A posteriori error estimates in finite element acoustic analysis

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Abstract

We present an a posteriori error estimator for the approximations of the acoustic vibration modes obtained by a finite element method which does not present spurious or circulation modes for non zero frequencies. We prove that the proposed estimator is equivalent to the error in the approximation of the eigenvectors up to higher order terms with constants independent of the eigenvalues. Numerical results for some test examples are presented which show the good behavior of the estimator when it is used as local error indicator for adaptive refinement. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In recent years, considerable interest has been shown in finite element methods to compute the vibration modes of a fluid based on the displacement formulation. One of these methods consists of using the lowest-order triangular Raviart–Thomas elements for fluid variables. It can be proved that this method does not present spurious or circulations modes for nonzero frequencies which are typical of displacement formulation. Also, it exhibits an optimal order of convergence. This methodology was introduced in [3] for fluid–structure interaction problems and good approximations were obtained for compressible and incompressible fluids [6,4].

From the computational point of view, many questions are still open. The necessity of using adequately refined meshes in order to take care of the singularities of the eigenmodes is among the most relevant ones. In this context, a posteriori error estimators play a fundamental role since they are used to know where a given mesh needs to be refined.

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There are not many references about a posteriori error estimators for eigenvalue problems. For classical finite element approximations one could cite a paper by Babuška and Rheinboldt [2], which includes as an example a simple one-dimensional eigenvalue problem, and a paper by Verfürth [14], who introduced a general framework to derive error estimators for nonlinear problems and applied it, in particular, to linear eigenvalue problems.

Very recently, an a posteriori error estimator for a mixed finite element approximation of the eigenvalues and the eigenvectors of a second-order elliptic problem was introduced in [9]. In this paper we adapt the techniques presented in [9] to derive an error estimator for the Raviart–Thomas approximations of the acoustic vibration problem. We prove that this error estimator is equivalent to the error up to higher-order terms with constants independent of the eigenvalues and of the mesh size. Finally, we present some numerical experiments which show the good behavior of the estimators when they are used as local error indicators for adaptive refinement.

2. The model problem and its discretization

We consider the problem of determining the vibration modes of an ideal inviscid barotropic fluid contained into a rigid cavity.

Let \( \Omega \subset \mathbb{R}^2 \) be the domain occupied by the fluid. We assume \( \Omega \) is a simply connected polygon.

The eigenmodes \( u \neq 0 \) and the eigenfrequencies \( \lambda \geq 0 \) are the solution of the following spectral problem:

\[
-\nabla (\rho c^2 \text{div } u) = \lambda \rho u \quad \text{in } \Omega,
\]

\[
u . n = 0 \quad \text{on } \partial \Omega,
\]

where \( u \) is the displacement vector in the fluid, \( \rho \) is the mass density, \( c \) is the acoustic speed in the fluid and \( n \) is the outward unit normal vector.

Introducing,

\[
p = -\rho c^2 \text{div } u,
\]

we obtain a mixed formulation of problem (2.1)

\[
\nabla p = \lambda \rho u \quad \text{in } \Omega,
\]

\[
\frac{1}{\rho c^2} p + \text{div } u = 0 \quad \text{in } \Omega,
\]

\[
u . n = 0 \quad \text{on } \partial \Omega.
\]

Let \( H_0(\text{div}, \Omega) := \{ v \in H(\text{div}, \Omega) : v.n|_{\partial \Omega} = 0 \} \).

The weak formulation of problem (2.3) is then: find \( \lambda \geq 0, u \in H_0(\text{div}, \Omega) \) and \( p \in L^2(\Omega) \), with \( p \neq 0 \), such that

\[
- \int_{\Omega} p \text{div } v = \lambda \rho \int_{\Omega} u . v, \quad \forall v \in H_0(\text{div}, \Omega),
\]

\[
- \int_{\Omega} q \text{div } u - \frac{1}{\rho c^2} \int_{\Omega} p q = 0, \quad \forall q \in L^2(\Omega).
\]
In order to ensure well-posedness, we consider the following modified eigenvalue problem: find \( \lambda \geq 0, \ u \in H_0(\text{div}, \Omega) \) and \( p \in L^2(\Omega) \), with \( p \neq 0 \), such that

\[
\rho \int_\Omega u \cdot v - \int_\Omega p \ \text{div} \ v = (\lambda + 1) \rho \int_\Omega u \cdot v, \quad \forall v \in H_0(\text{div}, \Omega),
\]

\[
- \int_\Omega q \ \text{div} \ u - \frac{1}{\rho c^2} \int_\Omega pq = 0, \quad \forall q \in L^2(\Omega).
\]

(2.5)

Clearly, problem (2.5) has the same eigenvalues and eigenfunctions as problem (2.4) and is a well-posed mixed problem in the sense that the bilinear forms involved satisfy the classical Brezzi's conditions (see [7]).

