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Duality in reconstruction systems

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ABSTRACT

We consider reconstruction systems (RS's), which are G-frames in a finite dimensional setting, and that includes the fusion frames as projective RS's. We describe the spectral picture of the set of RS operators for the projective systems with fixed weights. We also introduce a functional defined on dual pairs of RS's, called the joint potential, and study the structure of the minimizers of this functional. In the case of irreducible RS's the minimizers are characterize as the tight systems. In the general case we give spectral and geometric characterizations of the minimizers of the joint potential. At the end of the paper we show several examples that illustrate our results.

1. Introduction

Fusion frames (briefly FF's) arise naturally as a generalization of the usual frames of vectors for a Hilbert space $\mathcal{H}$. Several applications of FF's have been studied, for example, sensor networks [14], neurology [27], coding theory [5,6,22], among others. We refer the reader to [13] and the references therein for a detailed treatment of the FF theory. Further developments can be found in [3,8,9,11,12,28].

Given $m \in \mathbb{N}$ we denote by $\mathbb{I}_m = \{1, \dots, m\} \subseteq \mathbb{N}$. In the finite dimensional setting, a FF is a sequence $\mathcal{N}_w = (w_i\,,\,\mathcal{N}_i)_{i\in\mathbb{I}_m}$ where each $w_i \in \mathbb{R}_{>0}$ and the $\mathcal{N}_i \subseteq \mathbb{C}^d$ are subspaces that generate $\mathbb{C}^d$. The synthesis operator of $\mathcal{N}_w$ is usually defined as

$$T_{\mathcal{N}_w} : \mathcal{K}_{\mathcal{N}_w} \stackrel{\text{def}}{=} \bigoplus_{i\in\mathbb{I}_m} \mathcal{N}_i \to \mathbb{C}^d \quad \text{given by} \quad T_{\mathcal{N}_w}(x_i)_{i\in\mathbb{I}_m} = \sum_{i\in\mathbb{I}_m} w_i\, x_i .$$

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Its adjoint, the so-called analysis operator of $\mathcal{N}_w$, is given by $T^*_{\mathcal{N}_w} y = (w_i\, P_{\mathcal{N}_i}\, y)_{i\in\mathbb{I}_m}$ for $y \in \mathbb{C}^d$, where $P_{\mathcal{N}_i}$ denotes the orthogonal projection onto $\mathcal{N}_i$. The frame $\mathcal{N}_w$ induces a linear encoding–decoding scheme that can be described in terms of these operators.

The previous setting for the theory of FF's presents some technical difficulties. For example the domain of $T_{\mathcal{N}_w}$ relies strongly on the subspaces of the fusion frame. In particular, any change on the subspaces modifies the domain of the operators preventing smooth perturbations of these objects. Moreover, this kind of rigidity on the definitions implies that the notion of a dual FF is not clear.

An alternative approach to the fusion frame (FF) theory comes from the theory of G-frames [29] (see also [20,30,31,19] where operator valued frames are introduced and developed) and its variants, namely the theory of protocols introduced in [5] and the theory of reconstruction systems considered in [23] (see also [26]), which are finite dimensional G-frames.

In this context, we fix the dimensions $\dim \mathcal{N}_i = k_i$ and consider a universal space

$$\mathcal{K} = \mathcal{K}_{m\,,\,\mathbf{k}} \stackrel{\text{def}}{=} \bigoplus_{i\in\,\mathbb{I}_m} \mathbb{C}^{k_i}, \quad \text{where} \quad k = (k_1\,,\,\ldots\,,\,k_m) \in \mathbb{N}^m.$$

A reconstruction system (RS) is a sequence $\mathcal{V} = \{V_i\}_{i\in\,\mathbb{I}_m}$ such that $V_i \in L(\mathbb{C}^d\,,\,\mathbb{C}^{k_i})$ for every $i \in \mathbb{I}_m$, which allows the construction of an encoding–decoding algorithm (see Definition 2.1, for details). We denote by $\mathcal{RS} = \mathcal{RS}(m, \mathbf{k}, d)$ the set of all RS's with these fixed parameters. Observe that, if $\mathcal{N}_w = (w_i\,,\,\mathcal{N}_i)_{i\in\mathbb{I}_m}$ is a FF, it can be modeled as a system $\mathcal{V} = \{V_i\}_{i\in\,\mathbb{I}_m} \in \mathcal{RS}$ such that $V_i^*V_i = w_i^2\, P_{\mathcal{N}_i}$ for every $i \in \mathbb{I}_m$. These systems are called projective RS's. On the other hand, a general RS arises from a usual vector frame by grouping together the elements of the frame. Thus, the coefficients involved in the encoding–decoding scheme of RS are vector valued, and they lie in the space $\mathcal{K}$.

The main advantage of the RS (or more generally of the G-frame) framework with respect to the fusion frame formalism is that each (projective) RS has many RS's that are *dual* systems. In particular, the canonical dual RS remains a RS (for details and definitions see Section 2). In contrast, it is easy to give examples of a FF such that its canonical dual is not a fusion frame. There exists a notion of duality among fusion frames defined by Gavruta (see [17]), where the reconstruction formula of a fixed $\mathcal{V}$ involves the FF operator $S_\mathcal{V}$ of $\mathcal{V}$. Nevertheless, in the context of RS's, we show that the notion of dual systems can be described and characterized in a quite natural way. On the other hand, the RS framework allows to make not only a metric but also a differential geometric study of the set of RS's, which will be developed in Section 4 of this paper.

Let us fix the parameters $(m, \mathbf{k}, d)$ and the sequence $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ of weights. In this work we study some properties of the sets $\mathcal{RS} = \mathcal{RS}(m, \mathbf{k}, d)$ of RS's and $\mathcal{PRS}_\mathbf{v} = \mathcal{PRS}_\mathbf{v}\,(m, \mathbf{k}, d)$ of projective systems with fixed weights $\mathbf{v}$.

We are interested in those $\mathcal{V} \in \mathcal{PRS}_\mathbf{v}$ that admit a dual RS, $\mathcal{W} \in \mathcal{RS}$, with some additional structure and such that the pair $(\mathcal{V}, \mathcal{W})$ has some nice duality relations. Several other problems related with *dual pairs* that are optimal with respect to some criteria have been widely studied in the theory of frames and G-frames (see, for example [5–7,18,20,22,23,29–31]). We are interested in dual pairs $(\mathcal{V}, \mathcal{W}) \in \mathcal{PRS}_\mathbf{v} \times \mathcal{RS}$ such that $\mathcal{W}$ is also projective and such that the induced reconstruction formula is simple. The optimal solutions of this problem would be those $\mathcal{V} \in \mathcal{PRS}_\mathbf{v}$ that are tight (which are the analog of tight fusion frames), i.e., such that $(\mathcal{V}, \alpha \cdot \mathcal{V})$ is a dual pair for some $\alpha \in \mathbb{R}_{>0}$. Unfortunately, it is well known that there are choices of weights and dimensions $\mathbf{k}$ for which no projective tight system $\mathcal{V} \in \mathcal{PRS}_\mathbf{v}$ exists.

In order to study dual pairs $(\mathcal{V}, \mathcal{W}) \in \mathcal{PRS}_\mathbf{v} \times \mathcal{RS}$ that have the properties described above we introduce a functional that we call the *joint potential*. More explicitly, given a dual pair $(\mathcal{V}, \mathcal{W}) \in \mathcal{PRS}_\mathbf{v} \times \mathcal{RS}$ we consider its joint potential given by

$$\text{RSP}\,(\mathcal{V}, \mathcal{W}) \stackrel{\text{def}}{=} \text{tr }S^2_\mathcal{V} + \text{tr }S^2_\mathcal{W} \in \mathbb{R}_{>0},$$

where $S_\mathcal{V}$ and $S_\mathcal{W}$ denote the so-called RS operators of $\mathcal{V}$ and $\mathcal{W}$, respectively (see Definition 2.1). This functional has already been considered in [10] in the context of vector frames. We point out that since we focus on a problem dealing with a potential defined using the trace of invertible operators, we must restrict our study to the finite dimensional setting. Thus, instead of working with general G-frames we deal with RS's.

We study some properties of local minimizers of the joint potential (with respect to natural metrics in the set $\mathcal{PRS}_{\mathbf{v}}$). We show that local minimizers are also global and that optimal dual pairs with respect to the joint potential are of the form $(\mathcal{V}, \mathcal{V}^{\#})$, where $\mathcal{V}^{\#}$ denote the canonical dual RS of $\mathcal{V}$ (see Definition 2.3) and with an intrinsic spectral structure (that depends on the parameters $(m, \mathbf{k}, d)$ and the fixed weights $\mathbf{v}$). In order to obtain a detailed structure of the minimizers of the joint potential we present a geometrical description of $\mathcal{PRS}_{\mathbf{v}}$ and give a sufficient condition – the notion of irreducible systems – in order that the operation of taking RS operators $\mathcal{PRS}_{\mathbf{v}} \ni \mathcal{V} \mapsto S_{\mathcal{V}}$ (see Definition 2.1) has smooth local cross-sections. We show that given an irreducible system $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ then $(\mathcal{V}, \mathcal{V}^{\#})$ is a local minimizer if and only if $\mathcal{V}$ is a tight projective system, so that in this case $(\mathcal{V}, \mathcal{V}^{\#})$ is a global minimizer. Using these results and geometrical reduction arguments we deal with the general case and obtain the geometrical structure of local minimizers.

