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Cliques and extended triangles. A necessary condition for planar clique graphs

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Abstract

By generalizing the idea of extended triangle of a graph, we succeed in obtaining a common framework for the result of Roberts and Spencer about clique graphs and the one of Szwarcfiter about Helly graphs. We characterize Helly and 3-Helly planar graphs using extended triangles. We prove that if a planar graph G is a clique graph, then every extended triangle of G must be a clique graph. Finally, we show the extended triangles of a planar graph which are clique graphs. Any one of the obtained characterizations are tested in $O(n^2)$ time.

Keywords: Clique graphs; Helly graphs; Extended triangles; Planar graphs

1. Introduction and basic definitions

We consider simple, finite and undirected graphs. Given a graph G, $V(G)$ denotes its vertex set and $n = |V(G)|$. A *complete* of G is a subset of $V(G)$ inducing a complete subgraph. A *clique* is a maximal complete. We also use the terms complete and clique to refer to the corresponding subgraphs. A complete C *covers* the edge uv if the end vertices, u and v, belong to C. A *complete edge cover* of G is a family of completes covering all its edges.

Given $\mathscr{F} = (F_i)_{i\in I}$ a family of nonempty sets, the sets F_i are called *members* of the family. $\mathscr{F}$ is *pairwise intersecting* if the intersection of any two members is not the

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empty set. The *intersection* or *total intersection* of $\mathscr{F}$ is the set $\bigcap \mathscr{F} = \bigcap_{i \in I} F_i$. $\mathscr{F}$ obeys the *Helly* (*k-Helly*) *property* if the total intersection of any pairwise intersecting subfamily (with at most k members) is nonempty.

Let $\mathscr{C}(G)$ be the family of cliques of G. The *clique graph of* G, $K(G)$, is the intersection graph of $\mathscr{C}(G)$. G *is a clique graph* if there exists a graph H such that $G = K(H)$. The only general characterization for clique graphs so far known is the one given by the following theorem. Recognizing clique graphs through this characterization is in general difficult; it is an open problem determining the time complexity of clique graphs recognition [5].

Theorem 1 (Roberts and Spencer [3]). *A graph G is a clique graph if and only if there exists a complete edge cover of G satisfying the Helly property.*

A special family of completes of G that covers its edges is the family $\mathscr{C}(G)$. G is a *Helly* (*k-Helly*) *graph* if $\mathscr{C}(G)$ obeys the Helly (k-Helly) property ([2], it contains some related topics). It follows that Helly graphs are always clique graphs. Helly graphs can be recognized in polynomial time using the following characterization.

Theorem 2 (Szwarcfiter [4]). *A graph G is a Helly graph if and only if every extended triangle of G has a universal vertex.*

Since Helly graphs are clique graphs, and they have been characterized looking at its triangles, what can we say about the triangles of clique graphs? Is there a more general result than Theorem 2 about the triangles of clique graphs? In Section 2 we show an affirmative answer to this question. We present a generalized notion of extended triangle which allows a blending of the techniques of Roberts–Spencer and Szwarcfiter.

In Section 3 we obtain a characterization of Helly planar graphs and 3-Helly planar graphs by describing a simple family of admissible extended triangles. Section 4 contains our advance in the recognition of planar clique graphs; the main result provides a necessary condition for planar clique graphs: that any extended triangle must be a clique graph. The planar extended triangles which are clique graphs are totally characterized in Section 5.

2. Extended triangles generalization

A *triangle* T of a graph G is a complete containing exactly three vertices. The set of triangles of G is symbolized by $T(G)$. The *extended triangle of G relative to* the triangle T is defined in [4] as the subgraph induced in G by the vertices adjacent to at least two vertices of T and it is denoted by T'. It is easy to prove that the following definition is equivalent: T' is the subgraph induced in G by the vertices of the cliques of G containing at least two vertices of T. It follows the way we generalize the idea of extended triangle:

Definition 3. Let $\mathscr{F}$ be a complete edge cover of a graph G and $T \in T(G)$. The subfamily of $\mathscr{F}$ formed by the members containing at least two vertices of T is denoted by $\mathscr{F}_T$.

The extension—according to the family $\mathscr{F}$—of the triangle T is the subgraph $T_{\mathscr{F}}$ induced in G by the vertices belonging to the members of $\mathscr{F}_T$.

The extension—according to the family $\mathscr{C}(G)$—of T is called the extended triangle of G relative to T and it is simply denoted by T' instead of $T_{\mathscr{C}(G)}$.

Notice that given $\mathscr{F}$, any complete edge cover of G, $T_{\mathscr{F}}$ is an induced subgraph of the extended triangle T'.

The following lemmas give a useful relation between $\mathscr{F}_T$ and $T_{\mathscr{F}}$. They generalize previous works in [3,4].

Lemma 4. *Let $\mathscr{F}$ be a complete edge cover of G. The following conditions are equivalent*:

(i) *$\mathscr{F}$ has the Helly property.*
(ii) *For every $T \in T(G)$, the subfamily $\mathscr{F}_T$ has the Helly property.*
(iii) *For every $T \in T(G)$, the subfamily $\mathscr{F}_T$ has nonempty intersection.*

Proof. If $\mathscr{F}$ has the Helly property, then any subfamily has the Helly property, in particular $\mathscr{F}_T$ has the Helly property. On the other hand, if $\mathscr{F}_T$ has the Helly property, since $\mathscr{F}_T$ is pairwise intersecting, then it has no empty intersection. Now suppose the third condition is true but $\mathscr{F}$ has not the Helly property, then there must be a subfamily $\mathscr{F}' = (F_i)_{i \in I'}$ pairwise intersecting with empty intersection. We can consider it a minimal one, then for every $i_0 \in I'$, $\bigcap_{i \in I' - \{i_0\}} F_i \neq \emptyset$. Let v_{i_0} be a vertex belonging to that intersection. Since the total intersection of the subfamily is empty, then $i_0, i_1 \in I'$, $i_0 \neq i_1$ implies $v_{i_0} \neq v_{i_1}$.