Let \( \mathcal{T}_h \) be a regular family of triangulations of \( \Omega \), and let

\[
V_h = \{ v \in H_0(\text{div}, \Omega) : v|_T \in RT_0(T), \forall T \in \mathcal{T}_h \}
\]

and

\[
Q_h = \{ q \in L^2(\Omega) : q|_T \in P_0(T), \forall T \in \mathcal{T}_h \},
\]

where \( RT_0 = \{ v \in P_1(T)^d : v(x, y) = (a + bx, c + by) : a, b, c \in R \} \) is the lowest-order Raviart–Thomas space [12].

Then, the mixed finite element approximation of problem (2.5) is: find \( \lambda_h \geq 0, \ u_h \in V_h \) and \( p_h \in Q_h \), with \( p_h \neq 0 \), such that

\[
\rho \int_\Omega u_h \cdot v - \int_\Omega p_h \ \text{div} \ v = (\lambda_h + 1) \rho \int_\Omega u_h \cdot v, \quad \forall v \in V_h,
\]

\[
- \int_\Omega q \ \text{div} \ u_h - \frac{1}{\rho c^2} \int_\Omega p_h q = 0, \quad \forall q \in Q_h.
\]

(2.6)

The techniques in [5,13] can be adapted to problem (2.6) in order to prove that its eigenfrequencies converge to those of problem (2.5) and that nonzero frequency spurious modes do not arise in this discretization. More precisely, assume for simplicity that \( \lambda \) is a simple eigenvalue and take \( \| p \|_{0,\Omega} = \| p_h \|_{0,\Omega} = 1 \), then the following a priori estimates holds:

\[
\| u - u_h \|_{0,\Omega} + \| p - p_h \|_{0,\Omega} = O(h^\alpha),
\]

(2.7)

\[
| \lambda - \lambda_h | = O(h^{2\alpha}).
\]

(2.8)

The value of the constant \( \alpha \) in (2.7) and (2.8) depends on the regularity of the eigenmodes. In particular, if \( \Omega \) is a polygonal convex region, the eigenvalues of the discrete problem converge with order \( O(h^2) \).

Now, problem (2.1) can be rewritten in terms of the variable \( p \).

From (2.3) we get

\[
Ap = \lambda \rho \ \text{div} \ u
\]

By using (2.2) and for \( \lambda \neq 0 \), we obtain the eigenvalue problem

\[
-c^2 Ap = \lambda p \quad \text{in} \ \Omega,
\]

\[
\frac{\partial p}{\partial n} = 0 \quad \text{on} \ \partial \Omega.
\]

(2.9)
In [9], an error estimator of the residual type was introduced for the approximation of the eigenvectors of a second-order elliptic problem obtained by the mixed finite element method of Raviart–Thomas of the lowest order. Also in [9], it was proved that this estimator is equivalent to the norm of the error up to higher-order terms with constants that depend on the corresponding eigenvalue.

Following the techniques developed there, we obtain another error estimator. This new estimator contains the same terms than the one presented in [9], (putting $c = 1$ in (2.9)), but they are in a different weighted combination. As we will show in the section below, these weights are very important because they determine the behavior of the estimator when it is used as an error indicator for adaptive mesh refinement.

Moreover, our estimator is equivalent to the norm of the error up to higher-order terms with constants independent of the eigenvalues.

In the next section we adapt the ideas in [9] to our problem.