The main results with respect to both spectral and geometrical structure of the (local) minimizers of the joint potential can be summarized as follows:

- There exist $\lambda_{\mathbf{v}} = \lambda_{\mathbf{v}}(m, \mathbf{k}, d) \in \mathbb{R}^d_{>0}$ such that a pair $(\mathcal{V}\,,\ \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ is a (local) minimizer for the RSP if and only if $\mathcal{W} = \mathcal{V}^{\#}$ and the vector of eigenvalues $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$.
- Every such $\mathcal{V}$ can be decomposed as a orthogonal sum of tight projective RS's, where the quantity of components and their tight constants are the same for every minimizer.

The paper is organized as follows. In Section 2, we recall the basic framework of reconstruction systems. In Section 3, we introduce the joint potential of dual pairs of reconstruction systems. In order to obtain the spectral structure of local minimizers of this functional we consider first some consequences of Horn–Klyachko's theory on sums of hermitian matrices. In Section 4, we develop a geometric approach to some perturbation problems in RS's theory. We end Section 4 with the main result on the geometrical structure of local minimizers of the joint potential. In Section 5, we give some examples of these problems, showing sets of parameters for which the vector $\lambda_{\mathbf{v}}$ and all minimizers $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ can be explicitly computed. We also present a conjecture which suggest a way to compute the vector $\lambda_{\mathbf{v}}$, as the minimal element in the spectral picture $\Lambda(\mathcal{OP}_{\mathbf{v}})$ of $\mathcal{OP}_{\mathbf{v}}$ with respect to the majorization (see Conjecture 5.4).

1.1. General notations

Given $m \in \mathbb{N}$ we denote by $\mathbb{I}_m = \{1, \ldots, m\} \subseteq \mathbb{N}$ and $\mathbb{1} = \mathbb{1}_m \in \mathbb{R}^m$ denotes the vector with all its entries equal to 1. For a vector $x \in \mathbb{R}^m$ we denote by $x^{\downarrow}$ the rearrangement of x in a decreasing order, and $(\mathbb{R}^m)^{\downarrow} = \{x \in \mathbb{R}^m : x = x^{\downarrow}\}$ the set of ordered vectors.

Given $\mathcal{H} \cong \mathbb{C}^d$ and $\mathcal{K} \cong \mathbb{C}^n$, we denote by $L(\mathcal{H}, \mathcal{K})$ the space of linear operators $T : \mathcal{H} \to \mathcal{K}$. Given an operator $T \in L(\mathcal{H}, \mathcal{K})$, $R(T) \subseteq \mathcal{K}$ denotes the image of T, $\ker T \subseteq \mathcal{H}$ the null space of T and $T^* \in L(\mathcal{K}, \mathcal{H})$ the adjoint of T. If $d \leq n$ we say that $U \in L(\mathcal{H}, \mathcal{K})$ is an isometry if $U^*U = I_{\mathcal{H}}$. In this case, U^* is called a coisometry. If $\mathcal{K} = \mathcal{H}$ we denote by $L(\mathcal{H}) = L(\mathcal{H}\,,\ \mathcal{H})$, by $\mathcal{G}l\,(\mathcal{H})$ the group of all invertible operators in $L(\mathcal{H})$, by $L(\mathcal{H})^+$ the cone of positive operators and by $\mathcal{G}l\,(\mathcal{H})^+ = \mathcal{G}l\,(\mathcal{H}) \cap L(\mathcal{H})^+$. If $T \in L(\mathcal{H})$, we denote by $\sigma(T)$ the spectrum of T, by $\operatorname{rk} T$ the rank of T, and by $\operatorname{tr} T$ the trace of T. Given $A \in L(\mathcal{H})^+$, its vector of eigenvalues is denoted by $\lambda(A) = \big(\lambda_1(A)\,,\ \ldots\,,\ \lambda_d(A)\big) \in (\mathbb{R}^d_+)^{\downarrow}$ (counting multiplicities and in decreasing order). By fixing orthonormal basis (onb) of the Hilbert spaces involved, we shall identify operators with matrices, using the following notations:

By $\mathcal{M}_{n,d}(\mathbb{C}) \cong L(\mathbb{C}^d\,,\ \mathbb{C}^n)$ we denote the space of complex $n \times d$ matrices. If $n = d$ we write $\mathcal{M}_n(\mathbb{C}) = \mathcal{M}_{n,n}(\mathbb{C})$. $\mathcal{H}(n)$ is the $\mathbb{R}$-subspace of selfadjoint matrices, $\mathcal{G}l\,(n)$ the group of all invertible elements of $\mathcal{M}_n(\mathbb{C})$, $\mathcal{U}(n)$ the group of unitary matrices, $\mathcal{M}_n(\mathbb{C})^+$ the set of positive semidefinite matrices, and $\mathcal{G}l\,(n)^+ = \mathcal{M}_n(\mathbb{C})^+ \cap \mathcal{G}l\,(n)$. If $d \leq n$, we denote by $\mathcal{I}(d\,,\ n) \subseteq \mathcal{M}_{n\,,\,d}(\mathbb{C})$ the set of isometries, i.e., those $U \in \mathcal{M}_{n\,,\,d}(\mathbb{C})$ such that $U^*U = I_d$.

If $W \subseteq \mathcal{H}$ is a subspace we denote by $P_W \in L(\mathcal{H})^+$ the orthogonal projection onto W, i.e., $R(P_W) = W$ and $\ker P_W = W^{\perp}$. For vectors on $\mathbb{C}^n$ we shall use the euclidean norm. On the other hand, for matrices $T \in \mathcal{M}_n(\mathbb{C})$ we shall use both

1. The spectral norm $\|T\| = \|T\|_{sp} = \max\limits_{\|x\|=1} \|Tx\|$.
2. The Frobenius norm $\|T\|_2 = (\text{tr } T^*T)^{1/2} = \left(\sum_{i,j\in\mathbb{I}_n} |T_{ij}|^2\right)^{1/2}$. This norm is induced by the inner product $\langle A,\ B\rangle = \text{tr } B^*A$, for $A, B \in \mathcal{M}_n(\mathbb{C})$.

2. Basic framework of reconstruction systems

In this section, we fix the notations and define the usual objects related to reconstruction systems, following the well-known framework of G-frames, but adapted to the finite dimensional setting.

Definition 2.1. Let $m, d \in \mathbb{N}$ and $\mathbf{k} = (k_1\ , \ \dots\ , \ k_m) \in \mathbb{N}^m$.

1. We shall abbreviate the above description by saying that $(m, \mathbf{k}, d)$ is a set of parameters. We denote by $n = \text{tr } \mathbf{k} \overset{\text{def}}{=} \sum_{i\in\mathbb{I}_m} k_i$ and assume that $n \geq d$.
2. We denote by $\mathcal{K} = \mathcal{K}_{m\,,\,\mathbf{k}} \overset{\text{def}}{=} \bigoplus_{i\in\mathbb{I}_m} \mathbb{C}^{k_i} \cong \mathbb{C}^n$. We shall often write each direct summand by $\mathcal{K}_i = \mathbb{C}^{k_i}$.
3. Given a space $\mathcal{H} \cong \mathbb{C}^d$ we denote by
$$L(m, \mathbf{k}, d) \overset{\text{def}}{=} \bigoplus_{i\in\mathbb{I}_m} L(\mathcal{H}\ ,\ \mathcal{K}_i) \cong L(\mathcal{H}, \mathcal{K}) \cong \bigoplus_{i\in\mathbb{I}_m} \mathcal{M}_{k_i\,,\,d}(\mathbb{C}) \cong \mathcal{M}_{n,d}(\mathbb{C}).$$
A typical element of $L(m, \mathbf{k}, d)$ is a system $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m}$ such that each $V_i \in L(\mathcal{H}\ ,\ \mathcal{K}_i)$.
4. A family $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in L(m, \mathbf{k}, d)$ is an $(m, \mathbf{k}, d)$-reconstruction system (RS) for $\mathcal{H}$ if
$$S_{\mathcal{V}} \overset{\text{def}}{=} \sum_{i\in\mathbb{I}_m} V_i^* V_i \in \mathcal{G}l\,(\mathcal{H})^+\ , \tag{1}$$
i.e., if $S_{\mathcal{V}}$ is invertible. This $S_{\mathcal{V}}$ is called the **RS operator** of $\mathcal{V}$. In this case, the m-tuple $\mathbf{k} = (k_1\ , \ \dots\ , \ k_m) \in \mathbb{N}^m$ satisfies that $n = \text{tr}\,\mathbf{k} \geq d$.
 We shall denote by $\mathcal{RS} = \mathcal{RS}(m, \mathbf{k}, d)$ the set of all $(m, \mathbf{k}, d)$-RS's for $\mathcal{H} \cong \mathbb{C}^d$.
5. The system $\mathcal{V}$ is said to be **projective** if there exists a sequence $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}_+^m$ of positive numbers, the weights of $\mathcal{V}$, such that
$$V_i\, V_i^* = v_i^2\, P_{\mathcal{K}_i}, \quad \text{for every} \quad i \in \mathbb{I}_m.$$
In this case, the following properties hold:
 (a) The weights can be computed directly, since each $v_i = \|V_i\|_{sp}$.
 (b) Each $V_i = v_i U_i$ for a coisometry $U_i \in L(\mathcal{H}\ ,\ \mathcal{K}_i)$. Thus $V_i^* V_i = v_i^2\, P_{R(V_i^*)} \in L(\mathcal{H})^+$ for every $i \in \mathbb{I}_m$.
 (c) $S_{\mathcal{V}} = \sum_{i\in\mathbb{I}_m} v_i^2\, P_{R(V_i^*)}$ as in fusion frame theory.
 We shall denote by $\mathcal{PRS} = \mathcal{PRS}(m, \mathbf{k}, d)$ the set of all projective elements of $\mathcal{RS}$.
6. The **analysis** operator of the system $\mathcal{V}$ is defined by
$$T_{\mathcal{V}}\ :\ \mathcal{H} \to \mathcal{K} = \bigoplus_{i\in\mathbb{I}_m} \mathcal{K}_i \ \text{ given by } \ T_{\mathcal{V}}\, x = (V_1\, x\ , \ \dots\ , \ V_m\, x), \quad \text{for} \quad x \in \mathcal{H}.$$
7. Its adjoint $T_{\mathcal{V}}^*$ is called the **synthesis** operator of the system $\mathcal{V}$, and it satisfies that
$$T_{\mathcal{V}}^* : \mathcal{K} = \bigoplus_{i\in\mathbb{I}_m} \mathcal{K}_i \to \mathcal{H} \quad \text{is given by} \quad T_{\mathcal{V}}^*\left((y_i)_{i\in\mathbb{I}_m}\right) = \sum_{i\in\mathbb{I}_m} V_i^*\, y_i.$$
Using the previous notations and definitions we have that $S_{\mathcal{V}} = T_{\mathcal{V}}^*\, T_{\mathcal{V}}$.

