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Some noncommutative multi-instantons from vortices in curved space

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Abstract

We construct $U(2)$ noncommutative multi-instanton solutions by extending Witten's ansatz [1] which reduces the problem of cylindrical symmetry in four dimensions to that of a set of Bogomol'nyi equations for an Abelian Higgs model in two-dimensional curved space. Using the Fock space approach, we give explicit vortex solutions to the Bogomol'nyi equations and, from them, we present multi-instanton solutions.

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After the first instanton solution with topological charge $Q = 1$ was presented in [2], many efforts were devoted to the construction of $Q = n$ instantons, as well as to the analysis of free parameters of the general solution. A first successful result was reported in [1], where a cylindrically symmetric multi-instanton solution was constructed by relating the problem with that of vortex solutions in two-dimensional curved space. After 't Hooft proposal of a very simple ansatz [3], another family of multi-instantons was constructed [4]. Finally, a systematic method for finding instanton solutions and their moduli space, the so-called ADHM construction [5], was developed.

Instantons were rapidly recognized as a basic ingredient for studying non-perturbative aspects of quantum field theories. They are also relevant in the context of string theory and noncommutative geometry. Concerning this last issue, instantons in noncommutative \mathbf{R}^4 space were first presented in [6] where the ADHM construction was adapted to the noncommutative case. In this Letter the discussion was mainly centred in the $U(1)$ gauge group and $q = 1$ instanton. Prompted by this work, many other applications of the ADHM method in noncommutative space were presented [7–10].

The alternative 't Hooft ansatz approach to noncommutative instantons was also analysed in [6], but problems with selfduality were overlooked in that work. These problems were overcome in [11], where a regular solution for the $q = 1$ $U(2)$ instanton was explicitly constructed by an appropriate extension of the 't Hooft ansatz (related work on this issue was dis-

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cussed in [12,13]). Concerning multi-instantons, the analogous of 't Hooft solution for $U(N)$ instantons with $N > 1$ has not been found (some problems preventing their construction were already discussed in [11]).

As in the commutative case, there is still the possibility to look for noncommutative multi-instanton $U(2)$ solutions by connecting the problem with that of (noncommutative) vortex solutions in curved space, the analogous of the solution presented in [1] in ordinary space. It is the purpose of the present work to analyse this issue which has the additional interest of requiring the construction of noncommutative solitons in a nontrivial metric.

In order to extend the approach in [1], connecting an axially symmetric ansatz for the instanton gauge field with vortex solutions in 2-dimensional curved space-time, we shall consider the following commutation relations for cylindrical coordinates $((r, \vartheta, \varphi)$ and t) in Euclidean 4-dimensional space:

$$[r, t] = i\theta(r, t), \tag{1}$$

$$[r, \vartheta] = [r, \varphi] = [t, \vartheta] = [t, \varphi] = [\vartheta, \varphi] = 0. \tag{2}$$

Eq. (2) corresponds to the most natural commutation relations to impose when a problem with cylindrical symmetry is to be studied. Although $\theta(r, t)$ in (1) is in principle an arbitrary function, we shall see that, in the reduced 2-dimensional problem, a covariantly constant θ guarantees associativity of the noncommutative product of functions. As we shall see, this in turn implies

$$\theta(r, t) = \theta_0 r^2 \tag{3}$$

with θ_0 a dimensionless constant. We shall then take (3) as defining noncommutativity of coordinates r, t from here on.

Thanks to the fact that θ is covariantly constant in two-dimensional space, there is a change of coordinates that greatly simplifies calculations. Indeed, taking $(r, \vartheta, \varphi, t) \rightarrow (u = -1/r, \vartheta, \varphi, t)$, Eq. (1) reduces to

$$[u, t] = i\theta_0, \tag{4}$$

while (2) remain unchanged (with r replaced by $-1/u$). Note that in terms of curvilinear coordinates $(u = -1/r, \vartheta, \varphi, t)$, $\theta^{\mu\nu}$ is constant and hence the

Moyal product $*$ provides a realization of the noncommutative product of functions.

