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Clique-critical graphs: Maximum size and recognition

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Abstract

The clique graph of G, $K(G)$, is the intersection graph of the family of cliques (maximal complete sets) of G. Clique-critical graphs were defined as those whose clique graph changes whenever a vertex is removed. We prove that if G has m edges then any clique-critical graph in $K^{-1}(G)$ has at most $2m$ vertices, which solves a question posed by Escalante and Toft [On clique-critical graphs, J. Combin. Theory B 17 (1974) 170–182]. The proof is based on a restatement of their characterization of clique-critical graphs. Moreover, the bound is sharp. We also show that the problem of recognizing clique-critical graphs is NP-complete.

Keywords: Clique graphs; Clique-critical graphs; NP-complete problems

1. Introduction and basic definitions

We consider simple, finite, undirected graphs. Given a graph G, $V(G)$ and $E(G)$ denote, respectively, the vertex and edge sets of G. A complete set of G is a subset of $V(G)$ inducing a complete subgraph. A *clique* is a maximal complete set. Let $\mathscr{C}(G)$ be the family of cliques of G, the *clique graph of* G, $K(G)$, is the intersection graph of $\mathscr{C}(G)$. It is said that G *is a clique graph* if there exists H such that $K(H)=G$. Not every graph is a clique graph; characterizations of clique graphs are given in [4,1], however the time complexity of the problem of recognizing clique graphs is still open.

For a given G, let $K^{-1}(G)$ be the set of graphs H such that $K(H)=G$. The operation of adding to H a new vertex adjacent to all vertices of a given clique does not alter its clique graph, i.e. if H' is the resulting graph, then $H' \in K^{-1}(G)$ if and only if $H \in K^{-1}(G)$. It follows that if $K^{-1}(G)$ is not empty then it is an infinite set.

On studying $K^{-1}(G)$, it is natural not to take into consideration the graphs obtained by that or other *enlarging operation*. This motivated the notion of clique-critical graph introduced in [2] as minimal graphs in $K^{-1}(G)$, minimality in the sense that no induced subgraph belongs to $K^{-1}(G)$. Escalante and Toft proved that the number of clique-critical graphs in $K^{-1}(G)$ is always finite and they described the way of adding vertices to clique-critical graphs to obtain all graphs in $K^{-1}(G)$.

We present next a restatement of the characterization of clique-critical graphs given by Escalante and Toft and obtain a simpler description of the way of adding vertices to a graph without changing its clique graph. In Section 2, we prove that any clique-critical graph in $K^{-1}(G)$ has at most $2|E(G)|$ vertices. At the end of their paper [2], in a later note added in proof, Escalante and Toft suggest $3|E(G)|$ for this bound. We show that our bound is tight. In Section 3, we prove that the problem of determining if a graph is clique-critical is NP-complete.

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Let H be a graph and $v \in V(H)$. As usual, $H - v$ denotes the graph induced by $V(H)\backslash\{v\}$. The vertex v is *critical* (or *clique-critical*) if $K(H) \neq K(H - v)$. A graph H is *critical*(or *clique-critical*) if every one of its vertices is critical.

The following lemma is a reformulation of the characterization of critical vertex given by Escalante and Toft in (6) of [2] in terms of the cliques of the graph.

Lemma 1. *A vertex v of a graph H is critical if and only if there exist cliques of H, C_1 and C_2, such that either*

(i) $\{v\} = C_1\backslash C_2$, *or*
(ii) $\{v\} = C_1 \cap C_2$.

Corollary 2. *A graph H is critical if and only if for each vertex v of H there exist cliques of H, C_1 and C_2, such that either*

(i) $\{v\} = C_1\backslash C_2$, *or*
(ii) $\{v\} = C_1 \cap C_2$.

The way of adding vertices to a graph without changing its clique graph is described in the following corollary. For $x \notin V(H)$ and $V' \subseteq V(H)$, let $H + x_{V'}$ denote the graph obtained by adding to H the vertex x and making it adjacent to every vertex of V'; and let $H[V']$ be the subgraph of H induced by the vertices of V'.

Corollary 3. *The equality $K(H) = K(H + x_{V'})$ holds if and only if*

(i) *the cliques of $H[V']$ are cliques of H, and*
(ii) *the cliques of $H[V']$ are pairwise intersecting.*

2. Bound

The following lemma gives an upper bound for the number of vertices of any critical graph belonging to $K^{-1}(G)$. Notice as a consequence of it that a graph G with m edges is a clique graph if and only if there exists H with at most $2m$ vertices such that $K(H) = G$.

Lemma 4. *Let G be a clique graph with $m > 1$ edges. Any critical graph belonging to $K^{-1}(G)$ has at most $2m$ vertices.*

Proof. We can assume G is connected and non-trivial. Let H be a critical graph such that $K(H) = G$ and let C_u denote the clique of H corresponding to the vertex u of G. If H is a star, G is a complete, then the bound is true. Assume H is not a star and let A be the set of cardinality $2m$ whose elements are the ordered pairs (u, v) for $uv \in E(G)$. We claim that the following application f, from a subset of A into $V(H)$, is surjective, thus $|A| = 2m \geqslant |V(H)|$.

$$f(u, v) = \begin{cases} C_u\backslash C_v & \text{if } |C_u\backslash C_v| = 1, \\ C_u \cap C_v & \text{if } |C_u\backslash C_v| \neq 1 \text{ and } |C_u \cap C_v| = 1. \end{cases}$$

Indeed, if $x \in V(H)$, since H is critical, by Lemma 1, there exist C_u and C_v, cliques of H, such that $\{x\} = C_u\backslash C_v$ or $\{x\} = C_u \cap C_v$.