8. The frame constants in this context are the following: $\mathcal{V}$ is a RS if and only if
$$A_{\mathcal{V}}\,\|x\|^2 \;\le\; \langle S_{\mathcal{V}}\,x\,,\,x\rangle \;=\sum_{i\in\mathbb{I}_m}\;\|V_i\,x\|^2\;\le\;B_{\mathcal{V}}\,\|x\|^2 \tag{2}$$
for every $x\in\mathcal{H}$, where $0<A_{\mathcal{V}}=\lambda_{\min}(S_{\mathcal{V}})=\|S_{\mathcal{V}}^{-1}\|^{-1}\le\lambda_{\max}(S_{\mathcal{V}})=\|S_{\mathcal{V}}\|=B_{\mathcal{V}}$.
9. As usual, we say that $\mathcal{V}$ is **tight** if $A_{\mathcal{V}}=B_{\mathcal{V}}$. In other words, the system $\mathcal{V}\in\mathcal{RS}(m,\mathbf{k},d)$ is tight if and only if $S_{\mathcal{V}}=\frac{\tau}{d}\,I_{\mathcal{H}}$, where $\tau=\sum_{i\in\mathbb{I}_m}\operatorname{tr}(V_i^*V_i)$.
10. Given $U\in\mathcal{G}l\,(d)$, we define $\mathcal{V}\cdot U\overset{\text{def}}{=}\{V_i\,U\}_{i\in\mathbb{I}_m}\in\mathcal{RS}(m,\mathbf{k},d)$. $\square$

Remark 2.2. Let $\mathcal{V}=\{V_i\}_{i\in\mathbb{I}_m}\in\mathcal{RS}$ such that every $V_i\neq0$. In case that $\mathbf{k}=\mathbb{1}_m$, then $\mathcal{V}$ can be identified with a vector frame, since each $V_i:\mathbb{C}^d\to\mathbb{C}$ is in fact a vector $0\neq f_i\in\mathbb{C}^d$. In the same manner, the projective RS's can be seen as fusion frames. Here the identification is given by $V_i\simeq\big(\|V_i\|\,,\,R(V_i^*)\big)$ for every $i\in\mathbb{I}_m$. $\square$

Definition 2.3. For every $\mathcal{V}=\{V_i\}_{i\in\mathbb{I}_m}\in\mathcal{RS}(m,\mathbf{k},d)$, we define the system
$$\mathcal{V}^{\#}\overset{\text{def}}{=}\mathcal{V}\cdot S_{\mathcal{V}}^{-1}=\{V_i\,S_{\mathcal{V}}^{-1}\}_{i\in\mathbb{I}_m}\in\mathcal{RS}(m,\mathbf{k},d),$$
called the **canonical dual** RS associated to $\mathcal{V}$. $\square$

Remark 2.4. Given $\mathcal{V}=\{V_i\}_{i\in\mathbb{I}_m}\in\mathcal{RS}$ with $S_{\mathcal{V}}=\sum_{i\in\mathbb{I}_m}V_i^*V_i$, then
$$\sum_{i\in\mathbb{I}_m}S_{\mathcal{V}}{}^{-1}\,V_i^*V_i=I_{\mathcal{H}},\quad\text{and}\quad\sum_{i\in\mathbb{I}_m}V_i^*V_i\,S_{\mathcal{V}}{}^{-1}=I_{\mathcal{H}}. \tag{3}$$
Therefore, we obtain the reconstruction formulae
$$x=\sum_{i\in\mathbb{I}_m}S_{\mathcal{V}}^{-1}\,V_i^*\,(V_i\,x)=\sum_{i\in\mathbb{I}_m}V_i^*\,V_i(S_{\mathcal{V}}^{-1}\,x)\quad\text{for every}\quad x\in\mathcal{H}. \tag{4}$$
Observe that, by Eq. (3), we see that the canonical dual $\mathcal{V}^{\#}$ satisfies that
$$T_{\mathcal{V}^{\#}}^*\,T_{\mathcal{V}}=\sum_{i\in\mathbb{I}_m}S_{\mathcal{V}}{}^{-1}\,V_i^*V_i=I_{\mathcal{H}}\quad\text{and}\quad S_{\mathcal{V}^{\#}}=\sum_{i\in\mathbb{I}_m}S_{\mathcal{V}}^{-1}\,V_i^*\,V_i\,S_{\mathcal{V}}^{-1}=S_{\mathcal{V}}^{-1}. \tag{5}$$
Next we generalize the notion of dual RS's. $\square$

Definition 2.5. Let $\mathcal{V}=\{V_i\}_{i\in\mathbb{I}_m}$ and $\mathcal{W}=\{W_i\}_{i\in\mathbb{I}_m}\in\mathcal{RS}$. We say that $\mathcal{W}$ is a **dual** RS for $\mathcal{V}$ if $T_{\mathcal{W}}^*\,T_{\mathcal{V}}=I_{\mathcal{H}}$, or equivalently if $x=\sum_{i\in\mathbb{I}_m}W_i^*\,V_i\,x$ for every $x\in\mathcal{H}$.

We denote the set of all dual RS's for a fixed $\mathcal{V}\in\mathcal{RS}$ by $\mathcal{D}(\mathcal{V})\overset{\text{def}}{=}\{\mathcal{W}\in\mathcal{RS}:T_{\mathcal{W}}^*\,T_{\mathcal{V}}=I_{\mathcal{H}}\}$. Observe that $\mathcal{D}(\mathcal{V})\neq\emptyset$ since $\mathcal{V}^{\#}\in\mathcal{D}(\mathcal{V})$. $\square$

Remark 2.6. Let $\mathcal{V}\in L(m,\mathbf{k},d)$. Then $\mathcal{V}\in\mathcal{RS}\iff T_{\mathcal{V}}^*$ is surjective. In this case, a system $\mathcal{W}\in\mathcal{D}(\mathcal{V})$ if and only if its synthesis operator $T_{\mathcal{W}}^*$ is a pseudo-inverse of $T_{\mathcal{V}}$. Indeed, $\mathcal{W}\in\mathcal{D}(\mathcal{V})\iff T_{\mathcal{W}}^*\,T_{\mathcal{V}}=I_{\mathcal{H}}$. Observe that the map $\mathcal{RS}\ni\mathcal{W}\mapsto T_{\mathcal{W}}^*$ is one to one. Thus, in the context of RS's each $(m,\mathbf{k},d)$-RS has many duals that are $(m,\mathbf{k},d)$-RS's. This is one of the advantages of the RS's setting.

Moreover, the synthesis operator $T_{\mathcal{V}^{\#}}^*$ of the canonical dual $\mathcal{V}^{\#}$ corresponds to the Moore–Penrose pseudo-inverse of $T_{\mathcal{V}}$. Indeed, notice that $T_{\mathcal{V}}\,T_{\mathcal{V}^{\#}}^*=T_{\mathcal{V}}\,S_{\mathcal{V}}^{-1}T_{\mathcal{V}}^*\in L(\mathcal{K})^+$, so that it is an orthogonal projection. From this point of view, the canonical dual $\mathcal{V}^{\#}$ has some optimal properties that come from the theory of pseudo-inverses.

On the other hand the map $L(m,\mathbf{k},d)\ni\mathcal{W}\mapsto T_{\mathcal{W}}^*\in L(\mathcal{K},\mathcal{H})$ is $\mathbb{R}$-linear. Then, for every $\mathcal{V}\in\mathcal{RS}$, the set $\mathcal{D}(\mathcal{V})$ of dual systems is convex in $L(m,\mathbf{k},d)$, because the set of pseudoinverses of $T_{\mathcal{V}}$ is convex in $L(\mathcal{K},\mathcal{H})$. Moreover, as we show in Eq. (13) below, a system $\mathcal{W}\in\mathcal{D}(\mathcal{V})\iff T_{\mathcal{W}}^*=T_{\mathcal{V}^{\#}}^*+A$, for some $A \in L(\mathcal{K}, \mathcal{H})$ such that $A\,T_{\mathcal{V}} = A\,T_{\mathcal{V}^\#} = 0 \in L(\mathcal{H})$. Hence, $\mathcal{D}(\mathcal{V})$ is an affine submanifold of $L(m, \mathbf{k}, d)$. $\square$

3. Joint potential of projective RS's

In this section, we deal with the joint potential of dual pairs, as described in Section 1. We study the local minimizers of this functional, with respect to a natural pseudo-metric defined in the set $\mathcal{RS}$, and we describe their spectral structure. In the next section, we further show the geometrical structure of minimizers of the joint potential.

Throughout this section we fix the parameters $(m, \mathbf{k}, d)$ and the sequence $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ of weights. Note that the joint potential of a dual pair $(\mathcal{V}, \mathcal{W})$ is a function of the spectrum of the RS operators of $\mathcal{V}$ and $\mathcal{W}$. Hence we begin by developing some technical results which focus on what we call *the spectral picture* of RS operators, which is a key tool in order to find and characterize the minimizers.

