following:

- Relating graphs to families. Although the K operator takes graphs and returns graphs, it can be seen as a two-step process: first construct the family of cliques; then construct the intersection graph. The intermediate object is a family. Families are equivalent to hypergraphs, and they have

*The authors acknowledge the financial support of Argentinian agency Fund for Quality Improvement of High Education (FOMEC) and Brazilian agencies: the State of São Paulo Research Foundation (FAPESP), the National Council for Scientific and Technological Development (CNPq), and Vitae.

an important feature that is of great help: duality. Graphs lack duality.

- Defining basic operators. Thus, the K operator can be written as $K = LC$, where C is the operator that returns the family of cliques of a graph, and L is the intersection graph operator (takes a family and returns its intersection graph). Two other basic operators are the dual operator D for families (takes a family and returns another family where sets and points are interchanged, and the "belongs to" relation is inverted), and the two-section operator S defined as $S = LD$. These operators satisfy $D^2 = I$ (the identity for families) and $SC = I$ (the identity for graphs). Other operators can be defined, such as U, which adds singletons to a family, or M, which keeps only maximal sets in a family. Thus, in such a way we are able to build an "operator algebra" that helps to unify and automate arguments.

- Relating properties to the families and the operators. In the literature a number of important properties appear, such as being a *conformal* family (when $CS(\mathcal{F}) = M(\mathcal{F})$), being a *Helly* family (when $D(\mathcal{F})$ is conformal), being a *reduced* family (when $M(\mathcal{F}) = \mathcal{F}$), and being a *separating* family (when $D(\mathcal{F})$ is reduced). Besides, the operators themselves also carry useful properties. Operator C always returns conformal, reduced families. Operator U separates, while maintaining the image under S (i.e $SU = S$). Operator M reduces, while maintaining the image under S (i. e. $SM = S$). Relating these properties to the families and the operators also helps to unify and automate arguments.

As a result of our research we have reached the conclusion that many important results in clique graph theory rely on the fact that $CS = I$ for conformal, reduced families. This includes Hamelink's construction, Roberts and Spencer theorem, and Bandelt and Prisner's partial characterization of clique-fixed classes [2]. We demonstrate the power of our approach proving general results that lead to known polynomial recognition of certain classes.

The remaining of this paper is organized as follows. Section 2 contains the basic definitions, including those about the operators. Sections 3 and 4 contain definitions and results about several graph classes previously studied, but with proofs in the context of our new theory. Section 5 presents our concluding remarks.

2 Definitions and basic results

2.1 Graphs, families, and their classes

Next we introduce graphs, families, and their classes. To minimize misunderstanding throughtout the text, we try and use a somehow consistent notation, summarized as follows.

Vertices: lowercase roman letters (u, v, ...).

Graphs: uppercase roman letters (G, H,...).

Graph Classes: boldface names (**Chordal**, ...).

Elements: lowercase roman letters (u, v, ...).

Sets: uppercase roman letters (A, B,...).

Families: calligraphic letters ($\mathcal{F}$, $\mathcal{A}$, $\mathcal{B}$, ...).

Family Classes: slanted names (*Conformal*, ...).

In addition, the following diagram shows "hierarchical" roles of these objects.

```
 graph                          family
classes                         classes
   |                               |
graphs                          families
   |                               |
vertices                          sets
                                   |
                                elements
```

2.1.1 Graphs

For a set V, let $[V]^2$ denote the set of all two-element subsets of V. A *graph* is a pair (V, E) where V is a finite, nonempty set and $E \subseteq [V]^2$. The elements of V are *vertices* and the elements of E are *edges*. Notation for edges: $\{u, v\} = uv$ or vu.

Two graphs (V, E) and (V', E') are *isomorphic* when there is a bijection $a : V \mapsto V'$ such that $a(E) = E'$, where $a(E) = \{a(e) \mid e \in E\}$ and $a(uv) = a(u)a(v)$. This is an equivalence relation. We will not distinguish isomorphic graphs and will generally write $G = H$ when G and H are isomorphic.

A *class of graphs* is a subset of graphs closed under isomorphism.

A graph (V, E) is a *subgraph* of a graph (V', E') when $V \subseteq V'$ and $E \subseteq E'$. Notation: $(V, E) \leq (V', E')$. In addition, (V, E) is an *induced subgraph* of (V', E') when $E = E' \cap [V]^2$. Notation: $(V, E) \subseteq (V', E')$.

Notice that $G \subseteq H$ implies $G \leq H$.

A set C of vertices of a graph (V, E) is *complete* when any two vertices of C are adjacent. A maximal complete subset of V is called a *clique*.

2.1.2 Families

A *family* is a pair (I, F) where I is a finite, nonempty set and F is a mapping from I to the class of all sets such that $F(i)$ is a finite, nonempty set for all $i \in I$. We usually denote the set $F(i)$ by F_i and a family (I, F) by $(F_i)_{i \in I}$. We will call *elements* the elements of $\bigcup_{i \in I} F_i$ and *members* the sets F_i.

Two families $\mathcal{F} = (F_i)_{i \in I}$ and $\mathcal{A} = (A_j)_{j \in J}$ are *isomorphic* when there are two bijections $a : I \mapsto J$ and $b : \bigcup_{i \in I} F_i \mapsto \bigcup_{j \in J} A_j$ such that $b(F_i) = A_{a(i)}$ for all $i \in I$. This is an equivalence relation. We will not distinguish isomorphic families and will generally write $\mathcal{F} = \mathcal{A}$ when $\mathcal{F}$ and $\mathcal{A}$ are isomorphic.