Since $\mathscr{F}'$ has at least three members, we can consider three different vertices v_{i_0}, v_{i_1} and v_{i_2} in such conditions. These vertices form a triangle T of G. Clearly $\mathscr{F}'$ is a subfamily of $\mathscr{F}_T$, and by hypothesis $\mathscr{F}_T$ has no empty intersection, thus $\mathscr{F}'$ has no empty intersection. Contradiction. □

Lemma 5. *Let $\mathscr{F}$ be a family of completes of G and $T \in T(G)$. If the subfamily $\mathscr{F}_T$ has nonempty intersection then the subgraph $T_{\mathscr{F}}$ has a universal vertex. The converse is true if $\mathscr{F}$ is the family $\mathscr{C}(G)$ of cliques of G.*

Proof. Let $u \in \bigcap \mathscr{F}_T$. We claim that u is a universal vertex of $T_{\mathscr{F}}$, indeed: let $v \neq u$ and $v \in V(T_{\mathscr{F}})$. There exists $F \in \mathscr{F}_T$ such that $v \in F$. Thus u and v belong to the complete F, then u is adjacent to v.

The other assumption says that if the subgraph $T_{\mathscr{C}(G)} = T'$ has a universal vertex then the subfamily $\mathscr{C}(G)_T$ has no empty intersection. Let u be a universal vertex of T'. Let $C \in \mathscr{C}(G)_T$ and $v \in C$, $v \neq u$. Since $v \in V(T')$, then u is adjacent to v. Since C is a clique, then $u \in C$. It follows that $u \in \bigcap \mathscr{C}(G)_T$. □

We obtain Theorem 2 from these lemmas:

Theorem 6 (Theorem 2 generalization). *The following conditions are equivalent*:

(i) *G is a Helly graph.*
(ii) *The family $\mathscr{C}(G)$ has the Helly property.*
(iii) *For every $T \in T(G)$, the family $\mathscr{C}(G)_T$ has the Helly property.*
(iv) *For every $T \in T(G)$, the family $\mathscr{C}(G)_T$ has no empty intersection.*
(v) *For every $T \in T(G)$, the subgraph $T_{\mathscr{C}(G)} = T'$ has a universal vertex.*
(vi) *For every $T \in T(G)$, the subgraph $T_{\mathscr{C}(G)} = T'$ is a Helly graph.*

Using the previous lemmas we also can re-state Theorem 1 and relate it with Theorem 2.

Theorem 7 (Theorem 1 generalization). *The following conditions are equivalent*:

(i) *G is a Clique graph.*
(ii) *There exists a complete edge cover of G satisfying the Helly property.*
(iii) *There exists $\mathscr{F}$, a complete edge cover of G, such that for every $T \in T(G)$, the subfamily $\mathscr{F}_T$ has the Helly property.*
(iv) *There exists $\mathscr{F}$, a complete edge cover of G, such that for every $T \in T(G)$, the subfamily $\mathscr{F}_T$ has no empty intersection.*
(v) *There exists $\mathscr{F}$, a complete edge cover of G, such that for every $T \in T(G)$, the subgraph $T_{\mathscr{F}}$ has a universal vertex and this vertex belongs to every member of the subfamily $\mathscr{F}_T$.*

3. Helly and 3-Helly planar graphs

The well-known planar graphs (see [1]) are those admitting a representation on the plane such that two edges do not intersect except at common end vertex. Kuratowsky's theorem shows that a graph is planar if and only if it does not contains a subdivision of K_5 or $K_{3,3}$.

A planar graph G is a Helly graph if and only if it is a 4-Helly graph because its largest clique contains at most 4 vertices [3, Lemma 2]. Any 4-Helly graph is a 3-Helly graph but the converse is not true. Thus we can define the following subsets of planar graphs: planar Helly graphs = planar 4-Helly graphs $\subset$ planar 3-Helly graphs $\subset$ planar graphs. We will characterize them using the extended triangles.

Let G be any graph and $v, v' \in V(G)$. We write $v \sim v'$ to mean that v and v' are adjacent, otherwise we write $v \nsim v'$.

For a given triangle $T = \{x, y, z\}$ of G, we call:

$$V_{xy} = \{v \in V(G):\ v \sim x,\ v \sim y,\ v \nsim z\},$$

$$V_{xz} = \{v \in V(G):\ v \sim x,\ v \sim z,\ v \nsim y\},$$

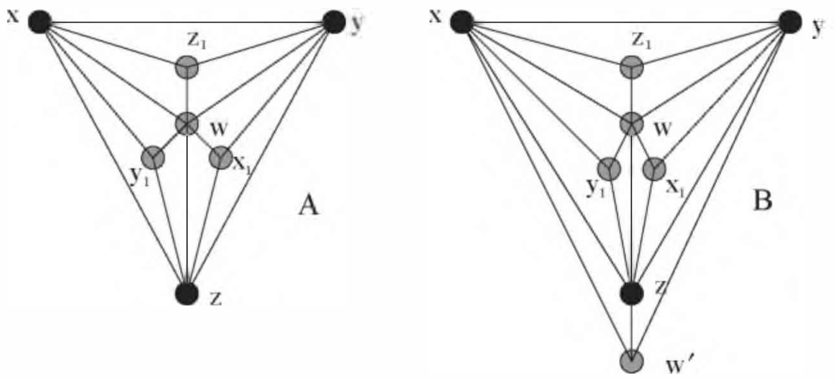

Fig. 1. Extended triangles of type 2 and 3.

$$V_{yz} = \{v \in V(G)\colon\ v \sim y,\ v \sim z,\ v \nsim x\},$$

$$V_{xyz} = \{v \in V(G)\colon\ v \sim x,\ v \sim y,\ v \sim z\}.$$

Definition 8. Let G be a graph and T' the extended triangle of G relative to the triangle $T = \{x, y, z\}$. Say that:

T' is of type 1 if at least one of the sets V_{xy}, V_{xz} or V_{yz} is empty.

T' is of type 2 if $V_{xy} = \{z_1\}$, $V_{xz} = \{y_1\}$, $V_{yz} = \{x_1\}$, $V_{xyz} = \{w\}$, $x_1 \sim w$, $y_1 \sim w$ and $z_1 \sim w$.

T' is of type 3 if $V_{xy} = \{z_1\}$, $V_{xz} = \{y_1\}$, $V_{yz} = \{x_1\}$, $V_{xyz} = \{w, w'\}$, $x_1 \sim w$, $y_1 \sim w$ and $z_1 \sim w$.