3. A posteriori error estimator

It can be proved that problem (2.6) is equivalent to a nonconforming approximation of the standard formulation of problem (2.9) (see [1,11]). Let us denote by

\[ X_h = \{ \phi \in L^2(\Omega): \phi|_T \in P_1(T), \forall T \in \mathcal{T}_h, \phi \text{ is continuous at interior midside points} \} , \]

\[ B_h = \{ b \in H_0^1(\Omega): b|_T \text{ is a cubic polynomial vanishing on } \partial T, \forall T \in \mathcal{T}_h \} , \]

\[ V^d_h = \{ v \in (L^2(\Omega))^2: v|_T \in RT_0(T), \forall T \in \mathcal{T}_h \} . \]

and

\[ W_h = X_h \oplus B_h. \]

Let $P$ and $\Pi$ be the $L^2$-projection operators onto $Q_h$ and $V^d_h$, respectively.

Then, it can be proved that problem (2.6) is equivalent to problem

\[ \sum_{T \in \mathcal{T}_h} \int_T \Pi(\nabla \phi_h).\nabla \varphi = \frac{\lambda_h}{c^2} \int_\Omega P \phi_h \varphi, \quad \forall \varphi \in W_h, \tag{3.10} \]

in the sense that they have the same eigenvalues $\lambda_h$ and the eigenvectors are related by

\[ \lambda_h \rho u_h = \Pi(\nabla \phi_h), \tag{3.11} \]

\[ p_h = P \phi_h. \tag{3.12} \]

For the eigenvectors of (3.10) the following a priori error estimate holds:

\[ \| P - \tilde{\phi}_h \|_{0, \Omega} = O(h^{2\alpha}) , \tag{3.13} \]

where $\tilde{\phi}_h = \phi_h/\| \phi_h \|_{0, \Omega}$.

The double order for convergence of eigenvectors (3.13) is one of the results presented in [9] and it will allow us to obtain an a posteriori estimator equivalent to the error in $H(\text{div})$-norm up to higher-order terms.
For each edge \( l \) of the triangulation \( T_h \), we define
\[
J_l = \begin{cases} 
[u_h, t]_l & \text{if } l \not\subset \partial \Omega, \\
0 & \text{if } l \subset \partial \Omega,
\end{cases}
\]
where \( t \) is a unit tangent on \( l \) in an arbitrary, but fixed, orientation.

For any \( T \in \mathcal{T}_h \), we define as a local error estimator
\[
\eta_T^2 = \frac{\lambda^2}{c^4} |T| \| u_h \|_{0,T}^2 + \sum_{l \subset \partial T} |l| \| J_l \|_{0,l}^2,
\]
where \( |T| \) and \( |l| \) are the area of \( T \) and the length of \( l \), respectively.

Let,
\[
\eta^2 = \sum_{T \in \mathcal{T}_h} \eta_T^2.
\]

**Theorem 3.1.** There exist positive constants \( C_1 \) and \( C_2 \), depending on the regularity of the mesh but independent of \( \lambda \), such that
\[
\| u - u_h \|_{H(\operatorname{div}, \Omega)} \leq C_1 \eta + \frac{C_2}{\rho c^2} \| p - \bar{\phi}_h \|_{0,\Omega}.
\]

**Proof.** Since \( u - u_h \in H(\operatorname{div}, \Omega) \), we can decompose it as
\[
u - u_h = \nabla s + \operatorname{curl} \xi,
\]
where \( s \in H^1(\Omega) \) is a solution of
\[
\Delta s = \operatorname{div}(u - u_h) \quad \text{in } \Omega,
\]
\[
\frac{\partial s}{\partial n} = 0 \quad \text{on } \partial \Omega
\]
and \( \xi \in H^1_0(\Omega) \).

(In what follows, \( C \) will denote a constant independent of \( h \) and \( \lambda \), but not necessarily the same at each occurrence.)

Because of the standard a priori estimate for the Neumann problem (see [10]),
\[
\| \nabla s \|_{0,\Omega} \leq C \| \operatorname{div}(u - u_h) \|_{0,\Omega}.
\]

Now,
\[
\| \operatorname{div}(u - u_h) \|_{0,\Omega} = \frac{1}{\rho c^2} \| p - p_h \|_{0,\Omega} \leq \frac{1}{\rho c^2} (\| p - \bar{\phi}_h \|_{0,\Omega} + \| \bar{\phi}_h - P\phi_h \|_{0,\Omega}),
\]
where we used (3.12) to replace \( p_h \).