3.1. Spectral picture of RS operators of projective systems

Given a fixed sequence of weights $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$, we define the set of projective RS's with fixed set of weights $\mathbf{v}$:

$$\mathcal{PRS}_{\mathbf{v}} \stackrel{\text{def}}{=} \Big\{ \mathcal{V} = \{V_i\}_{i\in\,\mathbb{I}_m} \in \mathcal{PRS} :\ \|V_i\|_{sp} = v_i \ \text{ for every } i \in \mathbb{I}_m \Big\}. \tag{6}$$

Denote by $\tau = \sum_{i\in\mathbb{I}_m} v_i^2\, k_i$. Observe that $\operatorname{tr}\, S_{\mathcal{V}} = \sum_{i\in\mathbb{I}_m} \operatorname{tr}\, V_i^* V_i = \tau$ for every $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$. Given a $d \times d$ matrix $A \in L(\mathcal{H})^+$, recall that its vector of eigenvalues is denoted by $\lambda(A) = \big(\lambda_1(A)\,,\ \dots\ ,\ \lambda_d(A)\,\big) \in (\mathbb{R}^d_+)^{\downarrow}$ (counting multiplicities and in decreasing order). We consider the set of operators $S_{\mathcal{V}}$ for $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ and its spectral picture:

$$\mathcal{OP}_{\mathbf{v}} \stackrel{\text{def}}{=} \{S_{\mathcal{V}} :\ \mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}\} \quad \text{and} \quad \Lambda(\mathcal{OP}_{\mathbf{v}}) \stackrel{\text{def}}{=} \{\lambda(S) : S \in \mathcal{OP}_{\mathbf{v}}\} \subseteq (\mathbb{R}^d_{>0})^{\downarrow}. \tag{7}$$

We shall give a characterization of the set $\Lambda(\mathcal{OP}_{\mathbf{v}})$ in terms of the Horn-Klyachko's theory of sums of hermitian matrices. In order to do this we shall describe briefly the basic facts about the spectral characterization obtained by Klyachko [21] and Fulton [16]. Let

$$\mathcal{K}^r_d = \{(j_1, \dots, j_r) \in (\mathbb{I}_d)^r :\ j_1 < j_2 \cdots < j_r\}.$$

For $J = (j_1, \dots, j_r) \in \mathcal{K}^r_d$, define the associated partition $\lambda(J) = (j_r - r, \dots, j_1 - 1)$. For $r \in \mathbb{I}_{d-1}$ denote by $LR^r_d(m)$ the set of $(m+1)$-tuples $(J_0, \dots, J_m) \in (\mathcal{K}^r_d)^{m+1}$, such that the Littlewood–Richardson coefficient of the associated partitions $\lambda(J_0), \dots, \lambda(J_m)$ is positive, i.e., one can generate the Young diagram of $\lambda(J_0)$ from those of $\lambda(J_1), \dots, \lambda(J_m)$ according to the Littlewood–Richardson rule (see [16]).

The theorem of Klyachko gives a characterization of the spectral picture of the set of all sums of m matrices in $\mathcal{H}(d)$ with fixed given spectra, in terms on a series of inequalities involving the $(m+1)$-tuples in $LR^r_d(m)$ (see [21] for a detailed formulation). In the following Lemma we give a description of this result in the particular case where these m matrices are multiples of projections. Let us first fix some notations: let $\operatorname{Gr}(k, d)$ denote the Grassmann manifold of orthogonal projections of rank k in $\mathbb{C}^d$ and let

$$\operatorname{Gr}\,(\mathbf{k}\,,\ d) \stackrel{\text{def}}{=} \bigoplus_{i\in\mathbb{I}_m} \operatorname{Gr}\,(k_i\,,\ d) \subseteq L(\mathcal{H})^m.$$

Lemma 3.1. Fix the parameters $(m, \mathbf{k}, d)$, the weights $\mathbf{v} \in \mathbb{R}_{>0}^m$ and a vector $\mu \in (\mathbb{R}_+^m)^\downarrow$. Then there exists a sequence $\{P_i\}_{i\in\mathbb{I}_m} \in \mathrm{Gr}\,(\mathbf{k}\,,\ d)$ such that $\mu = \lambda\left(\sum_{i\in\mathbb{I}_m} v_i^2 P_i\right)$ if and only if

$$\operatorname{tr}\ \mu = \sum_{i\in\mathbb{I}_m} v_i^2\, k_i \quad \text{and} \quad \sum_{i\in J_0} \mu_i \leqslant \sum_{i\in\,\mathbb{I}_m} v_i^2\, |J_i \cap \mathbb{I}_{k_i}|, \tag{8}$$

for every $r \in \mathbb{I}_{d-1}$ and every $(m+1)$-tuple $(J_0, \ldots, J_m) \in LR_d^r(m)$. $\square$

Proposition 3.2. *Fix the parameters $(m, \boldsymbol{k}, d)$ and the vector $\boldsymbol{v} \in \mathbb{R}_{>0}^m$ of weights. Fix also a positive matrix $S \in \mathcal{G}l\,(d)^+$. Then,*

$$S \in \mathcal{OP}_{\mathbf{v}} \iff \lambda(S) \in \Lambda(\mathcal{OP}_{\mathbf{v}}) \iff \mu = \lambda(S) \text{ satisfies Eq. (8)}.$$

Proof. The set $\mathcal{OP}_{\mathbf{v}} \subseteq \mathcal{G}l\,(d)^+$ is saturated by unitary equivalence. Indeed, if $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ and $U \in \mathcal{U}(d)$, then $\mathcal{V}\cdot U \stackrel{\text{def}}{=} \{V_i\, U\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}$ and $U^* S_{\mathcal{V}} U = S_{\mathcal{V}\cdot U} \in \mathcal{OP}_{\mathbf{v}}$. This shows the first equivalence. On the other hand, we can assure that an ordered vector $\mu \in \Lambda(\mathcal{OP}_{\mathbf{v}})$ if and only if $\mu_d > 0$ and there exists a sequence of projections $\mathcal{P} = \{P_i\}_{i\in\mathbb{I}_m} \in \mathrm{Gr}\,(\mathbf{k}\,,\ d)$ such that $\mu = \lambda\left(\sum_{i\in\mathbb{I}_m} v_i^2 P_i\right)$, by choosing isometries between each $\mathcal{K}_i$ and the respective $R(P_i)$. Hence, the second equivalence follows from Lemma 3.1. $\square$

Corollary 3.3. *For every set $(m, \boldsymbol{k}, d)$ of parameters and every vector $\boldsymbol{v} \in \mathbb{R}_{>0}^m$ of weights,*

1. *The set $\Lambda(\mathcal{OP}_{\mathbf{v}})$ is convex.*
2. *Its closure $\overline{\Lambda(\mathcal{OP}_{\mathbf{v}})}$ is compact.*
3. *A vector $\mu \in \overline{\Lambda(\mathcal{OP}_{\mathbf{v}})} \setminus \Lambda(\mathcal{OP}_{\mathbf{v}}) \iff \mu_d = 0$. In other words,*

$$\overline{\Lambda(\mathcal{OP}_{\mathbf{v}})} \cap \mathbb{R}_{>0}^m = \Lambda(\mathcal{OP}_{\mathbf{v}}). \tag{9}$$

Proof. Denote by $\mathcal{M}$ the set of vectors $\lambda \in (\mathbb{R}_+^d)^\downarrow$ which satisfies Eq. (8). It is clear that $\mathcal{M}$ is compact and convex. But Proposition 3.2 assures that $\Lambda(\mathcal{OP}_{\mathbf{v}}) = \mathcal{M} \cap \mathbb{R}_{>0}^d \subseteq \mathcal{M}$. This proves items 2 and 3. Item 1 follows by the fact that also $\mathbb{R}_{>0}^d$ is convex. $\square$

Remark 3.4. With the notations of Corollary 3.3, actually $\overline{\Lambda(\mathcal{OP}_{\mathbf{v}})} = \mathcal{M}$. This fact is not obvious from the inequalities of Eq. (8), but can be deduced using Lemma 3.1. Indeed, it is clear that if $\mathcal{P} \in \mathrm{Gr}\,(\mathbf{k}\,,\ d)$ and $S_{\mathbf{v}}(\mathcal{P}) \stackrel{\text{def}}{=} \sum_{i\in\mathbb{I}_m} v_i^2 P_i \notin \mathcal{G}l\,(d)^+$, then $S_{\mathbf{v}}(\mathcal{P})$ can be approximated by matrices $S_{\mathbf{v}}(\mathcal{Q}) \stackrel{\text{def}}{=} \sum_{i\in\mathbb{I}_m} v_i^2 Q_i$ for sequences $\mathcal{Q} \in \mathrm{Gr}\,(\mathbf{k}\,,\ d)$ such that $S_{\mathbf{v}}(\mathcal{Q}) \in \mathcal{G}l\,(d)^+$. Using Lemma 4.1 and Eq. (23) below, this means that these matrices $S_{\mathbf{v}}(\mathcal{Q}) \in \mathcal{OP}_{\mathbf{v}}$. $\square$

3.2. The joint potential and its minimizers

Fix the parameters $(m, \mathbf{k}, d)$. We consider the set of dual pairs associated to $\mathcal{PRS}_{\mathbf{v}}$:

$$\mathcal{DP}_{\mathbf{v}} = \mathcal{DP}_{\mathbf{v}}(m, \mathbf{k}, d) \stackrel{\text{def}}{=} \Big\{ (\mathcal{V},\ \mathcal{W}) \in \mathcal{PRS}_{\mathbf{v}} \times \mathcal{RS} \ :\ \mathcal{W} \in \mathcal{D}(\mathcal{V}) \Big\}.$$

Recall that, given $(\mathcal{V}\,,\ \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$, the joint potential of the pair is defined as

$$\mathrm{RSP}\,(\mathcal{V}, \mathcal{W}) = \operatorname{tr}\ S_{\mathcal{V}}^2 + \operatorname{tr}\ S_{\mathcal{W}}^2 \stackrel{\text{def}}{=} \mathrm{RSP}\,(\mathcal{V}) + \mathrm{RSP}\,(\mathcal{W}) \in \mathbb{R}_{>0}. \tag{10}$$

We will denote by

$$p_{\mathbf{v}} = p_{\mathbf{v}}(m, \mathbf{k}, d) \stackrel{\text{def}}{=} \inf\{\, \mathrm{RSP}\,(\mathcal{V}, \mathcal{W}) \ :\ (\mathcal{V},\ \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}} \,\}. \tag{11}$$

We shall need the following result from [10, Proposition 5]. We give a short proof of it in order to keep the text self-contained.