Notice that if T' is an extended triangle of type 2 (type 3) of a planar graph, then T' is isomorphic to the graph A (to the graph B) of Fig. 1, thus each class contains a unique planar graph. This is easy to prove since graphs A and B are maximal planar. On the other hand, there is an infinite number of planar extended triangles of type 1.

Lemma 9. *Let $T = \{x, y, z\}$ be a triangle of a planar graph G.*

(1) *If $w \in V_{xyz}$, $z_1, z_2 \in V_{xy}$ and $w \sim z_1$ then $w \nsim z_2$.*
(2) *If $w \in V_{xyz}$, $z_1 \in V_{xy}$, $y_1 \in V_{xz}$, $w \sim z_1$ and $w \sim y_1$ then $z_1 \nsim y_1$.*
(3) *If $w, w' \in V_{xyz}$ then $w \nsim w'$.*
(4) *If $w, w' \in V_{xyz}$, $z_1 \in V_{xy}$ and $z_1 \sim w$ then $z_1 \nsim w'$.*
(5) *If u_T is a universal vertex of the extended triangle of G relative to T, then $u_T \in T$ or $u_T \in V_{xyz}$. Moreover, if $u_T \in T$ then one of the sets V_{xy}, V_{xz} or V_{yz} is empty.*

Proof. (1) If $w \sim z_2$ then the vertices w, x, y and the vertices z, z_1, z_2 form a $K_{3,3}$, which is a contradiction because G is a planar graph. (2) The vertices w, x, y and z

form a K_4; if $z_1 \sim y_1$ then there is a subdivision of a K_5 considering y_1 the fifth vertex. (3) If $w \sim w'$ then the vertices w, w', x, y and z conform a K_5. (4) The vertices w, x, y and z form a K_4; if $w' \sim z_1$ then there is a subdivision of a K_5, considering w' or z_1 the fifth vertex. (5) It is clear because of the definition of the sets. □

Now, we give the characterization:

Theorem 10. *Let G be a planar graph.*

(1) *G is a Helly graph if and only if every extended triangle of G is of type* 1 *or type* 2.
(2) *G is a* 3-*Helly graph if and only if every extended triangle of G is of type* 1, *type* 2 *or type* 3.

Proof. If $F \subseteq V(G)$, $F \supseteq \{u, /v, \ldots\}$ means that the vertex u belongs to the set F and that the vertex v does not belong to it.

(1) If G is a Helly graph and T' is an extended triangle of G, by Theorem 2, there exists u_T, a universal vertex of T'. Suppose there is a triangle $T = \{x, y, z\}$ which is not type 1, then V_{xy}, V_{xz} and V_{yz} are not empty, so, by Lemma 9, item 5, $u_T \in V_{xyz}$. Since u_T must be adjacent to every vertex belonging to the subsets V_{xy}, V_{xz}, or V_{yz} and to any other vertex in V_{xyz}, then, by Lemma 9, items 1 and 3, every one of these sets contains at most one vertex, thus every one of them contains exactly one vertex; it follows that T' is a type 2 extended triangle.

It is clear that any extended triangle of type 1 or type 2 has a universal vertex, then the converse is true by Theorem 2.

(2) Let G be a 3-Helly planar graph and suppose there exists a triangle $T = \{x, y, z\}$ of G, such that the extended triangle T' is not type 1; then there are different vertices $z_1 \in V_{xy}$, $y_1 \in V_{xz}$ and $x_1 \in V_{yz}$. Thus, there are cliques $C_1 \supseteq \{x, y, z_1, /z, /x_1, /y_1\}$, $C_2 \supseteq \{x, y_1, z, /y, /x_1, /z_1\}$, $C_3 \supseteq \{x_1, y, z, /x, /y_1, /z_1\}$. Since G is 3-Helly and these three cliques are pairwise intersecting, then there exists w, a common vertex. It is clear that $w \notin \{x, y, z, x_1, y_1, z_1\}$. If T' has no more vertices, then T' is of type 2.

Now, assume there exists w', another vertex of T'; we claim that $w' \in V_{xyz}$ and so T' is of type 3, indeed: if $w' \in V_{xy}$, since the cliques C_1, C_2 and C_3 already contain four vertices, there must be another clique $C_4 \supseteq \{x, y, w', /z, /w\}$. Notice that $z \nsim w'$ because $w' \in V_{xy}$; and $w \nsim w'$ because of Lemma 9, item 1. Now $C_2 = \{x, y_1, z, w\}$, $C_3 = \{x_1, y, z, w\}$ and C_4 are pairwise intersecting and they have not a common vertex, contradiction. We conclude $w' \notin V_{xy}$ and by symmetry $w' \notin V_{xz}$ and $w' \notin V_{yz}$, thus $w' \in V_{xyz}$, as we claimed.

To prove the converse suppose G is a planar, not 3-Helly graph. Then there must be three cliques C_1, C_2 and C_3 pairwise intersecting with empty total intersection. Let the vertices belonging to the respective intersections be named x, y and z; and let T be the triangle that they form. Since these cliques must contain at least three vertices and they have not a common vertex, it follows that there exists $z_1 \nsim z$ and $C_1 \supseteq \{x, y, z_1, /z\}$; $y_1 \nsim y$ and $C_2 \supseteq \{x, y_1, z, /y\}$; $x_1 \nsim x$ and $C_3 \supseteq \{x_1, y, z, /x\}$. Now it is easy to see that the extended triangle relative to T is not type 1, not type 2 and not type 3. Contradiction. □

These characterizations lead to $O(n^2)$ recognition algorithms for Helly and 3-Helly planar graphs. Remember that the triangles of a planar graph can be listed in linear time [1].

4. Planar clique graphs

The following theorem shows a way to obtain from a Helly complete edge cover of a planar graph G, a Helly complete edge cover of every extended triangle of G. Thus if G is a planar clique graph, then every extended triangle of G is a clique graph.