Since we have taken \( \| p_h \|_{0,\Omega} = 1 \), a straightforward computation yields
\[
\| \bar{\phi}_h - P\phi_h \|_{0,\Omega} \leq 2 \| \phi_h - P\phi_h \|_{0,\Omega}.
\]

So, we have
\[
\| \operatorname{div}(u - u_h) \|_{0,\Omega} \leq \frac{1}{\rho c^2} \| p - \bar{\phi}_h \|_{0,\Omega} + \frac{2}{\rho c^2} \| \phi_h - P\phi_h \|_{0,\Omega}.
\]
(3.16)
Now, it is known that
\[ \|\phi_h - P\phi_h\|_{0,T} \leq C\|\nabla\phi_h\|_{0,T}|T|^{1/2}. \]
Using (3.11) we can obtain
\[ \|\phi_h - P\phi_h\|_{0,T} \leq C(\|\nabla\phi_h - \Pi\nabla\phi_h\|_{0,T} + \|\hat{\lambda}_h\rho u_h\|_{0,T})|T|^{1/2} \]
\[ \leq C(\|\nabla\phi_h\|_{1,T}|T| + \|\hat{\lambda}_h\rho u_h\|_{0,T}|T|^{1/2}), \tag{3.17} \]
where the last inequality follows from standard error estimates for \( \Pi \).

We are going to prove that the first term on the right-hand side of (3.17) is bounded by the second one.

Let
\[ \phi_h = \tilde{\phi}_h + \beta_h \]
with \( \tilde{\phi}_h \) piecewise linear and \( \beta_h = \sum_{T \in \mathcal{T}_h} c_T b_T \), where \( b_T = b|_T \). It is easy to see that
\[ |\nabla\phi_h|_{1,T} = |c_T| |b_T|_{2,T}. \]
Taking \( b_T \) as test function in (3.10), we have
\[ \int_T \Pi(\nabla \tilde{\phi}_h).\nabla b_T + \int_T \Pi(\nabla \beta_h).\nabla b_T = \frac{\hat{\lambda}_h}{c^2} \int_T p_h b_T. \tag{3.18} \]
Because \( \nabla \tilde{\phi}_h \in V^4_h \),
\[ \Pi(\nabla \tilde{\phi}_h) = \nabla \tilde{\phi}_h \]
and integrating by parts, we obtain
\[ \int_T \Pi(\nabla \tilde{\phi}_h).\nabla b_T = \int_T \nabla \tilde{\phi}_h.\nabla b_T = -\int_T \text{div}(\nabla \tilde{\phi}_h) b_T + \int_{\partial T} \nabla \tilde{\phi}_h . n b_T = 0. \]
Now, since
\[ \int_T \Pi(\nabla b_T).\nabla b_T = \|\Pi(\nabla b_T)\|_{0,T}^2 \]
from (3.18) it follows that
\[ \int_T \Pi(\nabla \beta_T).\nabla b_T = c_T \int_T \Pi(\nabla b_T).\nabla b_T = c_T \|\Pi(\nabla b_T)\|_{0,T}^2 = \frac{\hat{\lambda}_h}{c^2} \int_T p_h b_T \]
and so
\[ c_T = \frac{\hat{\lambda}_h}{c^2} p_h \|\Pi(\nabla b_T)\|_{0,T}^2. \]
Under the regularity assumption on the meshes, it can be proved that
\[ |c_T| \leq C \frac{\hat{\lambda}_h}{c^2} |p_h| \frac{\|b_T\|_{0,T}}{\|\Pi(\nabla b_T)\|_{0,T}^2} |T|^{1/2} \leq C \frac{\hat{\lambda}_h}{c^2} |p_h| |T| \leq C \frac{\hat{\lambda}_h}{c^2} p_h |0,T| |T|^{1/2} \tag{3.19} \]
and
\[ |b_T|_{2,T} \leq C |T|^{-1/2}. \tag{3.20} \]
Combining (3.19) and (3.20), we have
\[ |\nabla \phi_h|_{1,T} \leq C \frac{\lambda_h}{c^2} \| p_h \|_{0,T} = C \lambda_h \rho \| \text{div } u_h \|_{0,T} \]
and using an inverse estimate, we obtain
\[ |\nabla \phi_h|_{1,T} |T| \leq C \lambda_h \rho \| u_h \|_{0,T} |T|^{1/2}, \quad (3.21) \]
which gives the desired result.