We shall take the gauge group to be $U(2)$ and define

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \tag{5}$$

with

$$A_\mu = \vec{A}_\mu \cdot \frac{\vec{\sigma}}{2} + A_\mu^4 \frac{I}{2} \tag{6}$$

and $\vec{\sigma} = (\sigma^a)$ the Pauli matrices. The dual field strength $\tilde{F}_{\mu\nu}$ is defined as

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \sqrt{g} g^{(\alpha)} g^{(\beta)} F_{\alpha\beta}. \tag{7}$$

Here we have used that the metric tensor associated with curvilinear coordinates $(u = -1/r, \vartheta, \varphi, t)$ is diagonal. Its components read

$$g^{\mu\nu} = \text{diag}\left(u^4, u^2, \frac{u^2}{\sin^2 \vartheta}, 1\right) \\ \equiv (g^{(u)}, g^{(\vartheta)}, g^{(\varphi)}, g^{(t)}). \tag{8}$$

We shall look for multi-instanton solutions to the selfduality equations

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}. \tag{9}$$

To this end, we consider an ansatz for the gauge field components, which is the $U(2)$ noncommutative extension of the one solving the commutative case. For the $SU(2)$ sector we just take Witten's ansatz [1], which can be written as

$$\vec{A}_u = A_u(u, t) \vec{\Omega}(\vartheta, \varphi), \\ \vec{A}_t = A_t(u, t) \vec{\Omega}(\vartheta, \varphi), \\ \vec{A}_\vartheta = \phi_1(u, t) \partial_\vartheta \vec{\Omega}(\vartheta, \varphi) \\ + (1 + \phi_2(u, t)) \vec{\Omega}(\vartheta, \varphi) \wedge \partial_\vartheta \vec{\Omega}(\vartheta, \varphi), \\ \vec{A}_\varphi = \phi_1(u, t) \partial_\varphi \vec{\Omega}(\vartheta, \varphi) \\ + (1 + \phi_2(u, t)) \vec{\Omega}(\vartheta, \varphi) \wedge \partial_\varphi \vec{\Omega}(\vartheta, \varphi), \tag{10}$$

with

$$\vec{\Omega}(\vartheta, \varphi) = \begin{pmatrix} \sin \vartheta \cos \varphi \\ \sin \vartheta \sin \varphi \\ \cos \vartheta \end{pmatrix}. \tag{11}$$

Concerning the remaining $U(1)$ components, it is natural to propose the ansatz

$$A_u^4 = A_u^4(u, t), \\ A_t^4 = A_t^4(u, t), \\ A_\vartheta^4 = A_\vartheta^4 = 0. \tag{12}$$

With this ansatz, the selfduality equations (9) become

$$\begin{aligned}
 \partial_t A_u - \partial_u A_t + \frac{i}{2}[A_t, A_u^4] + \frac{i}{2}[A_t^4, A_u] &= 1 - \phi_1^2 - \phi_2^2, \\
 \partial_t A_u^4 - \partial_u A_t^4 + \frac{i}{2}[A_t^4, A_u^4] + \frac{i}{2}[A_t, A_u] &= -i[\phi_1, \phi_2], \\
 \partial_t \phi_1 + \frac{1}{2}[A_t, \phi_2]_+ + \frac{i}{2}[A_t^4, \phi_1] &= u^2 \left(\partial_u \phi_2 - \frac{1}{2}[A_u, \phi_1]_+ + \frac{i}{2}[A_u^x, \phi_2] \right), \\
 \partial_t \phi_2 - \frac{1}{2}[A_t, \phi_1]_+ + \frac{i}{2}[A_t^4, \phi_2] &= -u^2 \left(\partial_u \phi_1 + \frac{1}{2}[A_u, \phi_2]_+ + \frac{i}{2}[A_u^x, \phi_1] \right). \quad (13)
 \end{aligned}$$