Theorem 11. *Let $\mathscr{F} = (F_i)_{i\in I}$ be a Helly complete edge cover of a planar graph G, and T' an extended triangle of G. The family $\mathscr{F}' = (F_i \cap V(T'))_{i\in I'}$ where $I' = \{i\in I: |F_i \cap V(T')| \geqslant 3\}$ is a Helly complete edge cover of T'.*

Proof. For every $i\in I$, $F_i' = F_i \cap V(T')$ is a complete of T' because F_i is a complete of G and T' is an induced subgraph of G. Suppose there is an edge uv of T' which is covered by no member of $\mathscr{F}'$, thus for every $i \in I'$ if $u\in F_i'$ then $v \notin F_i'$; so for every $i\in I$ such that $|F_i \cap V(T')| \geqslant 3$ if $u \in F_i \cap V(T')$ then $v \notin F_i \cap V(T')$; so for every $i\in I$, if u and $v\in F_i$ then $|F_i\cap V(T')| < 3$; this means that

$$F_i \in \mathscr{F} \text{ and } u, v \in F_i \text{ implies } F_i \cap V(T') = \{u, v\}. \tag{1}$$

We will see that this is not possible. Let $T = \{x, y, z\}$.

Case 1: $u, v\in T$. In this case any vertex in a complete containing u and v belongs to $V(T')$, then by implication 1 any member of $\mathscr{F}$ containing u and v does not contain more vertices, then it is a K_2. This is not possible since $\mathscr{F}$ has the Helly property.

Case 2: $u\in T$ and $v \notin T$. Since $v\in V(T')$ we can assume $v \sim x$ and $u \neq x$. By implication 1, the triangle $\{u, v, x\}$ cannot be included in a member of $\mathscr{F}$ so there must be different members covering the edges: xv, vy and yx. These members are pairwise intersecting then they must contain a common vertex. Clearly, the common vertex belongs to $V(T')$. This contradicts implication 1.

Case 3: $u, v \notin T$. We will consider two subcases: when both vertices are adjacent to a same pair of vertices of T, and when they are adjacent to different pairs.

Subcase 3.1: u and v are adjacent to x and y (Fig. 2a). Again, by implication 1, the triangle $\{u, v, x\}$ cannot be included in any member of $\mathscr{F}$, so there must be completes $F_1 \supseteq \{u, v, /x, /y, /z\}$, $F_2 \supseteq \{u, x, /v\}$, and $F_3 \supseteq \{x, v, /u\}$. Since they are pairwise intersecting, they must contain a common vertex, say w. Notice that $w \notin \{x, y, z, v, u\}$, and that $w \notin V(T')$, then w is adjacent neither to y nor to z (Fig. 2b). Now, consider the triangle $\{u, v, y\}$, by the same reason there must be completes $F_4 \supseteq \{u, y, /v, /w\}$ and $F_5 \supseteq \{v, y, /u, /w\}$. Since F_1, F_4 and F_5 are pairwise intersecting, they must contain a common vertex $w' \notin \{x, y, z, v, w, u\}$ (Fig. 2c). Clearly $\{u, v, w, w'\}$ conform a K_4, so considering x or y as the fifth vertex there is a subdivision of a K_5. Contradiction.

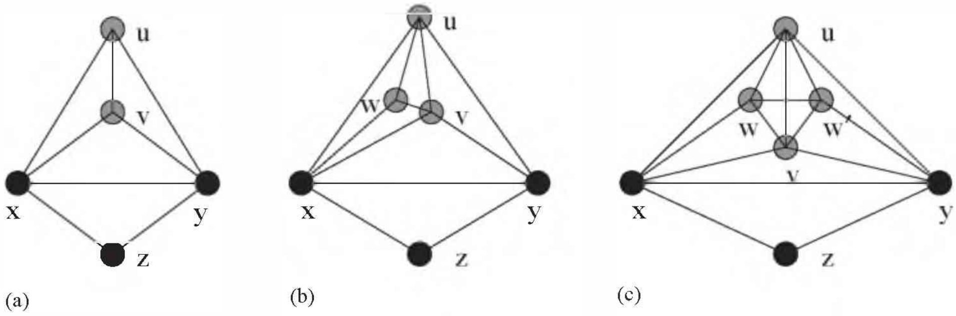

Fig. 2.

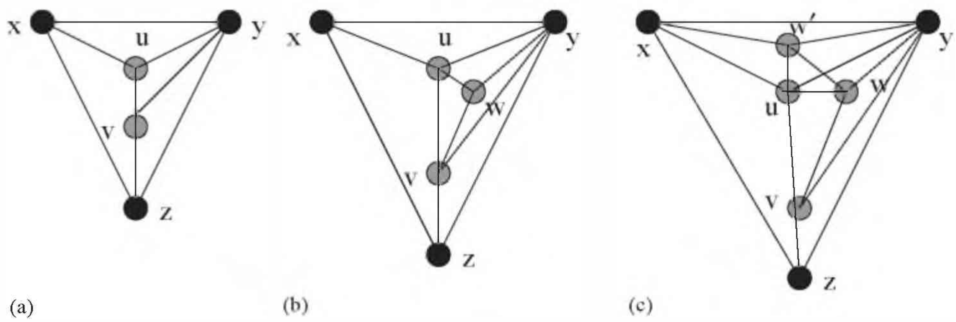

Fig. 3.

Subcase 3.2: u is adjacent to x and y, and v is adjacent to y and z (Fig. 3a). As in the previous subcase, because of implication 1, the triangle $\{u,v,y\}$ is not included in any member of $\mathscr{F}$; then there must exist completes of $\mathscr{F}$ $F_1 \supseteq \{u,v,w,/x,/y,/z\}$, $F_2 \supseteq \{u,y,w,/x,/z,/v\}$, and $F_3 \supseteq \{v,y,w,/x,/z,/u\}$, furthermore $w \notin V(T')$ (Fig. 3b). Now, suppose there exists $F \in \mathscr{F}$ such that $\{x,y,u\} \subseteq F$. Again, there must exist $w' \in F \cap F_1 \cap F_3$. Clearly $w' \notin \{x,y,z,u,v,w\}$ and $w' \in V(T')$, this contradicts implication 1 since $\{u,v,w'\} \subseteq F_1$. We get that the triangle $\{x,y,u\}$ is not included in any member of $\mathscr{F}$, then there must be members $F_4 \supseteq \{x,y,/u,/w\}$ and $F_5 \supseteq \{x,u,/y,/w\}$. Since they and F_2 are pairwise intersecting, they must contain a common vertex, say w', which clearly does not belong to $\{x,y,z,u,v,w\}$ (Fig. 3c). Notice that $\{u,y,w,w'\}$ conform a K_4, so there is a subdivision of a K_5 considering x or v as the fifth vertex. Contradiction. We have proved that $\mathscr{F}' = (F_i')_{i \in I'}$ is a complete edge cover of T', suppose it has not the Helly property, then there is a subfamily pairwise intersecting without a common vertex, let $(F_i')_{i \in J}$, $J \subset I'$ be a minimal one. Notice that $|J| \leqslant 4$ because the completes have at most four vertices. Since $\bigcap_{i \in J} F_i \neq \emptyset$, $\bigcap_{i \in J} F_i' = \emptyset$, and $3 \leqslant |F_i'| \leqslant 4$ then for each $i \in J$, $F_i = F_i' \cup \{h\}$ where $h \in V(G)$ and $h \notin V(T')$. Assume $|J| = 4$. Since the subfamily is minimal, any three members contain a common