Now, (3.16) together with (3.17) and (3.21) yields
\[ \| \nabla \phi_h \|_{0,\Omega} \leq C \left( \sum_{T \in \mathcal{T}_h} \frac{\lambda_h}{c^2} \| u_h \|_{0,\Omega} |T|^{1/2} + \frac{1}{\rho c^2} \| p - \phi_h \|_{0,\Omega} \right). \quad (3.22) \]

Finally, it remains to bound \( \| \text{curl } \xi \|_{0,\Omega} \).

Let \( \xi^I \in H^1_0(\Omega) \) be a continuous piecewise linear approximation of \( \xi \) such that
\[ \| \xi - \xi^I \|_{0,T} \leq C \| \xi \|_{1,T} |T|^{1/2}, \quad \forall l \subset \partial T, \quad (3.23) \]
where \( \tilde{T} \) is the union of the triangles sharing a vertex with \( T \) (see [8], for instance).

Since decomposition (3.15) is orthogonal in \( \| . \|_{0,\Omega} \) and \( \text{curl } \xi^I \in V_h \), by using (2.4) and (2.6) we have
\[ \| \text{curl } \xi \|_{0,\Omega}^2 = - \int_{\Omega} u_h \cdot \text{curl}(\xi - \xi^I) = - \sum_{T \in \mathcal{T}_h} \int_{\partial T} u_h \cdot t(\xi - \xi^I) = -\frac{1}{2} \sum_{T \in \mathcal{T}_h} \sum_{l \subset \partial T} \int_{\gamma_r} J_r(\xi - \xi^I). \]

Now, using (3.23) we obtain
\[ \| \text{curl } \xi \|_{0,\Omega} \leq C \sum_{T \in \mathcal{T}_h} \left( \sum_{l \subset \partial T} \| J_r \|_{0,l} |l|^{1/2} \right). \quad (3.24) \]

Therefore, (3.14) follows from (3.16), (3.22) and (3.24). \( \Box \)

In the next theorem we are going to prove that our error estimator gives a local lower bound for the true error in \( H^1_0(\text{div}) \)-norm.

**Theorem 3.2.** There exist positive constants \( C_3 \) and \( C_4 \), depending on the regularity of the mesh but independent of \( \lambda \), such that
\[ \eta_T \leq C_3 \| u - u_h \|_{H^1(\text{div},T^*)} + C_4 \| \lambda u - \lambda_h u_h \|_{0,T} \frac{|T|^{1/2}}{c^2}, \quad (3.25) \]
where \( T^* \) is the union of the triangles sharing an edge with \( T \).

**Proof.** Let \( T \in \mathcal{T}_h \) and put \( z = -v u_h b_T \). Observing that \( z \in H_0(\text{div}, \Omega) \), we have
\[ \int_T \text{div } z \ p_h = -\int_T z \cdot \nabla p_h + \int_{\partial T} p_h z \cdot n = 0. \quad (3.26) \]
Combining (2.4) and (2.6) and using (3.26) we may write
\[
\int_T (p - p_h) \text{div } z + \int_T \rho (\lambda u - \lambda_h u_h) \cdot z = -\lambda_h \int_T \rho u_h \cdot z.
\] (3.27)

Choosing \( v = (\lambda_h |T| || u_h ||^2) / (\rho c^4 \int_T | u_h |^2 b_T) \), we have
\[
- \lambda_h \rho \int_T u_h \cdot z = \frac{\lambda_h^2}{c^2} || u_h ||^2 |T|.
\] (3.28)

and using standard homogeneity arguments, we obtain after some computations
\[
|| z ||_{0,T} \leq C \frac{\lambda_h}{\rho c^4} || u_h ||_{0,T} |T|,
\] (3.29)
\[
| z |_{1,T} \leq C \frac{\lambda_h}{\rho c^4} || u_h ||_{0,T} |T|^{1/2}.
\] (3.30)

Thus, (3.27) with (3.28), (3.29) and (3.30) yields
\[
\frac{\lambda_h}{c^2} || u_h ||_{0,T} |T|^{1/2} \leq C \left( || \text{div} (u - u_h) ||_{0,T} + || \lambda u - \lambda_h u_h ||_{0,T} |T|^{1/2} \right),
\] (3.31)
where we have used that \( \text{div}(u - u_h) = (-1/\rho c^2)(p - p_h) \).

