Clique-critical graphs: Maximum size and recognition

Liliana Alcón

Departamento de Matemática. Universidad Nacional de La Plata. C.C. 172 (1900) La Plata, Argentina

Received 22 December 2004; received in revised form 10 May 2005; accepted 18 January 2006

Available online 18 May 2006

Abstract

The clique graph of $G$, $K(G)$, is the intersection graph of the family of cliques (maximal complete sets) of $G$. Clique-critical graphs were defined as those whose clique graph changes whenever a vertex is removed. We prove that if $G$ has $m$ edges then any clique-critical graph in $K^{-1}(G)$ has at most $2m$ vertices, which solves a question posed by Escalante and Toft [On clique-critical graphs. J. Combin. Theory B 17 (1974) 170–182]. The proof is based on a restatement of their characterization of clique-critical graphs. Moreover, the bound is sharp. We also show that the problem of recognizing clique-critical graphs is NP-complete.

Keywords: Clique graphs; Clique-critical graphs; NP-complete problems

1. Introduction and basic definitions

We consider simple, finite, undirected graphs. Given a graph $G$, $V(G)$ and $E(G)$ denote, respectively, the vertex and edge sets of $G$. A complete set of $G$ is a subset of $V(G)$ inducing a complete subgraph. A clique is a maximal complete set. Let $\mathcal{C}(G)$ be the family of cliques of $G$, the clique graph of $G$, $K(G)$, is the intersection graph of $\mathcal{C}(G)$. It is said that $G$ is a clique graph if there exists $H$ such that $K(H) = G$. Not every graph is a clique graph; characterizations of clique graphs are given in [4,1], however the time complexity of the problem of recognizing clique graphs is still open.

For a given $G$, let $K^{-1}(G)$ be the set of graphs $H$ such that $K(H) = G$. The operation of adding to $H$ a new vertex adjacent to all vertices of a given clique does not alter its clique graph, i.e. if $H'$ is the resulting graph, then $H' \in K^{-1}(G)$ if and only if $H \in K^{-1}(G)$. It follows that if $K^{-1}(G)$ is not empty then it is an infinite set.

On studying $K^{-1}(G)$, it is natural not to take into consideration the graphs obtained by that or other enlarging operation. This motivated the notion of clique-critical graph introduced in [2] as minimal graphs in $K^{-1}(G)$, minimality in the sense that no induced subgraph belongs to $K^{-1}(G)$. Escalante and Toft proved that the number of clique-critical graphs in $K^{-1}(G)$ is always finite and they described the way of adding vertices to clique-critical graphs to obtain all graphs in $K^{-1}(G)$.

We present next a restatement of the characterization of clique-critical graphs given by Escalante and Toft and obtain a simpler description of the way of adding vertices to a graph without changing its clique graph. In Section 2, we prove that any clique-critical graph in $K^{-1}(G)$ has at most $2|E(G)|$ vertices. At the end of their paper [2], in a later note added in proof, Escalante and Toft suggest $3|E(G)|$ for this bound. We show that our bound is tight. In Section 3, we prove that the problem of determining if a graph is clique-critical is NP-complete.

E-mail address: liliana@mate.unlp.edu.ar.

0166-218X/© 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2006.03.024
Let $H$ be a graph and $v \in V(H)$. As usual, $H - v$ denotes the graph induced by $V(H) \setminus \{v\}$. The vertex $v$ is critical (or clique-critical) if $K(H) \neq K(H - v)$. A graph $H$ is critical (or clique-critical) if every one of its vertices is critical.

The following lemma is a reformulation of the characterization of critical vertex given by Escalante and Toft in (6) of [2] in terms of the cliques of the graph.

**Lemma 1.** A vertex $v$ of a graph $H$ is critical if and only if there exist cliques of $H$, $C_1$ and $C_2$, such that either

(i) $\{v\} = C_1 \setminus C_2$, or
(ii) $\{v\} = C_1 \cap C_2$.

**Corollary 2.** A graph $H$ is critical if and only if for each vertex $v$ of $H$ there exist cliques of $H$, $C_1$ and $C_2$, such that either

(i) $\{v\} = C_1 \setminus C_2$, or
(ii) $\{v\} = C_1 \cap C_2$.

The way of adding vertices to a graph without changing its clique graph is described in the following corollary. For $x \notin V(H)$ and $V' \subseteq V(H)$, let $H + x_{V'}$ denote the graph obtained by adding to $H$ the vertex $x$ and making it adjacent to every vertex of $V'$; and let $H[V']$ be the subgraph of $H$ induced by the vertices of $V'$.

**Corollary 3.** The equality $K(H) = K(H + x_{V'})$ holds if and only if

(i) the cliques of $H[V']$ are cliques of $H$, and
(ii) the cliques of $H[V']$ are pairwise intersecting.

### 2. Bound

The following lemma gives an upper bound for the number of vertices of any critical graph belonging to $K^{-1}(G)$. Notice as a consequence of it that a graph $G$ with $m$ edges is a clique graph if and only if there exists $H$ with at most $2m$ vertices such that $K(H) = G$.