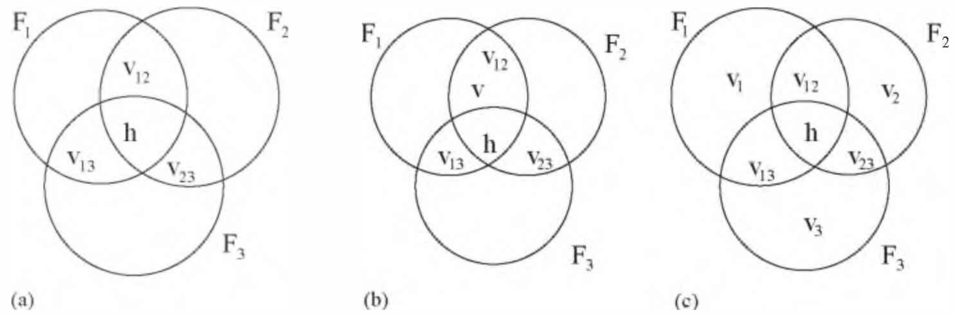

Fig. 4.

vertex, then there are four vertices mutually adjacent; these vertices with the vertex h conform a K_5, which is a contradiction.

If $|J|=3$, say $J=\{1,2,3\}$, call $v_{ij}=v_{ji}$ a vertex belonging to the intersection of F_i' and F_j', then we have $F_1 \supseteq \{v_{12}, v_{13}, h\}$; $F_2 \supseteq \{v_{12}, v_{23}, h\}$ $F_3 \supseteq \{v_{13}, v_{23}, h\}$ (Fig. 4a). Since $h \notin V(T')$ and every set must contain at least three vertices of T', then every one of these sets must contain another vertex of T', and it cannot be the same vertex for the three sets. Then there are two possibilities: (a) One of the three fourth vertices belongs to one intersection, for instance suppose there is another vertex $v \in F_1 \cap F_2$ (Fig. 4b), then $v_{12}, v_{13}, v_{23}, v, h$ conform a K_5, which contradicts planarity. (b) None of the three fourth vertices is in one intersection, then they are different vertices: v_1, v_2 and v_3, and the situation is $F_1=\{v_1, v_{12}, v_{13}, h\}$, $F_2=\{v_2, v_{12}, v_{23}, h\}$, and $F_3=\{v_3, v_{13}, v_{23}, h\}$ (Fig. 4c).

Since the vertex h is not in T', at most one of the vertices $v_1, v_2, v_3, v_{12}, v_{13}, v_{23}$ is a vertex of the triangle T. The remaining vertices are adjacent to at least two vertices of the triangle T, then it is easy to see that there is a subdivision of a K_5. Contradiction. □

Corollary 12. *Let G be a planar graph. If G is a clique graph then every extended triangle of G is a clique graph.*

5. Planar extended triangles which are clique graphs

We have obtained, for a given planar graph, a necessary condition to be a clique graph: that every extended triangle of the given graph must be a clique graph. Then it is natural to ask: is it easy to know if an extended triangle of a planar graph is a clique graph? The answer is yes. In Theorem 14 we present a total characterization of the extended triangles of a planar graph which are clique graphs. This characterization leads to an $O(n^2)$ algorithm to decide if a planar extended triangle is a clique graph.

Before enunciating the theorem we will prove the following useful lemma about Helly complete edge covers of an extended triangle of a planar graph.

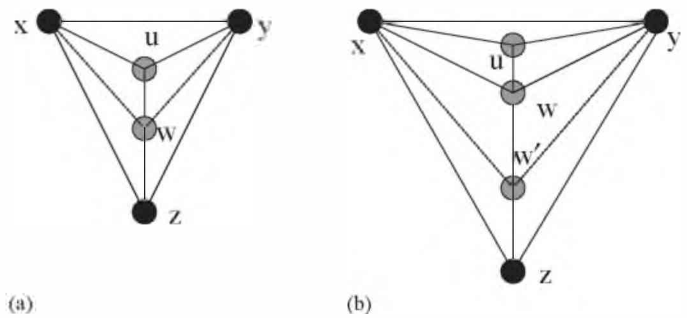

Fig. 5. Item of Lemma 13

Lemma 13. *Let G be a planar graph and T' the extended triangle of G relative to the triangle $T=\{x,y,z\}$. Let $\mathscr{F}$ be a Helly complete edge cover of T' and $u_T \in \bigcap \mathscr{F}_T$, then*:

(1) *If $u \in V(T_{\mathscr{F}})$ and $u \neq u_T$, then $u \sim u_T$.*
(2) *Either $u_T \in T$ or $u_T \in V_{xyz}$.*
(3) *If $w \in V_{xyz}$ then $w \in V(T_{\mathscr{F}})$.*
(4) *If $|V_{xyz}| = 2$ then $u_T \in T$.*
(5) *If $u \in V_{xy}$ and $u \notin V(T_{\mathscr{F}})$, then either*
  (i) *$V_{xy} = \{u\}$, and there exists $w \in V_{xyz}$ such that $u \sim w$* (Fig. 5a), *or*
  (ii) *there exists w such that $V_{xy} = \{u, w\}$, and $w' \in V_{xyz}$ such that $u \sim w \sim w'$. Furthermore, $w \in V(T_{\mathscr{F}})$ and $w' = u_T$ (Fig. 5b).*
(6) *If $|V_{xy}| > 2$ then either $u_T = x$ or $u_T = y$.*

Notice that we can obtain results analogous to items 5 and 6, *beginning from V_{xz} or V_{yz} instead of V_{xy}.*

Proof. (1) It is clear since $V(T_{\mathscr{F}}) = \bigcup \mathscr{F}_T$ and u_T belongs to every member of $\mathscr{F}_T$, which are completes.