A *class of families* is a subset of families closed under isomorphism.

A family $(F_i)_{i \in I}$ is a *subfamily* of (or: is contained in) a family $(A_j)_{j \in J}$ when $I \subseteq J$ and $F_i = A_i$ for all $i \in I$. Notation: $(F_i)_{i \in I} \subseteq (A_j)_{j \in J}$.

There is still another important relation among families:

A family $(F_i)_{i \in I}$ is *below* another family $(A_j)_{j \in J}$ when there is a mapping $a : I \mapsto J$ such that $F_i \subseteq A_{a(i)}$ for all $i \in I$. Notation: $(F_i)_{i \in I} \leq (A_j)_{j \in J}$.

In this case we also say that $(A_j)_{j \in J}$ is *above* $(F_i)_{i \in I}$.

This relation is a preorder, that is, it is reflexive and transitive, but not antisymmetric. However, it is antisymmetric (and hence a partial order) for *reduced* families, defined later in Section 2.3.

Notice that $\mathcal{F} \subseteq \mathcal{A}$ implies $\mathcal{F} \leq \mathcal{A}$.

2.2 Operators

Let **Graph** be the class of all graphs and *Family* be the class of all families.

We define the *intersection* operator L : *Family* $\mapsto$ **Graph** as follows. Given a family $\mathcal{F} = (F_i)_{i \in I}$, $L(\mathcal{F})$ is the graph (V, E) where $V = I$ and $E = \{ij \mid i \neq j \text{ and } F_i \cap F_j \neq \emptyset\}$.

We define the *family-of-cliques* operator C : **Graph** $\mapsto$ *Family* as follows. Given a graph $G = (V, E)$, $C(G)$ is the family $(F_i)_{i \in I}$ where I is the set of all cliques of (V, E) and $F_i = i$ for all $i \in I$.

We define the *dual* operator D : *Family* $\mapsto$ *Family* as follows. Given a family $\mathcal{F} = (F_i)_{i \in I}$, $D(\mathcal{F})$ is the family $(A_j)_{j \in J}$ where $J = \bigcup_{i \in I} F_i$ and $A_j = \{i \in I \mid j \in F_i\}$.

It is important that families do not have empty members. Otherwise all we would get is that $D^2(\mathcal{F})$ would be a subfamily of $\mathcal{F}$ instead of the stronger result in Theorem 1.

We define the *two-section* operator S : *Family* $\mapsto$ **Graph** as follows. Given a family $\mathcal{F} = (F_i)_{i \in I}$, $S(\mathcal{F})$ is the graph (V, E) where $V = \bigcup_{i \in I} F_i$ and $E = \{uv \mid \text{there is } i \in I \text{ such that } u, v \in F_i\}$.

Theorem 1 *If $\mathcal{A}$, $\mathcal{F}$ are families, G and H are graphs, we have:*

- $\mathcal{F} \subseteq \mathcal{A} \Longrightarrow L(\mathcal{F}) \subseteq L(\mathcal{A})$
- $\mathcal{F} \leq \mathcal{A} \not\Longrightarrow L(\mathcal{F}) \leq L(\mathcal{A})$
- $G \subseteq H \not\Longrightarrow C(G) \subseteq C(H)$
- $G \leq H \Longrightarrow C(G) \leq C(H)$.
- $\mathcal{F} \subseteq \mathcal{A} \Longrightarrow D(\mathcal{F}) \leq D(\mathcal{A})$
- $\mathcal{F} \leq \mathcal{A} \not\Longrightarrow D(\mathcal{F}) \leq D(\mathcal{A})$
- $D^2(\mathcal{F}) = \mathcal{F}$ *for every family* $\mathcal{F}$.
- $\mathcal{F} \subseteq \mathcal{A} \not\Longrightarrow S(\mathcal{F}) \subseteq S(\mathcal{A})$
- $\mathcal{F} \leq \mathcal{A} \Longrightarrow S(\mathcal{F}) \leq S(\mathcal{A})$
- $SC(G) = G$ *for all graphs* G.
- $LD = S$ *and* $SD = L$

2.3 Reduced and separating families

A family $(F_i)_{i \in I}$ is *reduced* when $i \neq j \Longrightarrow F_i \not\subseteq F_j$ for all pairs $i, j \in I$.

It is straightforward to verify that this property is invariant under isomorphism, so we can speak of the class of all reduced families, namely, *Reduced*.

Theorem 2 *If $\mathcal{A}$ and $\mathcal{F}$ are reduced families with $\mathcal{A} \leq \mathcal{F}$ and $\mathcal{F} \leq \mathcal{A}$, then $\mathcal{A} = \mathcal{F}$.*

Proof: Let $\mathcal{F} = (F_i)_{i \in I}$ and $\mathcal{A} = (A_j)_{j \in J}$. If $\mathcal{F} \leq \mathcal{A}$ and $\mathcal{A} \leq \mathcal{F}$ there are mappings $a : I \mapsto J$ and $a' : J \mapsto I$ such that $F_i \subseteq A_{a(i)}$ for all $i \in I$, and $A_j \subseteq F_{a'(j)}$ for all $j \in J$.