To conclude the theorem, it is enough to prove that
\[
|| J ||_{0,T}^{1/2} \leq C || u - u_h ||_{0,T*}.
\] (3.32)

The proof will not be given here because it is essentially identical to that in [9].

Finally, (3.25) follows from (3.31) and (3.32). \( \square \)

**Remark.** It is not difficult to show that the second term appearing on the right-hand side of (3.25) is a higher-order term. In fact, we have
\[
|| \lambda u - \lambda_h u_h ||_{0,T} \leq ||\lambda - \lambda_h||_{0,T} || u ||_{0,T} + \lambda_h || u - u_h ||_{0,T}
\]
and observing that
\[
|| u - u_h ||_{0,T} \leq || u - u_h ||_{H(\text{div}, T*)},
\]

it follows that \( |T|^{1/2} || u - u_h ||_{0,T} \) is of higher-order than \( || u - u_h ||_{H(\text{div}, T*)} \). Finally, we can use the a priori estimate (2.8) for the term \( |\lambda - \lambda_h| \).

### 4. Numerical results

In this section we present the results of some numerical computation. We consider the problem of a rigid L-shaped cavity with air inside.

The geometrical data can be seen in Fig. 1.

We have taken the following values of the physical parameters:
\[
\rho = 1 \text{ kg/m}^3, \\
c = 340 \text{ m/s}.
\]
Table 1
Eigenfrequencies computed by using uniform refinement

<table>
<thead>
<tr>
<th>Mode</th>
<th>$N = 5280$</th>
<th>$N = 9344$</th>
<th>$N = 14560$</th>
<th>$N = 20928$</th>
<th>Order</th>
<th>$\lambda_{\text{exact}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>412.293</td>
<td>412.527</td>
<td>412.654</td>
<td>412.733</td>
<td>1.36</td>
<td>413.013</td>
</tr>
<tr>
<td>$F_2$</td>
<td>639.155</td>
<td>639.160</td>
<td>639.162</td>
<td>639.164</td>
<td>1.72</td>
<td>639.167</td>
</tr>
<tr>
<td>$F_3$</td>
<td>1067.682</td>
<td>1067.833</td>
<td>1067.976</td>
<td>1068.027</td>
<td>2.00</td>
<td>1068.141</td>
</tr>
<tr>
<td>$F_4$</td>
<td>1068.093</td>
<td>1068.114</td>
<td>1068.124</td>
<td>1068.129</td>
<td>2.01</td>
<td>1068.142</td>
</tr>
<tr>
<td>$F_5$</td>
<td>1147.241</td>
<td>1147.329</td>
<td>1147.370</td>
<td>1147.392</td>
<td>1.99</td>
<td>1147.442</td>
</tr>
<tr>
<td>$F_6$</td>
<td>1204.868</td>
<td>1205.079</td>
<td>1205.197</td>
<td>1205.272</td>
<td>1.25</td>
<td>1205.566</td>
</tr>
<tr>
<td>$F_7$</td>
<td>1510.938</td>
<td>1510.782</td>
<td>1510.709</td>
<td>1510.670</td>
<td>1.99</td>
<td>1510.580</td>
</tr>
<tr>
<td>$F_8$</td>
<td>1571.465</td>
<td>1572.302</td>
<td>1572.728</td>
<td>1572.978</td>
<td>1.63</td>
<td>1573.700</td>
</tr>
</tbody>
</table>

In Table 1 we present the computed lowest eigenfrequencies for several embedded meshes. We denote by $N$ the number of the unknowns. Because no analytical expression for the eigenvalues is known, we have extrapolated the computed ones to obtain what will be denoted by $\lambda_{\text{exact}}$. We also used this extrapolation technique to get an estimation of the order of convergence in powers of $h = O(1/\sqrt{N})$.

Since $\Omega$ has reentrant corners, eigenfunctions with singularities are expected. In these cases, the order of convergence is less than 2 as predicted by the theory for regular eigenfunctions.

Fig. 2 presents some computed eigenmodes. It shows the displacement in the fluid corresponding to the eigenvalues $F_1, F_2, F_5$ and $F_6$. Observe the singular behavior of the displacement near the corner in modes $F_1$ and $F_6$.

Mode $F_2$ represents a particular case: although its corresponding displacement field is regular, it converges much slower than regular eigenmodes.