(2) By definition the members of $\mathscr{F}$ covering the edges of T are members of $\mathscr{F}_T$, then $x, y, z \in V(T_{\mathscr{F}})$, thus if $u_T \notin T$, it follows from the previous item that u_T is adjacent to x, y and z, then $u_T \in V_{xyz}$.

(3) Suppose $w \in V_{xyz}$ and $w \notin V(T_{\mathscr{F}})$. Then there must be members of $\mathscr{F}$ satisfying: $F_1 \supseteq \{w, x, /y, /z\}$, $F_2 \supseteq \{w, y, /x, /z\}$, $F_3 \supseteq \{w, z, /x, /y\}$. There are two possibilities: (a) there exists a member of $\mathscr{F}$ containing the triangle $\{x, y, z\}$: let it be $F_4 \supseteq \{x, y, z, /w\}$ (notices that $w \notin F_4$ because $w \notin V(T_{\mathscr{F}})$). Since the four completes are pairwise intersecting, they must contain a common vertex: $h \notin \{x, y, z, w\}$. Then there is a K_5. Contradiction.

(b) There is not a member of $\mathscr{F}$ containing the triangle $\{x, y, z\}$, so there must be different completes covering its edges: $F_4 \supseteq \{x, y, /z, /w\}$, $F_5 \supseteq \{x, z, /y, /w\}$, $F_6 \supseteq \{y, z, /x, /w\}$. It is easy to see that these completes cannot be the previous ones, and, since every one of them contains two vertices of T, then they must contain u_T. It follows that u_T cannot be x, y, z or w, then we have to add to F_4, F_5 and F_6 the vertex $u_T \in V_{xyz}$. On the other hand, F_1, F_2 and F_4 are pairwise intersecting, then they must contain a common vertex $h \notin \{x,y,z,w\}$. By Lemma 9, item 3, $u_T \nsim w$, then $h \neq u_T$. Thus h is adjacent to x, y, w and u_T; again we contradict planarity.

(4) Let $w, w' \in V_{xyz}$. By Lemma 9, item 3, they are not adjacent. In accordance with the previous item $w' \in V(T_{\mathscr{F}})$, and since $w' \nsim w$, then $u_T \neq w$. Analogously, $u_T \neq w'$. We conclude that $u_T \notin V_{xyz}$. It follows from the second item that $u_T \in T$.

(5) Let $u \in V_{xy}$ and suppose that $u \notin V(T_{\mathscr{F}})$, i.e. u does not belong to any member of $\mathscr{F}$ containing at least two vertices of T. Since every edge is covered by a member of the family $\mathscr{F}$, there are completes $F_1 \supseteq \{u,x,/y,/z\}$, $F_2 \supseteq \{u,y,/x,/z\}$, $F_3 \supseteq \{x,y,/u\}$. Since they are pairwise intersecting and $\mathscr{F}$ has the Helly property, they contain a common vertex w which is not x,y,z, or u; actually these completes satisfy:

$$F_1 \supseteq \{w,u,x,/y,/z\}, \quad F_2 \supseteq \{w,u,y,/x,/z\}, \quad F_3 \supseteq \{w,x,y,/u\}.$$

Let us see that in this conditions,

$$F \in \mathscr{F}, \quad x,y \in F \quad \text{implies } w \in F, \tag{2}$$

we will use it later. Suppose $F \in \mathscr{F}$ and $F \supseteq \{x,y,/w\}$, clearly F is not F_1, nor F_2 and nor F_3. The four completes F, F_1, F_2 and F_3 are pairwise intersecting so they contain a common vertex which is not x,y,z,u or w, then the common vertex must be a vertex h which is adjacent to x,y,u and w, so there exists a K_5. This contradicts planarity. We have proved implication 2.

Now, let us consider two cases: when the vertex w is adjacent to z and when it is not. (i) Assume $w \sim z$, then $w \in V_{xyz}$. We only need to prove that $V_{xy} = \{u\}$. Suppose there exists $u' \in V_{xy}$. By Lemma 9, item 1, $u' \nsim w$, then by implication 2, u' does not belong to any member of $\mathscr{F}$ containing x and y, thus $u' \notin V(T_{\mathscr{F}})$. It follows that there must be completes $F_4 \supseteq \{x,u',/y,/z,/w\}$ and $F_5 \supseteq \{y,u',/x,/z,/w\}$. Again, these completes and $F_3 \supseteq \{x,y,w,/u,/u'\}$, must contain a common vertex, say h. Clearly $h \notin \{x,y,z,w,u,u'\}$ and h is adjacent to x, y and w. Notice that u and z are also adjacent to these three vertices, then there is a $K_{3,3}$. Contradiction. We have proved that $V_{xy} = \{u\}$ and $u \sim w \in V_{xyz}$.

(ii) If $w \nsim z$, then, by implication 2, z does not belong to any member of $\mathscr{F}$ containing x and y, then there must be completes $F_4 \supseteq \{x,z,/y,/w\}$ and $F_5 \supseteq \{y,z,/x,/w\}$. These completes and $F_3 \supseteq \{x,y,w,/u,/z\}$ are pairwise intersecting, then there exists $w' \in F_3 \cap F_4 \cap F_5$. Clearly $w' \notin \{x,y,z,u,w\}$. Notice that $w' \in V_{xyz}$, $w \in V_{xy}$ and $u \sim w \sim w'$. On the other hand, by Lemma 9, item 1, $w' \nsim u$, then actually the completes satisfy $F_1 \supseteq \{w,u,x,/y,/z,/w'\}$, $F_2 \supseteq \{w,u,y,/x,/z,/w'\}$, $F_3 = \{w',w,x,y\}$, $F_4 \supseteq \{w',x,z,/y,/w,/u\}$ and $F_5 \supseteq \{w',y,z,/x,/w,/u\}$. Since $F_3 = \{w',w,x,y\}$ then $w \in V(T_{\mathscr{F}})$, as we wanted to prove. Since the completes $F_3 = \{w',w,x,y\}$, $F_4 \supseteq \{w',x,z,/y,/w,/u\}$ and $F_5 \supseteq \{w',y,z,/x,/w,/u\}$ are members of $\mathscr{F}_T$ (every one of them has two vertices of T), then each one must contain the vertex u_T, it follows that $u_T = w'$.