Then we have that

$$|\bigcup_{i \in I} F_i| = |\bigcup_{j \in J} A_j|.$$

and

$$F_i \subseteq A_{a(i)} \subseteq F_{a'(a(i))}. \tag{1}$$

But $\mathcal{F}$ is a reduced family then $i = a'(a(i))$ for all $i \in I$. Exchanging the roles of $\mathcal{F}$ and $\mathcal{A}$ we can also obtain that aa' is the identity. Hence a and a' are bijections. By (1), $F_i = A_{a(i)}$. Therefore $(F_i)_{i\in I}$ and $(A_j)_{j\in J}$ are isomorphic. Hence $\mathcal{F} = \mathcal{A}$. □

Let $(F_i)_{i\in I}$ be a family. We say that $u \in \cup_{i\in I}F_i$ is *separated* by the family when $\bigcap_{i\in I, u\in F_i} F_i = \{u\}$. A family is *separating* when it separates every element in $\cup_{i\in I}F_i$.

It is straightforward to verify that this property is invariant under isomorphism, so we can speak of the class of all separating families, namely, *Separating*.

The following theorem tells us that both properties, reduced and separating are dual.

Theorem 3 *A family is separating if and only if its dual is reduced.*

2.4 Operators for reduction and separation

In this section we introduce operators that make a family either reduced or separating, without changing its image under either S or L. So, we are looking at four operators: one that reduces maintaining the image under S, another that separates maintaining the image under S, and two others that do these things maintaining the image of the family under L.

The first operator, called M (for "maximal sets"), acts in the following way. Given $\mathcal{F} = (F_i)_{i\in I}$, throw away all F_i properly contained in another $F_{i'}$, then remove duplicates (if any), creating a subfamily. This operator has the following properties:

Theorem 4 *For any family $\mathcal{F}$ we have:*

- $M(\mathcal{F})$ *is reduced.*
- $M(\mathcal{F}) \subseteq \mathcal{F} \leq M(\mathcal{F})$
- $SM(\mathcal{F}) = S(\mathcal{F})$.

Thus M reduces a family, maintaining its image under S.

The second operator, called U, acts as follows. Given a family $F = (F_i)_{i\in I}$, it adds members of the form $\{u\}$ for each $u \in \bigcup_{i\in I} F_i$. It has the following properties:

Theorem 5 *For any family $\mathcal{F}$ we have:*

- $U(\mathcal{F})$ *is separating.*
- $U(\mathcal{F}) \leq \mathcal{F} \subseteq U(\mathcal{F})$
- $SU(\mathcal{F}) = S(\mathcal{F})$.

Thus U separates a family, maintaining its image under S.

The analogous operators for L can be readily obtained from M and U since $LD = S$ and $SD = L$.

Theorem 6 *For any family $\mathcal{F}$ we have:*

- $DMD(\mathcal{F})$ *is separating.*
- $LDMD(\mathcal{F}) = L(\mathcal{F})$
- $DUD(\mathcal{F})$ *is reduced.*
- $LDUD(\mathcal{F}) = L(\mathcal{F})$.

2.5 Helly and conformal families

A family $(F_i)_{i\in I}$ is called *intersecting* when $F_i \cap F_j \neq \emptyset$ for all pairs $i, j \in I$. A family $(F_i)_{i\in I}$ is *Helly* or *has the Helly property* when all its intersecting subfamilies of the form $(F_i)_{i\in I'}$, for $\emptyset \neq I' \subseteq I$, have a nonempty intersection.

It is straightforward to verify that this property is invariant under isomorphism, so we can speak of the class of Helly families. We denote by *Helly* this class.

Theorem 7 (Three-Point Condition) *A family $(F_i)_{i\in I}$ is Helly if and only if for each triple $x, y, z \in \bigcup_{i\in I} F_i$ there is an element $w \in \bigcup_{i\in I} F_i$ with*

$$|F_i \cap \{x, y, z\}| \geq 2 \Longrightarrow w \in F_i,$$

for all $i \in I$.

A family $\mathcal{F}$ is *conformal* when its dual is a Helly family. We denote by *Conformal* the class of all conformal families.

Theorem 8 (Three-Set Condition) *A family $(F_i)_{i\in I}$ is conformal if and only if for each triple $i, j, k \in I$ there is an index $l \in I$ with*

$$(F_i \cap F_j) \cup (F_j \cap F_k) \cup (F_k \cap F_i) \subseteq F_l. \qquad (2)$$

2.6 Operators for conformalization and hellization

The reader may have noticed that a family can be made Helly by addition of elements, and can be made conformal by addition of sets. This section formalizes these results. It turns out that one can always make a family Helly maintaining its image under L, but not always under S. For conformal, the situation is dual:

one can easily conformize maintaining the image under S, but not, in general, maintaing the image under L.

Let us start with conformization. The composite operator CS performs the desired task. First, a useful lemma.

Lemma 1 *If $\mathcal{F}$ is conformal then $CS(\mathcal{F}) \subseteq \mathcal{F}$.*

Proof: Notice that a clique R in $S(\mathcal{F})$ is a clique in $LD(\mathcal{F})$. Since $D(\mathcal{F})$ is Helly, and R is a maximal intersecting subfamily of $D(\mathcal{F})$, there is an element in its common intersection, which is a set in $\mathcal{F}$. This set contains R, and, by maximality of R, this set is actually equal to R. Then R is a member of $\mathcal{F}$. □

Theorem 9 *For any family $\mathcal{F}$ we have:*

- *$CS(\mathcal{F})$ is conformal and reduced.*
- *$SCS(\mathcal{F}) = S(\mathcal{F})$*
- *Family $\mathcal{F}$ is conformal if and only if $CS(\mathcal{F}) = M(\mathcal{F})$*
- *If $\mathcal{F}$ is conformal and reduced, then $CS(\mathcal{F}) = \mathcal{F}$*
- *If $\mathcal{F}'$ is any conformal family with $S(\mathcal{F}') = S(\mathcal{F})$, then $M(\mathcal{F}') = CS(\mathcal{F})$.*

Proof: The first statement is true because the family of cliques of any graph is conformal and reduced. In fact, it satisfies the Three-Set Condition, because $(F_i \cap F_j) \cup (F_j \cap F_k) \cup (F_k \cap F_i)$ is a complete set whenever F_i, F_j, and F_k are cliques. And every complete set is contained in some clique. Also, cliques form a reduced family because they are maximal complete sets.