Lemma 3.5. *Let $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{RS}$. If $\mathcal{W} \in \mathcal{D}(\mathcal{V})$ then*

$$\operatorname{RSP}(\mathcal{W}) \overset{def}{=} \operatorname{tr} S_{\mathcal{W}}^2 \geqslant \operatorname{tr} S_{\mathcal{V}}^{-2} = \sum_{i=1}^{d} \lambda_i(S_{\mathcal{V}})^{-2} = \operatorname{RSP}(\mathcal{V}^{\#}). \tag{12}$$

Moreover, $\mathcal{V}^{\#}$ is the unique *element of $\mathcal{D}(\mathcal{V})$ which attains the lower bound in (12).*

Proof. Fix another $\mathcal{W} \in \mathcal{D}(\mathcal{V})$. Then the equalities $T_{\mathcal{W}}^* T_{\mathcal{V}} = T_{\mathcal{V}^\#}^* T_{\mathcal{V}} = I_{\mathcal{H}}$ imply that

$$T_{\mathcal{W}}^* = T_{\mathcal{V}^\#}^* + A \quad \text{for some} \quad A \in L(\mathcal{K}, \mathcal{H}) \quad \text{that satisfies} \quad A\, T_{\mathcal{V}} = 0. \tag{13}$$

Note that $T_{\mathcal{V}^\#} = T_{\mathcal{V}} S_{\mathcal{V}}^{-1}$, so that also $A\, T_{\mathcal{V}^\#} = 0$. Thus, $S_{\mathcal{W}} = T_{\mathcal{W}}^* T_{\mathcal{W}} = S_{\mathcal{V}^\#} + AA^*$ and hence

$$\operatorname{tr} S_{\mathcal{W}}^2 = \operatorname{tr} S_{\mathcal{V}^\#}^2 + \operatorname{tr}(AA^*)^2 + 2 \operatorname{Re} \operatorname{tr}(S_{\mathcal{V}^\#} AA^*) \geqslant \operatorname{tr} S_{\mathcal{V}^\#}^2, \tag{14}$$

since $\operatorname{tr}(S_{\mathcal{V}^\#} AA^*) \geqslant 0$. Moreover, if the lower bound in Eq. (12) is attained at $\mathcal{W}$ then the previous computation forces that in this case $A = 0$ and hence $\mathcal{W} = \mathcal{V}^{\#}$. $\square$

Now we can give a RS-version of the known result [10, Proposition 6] about vector frames.

Proposition 3.6. *For every set $(m, \boldsymbol{k}, d)$ of parameters, the following properties hold:*

1. *The infimum $p_{\mathbf{v}}$ in Eq. (11) is actually a* **minimum**.
2. *Let $\tau = \sum_{i\in\mathbb{I}_m} v_i^2 k_i$. For every pair $(\mathcal{V}, \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ we have that*

$$\operatorname{RSP}(\mathcal{V}, \mathcal{W}) \geq p_{\mathbf{v}} \geqslant \frac{\tau^4 + d^4}{d\,\tau^2}, \tag{15}$$

3. *This lower bound is attained if and only if $\mathcal{V}$ is tight ($S_{\mathcal{V}} = \frac{\tau}{d} I_d$) and $\mathcal{W} = \frac{d}{\tau} \mathcal{V} = \mathcal{V}^{\#}$.*

Proof. Given $(\mathcal{V}, \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$, Lemma 3.5 asserts that $\operatorname{RSP}(\mathcal{V}, \mathcal{V}^{\#}) \leq \operatorname{RSP}(\mathcal{V}, \mathcal{W})$ and also that equality holds only if $\mathcal{W} = \mathcal{V}^{\#}$. Thus

$$p_{\mathbf{v}} = \inf_{\mathcal{V}\in\mathcal{PRS}_{\mathbf{v}}} \operatorname{RSP}(\mathcal{V}, \mathcal{V}^{\#}) \overset{(5)}{=} \inf_{\mathcal{V}\in\mathcal{PRS}_{\mathbf{v}}} \sum_{i=1}^{d} \lambda_i(S_{\mathcal{V}})^2 + \lambda_i(S_{\mathcal{V}})^{-2}. \tag{16}$$

Consider the **strongly convex** map $F : \mathbb{R}^d_{>0} \to \mathbb{R}_{>0}$ given by $F(x) = \sum_{i=1}^d x_i^2 + x_i^{-2}$, for $x \in \mathbb{R}^d_{>0}$. Observe that $\operatorname{RSP}(\mathcal{V}, \mathcal{V}^{\#}) = F(\lambda(S_{\mathcal{V}}))$ for every $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$. By Corollary 3.3 we know that $\Lambda(\mathcal{OP}_{\mathbf{v}})$ is a convex subset of $(\mathbb{R}^d_{>0})^{\downarrow}$, and it becomes also compact under a restriction of the type $\lambda_d \geq \varepsilon$ (for any $\varepsilon > 0$). Since a strongly convex function defined in a compact convex set attains its local (and therefore global) minima at a unique point, it follows that there exists a unique $\lambda_{\mathbf{v}} = \lambda_{\mathbf{v}}(m, \mathbf{k}, d) \in \Lambda(\mathcal{OP}_{\mathbf{v}})$ such that

$$F(\lambda_{\mathbf{v}}) = \min_{\lambda\in\Lambda(\mathcal{OP}_{\mathbf{v}})} F(\lambda) = p_{\mathbf{v}}. \tag{17}$$

This proves item 1. Moreover, using Lagrange multipliers it is easy to see that the restriction of F to the set $(\mathbb{R}^d_{>0})_\tau := \{\mathbf{x} \in \mathbb{R}^d_{>0} : \operatorname{tr}(\mathbf{x}) = \tau\}$ reaches its minimum in $\mathbf{x} = \frac{\tau}{d} \cdot \mathbb{1}$. Since $\Lambda(\mathcal{OP}_{\mathbf{v}}) \subset (\mathbb{R}^d_{>0})_\tau$ we get that

$$\operatorname{RSP}(\mathcal{V}, \mathcal{V}^{\#}) = F(\lambda(S_{\mathcal{V}})) \geq F\left(\frac{\tau}{d} \cdot \mathbb{1}\right) = \frac{\tau^4 + d^4}{d\,\tau^2} \quad \text{for every} \quad \mathcal{V} \in \mathcal{PRS}_{\mathbf{v}},$$

and this lower bound is attained if and only if $\lambda(S_{\mathcal{V}}) = \frac{\tau}{d} \cdot \mathbb{1}_d$. Note that in this case $S_{\mathcal{V}} = \frac{\tau}{d} I_d$, and therefore $\mathcal{V}^{\#} = \frac{d}{\tau}\,\mathcal{V}$. $\square$

In order to compute "local" minimizers for different functions defined on $\mathcal{RS}$ or some of its subsets, we shall consider two different (pseudo) metrics: Given $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m}$ and $\mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}$, we recall the (punctual) metric:

$$d_P(\mathcal{V},\ \mathcal{W}) = \left(\sum_{i\in\mathbb{I}_m} \|V_i - W_i\|_2^2\right)^{1/2} = \|T_{\mathcal{V}} - T_{\mathcal{W}}\|_2 = \|T_{\mathcal{V}}^* - T_{\mathcal{W}}^*\|_2 .$$

We consider also the pseudo-metric defined by $d_S(\mathcal{V},\ \mathcal{W}) = \|S_{\mathcal{V}} - S_{\mathcal{W}}\|$.