Finally, we have to prove that $V_{xy} = \{u,w\}$. Suppose there exists other vertex $u' \in V_{xy}$. We claim that $u' \notin V(T_{\mathscr{F}})$. Indeed, in the opposite case, there exist $F \in \mathscr{F}_T$ such that $\{x,y,u'\} \subseteq F$, then, by implication 2, $w \in F$ and so $w \sim u$. This contradicts planarity.

Now, since $u' \notin V(T_{\mathscr{F}})$, there must be completes $F_6 \supseteq \{x, u', /y, /z, /w, /w'\}$ and $F_7 \supseteq \{y, u', /x, /z, /w, /w'\}$ (it is easy to see that these completes cannot be the preceding ones, and that $u' \nsim w'$). Again these completes and $F_3 = \{x, y, w, w'\}$ must contain a common vertex which clearly does not belong to $\{x, y, w, w'\}$. Contradiction: F_3 cannot be a K_5.

(6) If $|V_{xy}| > 2$, since the previous item, every vertex in V_{xy} must belong to $V(T_{\mathscr{F}})$, then by item 1 every vertex in V_{xy} must be adjacent to u_T. It follows that $u_T \neq z$. By Lemma 9, item 1, at most one vertex of V_{xy} could be adjacent to a vertex of V_{xyz}, then in the present case $u_T \notin V_{xyz}$. We conclude, because of item 2, that u_T must be x or y, as we wanted to prove. □

Theorem 14. *Let G be a planar graph and T' the extended triangle relative to the triangle $T = \{x, y, z\}$ of G. T' is a clique graph if and only if at least one of the following conditions is satisfied*:

(1) $V_{xy} = \emptyset$ *or* $V_{xz} = \emptyset$ *or* $V_{yz} = \emptyset$.
(2) $V_{xy} = \{z_1\}$ *and* $z_1 \sim w \in V_{xyz}$, *or*
$V_{xz} = \{y_1\}$ *and* $y_1 \sim w \in V_{xyz}$, *or*
$V_{yz} = \{x_1\}$ *and* $x_1 \sim w \in V_{xyz}$.
(3) $V_{xy} = \{z_1, z_2\}$, $V_{xz} = \{y_1, y_2\}$, $V_{yz} = \{x_1, x_2\}$, $V_{xyz} = \{w\}$, *and*
$w \sim z_1 \sim z_2$, $w \sim y_1 \sim y_2$, $w \sim x_1 \sim x_2$.

Proof. Suppose that T', the extended triangle relative to the triangle $T = \{x, y, z\}$ of the planar graph G, is a clique graph, and that T' satisfies neither condition 1 (*Remark* 1: the subset V_{xy}, V_{xz} and V_{yz} are nonempty) nor condition 2 (*Remark* 2: if V_{xy}, V_{xz} or V_{yz} contains exactly one vertex, then the vertex is adjacent to non vertex of V_{xyz}), we are going to show that T' satisfies condition 3.

Since T' is a clique graph, there is a Helly complete edge cover $\mathscr{F}$ of T', then we can consider $\mathscr{F}_T$, $T_{\mathscr{F}}$, and u_T as in the previous lemma. Item 2 of that lemma says that $u_T \in T$ or $u_T \in V_{xyz}$, let us show that in the actually conditions $u_T \notin T$. Suppose $u_T \in T$, for instance $u_T = z$. By Remark 1, there exists $z_1 \in V_{xy}$. Since $z_1 \nsim z = u_T$ then $z_1 \notin V(T_{\mathscr{F}})$. Because of item 5 of Lemma 13 there are two possibilities: (i) $V_{xy} = \{z_1\}$ and there exists $w \in V_{xyz}$ such that $z_1 \sim w$. This is not possible because of Remark 2; or (ii) there exists $w' \in V_{xyz}$ such that $u_T = w'$. This is not possible since we have supposed $u_T \in T$.

We conclude that $u_T \notin T$, then $u_T \in V_{xyz}$. By Lemma 13, items 3 and 1, and by Lemma 9, item 3, $V_{xyz} = \{u_T\}$. On the other hand, it follows from item 6 of the previous lemma, that every one of the sets V_{xy}, V_{xz} and V_{yz} contains at most two vertices. Let us see that none of them contains exactly one vertex. Suppose $V_{xy} = \{z_1\}$. By Remark 2, z_1 cannot be adjacent to u_T, then $z_1 \notin V(T_{\mathscr{F}})$. Actually we have $V_{xy} = \{z_1\}$ and $z_1 \notin V(T_{\mathscr{F}})$, then item 5(i) of the previous lemma must be true, but this contradicts Remark 2.

We conclude that every one of the sets V_{xy}, V_{xy} and V_{yz} contains exactly two vertices. Both vertices cannot be vertices of $T_{\mathscr{F}}$ since they ought to be adjacent to u_T and this contradicts Lemma 9, item 1, then in each case at least one of them is not in $V(T_{\mathscr{F}})$. It follows from 5(ii) of Lemma 13, that condition 3 must be true, as we wanted to prove.

The converse says that T' must be a clique graph if it satisfies 1, 2 or 3.