The second statement is an immediate consequence of $SC = I$. For the third one, first note that $\mathcal{F} \leq CS(\mathcal{F})$ for every family $\mathcal{F}$. Together with Lemma 1 and Theorem 2, this shows that families $\mathcal{F}$ and $CS(\mathcal{F})$ are each one below the other one, that is, differ only by contained sets. Hence they have the same reduction, but since $CS(\mathcal{F})$ is reduced, this implies $CS(\mathcal{F}) = M(\mathcal{F})$. The reverse follows by noticing that $\mathcal{F}$ is conformal if and only if $M(\mathcal{F})$ is conformal, by the Three-Set Condition. Of course, if $\mathcal{F}$ is also reduced, then $CS(\mathcal{F}) = \mathcal{F}$, which is the fourth statement.

For the last one, notice that $S(\mathcal{F}') = S(\mathcal{F})$ implies $CS(\mathcal{F}') = CS(\mathcal{F})$. Since $\mathcal{F}'$ is conformal, CS becomes M, and the result follows. □

We make a little digression here to offer the converse of the result $\mathcal{F} \leq \mathcal{A} \Longrightarrow S(\mathcal{F}) \leq S(\mathcal{A})$.

Theorem 10 *If $\mathcal{A}$ is conformal, then $S(\mathcal{F}) \leq S(\mathcal{A})$ implies $\mathcal{F} \leq \mathcal{A}$.*

Proof: We have

$$\mathcal{F} \leq CS(\mathcal{F}) \leq CS(\mathcal{A}) \subseteq \mathcal{A}.$$

The first relation is a general property of CS, the second is true by hypothesis and because C preserves $\leq$, and the third is a consequence of the conformality of $\mathcal{A}$. □

Coming back to our main theme, we have seen that CS produces actually a conformal, reduced family while maintaining S. This shows that S(*Conformal*) = **Graph**. Moreover, unlike operators M and U, this is essentially the *only* way to make a family conformal maintaining S, since any other such family differs from the one given by CS by contained sets.

On the other hand, there is no general technique for conformalization maintaining L. Not all families can be fixed in this way.

Let us now look at hellization procedures. The situation is analogous. The operator $DCSD = DCL$ produces a Helly family with the same L (showing that L(*Helly*) = **Graph**), and there is no way, in general, to produce a Helly family with the same S as a given family. Deciding which families are hellizable maintaining S is equivalent to deciding which graphs are clique graphs.

2.7 Composition of operators

Compositions of the operators L, C, D, S have several important properties. We already had the opportunity to see some trivial: DD is the identity for families, SC is the identity for graphs, $LD = S$, and $SD = L$. Table 1 shows the other possible compositions of these operators and what we know about them.

Table 1: Compositions of operators. A dash ("-") indicates impossible composition.

	L	C	S	D
L	-	$LC = K$	-	$LD = S$
C	$CL = D^a$	-	$CS = I^b$	-
S	-	$SC = I$	-	$SD = L$
D	-	DC	-	$DD = I$

[a] Only for Helly, separating families.
[b] Only for conformal, reduced families.

In this table, I denotes the identity operador. We use the same letter for the identity in **Graph** as in *Family*.

The composition LC is K by definition, of course.

The compositions CS and CL are very interesting. Under certain conditions they can be simplified to I and D, respectively, and this will play a key role in determining the image under K of several graph classes. In Theorem 9 we saw that for CS previously. The result can be easily passed to CL by composing with D.

Theorem 11 *We have*

- *Family $\mathcal{F}$ is Helly if and only if $CL(\mathcal{F}) = MD(\mathcal{F})$.*
- *If $\mathcal{F}$ is Helly and separating then $CL(\mathcal{F}) = D(\mathcal{F})$.*

The behavior of these compositions leads to a trivial result about the injectivity of S and L.

Theorem 12 *S is an injective operator for conformal, reduced families. L is an injective operator for Helly, separating families.*

In addition, the following is true: $C(\mathbf{Graph}) =$ *Conformal* $\cap$ *Reduced*.

2.8 Complete edge covers

During our study of the clique operator, we often need to search for "a family of complete sets that covers all the edges of a graph". For instance, Roberts and Spencer [16] look for such a family that has the Helly property. It is interesting to note that there are other ways of expressing this concept, one of them using the $\leq$ relation for families, and the other using the S operator.

A technical detail: to handle isolated vertices correctly, we require that the family covers the edges *and* the vertices. For graphs without isolated vertices, covering the edges is enough.

Let G be a graph and $\mathcal{F}$ a family. We say that $\mathcal{F} = (F_i)_{i \in I}$ is a *complete edge cover* of $G = (V, E)$ when $V = \bigcup_{i \in I} F_i$, F_i is a complete set of G for each $i \in I$, and $uv \in E \Longrightarrow \exists i \in I$ with $u, v \in F_i$.

Define also $A(G)$ for G a graph as follows. If $G = (V, E)$, $A(G)$ is the family $(F_i)_{i \in I}$ where $I = V \cup E$, $F_i = \{v\}$ for $i \in V$, and $F_i = \{u, v\}$ for $i = uv \in E$. The family $A(G)$ is the family of all vertices and edges of G.