Lemma 3.7. *If a pair $(\mathcal{V},\ \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ is local d_P-minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$, then $\mathcal{W} = \mathcal{V}^{\#}$.*

Proof. By Remark 2.6 the set $\mathcal{D}(\mathcal{V})$ is convex. If $\mathcal{W} \in \mathcal{D}(\mathcal{V})$, Eq. (13) assures that $T_{\mathcal{W}}^* = T_{\mathcal{V}^{\#}}^* + A$, for some $A \in L(\mathcal{K}, \mathcal{H})$ such that $A\,T_{\mathcal{V}} = A\,T_{\mathcal{V}^{\#}} = 0$. Then the line segment $\mathcal{W}_t = t\mathcal{W} + (1-t)\mathcal{V}^{\#} \in \mathcal{D}(\mathcal{V})$ satisfies that $T_{\mathcal{W}_t}^* = T_{\mathcal{V}^{\#}}^* + tA$ for every $t \in [0, 1]$. Then $S_{\mathcal{W}_t} = S_{\mathcal{V}^{\#}} + t^2\,AA^*$ and

$$K(t) \stackrel{\text{def}}{=} \mathrm{RSP}\,(\mathcal{V}\,,\ \mathcal{W}_t) = \mathrm{RSP}\,(\mathcal{V}\,,\ \mathcal{V}^{\#}) + t^4\,\mathrm{tr}\,(AA^*)^2 + 2\,t^2\,\mathrm{tr}\ T_{\mathcal{V}^{\#}}AA^*T_{\mathcal{V}^{\#}}^* ,$$

for every $t \in [0, 1]$. Observe that $K(1) = \mathrm{RSP}\,(\mathcal{V}\,,\ \mathcal{W})$. But by taking one derivative of K one gets that if $A \neq 0$ then K is strictly increasing near $t = 1$, which contradicts the local d_P-minimality for $(\mathcal{V}\,,\ \mathcal{W})$. Therefore $T_{\mathcal{W}_t}^* = T_{\mathcal{V}^{\#}}^*$ and $\mathcal{W} = \mathcal{V}^{\#}$. $\square$

Remark 3.8. Let $\mathcal{A} \subseteq \mathcal{RS} \times \mathcal{RS}$ and $f : \mathcal{A} \to \mathbb{R}$ a d_S-continuous map. Fix $(\mathcal{V}\,,\ \mathcal{W}) \in \mathcal{A}$. Since the map $\mathcal{V} \mapsto S_{\mathcal{V}}$ is d_P-continuous, it is easy to see that f is also d_P-continuous and, if $(\mathcal{V}\,,\ \mathcal{W})$ is a local d_S-minimizer of f over $\mathcal{A}$, then $(\mathcal{V}\,,\ \mathcal{W})$ is also a local d_P-minimizer.

Theorem 3.9. *For every set $(m, \boldsymbol{k}, d)$ of parameters there exists $\lambda_{\mathbf{v}} = \lambda_{\mathbf{v}}(m, \boldsymbol{k}, d) \in \Lambda(\mathcal{OP}_{\mathbf{v}}) \subseteq (\mathbb{R}_{>0}^d)^{\downarrow}$ such that the following conditions are equivalent for pair $(\mathcal{V},\ \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$:*

1. *$(\mathcal{V},\ \mathcal{W})$ is local d_S-minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$.*
2. *$(\mathcal{V},\ \mathcal{W})$ is global minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$.*
3. *It holds that $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$ and $\mathcal{W} = \mathcal{V}^{\#}$.*

Proof. Take the vector $\lambda_{\mathbf{v}}$ defined in Eq. (17). In the proof of Proposition 3.6 we have already seen that a pair $(\mathcal{V},\ \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ is a global minimizer for RSP $\iff$ $\mathcal{W} = \mathcal{V}^{\#}$ and $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$. This means that $2 \iff 3$.

Suppose now that $(\mathcal{V},\ \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ is a local d_S-minimizer. By Remark 3.8 we know that it is also a local d_P-minimizer and by Lemma 3.7 we have that $\mathcal{W} = \mathcal{V}^{\#}$. In this case, denote $\lambda = \lambda(S_{\mathcal{V}})$ and take $U \in \mathcal{U}(d)$ such that $U^*D_{\lambda}U = S_{\mathcal{V}}$. Consider the line segment

$$h(t) = t\,\lambda_{\mathbf{v}} + (1-t)\,\lambda \quad \text{for every} \quad t \in [0, 1].$$

Then $h(t) \in \Lambda(\mathcal{OP}_{\mathbf{v}})$ for every $t \in [0, 1]$, since $\Lambda(\mathcal{OP}_{\mathbf{v}})$ is a convex set (Corollary 3.3). Consider the continuous curve $S_t = U^*D_{h(t)}U$ in $\mathcal{OP}_{\mathbf{v}}$ and a (not necessarily continuous) curve $\mathcal{V}_t \in \mathcal{PRS}_{\mathbf{v}}$ such that $S_0 = S_{\mathcal{V}}$, $\mathcal{V}_0 = \mathcal{V}$ and $S_{\mathcal{V}_t} = S_t$ for every $t \in [0, 1]$. Nevertheless, since the curve S_t is continuous, we can assure that the map $t \mapsto \mathcal{V}_t$ is d_S-continuous.

Finally, we can consider the map $G : [0, 1] \to \mathbb{R}$ given by

$$G(t) = \mathrm{RSP}\,(\mathcal{V}_t\,,\ \mathcal{V}_t^{\#}) = \mathrm{tr}\ S_t^2 + \mathrm{tr}\ S_t^{-2} = \sum_{i=1}^{d} h_i(t)^2 + h_i(t)^{-2} = F(h(t)),$$

for $t \in [0, 1]$, where F is the map defined after Eq. (16). Observe that $G(0) = \text{RSP}(\mathcal{V}, \mathcal{V}^{\#})$ and $G(1) = p_{\mathbf{v}}$, by Eq. (17). Then G has local minima at $t = 0$ and $t = 1$. By computing the second derivative of G in terms of the Hessian of F, we deduce that G must be constant, because otherwise it would be strictly convex. From this fact we can see that the map h is also constant, so that $\lambda_{\mathbf{v}} = \lambda$. Therefore $(\mathcal{V}, \mathcal{W}) = (\mathcal{V}, \mathcal{V}^{\#})$ is a global minimizer. □

4. The structure of minimizers

In order to obtain a detailed structure of the minimizers of the joint potential we present first subsection which includes a geometrical description of $\mathcal{PRS}_{\mathbf{v}}$ and give a sufficient condition - the notion of irreducible systems – in order that the operation of taking RS operators $\mathcal{PRS}_{\mathbf{v}} \ni \mathcal{V} \mapsto S_{\mathcal{V}}$ (see Definition 2.1) has smooth local cross-sections. We show that given an irreducible system $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ then $(\mathcal{V}, \mathcal{V}^{\#})$ is a local minimizer if and only if $\mathcal{V}$ is a tight projective system, so that in this case $(\mathcal{V}, \mathcal{V}^{\#})$ is a global minimizer. Using these results and geometrical reduction arguments we shall deal with the general case and obtain the geometrical structure of local minimizers.

4.1. Geometric presentation of Projective RS's with fixed weights

In this section, we shall study several objects related with the set $\mathcal{RS}$ from a geometrical point of view. In what follows we shall denote by

$$\mathcal{M}_d(\mathbb{C})^+_\tau \stackrel{\text{def}}{=} \{A \in \mathcal{M}_d(\mathbb{C})^+ \,:\, \operatorname{tr} A = \tau\} \quad \text{and} \quad \mathcal{G}l\,(d)^+_\tau \stackrel{\text{def}}{=} \mathcal{M}_d(\mathbb{C})^+_\tau \cap \mathcal{G}l\,(d),$$

the set of $d \times d$ positive and positive invertible operators with fixed trace τ, endowed with the metric and geometric structure induced by those of $\mathcal{G}l\,(d)$.

Recall that the set of projective RS's with fixed set of weights $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ is

$$\mathcal{PRS}_{\mathbf{v}} = \mathcal{PRS}_{\mathbf{v}}\,(m, \mathbf{k}, d) = \left\{ \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS} :\ \|V_i\|_{sp} = v_i \text{ for every } i \in \mathbb{I}_m \right\}.$$

Denote by $\tau = \sum\limits_{i\in\mathbb{I}_m} v_i^2\, k_i$. Observe that $\operatorname{tr} S_{\mathcal{V}} = \sum\limits_{i\in\mathbb{I}_m} \operatorname{tr} V_i^* V_i = \tau$ for every $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$. In this section, we look for conditions which assure that the smooth map

$$\text{RSO} : \mathcal{PRS}_{\mathbf{v}} \to \mathcal{G}l\,(d)^+_\tau \quad \text{given by} \quad \text{RSO}(\mathcal{V}) = S_{\mathcal{V}} = \sum_{i\in\mathbb{I}_m} V_i^* V_i, \tag{18}$$

for every $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}$, has smooth local cross-sections. Before giving these conditions and the proof of their sufficiency, we need some notations and two geometrical lemmas: fix $d \in \mathbb{N}$. For every $k \in \mathbb{I}_d$, we denote by $\mathcal{I}(k, d) = \{U \in L(\mathbb{C}^k, \mathbb{C}^d) : U^*U = I_k\}$ the set of isometries. Given an m-tuple $\mathbf{k} = (k_i)_{i\in\mathbb{I}_m} \in \mathbb{I}_d^m \subseteq \mathbb{N}^m$, we denote by

$$\mathcal{I}(\mathbf{k}, d) \stackrel{\text{def}}{=} \bigoplus_{i\in\mathbb{I}_m} \mathcal{I}(k_i, d) \subseteq \bigoplus_{i\in\mathbb{I}_m} L(\mathcal{K}_i, \mathcal{H}) \cong L(\mathcal{K}, \mathcal{H}),$$

endowed with the product (differential, metric) structure (see [1] for a description of the geometrical structure). Recall that $\text{Gr}(k, d)$ denotes the Grassmann manifold of orthogonal projections of rank k in $\mathbb{C}^d$. As before we shall denote by

$$\text{Gr}\,(\mathbf{k}, d) \stackrel{\text{def}}{=} \bigoplus_{i\in\mathbb{I}_m} \text{Gr}\,(k_i, d) \subseteq L(\mathcal{H})^m,$$

with the product smooth structure (see [15]).

Lemma 4.1. *Consider the smooth map $\Phi : \mathcal{I}(\boldsymbol{k}\,,\ d) \to \operatorname{Gr}(\boldsymbol{k}\,,\ d)$ given by*

$$\Phi(\mathcal{W}) = (W_1\,W_1^*\,,\ \ldots\,,\ W_m\,W_m^*) \quad \textit{for every} \quad \mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m} \in \mathcal{I}(\boldsymbol{k}\,,\ d).$$

Then Φ has smooth local cross-sections around any point $\mathcal{P} = (P_i)_{i\in\mathbb{I}_m} \in \operatorname{Gr}(\boldsymbol{k}\,,\ d)$ toward every $\mathcal{W} \in \mathcal{I}(\boldsymbol{k}\,,\ d)$ such that $\Phi(\mathcal{W}) = \mathcal{P}$. In particular, Φ is open and surjective.