Assume first that T' satisfies condition 1, say $V_{xy}=\emptyset$. Then z is a universal vertex of T', so T' is a Helly graph and hence T' is a clique graph. A special case will be important in what follows: Assume that $V_{xy}=\emptyset$ and that $w\in V_{xyz}$ has degree 3 in T'. Then $F_w=\{x,y,z,w\}$ is the only clique of T' containing w. There are at most two cliques of T' containing both x and y: one is certainly F_w and the other is $F_{w'}=\{x,y,z,w'\}$ if $V_{xyz}=\{w,w'\}$: indeed, the common vertex neighbours of x and y are $w\sim z\sim w'$ and this is an induced path (henceforth, every reference to w' and objects related to it must be disregarded if $V_{xyz}=\{w\}$). Let $\mathscr{F}=(\mathscr{C}(T')-F_{w'})\cup\{F_4,F_5\}$ where $F_4=\{x,z,w'\}$ and $F_5=\{y,z,w'\}$. Thus $\mathscr{F}$ is a complete edge cover of T' and satisfies Helly property since $z\in\bigcap\mathscr{F}$. Notice that F_w is the only member of $\mathscr{F}$ containing the vertex w or the edge xy.

Assume now that T' satisfies condition 2, say $V_{xy}=\{z_1\}$ and $z_1\sim w\in V_{xyz}$. By Lemma 9, items 1 and 3, besides z_1 there are at most two neighbours of w in $T'-T$, say $x_1\in V_{yz}$ and $y_1\in V_{xz}$ (again, references to them will be conditioned to their existence). Let $T''=(T'-z_1)-\{wx_1,wy_1\}$. Then T'' falls within the special case discussed above, so consider its Helly complete edge cover $\mathscr{F}=(\mathscr{C}(T'')-F_{w'})\cup\{F_4,F_5\}$. Define $F_0=\{x,w,z_1\}$, $F_1=\{y,w,z_1\}$, $F_2=\{x,w,y_1\}$ and $F_3=\{y,w,x_1\}$. Therefore, $\mathscr{F}_1=\mathscr{F}\cup\{F_0,F_1,F_2,F_3\}$ is a complete edge cover of T'. Note that F_0 and F_1 are the only member of $\mathscr{F}_1$ containing z_1, and that w is only in F_w,F_0,F_1,F_2 and F_3. We still have that $x,y\in F\in\mathscr{F}_1$ implies $F=F_w$.

We will show that $\mathscr{F}_1$ has the Helly property. Let $\mathscr{F}'_1$ be a pairwise intersecting subfamily of $\mathscr{F}$. We can assume that $\mathscr{F}'_1$ is not a subfamily of $\mathscr{F}$, and by symmetry we need to consider only the following two cases:

Case 1: $F_0\in\mathscr{F}'_1$. There are two subcases:

(A) $F_1\in\mathscr{F}'_1$. Suppose there is an $F\in\mathscr{F}'_1$ such that $w\notin F$. Then $F\in\mathscr{F}$, $F\cap F_0=\{x\}$ and $F\cap F_1=\{y\}$, so $x,y\in F$ and then $w\in F$ after all. Contradiction.

(B) $F_1\notin\mathscr{F}'_1$, so $F\cap F_0\subseteq\{x,w\}$ for all $F\in\mathscr{F}'_1$, $F\neq F_0$. If $\bigcap\mathscr{F}'_1=\emptyset$, there exist $F,G\in\mathscr{F}'_1$ such that $F\cap F_0=\{x\}$ and $G\cap F_0=\{w\}$. Then $G=F_3$, and $w\notin F$ implies $F\cap G\subseteq\{y,x_1\}$. Since $x\in F$, then $x_1\notin F$ and $F\cap G=\{y\}$, but so $x,y\in F$ implies $F=F_w$, a contradiction.

Case 2: $F_2\in\mathscr{F}'_1$, but $F_0,F_1\notin\mathscr{F}'_1$. Again, two subcases:

(A) $F_3\in\mathscr{F}'_1$. Assuming that there is an $F\in\mathscr{F}'_1$ such that $w\notin F$, we get $F\cap F_2\subseteq\{x,y_1\}$, and $F\cap F_3\subseteq\{y,x_1\}$. But then $x,y\in F$, $F=F_w$ and $w\in F$. Contradiction.

(B) $F_3\notin\mathscr{F}'_1$. Suppose that there is an $F\in\mathscr{F}'_1$ such that $x\notin F$. It follows that $F\notin\{F_w,F_0,F_1,F_2,F_3,F_4,F_5\}$, so $F\in\mathscr{C}(T'')$ and $w\notin F$. In particular, $F\cap F_2=\{y_1\}$. By Lemma 9, items 2 and 4 the neighbours in T' of y_1 are in $V_{xz}\cup\{x,z,w\}$. Hence, the neigbours in T'' of y_1 are in $V_{xz}\cup\{x,z\}$. Thus, $F\in\mathscr{C}(T'')$ and $y_1\in F$ imply $x\in F$, a contradiction. We conclude that $x\in\bigcap\mathscr{F}'_1$, in this subcase.

Finally consider that T' satisfies condition 3. It is easy to see that in this case the family depicts in following is a Helly complete edge cover of T', thus it is a clique graph:

$$\{x,z_1,z_2\},\quad \{y,z_1,z_2\},\quad \{x,y,z_1,w\},$$
$$\{x,y_1,y_2\},\quad \{z,y_1,y_2\},\quad \{x,z,y_1,w\},$$
$$\{y,x_1,x_2\},\quad \{z,x_1,x_2\},\quad \{y,z,x_1,w\}.$$

Corollary 15. *Let T' be an extended triangle of a planar graph G. If T' is of type* 1, 2 *or* 3 *then T' is a clique graph.*

6. Remarks

It is known that a graph G is a clique graph (Helly graph, k-Helly graph) if and only if the graph obtained from G by removing the edges which are cliques of G, is a clique graph (Helly graph, k-Helly graph), therefrom, the results presented in this work hold for a class of graphs wider than planar.

We have proved that if a planar graph is a clique graph, then its extended triangles are clique graphs. We have found counterexamples that show that the converse is not true, i.e. there exists a planar graph such that every one of its extended triangles is clique graph but the whole graph is not a clique graph. However, Theorem 11 says that if a planar graph G is a clique graph then every extended triangle of G admits a Helly complete edge cover coming from a same Helly complete edge cover of the entirely graph G, this means that every extended triangle of G must be a clique graph and every extended triangle *must admit a Helly complete edge cover "compatible" with the one of the other extended triangle*. Then we think that the existence or not of a Helly complete edge cover of a planar graph G could be determined knowing the different possible Helly complete edge covers of each extended triangle of G.

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