**Lemma 4.** Let $G$ be a clique graph with $m > 1$ edges. Any critical graph belonging to $K^{-1}(G)$ has at most $2m$ vertices.

**Proof.** We can assume $G$ is connected and non-trivial. Let $H$ be a critical graph such that $K(H) = G$ and let $C_u$ denote the clique of $H$ corresponding to the vertex $u$ of $G$. If $H$ is a star, $G$ is a complete, then the bound is true. Assume $H$ is not a star and let $A$ be the set of cardinality $2m$ whose elements are the ordered pairs $(u, v)$ for $u, v \in E(G)$. We claim that the following application $f$, from a subset of $A$ into $V(H)$, is surjective, thus $|A| = 2m \geq |V(H)|$.

$$f(u, v) = \begin{cases} C_u \setminus C_v & \text{if } |C_u \setminus C_v| = 1, \\ C_u \cap C_v & \text{if } |C_u \setminus C_v| \neq 1 \text{ and } |C_u \cap C_v| = 1. \end{cases}$$

Indeed, if $x \in V(H)$, since $H$ is critical, by Lemma 1, there exist $C_u$ and $C_v$, cliques of $H$, such that $\{x\} = C_u \setminus C_v$ or $\{x\} = C_u \cap C_v$.

If $\{x\} = C_u \setminus C_v$, then $f(u, v) = C_u \setminus C_v = \{x\}$.

If $\{x\} = C_u \cap C_v$ and $|C_u \setminus C_v| = 0$, then $C_u \subseteq C_v$, this is a contradiction since they are maximal complete sets.

If $\{x\} = C_u \cap C_v$ and $|C_u \setminus C_v| > 1$, then $f(u, v) = C_u \cap C_v = \{x\}$.

If $\{x\} = C_u \setminus C_v$ and $|C_u \setminus C_v| = 1$, then there are two possibilities: first, $|C_v \setminus C_u| > 1$, in this case $f(v, u) = C_u \cap C_v = \{x\}$; and second, $|C_v \setminus C_u| = 1$, in this case, both cliques have exactly two vertices and, since $m > 1$ and $G$ is connected, there exists another clique $C_h$ intersecting $C_u$ or $C_v$, moreover, the intersection contains exactly one vertex. If this vertex is not $x$, (Fig. 1a), then $\{x\} = C_h \setminus C_b$ and thus $f(u, h) = C_y \setminus C_b = \{x\}$. If the vertex is $x$, since $H$ is not a star, we can assume either $|C_h \setminus C_u| > 1$, (Fig. 1b), in this case $f(h, u) = C_h \cap C_u = \{x\}$; or $|C_h \setminus C_u| = 1$ and there exists $C_w$ such that $C_w \cap C_h \neq \emptyset$ and $x \notin C_w$, (Fig. 1c), in this case $f(h, w) = C_h \cap C_w = \{x\}$. The proof is completed. \[\square\]
To show that the bound is sharp, we will exhibit, for each positive integer \( m > 1 \), a graph \( G \) with \( m \) edges and a critical graph \( H \in K^{-1}(G) \) with \( 2m \) vertices.

The graph \( G \) is the bipartite graph \( K_{1,m} \) which, clearly, has \( m \) edges. The graph \( H \) can be depicted as the complete graph \( K_m \) plus a vertex \( v' \) and an edge \( vv' \) for each vertex \( v \) of \( K_m \). Trivially, \( |V(H)| = 2m \); by Corollary 2, \( H \) is critical; and, clearly, \( K(H) = K_{1,m} \).

3. Recognizing clique-critical graphs

In this section, we study the time complexity of recognizing clique-critical graphs.

**Theorem 5.** The problem of recognizing clique-critical graphs is NP-complete.

**Proof.** Let \( H \) be any graph. A certificate of \( H \) being a critical graph is, for each vertex of \( H \), a pair of cliques satisfying (i) or (ii) of Corollary 2. Verifying the exactness of this certificate requires polynomial time, thus the problem belongs to NP.

In [3], it was proved that determining if a connected graph has two disjoint cliques is NP-complete, we will reduce our problem from that one.

Given a non-trivial connected graph \( G \) and \( x \notin V(G) \), let \( G' \) be the graph obtained from \( G + x V(G) \) by adding a vertex \( v' \) and one edge \( vv' \) for each of the vertices \( v \in V(G) \), (Fig. 2). We claim that \( G \) has two disjoint cliques if and only if \( G' \) is critical. Indeed, clearly, any vertex \( v' \) is a clique difference and any vertex \( v \) is a clique intersection, then, by Corollary 2, we need only see what happens with \( x \). In no case, since \( G \) is connected and non-trivial, \( x \) can be a clique difference and, on the other hand, \( x \) is a clique intersection if and only if \( G \) has two disjoint cliques. The proof is complete. \( \square \)
References