Theorem 13 *For a graph G and a family $\mathcal{F}$ the following are equivalent:*

1. *$\mathcal{F}$ is a complete edge cover of G.*
2. *$A(G) \leq \mathcal{F} \leq C(G)$.*
3. *$G = S(\mathcal{F})$.*

Proof: 1) $\Rightarrow$ 2): $A(G) \leq \mathcal{F}$ because all vertices and edges of G must be contained in a member of $\mathcal{F}$. On the other hand each member of $\mathcal{F}$ is a complete set of G and it is contained in a clique of G. Hence $\mathcal{F} \leq C(G)$.

2) $\Rightarrow$ 3): By Theorem 1 we have that $SA(G) \leq S(\mathcal{F}) \leq SC(G)$. But both extremes of the inequality are G, and then $S(\mathcal{F}) = G$.

3) $\Rightarrow$ 1): Since $G \leq S(\mathcal{F})$ all vertices and edges of G are contained in some member of $\mathcal{F}$. But also $S(\mathcal{F}) \leq G$ then all members of $\mathcal{F}$ are complete sets of G. Hence $\mathcal{F}$ is a complete edge cover of G. □

3 Classes

In this work we will be interested in several particular classes of graphs and families. This section summarizes the definitions and some properties of these classes.

The operators of Section 2.2 were defined for graphs and families, but they can be extended to classes in the usual way. For instance,

$$L(\mathit{Class}) = \{L(\mathcal{F}) \mid \mathcal{F} \in \mathit{Class}\},$$

and so on.

3.1 Family classes

Table 2 presents most of the family classes we study. We have the following class containments:

$$\mathit{Interval} \subseteq \mathit{RDTP\text{-}V} \subseteq \mathit{DTP\text{-}V} \subseteq \mathit{TP\text{-}V} \subseteq \mathit{Subtree}$$

$$\mathit{Interval} \subseteq \mathit{RDTP\text{-}E} \subseteq \mathit{DTP\text{-}E} \subseteq \mathit{TP\text{-}E} \cap \mathit{Helly} \subseteq \mathit{TP\text{-}E}$$

$$\mathit{Interval} \subseteq \mathit{CircularArc}$$

Apart from these we have the already defined classes *Reduced*, *Separating*, *Helly*, and *Conformal*. It is interesting to see how these classes relate to one another. Table 3 shows some of this information.

3.2 Graph classes

Each family class defined above can potentially lead to two graph classes, by taking its image under the operators S and L. Sometimes $S(\mathit{Class}) = L(\mathit{Class})$, but this is not true for any of the above families.

Table 4 shows the corresponding graph classes.

Table 2: Family Classes.

Class	A family belongs to this class when its instances $(F_i)_{i\in I}$ satisfy ...
Subtree	... there is a tree T such that $V(T) = \bigcup_{i\in I} F_i$ and each F_i induces a subtree of T.
TP-V	... there is a tree T such that $V(T) = \bigcup_{i\in I} F_i$ and each F_i induces a path of T.
TP-E	... there is a tree T such that $E(T) = \bigcup_{i\in I} F_i$ and each F_i induces a path of T.
DTP-V	... there is a directed tree T such that $V(T) = \bigcup_{i\in I} F_i$ and each F_i induces a directed path of T.
DTP-E	... there is a directed tree T such that $E(T) = \bigcup_{i\in I} F_i$ and each F_i induces a directed path of T.
RDTP-V	... there is a rooted directed tree T such that $V(T) = \bigcup_{i\in I} F_i$ and each F_i induces a path of T.
RDTP-E	... there is a rooted directed tree T such that $E(T) = \bigcup_{i\in I} F_i$ and each F_i induces a path of T.
Interval	... there exists a total order on $\bigcup_{i\in I} F_i$ such that each F_i is an interval with respect to this order.
CircularArc	... there exists a circular order on $\bigcup_{i\in I} F_i$ such that each F_i is an interval (arc) with respect to this order.

Table 3: The first table indicates which families are reduced, closed under M, closed under DUD, separating, closed under U, and closed under DMD; the second table indicates which families are Helly, closed under DCL, conformal, and closed under CS.

Class	R	M	DUD	S	U	DMD
Subtree	N	Y	Y	N	Y	Y
TP-V	N	Y	Y	N	Y	Y
TP-E	N	Y	Y	N	Y	Y
DTP-V	N	Y	Y	N	Y	Y
DTP-E	N	Y	Y	N	Y	Y
RDTP-V	N	Y	Y	N	Y	Y
RDTP-E	N	Y	Y	N	Y	Y
Interval	N	Y	N	N	Y	Y
CircularArc	N	Y	N	N	Y	Y
Reduced	Y	Y	Y	N	N	N
Separating	N	N	N	Y	Y	Y
Helly	N	Y	Y	N	Y	Y
Conformal	N	Y	Y	N	Y	Y

Class	H	DCL	C	CS
Subtree	Y	Y	N	Y
TP-V	Y	Y	N	N
TP-E	N	N	N	N
DTP-V	Y	Y	Y	Y
DTP-E	Y	Y	Y	Y
RDTP-V	Y	Y	Y	Y
RDTP-E	Y	Y	Y	Y
Interval	Y	Y	Y	Y
CircularArc	N	N	N	Y
Reduced	N	N	N	Y
Separating	N	Y	N	N
Helly	Y	Y	N	N
Conformal	N	N	Y	Y

Table 4: Graph Classes.