As in other instanton analysis [11] one could restrict even more the ansatz for the $U(1)$ sector so that the form of the equations for the $SU(2)$ components become the natural noncommutative generalization of those in ordinary space. We then propose the following identification:

$$\begin{aligned}
 A_t^4(u, t) &= A_t(u, t), \\
 A_u^4(u, t) &= A_u(u, t). \quad (14)
 \end{aligned}$$

With this, and introducing the notation

$$\begin{aligned}
 \phi &= \phi_1 - i\phi_2, & D\phi &= \partial\phi + iA\phi, \\
 F_{tu} &= \partial_t A_u - \partial_u A_t + i[A_t, A_u], \quad (15)
 \end{aligned}$$

system (13) reduces to

$$F_{tu} = -\frac{1}{2}[\phi, \bar{\phi}], \quad (16)$$

$$F_{tu} = 1 - \frac{1}{2}[\phi, \bar{\phi}]_+, \quad (17)$$

$$D_t \phi = iu^2 D_u \phi. \quad (18)$$

This system of equations is one of the main steps in our task of constructing multi-instantons and deserves some comments. It is an overconstrained system but, as we shall see, nontrivial solutions can be found. The two last equations resemble the Bogomol’nyi equations arising in ordinary two dimensional *curved* space. In fact, they coincide (in the commutative limit) with those discussed in [1], with coordinates $u =$

$-1/r, t$ and a metric

$$g^{\mu\nu} = \begin{pmatrix} u^2 & 0 \\ 0 & u^{-2} \end{pmatrix} \quad (19)$$

with $\det g(x) = 1$. Hence, we succeeded in connecting, also in the noncommutative case, instanton selfdual equations in 4-dimensional Euclidean space working in curvilinear coordinates ($u = -1/r, \vartheta, \varphi, t$) with vortex Bogomol’nyi equations in 2-dimensional curved space with coordinates (u, t) . Were we able to find noncommutative multivortex solutions, we then could explicitly write noncommutative multi-instanton solutions, as done in [1] for the commutative theory.

There is, however, the third equation (16) in the coupled system (16)–(18), a remnant of the $U(1)$ sector necessarily present in the noncommutative case. We shall see, however, that this equation becomes identical to Eq. (17) for a particular family of scalar field solutions. We then pass analysing this and we obtain vortex solutions in curved noncommutative space.

At this point, we have to handle a noncommutative field theory in curved space. This problem can be related to that of defining a noncommutative product with θ depending on the coordinates. Let us briefly recall this last problem. Consider a general noncommutative product \circ such that

$$[y^m, y^n]_\circ = y^m \circ y^n - y^n \circ y^m = i\theta^{mn}(y). \quad (20)$$

Associativity of the \circ product can be seen to impose the following condition on θ^{ab} [14–16],

$$\theta^{mn} \partial_n \theta^{ab} + \theta^{an} \partial_n \theta^{bm} + \theta^{bn} \partial_n \theta^{ma} = 0, \quad (21)$$

which is equivalent to

$$\theta^{mn} \nabla_n \theta^{ab} + \theta^{an} \nabla_n \theta^{bm} + \theta^{bn} \nabla_n \theta^{ma} = 0 \quad (22)$$

with

$$\nabla_m \theta^{ab} = \partial_m \theta^{ab} + \Gamma_{ms}^a \theta^{sb} + \Gamma_{ms}^b \theta^{as}, \quad (23)$$