Proof. Since both spaces have a product structure, it suffices to consider the case $m = 1$. It is clear that the map Φ is surjective.

For every $P \in \operatorname{Gr}(k\,,\ d)$, the C^∞ map $\pi_P : \mathcal{U}(d) \to \operatorname{Gr}(k\,,\ d)$ given by $\pi_P(U) = UPU^*$ for $U \in \mathcal{U}(d)$ is a submersion with a smooth local cross-section (see [15])

$$h_P : U_P \overset{\text{def}}{=} \{Q \in \operatorname{Gr}(k\,,\ d) : \|Q - P\| < 1\} \to \mathcal{U}(d) \quad \text{such that} \quad h_P(P) = I_d.$$

For completeness we recall that, for every $Q \in U_P$, the matrix $h_P(Q)$ is the unitary part in the polar decomposition of the invertible matrix $QP + (I_d - Q)(I_d - P)$. Then, fixed $W \in \mathcal{I}(k\,,\ d)$ such that $\Phi(W) = P$, we can define the following smooth local cross-section for Φ :

$$s_{P\,,\,W} : U_P \to \mathcal{I}(k\,,\ d) \quad \text{given by} \quad s_{P\,,\,W}(Q) = h_P(Q)\,W, \quad \text{for every} \quad Q \in U_P. \quad \square \tag{19}$$

We shall need the following result from [25]. In order to state it we recall that, given a set $\mathcal{P} = \{P_j : j \in \mathbb{I}_m\} \subseteq \mathcal{M}_d(\mathbb{C})^+$, we denote the commutant of $\mathcal{P}$ as

$$\mathcal{P}' = \{P_j : j \in \mathbb{I}_m\}' = \{A \in \mathcal{M}_d(\mathbb{C}) : AP_j = P_j\,A \quad \text{for every} \quad j \in \mathbb{I}_m\}. \tag{20}$$

Note that $\mathcal{P}'$ is a (closed) unital selfadjoint subalgebra of $\mathcal{M}_d(\mathbb{C})$. Therefore,

$$\mathcal{P}' = \mathbb{C}\,I_d \iff \text{there is no non-trivial orthogonal projection } Q \in \mathcal{P}'\,. \tag{21}$$

Lemma 4.2 [25, Theorem 4.2.1]. *Let $\boldsymbol{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ and $\mathcal{P} = \{P_i\}_{i\in\mathbb{I}_m} \in \operatorname{Gr}(\boldsymbol{k}\,,\ d)$. Denote by $\tau = \sum_{i\in\mathbb{I}_m} v_i^2\,k_i$. Then the map $S_{\boldsymbol{v}} : \operatorname{Gr}(\boldsymbol{k}\,,\ d) \to \mathcal{M}_d(\mathbb{C})^+_\tau$ given by*

$$S_{\boldsymbol{v}}(\mathcal{Q}) \overset{\text{def}}{=} \sum_{i\in\mathbb{I}_m} v_i^2\,Q_i \quad \textit{for} \quad \mathcal{Q} = \{Q_i\}_{i\in\mathbb{I}_m} \in \operatorname{Gr}(\boldsymbol{k}\,,\ d) \tag{22}$$

is smooth and, if $\mathcal{P}$ satisfies that $\mathcal{P}' = \mathbb{C}\,I_d$, then

1. *The matrix $S_{\boldsymbol{v}}(\mathcal{P}) \in \mathcal{G}l\,(d)^+_\tau$.*
2. *The image of $S_{\boldsymbol{v}}$ contains an open neighborhood of $S_{\boldsymbol{v}}(\mathcal{P})$ in $\mathcal{M}_d(\mathbb{C})^+_\tau$.*
3. *Moreover, $S_{\boldsymbol{v}}$ has a smooth local cross-section around $S_{\boldsymbol{v}}(\mathcal{P})$ towards $\mathcal{P}$.* $\square$

4.3. The set $\mathcal{I}_0(\mathbf{k}\,,\ d) = \{\mathcal{W} \in \mathcal{I}(\mathbf{k}\,,\ d) : S_{\mathbf{v}} \circ \Phi(\mathcal{W}) \in \mathcal{G}l\,(d)^+\}$ is open in $\mathcal{I}(\mathbf{k}\,,\ d)$. Observe that its definition does not depend on the sequence $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ of weights. Moreover, the map $\gamma : \mathcal{I}_0(\mathbf{k}\,,\ d) \to \mathcal{PRS}_{\mathbf{v}}$ given by

$$\gamma(\mathcal{W}) = \{v_i\,W_i^*\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}} \quad \text{for every} \quad \mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m} \in \mathcal{I}_0(\mathbf{k}\,,\ d), \tag{23}$$

is a homeomorphism. Hence, using this map γ we can endow $\mathcal{PRS}_{\mathbf{v}}$ with the differential structure which makes γ a diffeomorphism. With this structure, each space $\mathcal{PRS}_{\mathbf{v}}$ becomes a submanifold of $\mathcal{RS}$. It is in this sense in which the map $\text{RSO} : \mathcal{PRS}_{\mathbf{v}} \to \mathcal{G}l\,(d)^+_\tau$ defined in Eq. (18) is smooth. Indeed, we have that

$$\text{RSO} = S_{\mathbf{v}} \circ \Phi \circ \gamma^{-1}, \tag{24}$$

where $\Phi : \mathcal{I}(\mathbf{k}\,,\ d) \to \operatorname{Gr}(\mathbf{k}\,,\ d)$ is the smooth map defined in Lemma 4.1. Now we can give an partial answer to the problem about the existence of local cross-section for RSO posed in the beginning of this section: as in the proof of Lemma 4.2 (see [25]), in order to get local cross-sections for RSO near some system $\mathcal{V}$, one needs the derivative of RSO to be surjective at $\mathcal{V}$. Since this fact is equivalent to condition (21), we introduce the notion of irreducible systems.

Definition 4.4. Let $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ and $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}(m, \mathbf{k}, d)$. We say that the system $\mathcal{V}$ is **irreducible** if $C_{\mathcal{V}} \overset{\text{def}}{=} \{V_i^*V_i : i \in \mathbb{I}_m\}' = \mathbb{C}\, I_d$.

In Section 5, we show examples of reducible and irreducible systems. See also Remark 4.7.

Theorem 4.5. *Let $\boldsymbol{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ and $\tau = \sum\limits_{i\in\mathbb{I}_m} v_i^2 k_i$. If we fix an* **irreducible** *system $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}(m, \boldsymbol{k}, d)$, then the map RSO $: \mathcal{PRS}_{\mathbf{v}} \to \mathcal{G}l\,(d)^+_\tau$ defined in Eq. (18) has a smooth local cross-section around $S_{\mathcal{V}}$ which sends $S_{\mathcal{V}}$ to $\mathcal{V}$.*

Proof. We have to prove that there exists an open neighborhood A of $S_{\mathcal{V}}$ in $\mathcal{G}l\,(d)^+_\tau$ and a smooth map $\rho : A \to \mathcal{PRS}_{\mathbf{v}}$ such that RSO$(\rho(S)) = S$ for every $S \in A$ and $\rho(S_{\mathcal{V}}) = \mathcal{V}$.

Denote by $P_i = P_{R(V_i^*)}$ for every $i \in \mathbb{I}_m$, and consider the system

$$\gamma^{-1}(\mathcal{V}) = \mathcal{U} = \{U_i\}_{i\in\mathbb{I}_m} \in \mathcal{I}(\mathbf{k},\, d) \quad \text{given by} \quad U_i = v_i^{-1}\, V_i^* \in I(k_i\,,\, d)\;\; i \in \mathbb{I}_m\,.$$

Observe that $\Phi(\mathcal{U}) = \mathcal{P} = \{P_i\}_{i\in\mathbb{I}_m} \in \mathrm{Gr}\,(\mathbf{k}\,,\, d)$ and $S_{\mathbf{v}}(\mathcal{P}) = S_{\mathcal{V}}$. By our hypothesis, we know that $\mathcal{P}' = \{V_i^*V_i : i \in \mathbb{I}_m\}' = \mathbb{C}\, I_d$. Let $\alpha : A \to \mathrm{Gr}\,(\mathbf{k}\,,\, d)$ be the smooth section for the map $S_{\mathbf{v}} : \mathrm{Gr}\,(\mathbf{k}\,,\, d) \to \mathcal{M}_d(\mathbb{C})^+_\tau$ given by Lemma 4.2. Hence A is an open neighborhood of $S_{\mathcal{V}} = S_{\mathbf{v}}(\mathcal{P})$ in $\mathcal{G}l\,(d)^+_\tau$, and $\alpha(S_{\mathcal{V}}) = \mathcal{P}$.

Take now the cross-section $\beta : B \to \mathcal{I}(\mathbf{k}\,,\, d)$ for the map $\Phi : \mathcal{I}(\mathbf{k}\,,\, d) \to \mathrm{Gr}\,(\mathbf{k}\,,\, d)$ given by Lemma 4.1, such that B is an open neighborhood of $\mathcal{P}$ in $\mathrm{Gr}\,(\mathbf{k}\,,\, d)$, and that $\beta(\mathcal{P}) = \mathcal{U}$.