Class	L(*Class*)	S(*Class*)
Subtree	**Chordal**	**DuallyChordal**
TP-V	**UV**	**DuallyChordal**
TP-E	**UE**	**DuallyUE**
TP-E $\cap$ *Helly*	**UEH**	**DuallyUEH**
DTP-V	**DV**	**DuallyDV**
DTP-E	**DE**	**DuallyDE**
RDTP-V	**RDV**	**DuallyRDV**
RDTP-E	**RDE**	**DuallyRDE**
Interval	**Interval**	**Indifference**
CircularArc	**CA**	**DuallyCA**
Helly	**Graph**	S(*Helly*)
Conformal	S(*Helly*)	**Graph**
Helly $\cap$ *Conformal*	**Helly**	**Helly**

In general by **A** (respectively **DuallyA**) we denote the class of graphs $L(A)$ (respectively $S(A)$).

There is also a class **Helly** of graphs whose family of cliques satisfies the Helly property, that is, **Helly** $= C^{-1}(Helly)$.

4 Using the operators

Now we will prove several known results in the setting of the present paper, and relate the constructions with the operators.

4.1 Intersection and two-section classes

We try to unify properties of all classes of graphs defined as intersection or two-section graphs of a family class. Let A be a family class and $L(A)$ (resp. $S(A)$) the class of all graphs obtained by intersection (resp. two-section) of families in A. First, we will obtain trivial characterizations for classes $L(A)$ and $S(A)$.

As we saw in Theorem 13, $G = S(F)$ is equivalent to "F is a complete edge cover of G". It follows that $G \in S(A)$ is equivalent to "there is a complete edge cover of G in A."

Theorem 14 *If $A \subseteq$ Conformal and $M(A) \subseteq A$ then*

$$G \in S(A) \Longleftrightarrow C(G) \in A.$$

Proof: If $G = S(\mathcal{F})$ with $\mathcal{F} \in A$, then $C(G) = CS(\mathcal{F}) = M(\mathcal{F})$, because $\mathcal{F}$ is conformal (see Theorem 9). However, this last family is in A, since A is closed under M. The converse is true because $SC = I$. □

The dual of this Theorem is:

Theorem 15 *If $A \subseteq$ Helly and $DMD(A) \subseteq A$ then*

$$G \in L(A) \Longleftrightarrow DC(G) \in A.$$

Example 1 *It is easy to see that Helly, Helly $\cap$ Conformal, Subtree, TP-E $\cap$ Helly, DTP-V, DTP-E, RDTP-V, and Interval are Helly. To prove that all of them are closed under DMD, a hint is to think that $DMD(\mathcal{F})$ is the family obtained from $\mathcal{F}$ by deleting all dominated elements in $\mathcal{F}$. Then, applying Theorem 15 we obtain several results that have been obtained in different works by different authors:*

Fulkerson-Gross's Theorem [4], which characterizes interval graphs, Gavril's results [5, 6, 7] for chordal graphs, UV-graphs, and RDV-graphs, and the Clique-tree Theorem [13] for DV-graphs and other families.

4.2 The behaviour of K

In this section we will show how we can obtain results related to the clique operator. First, we will obtain a trivial result about clique-families of graphs in $L(A)$, from Theorem 11.

Theorem 16 *If $A \subseteq$ Helly then $CL(A) = MD(A)$.*

The following corollary gives a sufficient condition for all the graphs obtained as intersection graphs of families in A to have the Helly property in their clique-family.

Corollary 1 *If $A \subseteq$ Helly $\cap$ Conformal then $L(A) \subseteq$* **Helly**.

Proof: By Theorem 16 we have $CL(A) = MD(A)$. But $D(A)$ is Helly because A is conformal, and subfamilies of Helly families are also Helly families. □

Example 2 *Since DTP-V, DTP-E, RDTP-V and Interval are Helly and conformal families, we obtain that the clique family of graphs in each of* **DV**, **DE**, **RDV** *and* **Interval** *have the Helly property. In other words, the classes* **DV**, **DE**, **RDV** *and* **Interval** *are contained in* **Helly**.

In the following theorem we will see how the behaviour of the clique operator is similar in some classes of intersection graphs.

Theorem 17 *[8, 9] If (i) $A \subseteq$ Helly, (ii) $DMD(A) \subseteq A$, and (iii) $U(A) \subseteq A$ then $KL(A) = S(A)$.*

Proof:

$$
\begin{array}{rcll}
KL(A) & = & LCL(A) & \text{because } K = LC \\
 & = & LMD(A) & \text{by Theorem 16} \\
 & = & SDMD(A) & \text{because } L = SD \\
 & \subseteq & S(A) & \text{by } (ii)\ .
\end{array}
$$

Conversely

$$
\begin{array}{rcll}
S(A) & = & SU(A) & \text{because } SU = S \\
 & = & LDU(A) & \text{since } S = LD \\
 & = & LCSDU(A) & DU(A) \text{ reduc., conf. } (i) \\
 & = & KLU(A) & K = LC \text{ and } L = SD \\
 & \subseteq & KL(A) & \text{by } (iii)\ .
\end{array}
$$

□

Example 3 *As we said previously Helly, Helly ∩ Conformal, Subtree, TP-E ∩ Helly, DTP-V, DTP-E, RDTP-V , and Interval are Helly families closed under DMD. It is trivial that all of them are also closed under U. Then, applying Theorem 17 we obtain several results that have been obtained in different works by different authors.*

- K(**Graph**) = S(*Helly*)
- K(**Helly**) = **Helly** [3]
- K(**Chordal**) = **DuallyChordal** [1, 17]
- K(**UEH**) = **DuallyUEH** [9].
- K(**DV**) = **DuallyDV** [15]
- K(**DE**) = **DuallyDE** [10]
- K(**RDV**) = **DuallyRDV** [15]
- K(**Interval**) = **Indifference** [12]

Clearly we can obtain a dual result from Theorem 17.