We present below the results obtained with meshes generated by the following adaptive method. The process starts with a uniform triangulation $\mathcal{T}_0$. By using $\eta_T$ as an error indicator at the element
Fig. 2. Fluid displacements in a rigid L-shaped cavity: (a) eigenmode \( F_1 \); (b) eigenmode \( F_2 \); (c) eigenmode \( F_5 \); (d) eigenmode \( F_8 \).

\( T, \mathcal{I}_{k+1} \) is obtained from \( \mathcal{I}_k \) refining all \( T \in \mathcal{I}_k \) with

\[
\eta_T \geq \gamma \max_{T' \in \mathcal{I}_k} \eta_{T'}.
\]

In our experiments we have taken \( \gamma = 0.5 \) and we have started the process from a very coarse mesh.

Tables 2–4 show the results obtained in six steps of the refinement procedure for eigenmodes \( F_1, F_6 \) and \( F_8 \), respectively. Now, the order of convergence is computed in powers of \( 1/N \) and it is almost exactly 1 for all these eigenfrequencies. In another words, the optimal order of convergence with respect to the number of nodes was obtained for these singular eigenmodes. It is interesting to observe the significant reduction of the necessary computational effort to obtain a solution with a prescribed accuracy.

Numerical experiments for the regular eigenmodes were also performed. In Tables 5–7 we present the results of these experiments for eigenmodes \( F_2, F_5 \) and \( F_7 \), respectively.

A not monotone convergence can be observed for mode \( F_2 \). This could be a consequence of the particular behavior of this eigenmode already indicated. However, after five steps of the refinement procedure, the error in \( \lambda \) was reduced by 10, approximately.
Table 2
Eigenmode $F_1$ computed by using adaptive refinement

| $k$ | $N$  | $\lambda_h$ | $|\lambda_{\text{exact}} - \lambda_h|$ |
|-----|------|-------------|----------------------------------|
| 0   | 160  | 404.489     | 8.524                            |
| 1   | 208  | 407.793     | 5.220                            |
| 2   | 469  | 410.879     | 2.134                            |
| 3   | 628  | 411.676     | 1.337                            |
| 4   | 925  | 412.039     | 0.974                            |
| 5   | 1752 | 412.556     | 0.457                            |
| 6   | 2578 | 412.709     | 0.304                            |
|     |      |             | **Order = 1.13**                  |

Table 3
Eigenmode $F_6$ computed by using adaptive refinement

| $k$ | $N$  | $\lambda_h$ | $|\lambda_{\text{exact}} - \lambda_h|$ |
|-----|------|-------------|----------------------------------|
| 0   | 160  | 1198.613    | 6.952                            |
| 1   | 431  | 1199.141    | 6.425                            |
| 2   | 1103 | 1202.844    | 2.722                            |
| 3   | 1994 | 1204.271    | 1.295                            |
| 4   | 4097 | 1204.787    | 0.778                            |
| 5   | 6203 | 1205.170    | 0.396                            |
| 6   | 10200| 1205.328    | 0.238                            |
|     |      |             | **Order = 1.04**                  |

Table 4
Eigenmode $F_8$ computed by using adaptive refinement

| $k$ | $N$  | $\lambda_h$ | $|\lambda_{\text{exact}} - \lambda_h|$ |
|-----|------|-------------|----------------------------------|
| 0   | 160  | 1526.081    | 47.619                           |
| 1   | 516  | 1561.311    | 12.389                           |
| 2   | 859  | 1566.104    | 7.596                            |
| 3   | 2202 | 1570.848    | 2.852                            |
| 4   | 3251 | 1571.776    | 1.923                            |
| 5   | 8254 | 1572.991    | 0.709                            |
| 6   | 11134| 1573.235    | 0.464                            |
|     |      |             | **Order = 1.07**                  |

With the exception of mode $F_2$, the obtained results show that the adaptive mesh-refinement procedure does not deteriorate the order of convergence of the regular modes, i.e. these eigenmodes converge with optimal order.