If $\{x\} = C_u\backslash C_v$, then $f(u, v) = C_u\backslash C_v = \{x\}$.

If $\{x\} = C_u \cap C_v$ and $|C_u\backslash C_v| = 0$, then $C_u \subseteq C_v$, this is a contradiction since they are maximal complete sets.

If $\{x\} = C_u \cap C_v$ and $|C_u\backslash C_v| > 1$, then $f(u, v) = C_u \cap C_v = \{x\}$.

If $\{x\}=C_u\cap C_v$ and $|C_u\backslash C_v|=1$, then there are two possibilities: first, $|C_v\backslash C_u| > 1$, in this case $f(v, u)=C_v\cap C_u=\{x\}$; and second, $|C_v\backslash C_u| = 1$, in this case, both cliques have exactly two vertices and, since $m > 1$ and G is connected, there exists another clique C_h intersecting C_u or C_v, moreover, the intersection contains exactly one vertex. If this vertex is not x, (Fig. 1a), then $\{x\} = C_u\backslash C_h$ and thus $f(u, h) = C_u\backslash C_h = \{x\}$. If the vertex is x, since H is not a star, we can assume either $|C_h\backslash C_u| > 1$, (Fig. 1b), in this case $f(h, u) = C_h \cap C_u = \{x\}$; or $|C_h\backslash C_u| = 1$ and there exists C_w such that $C_w \cap C_h \neq \emptyset$ and $x \notin C_w$, (Fig. 1c), in this case $f(h, w) = C_h\backslash C_w = \{x\}$. The proof is completed. □

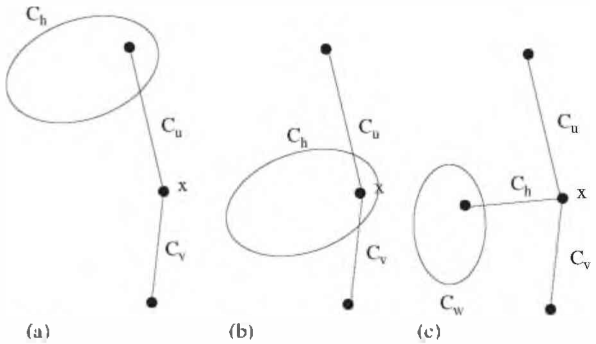

Fig. 1. The cliques C_u, C_v, C_h, and C_w.

To show that the bound is sharp, we will exhibit, for each positive integer $m > 1$, a graph G with m edges and a critical graph $H \in K^{-1}(G)$ with $2m$ vertices.

The graph G is the bipartite graph $K_{1,m}$ which, clearly, has m edges. The graph H can be depicted as the complete graph K_m plus a vertex v' and an edge vv' for each vertex v of K_m. Trivially, $|V(H)| = 2m$; by Corollary 2, H is critical; and, clearly, $K(H) = K_{1,m}$.

3. Recognizing clique-critical graphs

In this section, we study the time complexity of recognizing clique-critical graphs.

Theorem 5. *The problem of recognizing clique-critical graphs is NP-complete.*

Proof. Let H be any graph. A certificate of H being a critical graph is, for each vertex of H, a pair of cliques satisfying (i) or (ii) of Corollary 2. Verifying the exactness of this certificate requires polynomial time, thus the problem belongs to NP.

In [3], it was proved that determining if a connected graph has two disjoint cliques is NP-complete, we will reduce our problem from that one.

Given a non-trivial connected graph G and $x \notin V(G)$, let G' be the graph obtained from $G + x_{V(G)}$ by adding a vertex v' and one edge vv' for each of the vertices $v \in V(G)$, (Fig. 2). We claim that G has two disjoint cliques if and only if G' is critical. Indeed, clearly, any vertex v' is a clique difference and any vertex v is a clique intersection, then, by Corollary 2, we need only see what happens with x. In no case, since G is connected and non-trivial, x can be a clique difference and, on the other hand, x is a clique intersection if and only if G has two disjoint cliques. The proof is complete. □

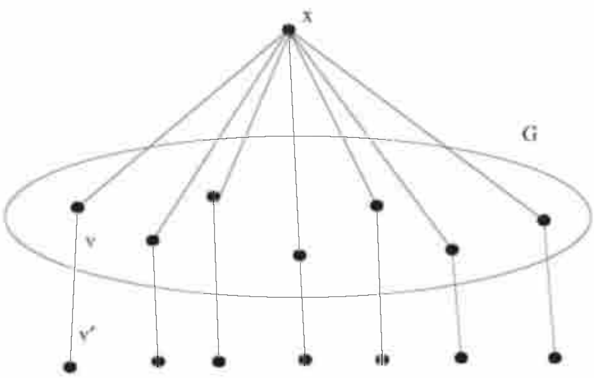

Fig. 2. The graph G'.

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