Theorem 18 *[8] If $A \subseteq$ Conformal, $M(A) \subseteq A$, and $DUD(A) \subseteq A$ then $KS(A) = L(A)$.*

Then we can obtain the Bandelt-Prisner result about clique fixed classes [2], as well as prove the behaviour of the K operator in several classes.

- K(**DuallyUEH**) = **UEH**.
- K(**DuallyDV**) = **DV** [15]
- K(**DuallyDE**) = **DE**
- K(**DuallyRDV**) = **RDV** [15]

$$
\begin{tikzcd}
\mathit{Subtree} \arrow[r, "D"] \arrow[d, "L"'] & D(\mathit{Subtree}) \arrow[d, "L"] \\
\textbf{Chordal} \arrow[r, "K"] & \textbf{DuallyChordal} \arrow[d, leftrightarrow, "K"] \\
 & \textbf{Chordal} \cap \textbf{Helly}
\end{tikzcd}
$$

Figure 1: Computing K(**Chordal**) and iterations.

$$
\begin{tikzcd}
\mathit{Interval} \arrow[r, "D"] \arrow[d, "L"'] & D(\mathit{Interval}) \arrow[d, "L"] \\
\textbf{Interval} \arrow[r, "K"] & \textbf{Indifference} \arrow[loop right, "K"]
\end{tikzcd}
$$

Figure 2: Computing K(**Interval**) and iterations.

$$
\begin{tikzcd}
\mathit{DTP\text{-}V} \arrow[r, "D"] \arrow[d, "L"'] & D(\mathit{DTP\text{-}V}) \arrow[d, "L"] \\
\textbf{DV} \arrow[r, leftrightarrow, "K"] & \textbf{DuallyDV}
\end{tikzcd}
$$

Figure 3: Computing K(**DV**) and iterations.

- K(**Indifference**) = **Indifference** [12]

In Figures 1, 2, and 3 we show the effect of K and its iterations on some classes.

Some of these new classes, such as **DuallyA**, have been characterized by properties of a complete edge cover of their graphs using the equivalence between $G \in S(A)$ and "there is a complete edge cover of G in A." In particular, since K(**Graph**) $= S(Helly)$, we can also obtain Roberts and Spencer's Theorem for clique graphs [16].

Unfortunately this characterization does not lead to a good algorithm for recognition of these new classes of graphs. Nevertheless, for some particular classes a general polynomial time algorithm works, as we will see in the following section.

4.3 An algorithm for recognizing two-sections

In this section we rephrase the techniques of Prisner and Szwarcfiter [15] in terms of operators and apply them to a generic class of graphs **A**. Prisner and Szwarcfiter define the graph G' obtained from G by adding a new vertex v' and an edge vv' for each $v \in V(G)$. The result we are interested in focuses on class **DV**, the class of intersection graphs of paths of directed trees, viewed as sets of vertices [13], and on **DuallyDV**, its image under K. Prisner and Szwarcfiter show that

$$G \in \mathbf{DuallyDV} \iff \begin{cases} G \text{ is clique-Helly and} \\ K(G') \in \mathbf{DV}. \end{cases} \qquad (3)$$

This result in some sense reduces the recognition of **DuallyDV** to the recognition of **DV**. We will try to generalize the idea for the other classes that appear in the last column of Table 2. All these classes, with the exception of **Indifference** and **Helly**, lack a polynomial time recognition algorithm. Since they were defined as the image under the clique operator K of a recognized class, a natural idea is study an "inverse" of K. More clearly, if we want to know whether a graph G is in **DuallyA** it is sufficient to find a graph H in **A** such that $K(H) = G$. However, the inverse image of each graph is an infinite set. What element of this inverse image is convenient to select?

Prisner and Szwarcfiter used $K(G')$ for this purpose. In fact, it is not difficult to prove directly that $KK(G')$ is G for every clique-Helly graph G. We will use the same construction, but we would like to point out two interesting facts here. First, $K(G')$ can be written in terms of operators as $LUC(G)$. Second, the construction of $K(G')$, or $LUC(G)$, as we will call it from here on, is the very one used by Hamelink in his celebrated proof that all clique-Helly graphs are clique graphs [11].

In the sequel we rewrite result (3) above, but replacing **DV** by a generic class **A** which shares with **DV** some fundamental properties. The proofs use operator techniques, and the fundamental properties mentioned were deduced from the fact that they are the ones needed for the proofs to work.

Theorem 19 *If (i) $A \subseteq Helly$, (ii) $CS(A) \subseteq A$, (iii) $DMD(A) \subseteq A$, and (iv) $U(A) \subseteq A$ then*

$$G \in \mathbf{DuallyA} \iff \begin{cases} G \text{ is clique-Helly and} \\ LUC(G) \in \mathbf{A} \end{cases}$$

Proof: Since $G \in$ **DuallyA** $= S(A)$ there is a family $\mathcal{F} \in A$ such that $G = S(\mathcal{F})$. But $C(G) = CS(\mathcal{F}) \in A$ by (ii). Hence $C(G)$ is Helly by (i), and conformal and reduced like all families of cliques. Then $LUC(G) \in L(A)$, by (iv).