Here Γ_{ms}^a is the Christoffel symbol associated with the two-dimensional metric $g_{ab}(y)$. Now, as exploited in [16], a covariantly constant θ^{ab} ,

$$\nabla_m \theta^{ab} = 0, \quad (24)$$

trivially verifies (22), and hence leads to an associative product. One can see that, in two dimensions, Eq. (24)

reduces to

$$\frac{1}{\sqrt{g}} \partial_m (\sqrt{g} \theta^{ab}) = 0, \tag{25}$$

where $g = \det g_{ab}$. Then, the most general covariantly constant θ^{ab} takes the form

$$\theta^{mn} = \frac{\varepsilon^{mn}}{\sqrt{g}}. \tag{26}$$

Let us consider at this point the two-dimensional metric relevant to the vortex problem in ordinary space. It corresponds, in coordinates (r, t) to $g^{ab} = r^2 \delta^{ab}$. According to Eq. (26), a covariantly constant $\theta^{ab} = \varepsilon^{ab} \theta(r, t)$ takes, in such a metric, the form

$$\theta(r, t) = \theta_0 r^2. \tag{27}$$

This is precisely the form chosen for $\theta(r, t)$ in Eqs. (1), (3). Now, in the coordinate system $(u = -1/r, t)$ the \circ commutator becomes the ordinary Moyal commutator since

$$[u, t]_\circ = i \theta_0. \tag{28}$$

Going back to the problem of solving the Bogomol’nyi system (16)–(18), it is convenient to define complex variables

$$z = \frac{1}{\sqrt{2}}(u + it), \quad \bar{z} = \frac{1}{\sqrt{2}}(u - it), \tag{29}$$

in terms of which Eqs. (16)–(18) become

$$\left(1 - \frac{1}{2}(z + \bar{z})^2\right) D_z \phi = \left(1 + \frac{1}{2}(z + \bar{z})^2\right) D_{\bar{z}} \phi, \tag{30}$$

$$i F_{z\bar{z}} = 1 - \frac{1}{2} [\phi, \bar{\phi}]_+, \tag{31}$$

$$i F_{z\bar{z}} = -\frac{1}{2} [\phi, \bar{\phi}]. \tag{32}$$

In view of equation (28), if one defines

$$z \rightarrow \sqrt{\theta_0} a, \quad \bar{z} \rightarrow \sqrt{\theta_0} a^\dagger, \tag{33}$$

one then has

$$[a, a^\dagger] = 1, \tag{34}$$

and hence one is lead to follow the alternative Fock space approach to noncommutative field theories, taking a and a^\dagger as annihilation and creation operators generating the Fock space $\{|n\rangle\}$. Concerning derivatives, one has

$$\partial_z \rightarrow -\frac{1}{\sqrt{\theta_0}} [a^\dagger,], \quad \partial_{\bar{z}} \rightarrow \frac{1}{\sqrt{\theta_0}} [a,]. \tag{35}$$

One should notice at this point that, in the case at hand, the complex variable z is defined in the half-lower semiplane and this could cause problems when connecting the product of operators in Fock space with Moyal products in coordinate representation. In fact, this connection can be established through an isomorphism which results in a mapping between operators in Fock space and functions in R^{2n} . If the two-dimensional manifold is not R^2 but a half plane one should analyse whether the isomorphism is modified. Instead, we shall follow an alternative approach which consist in doubling the space manifold so as to work in R^2 and exploit the ordinary connection. Afterwards, we shall restrict the solutions to the relevant domain. Having in mind the features of Witten’s solutions in ordinary space, with an even magnetic field associated as a function of $u = -1/r$, we shall seek for solutions with such a magnetic field behavior in R^2 .

Note that Eq. (31) coincides with the corresponding *flat space* original Bogomol’nyi equation for the magnetic field. It is Eq. (30) governing the scalar field dynamics where the curved space metric plays a role. As discussed before, there is also the new third equation (32) arising from the additional $U(1)$ sector.