All these results allow us to conclude that our error estimator efficiently detects the regions where the mesh must be refined.
Table 5
Eigenmode \( F_2 \) computed by using adaptive refinement

| \( k \) | \( N \) | \( \lambda_h \) | \( |\lambda_{\text{exact}} - \lambda_h| \) |
|-------|------|-------------|-----------------|
| 0     | 160  | 639.131     | 0.0360          |
| 1     | 573  | 639.117     | 0.0502          |
| 2     | 1504 | 639.151     | 0.0161          |
| 3     | 2433 | 639.168     | 0.0014          |
| 4     | 6673 | 639.160     | 0.0073          |
| 5     | 9607 | 639.166     | 0.0025          |
| Order |      | -           | -               |

Table 6
Eigenmode \( F_3 \) computed by using adaptive refinement

| \( k \) | \( N \) | \( \lambda_h \) | \( |\lambda_{\text{exact}} - \lambda_h| \) |
|-------|------|-------------|-----------------|
| 0     | 160  | 1140.485    | 6.957           |
| 1     | 570  | 1146.060    | 1.382           |
| 2     | 2050 | 1147.108    | 0.334           |
| 3     | 2552 | 1147.193    | 0.249           |
| 4     | 8120 | 1147.344    | 0.098           |
| 5     | 10331| 1147.378    | 0.064           |
| Order |      | 1.09        | -               |

Table 7
Eigenmode \( F_7 \) computed by using adaptive refinement

| \( k \) | \( N \) | \( \lambda_h \) | \( |\lambda_{\text{exact}} - \lambda_h| \) |
|-------|------|-------------|-----------------|
| 0     | 160  | 1521.421    | 10.841          |
| 1     | 586  | 1514.159    | 3.580           |
| 2     | 2190 | 1511.595    | 1.015           |
| 3     | 8610 | 1510.777    | 0.197           |
| 4     | 28208| 1510.596    | 0.017           |
| 5     | 36540| 1510.592    | 0.013           |
| Order |      | 1.21        | -               |

Fig. 3 shows, for eigenmode \( F_1 \), the initial triangulation together with three refined triangulations.

We also computed the eigenmodes of the same problem and using the same adaptive mesh-refinement procedure as above but assuming

\[
\hat{\eta}_T^2 = \|u_h\|_{0,T}^2 + \sum_{T \subset \Omega} |J_T|^2 \|u_h\|_{0,T}^2
\]  
(4.33)

as the error indicator at the element \( T \). The a posteriori local error estimator (4.33) was introduced in [9].

In Tables 8–10 we show some vibration frequencies computed in this way.
Now, we observe that modes $F_2$ and $F_8$ present a not monotone convergence. Moreover, for mode $F_2$ the error in $\lambda$ remains almost the same after five refinements.

Finally, the order of convergence obtained for mode $F_7$ is not optimal.

Other numerical experiments were performed starting from finer triangulations and using different values of $\gamma$. In all these experiments, approximated eigenmodes showing a not monotone convergence or a poor order of convergence were presented.
Table 9
Eigenmode $F_7$ computed by using adaptive refinement and $\tilde{\eta}_T$

| $k$ | $N$  | $\lambda_h$ | $|\lambda_{\text{exact}} - \lambda_h|$ |
|-----|------|-------------|-------------------------------|
| 0   | 160  | 1521.421    | 10.841                        |
| 1   | 466  | 1519.211    | 8.631                         |
| 2   | 1298 | 1519.071    | 8.492                         |
| 3   | 4302 | 1512.540    | 1.960                         |
| 4   | 8378 | 1511.770    | 1.190                         |
| 5   | 19426| 1510.933    | 0.354                         |

Order = 0.71

Table 10
Eigenmode $F_8$ computed by using adaptive refinement and $\tilde{\eta}_T$

| $k$ | $N$  | $\lambda_h$ | $|\lambda_{\text{exact}} - \lambda_h|$ |
|-----|------|-------------|-------------------------------|
| 0   | 160  | 1526.081    | 47.619                        |
| 1   | 428  | 1566.524    | 7.176                         |
| 2   | 632  | 1576.632    | 2.932                         |
| 3   | 1638 | 1574.554    | 0.854                         |
| 4   | 1717 | 1574.856    | 1.156                         |
| 5   | 3072 | 1575.722    | 2.022                         |

Order = -

Since the weights of the terms in our estimator are the only differences between this one and $\tilde{\eta}_T$, we can conclude that these weights are very important in order to obtain good local indicators of the error and so, a good performance of an adaptive mesh-refinement process.

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References