To prove the converse we will show that $KLUC(G) = G$ and thus G will be a graph in $S(A)$ by Theorem 17. Indeed

$$\begin{array}{rcll} KLUC(G) & = & LCSDUC(G) & K = LC, L = SD \\ & = & LDUC(G) & CS = I \text{ here} \\ & = & SUC(G) & \text{because } LD = S \\ & = & SC(G) & \text{because } SU = S \\ & = & G & \text{because } SC = I \end{array}$$

□

The classes *Interval*, *RDTP-V*, *DTP-V*, *RDTP-E*, *DTP-E*, *Subtree* and *TP-E*$\cap$*Helly* satisfy the hypotheses of this theorem. But in order to obtain a polynomial time algorithm to recognize $S(A)$ we need a polynomial upper bound on the number of cliques of these graphs.

Theorem 20 *If class A is Interval, RDTP-V, DTP-V, RDTP-E, DTP-E, or TP-E$\cap$Helly, then we have that:*

$$G \in S(A) \Rightarrow |C(G)| \leq n(n+1)/2,$$

where $n = |V(G)|$.

Proof: Recall that all these classes are conformal. Hence if $G \in S(A)$, there is a family $\mathcal{F}$ in A such that $G = S(\mathcal{F})$. Then $C(G) = CS(\mathcal{F}) = M(\mathcal{F})$ since A is conformal. In other words, the maximal members of $\mathcal{F}$ are exactly the cliques of $S(\mathcal{F})$. But, in these particular cases, it is clear that there are at most $n(n+1)/2$ different maximal members of $\mathcal{F}$ because each one is a path and therefore is determined by two elements: the end points. □

Unfortunately $S(Subtree)$ is a graph class for which there is no polynomial bound on the number of cliques, because any graph with a universal vertex is in this class.

Thus, these results show that if $\mathbf{A}$ is a class of graphs recognizable in polynomial time that fulfills the hypotheses of Theorems 19 and 20, then $K(\mathbf{A}) =$ **DuallyA** will be recognizable in polynomial time as well. As seen, this is the case of the classes **UEH**, **DV**, **DE**, **RDV** and **Interval**.

5 Conclusion

In this text we attempted to unify many results about the K operator based on a new theory involving graphs, families and operators. We were able to build an "operator algebra" that helps to unify and automate arguments. In addition, we related well-known properties, such as the Helly property, to the families and the operators.

As a result, we deduced many classic results in clique graph theory from the basic fact that $CS = I$ for conformal, reduced families. This includes Hamelink's construction, Roberts and Spencer theorem, and Bandelt and Prisner's partial characterization of clique-fixed classes [2]. Furthermore, we showed the power of our approach proving general results that lead to polynomial recognition of certain graph classes.

6 Acknowledgments

We acknowledge the financial support of Argentinian agency Fund for Quality Improvement of High Education (FOMEC) and of Brazilian agencies: the State of Sao Paulo Research Foundation (FAPESP), the National Council for Scientific and Technological Development (CNPq), and Vitae Foundation. Vitae does not necessarily share the concepts and opinions expressed in this work, for which the authors are solely responsible.

References

[1] V. D. Chepoi A. Brandstadt, F. F. Dragan and V. I. Voloshim. Dually chordal graphs. *Proccedigs of the 19th. International Workshop on Graph Theoretic Concepts in Computer Science, Lecture Notes in Computer Science*, 790:237–251, 1994.

[2] H. J. Bandelt and E. Prisner. Clique graphs and Helly graphs. *J. Combin. Theory B*, 51:34–45, 1991.

[3] F. Escalante. Über iterierte Clique-Graphen. *Abh. Math. Sem. Univ. Hamburg*, 39:59–68, 1973.

[4] D. R. Fulkerson and O. A. Gross. Incidence matrices and interval graphs. *Pacific J. of Mathematics*, 15:835–855, 1965.

[5] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *J. Combinat. Theory B*, 16:47–56, 1974.

[6] F. Gavril. A recognition algorithm for the intersection graphs of directed paths in directed trees. *Discrete Math.*, 13:237–249, 1975.

[7] F. Gavril. A recognition algorithm for the intersection graphs of paths in trees. *Discrete Math.*, 23:211–227, 1978.

[8] M. Gutierrez. Intersection graphs and clique operator. *Graphs and Combinatorics*, 2001. To appear.

[9] M. Gutierrez and J. Meidanis. An operator calculus for clique problems. Submitted to Discrete Applied Math, 2001.

[10] M. Gutierrez and J. Meidanis. Recognizing clique graphs of directed edge path graphs. *Discrete Applied Math*, 2002. To appear.

[11] R. C. Hamelink. A partial characterization of clique graphs. *J. Combin. Theory*, 5:192–197, 1968.

[12] B. Hedman. Clique graphs of time graphs. *J. of Combin. Theory*, 3:270–278, 1984.

[13] C. L. Monma and V. K. Wei. Intersection graphs of paths in a tree. *J. Combin. Theory B*, 41:141–181, 1986.

[14] E. Prisner. *Graph Dynamics*. Number 338 in Pitman Research Notes in Mathematical Series. Longman, 1995.

[15] E. Prisner and J. L. Szwarcfiter. Recognizing clique graphs of directed and rooted path graphs. *Discrete Applied Math.*, 94:321–328, 1999.

[16] F. S. Roberts and J. H. Spencer. A characterization of clique graphs. *J. Combin. Theory B*, 10:102–108, 1971.

[17] J. L. Szwarcfiter and C. F. Bornstein. Clique graphs of chordal graphs and path graphs. *SIAM J. Discrete Math.*, 7:331–336, 1994.