One can easily see that compatibility of (31) and (32), implies

$$\bar{\phi} \phi = 1 \tag{36}$$

and hence the only kind of nontrivial solutions following our ansatz should have the form

$$\phi = \sum_{n=0} |n + q\rangle \langle n|. \tag{37}$$

With this, it is easy now to construct a class of solutions analogous to those found in [17–20] for noncommutative Nielsen–Olesen vortices in flat space. Indeed, take

$$\begin{aligned} \phi &= \sum_{n=0} |n + q\rangle \langle n|, \\ A_z &= -\frac{i}{\sqrt{\theta_0}} \sum_{n=0}^{q-1} (\sqrt{n+1}) |n+1\rangle \langle n| \\ &\quad + \frac{i}{\sqrt{\theta_0}} \sum_{n=q} (\sqrt{n+1-q} - \sqrt{n+1}) |n+1\rangle \langle n|. \end{aligned} \tag{38}$$

One can trivially verify that functions (38) satisfy Eqs. (30)–(32) provided $\theta_0 = 2$. In particular, both the

l.h.s. and r.h.s of Eq. (30) vanish separately. Regarding the particular value of θ_0 for which we find a solution, let us recall that also for vortices in flat space it was necessary to fix $\theta_0 = 1$ in order to satisfy the corresponding Bogomol'nyi equations [19,20].

The magnetic field $B_{\theta_0} = iF_{z\bar{z}}$ associated with solution (38) takes the form

$$B_{\theta_0=2} = -\frac{1}{2}(|0\rangle\langle 0| + \dots + |q-1\rangle\langle q-1|) \equiv B \quad (39)$$

with associated magnetic flux

$$\Phi = \pi\theta_0 \text{Tr} B_{\theta_0} = -\pi q. \quad (40)$$

Note the factor $\pi\theta_0$ in the definition of the magnetic flux. It is one half of the usual factor, since our actual problem corresponds to the half plane.

We can now easily write the self-dual multi-instanton solution in 4-dimensional space by inserting the solution (38) in ansatz (10). The resulting selfdual field strength reads

$$\begin{aligned} \vec{F}_{tu} &= B\vec{\Omega}, & \vec{F}_{\vartheta\varphi} &= B \sin \vartheta \vec{\Omega}, \\ F_{tu}^4 &= B, & F_{\vartheta\varphi}^4 &= B \sin \vartheta, \end{aligned} \quad (41)$$

with the other field-strength components vanishing. The instanton number is then given by

$$\begin{aligned} Q &= \frac{1}{32\pi^2} \text{tr} \int d^4x \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \\ &= \frac{1}{\pi} \int_{-\infty}^0 du \int_{-\infty}^{\infty} dt B^2 = 2 \text{Tr} B^2 = \frac{q}{2}. \end{aligned} \quad (42)$$

We thus see that Q can be in principle integer or semi-integer, and this for ansatz (10) which is formally the same as that proposed in [1] for ordinary space and which yielded in that case to an integer. The origin of this difference between the commutative and the non-commutative cases can be traced back to the fact that in the former case, boundary conditions imposed on the half-plane force the solution to have an associated integer number. We were not able to find boundary conditions such that semi-integer configurations were excluded. Then, in order to have regular instanton solutions with integer number we just restrict vortex configurations (38) to those with $q = 2n$ so that the instanton number $Q = n \in \mathbb{Z}$. In fact, if one plots Witten's vortex solution in ordinary space in the whole (u, t)

plane, the magnetic flux has two peaks and the corresponding vortex number is even. Our choice then corresponds, in Fock space, to selecting the analogous of that solution in noncommutative space.

In summary, we have presented multi-instanton solutions to the $U(2)$ selfduality equations (9) in a non-commutative space where the commutation relations for coordinates are those defined by Eqs. (1), (2). In such a noncommutative space the problem can be reduced to that of solving Bogomol'nyi equations for vortices in curved space. Using the Fock space approach one can easily find a family of solutions for the latter, Eqs. (38), (39), and, from them, to explicitly construct instanton solutions, (41), with arbitrary charge. Let us end by recalling that axially symmetric instantons in ordinary 4-dimensional space-time can be employed to seek for multimonopole solutions in 3-dimensional space [21–24]. One could then follow a similar approach in the noncommutative case in the search of explicit magnetic monopole solutions. We hope to discuss this issue in a future work.

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