Finally we recall the diffeomorphism $\gamma : \mathcal{I}_0(\mathbf{k}\,,\, d) \to \mathcal{PRS}_{\mathbf{v}}$ defined in Eq. (23), where $\mathcal{I}_0(\mathbf{k}\,,\, d) = \{\mathcal{W} \in \mathcal{I}(\mathbf{k}\,,\, d) : S_{\mathbf{v}} \circ \Phi(\mathcal{W}) \in \mathcal{G}l\,(d)^+\}$ is an open subset of $\mathcal{I}(\mathbf{k}\,,\, d)$ such that $\mathcal{U} \in \mathcal{I}_0(\mathbf{k}\,,\, d)$. Note that $\gamma(\mathcal{U}) = \mathcal{V}$. Changing the first neighborhood A by some smaller open set, we can define the announced smooth cross-section for the map RSO by

$$\rho = \gamma \circ \beta \circ \alpha : A \subseteq \mathcal{G}l\,(d)^+_\tau \to \mathcal{PRS}_{\mathbf{v}}\,.$$

Following our previous steps, we see that $\rho(S_{\mathcal{V}}) = \mathcal{V}$ and that

$$\mathrm{RSO} \overset{(24)}{=} S_{\mathbf{v}} \circ \Phi \circ \gamma^{-1} \implies \mathrm{RSO}(\rho(S)) = S \quad \text{for every} \quad S \in A. \qquad \square \tag{25}$$

4.2. Orthogonal sums of tight systems

Recall that we use in $\mathcal{RS}$ the metric $d_P(\mathcal{V},\, \mathcal{W}) = \left(\sum\limits_{i\in\mathbb{I}_m} \|V_i - W_i\|_2^2\right)^{1/2} = \|T_{\mathcal{V}}^* - T_{\mathcal{W}}^*\|_2$ and the pseudometric $d_S(\mathcal{V},\, \mathcal{W}) = \|S_{\mathcal{V}} - S_{\mathcal{W}}\|$ for pairs $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m}$ and $\mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}$.

Theorem 3.9 assures that if a pair $(\mathcal{V},\, \mathcal{W})$ is local d_S-minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$, then the minimality of $(\mathcal{V},\, \mathcal{W})$ is global, so that it is also a local d_P minimizer. The converse needs not to be true.

Nevertheless, it is true under some assumptions: using Theorem 4.5 it can be deduced that if $(\mathcal{V}\,,\, \mathcal{V}^\#)$ is a local d_P minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$ in order to assure that it is also a local d_S minimizer it suffices to assume that $C_{\mathcal{V}} = \{V_i^*V_i : i \in \mathbb{I}_m\}' = \mathbb{C}\, I_d$, i.e., that $\mathcal{V}$ is irreducible. Moreover:

Lemma 4.6. *Fix the set $(m, \boldsymbol{k}, d)$ of parameters and the weights $\boldsymbol{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$. Assume that $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ is irreducible. Then the following conditions are equivalent:*

1. *The pair* $(\mathcal{V}\,,\ \mathcal{V}^{\#})$ *is local* d_P*-minimizer of the joint potential in* $\mathcal{DP}_{\mathbf{v}}$ *.*
2. *The pair* $(\mathcal{V}\,,\ \mathcal{V}^{\#})$ *is global minimizer of the joint potential in* $\mathcal{DP}_{\mathbf{v}}$ *.*
3. *The system* $\mathcal{V}$ *is tight, i.e.,* $S_{\mathcal{V}} = \frac{\tau}{d}\, I_d$ *.*

Therefore in this case the vector $\lambda_{\mathbf{v}}$ *of Theorem 3.9 is* $\lambda_{\mathbf{v}} = \frac{\tau}{d}\,\mathbb{1}_d$ *.*

Proof. Since $C_{\mathcal{V}} = \mathbb{C}\, I_d$, we can apply Theorem 4.5. Then the map $\mathrm{RSO} : \mathcal{PRS}_{\mathbf{v}} \to \mathcal{G}l\,(d)^+_\tau$ defined in Eq. (18) has a smooth local cross-section around $S_{\mathcal{V}}$ which sends $S_{\mathcal{V}}$ to $\mathcal{V}$. Assume that there exists no $\sigma \in \mathbb{R}_{>0}$ such that $S_{\mathcal{V}} = \sigma\, I_d$. In this case there exist $\alpha,\ \beta \in \sigma(S_{\mathcal{V}})$ such that $\beta > \alpha > 0$. Consider the map $g : \left[0,\ \frac{\beta-\alpha}{2}\right] \to \mathbb{R}_{>0}$ given by

$$g(t) = (\alpha + t)^2 + (\alpha + t)^{-2} + (\beta - t)^2 + (\beta - t)^{-2}.$$

Then $g'(0) = 2(\alpha - \beta) - 2(\frac{1}{\beta} - \frac{1}{\alpha}) < 0$, which shows that we can construct a continuous curve $M : [\,0\,,\ \varepsilon\,] \to \mathcal{G}l\,(d)^+_\tau$ such that $M(0) = S_{\mathcal{V}}$ and

$$\operatorname{tr}\, M(t)^2 + \operatorname{tr}\, M(t)^{-2} < \operatorname{tr}\, S_{\mathcal{V}}^2 + \operatorname{tr}\, S_{\mathcal{V}}^{-2} = \mathrm{RSP}\,(\mathcal{V}\,,\ \mathcal{V}^{\#}) \quad \text{for every} \quad t \in (0\,,\ \varepsilon\,].$$

Hence, using the continuous local cross-section mentioned before, we can construct a d_P-continuous curve $\mathcal{M} : [0\,,\ \delta\,] \to \mathcal{PRS}_{\mathbf{v}}$ such that $\mathrm{RSO} \circ \mathcal{M} = M$, $\mathcal{M}(0) = \mathcal{V}$ and

$$\mathrm{RSP}\,(\mathcal{M}(t)\,,\ \mathcal{M}(t)^{\#}) = \operatorname{tr}\, M(t)^2 + \operatorname{tr} M(t)^{-2} < \mathrm{RSP}\,(\mathcal{V}\,,\ \mathcal{V}^{\#}) \quad \text{for} \quad t \in (0\,,\ \delta\,].$$

This shows that $(\mathcal{V}\,,\ \mathcal{V}^{\#})$ is not a local d_P-minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$. We have proved that $1 \implies 3$. Note that $3 \implies 2$ follows from (15) and $2 \implies 1$ is trivial. □

Remark 4.7. It is easy to see that, if the parameters $(m\,,\ \mathbf{k}\,,\ d)$ allow the existence of at least one irreducible projective RS, then the set of irreducible systems becomes open and dense in $\mathcal{PRS}_{\mathbf{v}}\,(m\,,\ \mathbf{k}\,,\ d)$. Nevertheless, it is not usual that the minimizers are irreducible, even if they are tight (see Section 4.3 and Examples 5.1 and 5.2).

On the other hand, if the system $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ is reducible, there exists a system $\mathcal{Q} = \{Q_j\}_{j\in\mathbb{I}_p}$ of minimal projections of the unital C^*-algebra $C_{\mathcal{V}}$ of Definition 4.4 (with $p > 1$). This means that

- Each $Q_j \in C_{\mathcal{V}}$, and $Q_j^2 = Q_j^* = Q_j$.
- $\mathcal{Q}$ is a system of projections: $Q_j\, Q_k = 0$ if $j \neq k$ and $\sum_{j\in\mathbb{I}_p} Q_j = I_{\mathcal{H}}$.
- *Minimality:* The algebra $C_{\mathcal{V}}$ has no proper sub projection of any Q_j .

By compressing the system $\mathcal{V}$ to each subspace $\mathcal{H}_j = R(Q_j)$ in the natural way (see Section 4.3 below), it can be shown that every $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ is an "orthogonal sum" of irreducible subsystems.

Another system of projections associated with $\mathcal{V}$ are the spectral projections of $S_{\mathcal{V}}$: If $\sigma(S_{\mathcal{V}}) = \{\sigma_1\,, \ldots,\ \sigma_r\}$, we denote these projections by

$$P_{\sigma_j} = P_{\sigma_j}(S_{\mathcal{V}}) \overset{\text{def}}{=} P_{\ker\,(S-\sigma_j\, I_d)} \in \mathcal{M}_d(\mathbb{C})^+, \quad \text{for} \quad j \in \mathbb{I}_r.$$

Recall that $S_{\mathcal{V}}\, P_{\sigma_j} = \sigma_j\, P_{\sigma_j}$ and $\sum_{j=1}^r P_{\sigma_j} = I_d$, so that $S_{\mathcal{V}} = \sum_{j=1}^r \sigma_j\, P_{\sigma_j}$.

Theorem 4.8. *Fix* $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$ *. Let* $(\mathcal{V},\ \mathcal{W}) \in \mathcal{DP}_{\mathbf{v}}$ *be a* d_P*-local minimizer of the joint potential in* $\mathcal{DP}_{\mathbf{v}}$ *with* $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m}$ *. Then*

1. *The RS operator* $S_{\mathcal{V}} \in C_{\mathcal{V}} = \{V_i^* V_i :\ i \in \mathbb{I}_m\}'$*.*
2. *If* $\sigma(S_{\mathcal{V}}) = \{\sigma_1, \ldots, \sigma_r\}$*, then also* $P_{\sigma_i} = P_{\sigma_i}(S_{\mathcal{V}}) \in C_{\mathcal{V}}$ *for every* $i \in \mathbb{I}_r$ *.*

Proof. Recall that $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}} \subseteq \mathcal{RS}$ and hence $0 \notin \sigma(S_{\mathcal{V}})$. On the other hand, we have already seen in Lemma 3.7 that $\mathcal{W}$ must be $\mathcal{V}^{\#}$. Let $\mathcal{Q} = \{Q_j\}_{j\in\mathbb{I}_p}$ be a system of minimal projections of the unital C^*-algebra $C_{\mathcal{V}}$, as in Remark 4.7.

Fix $j \in \mathbb{I}_p$ and denote by $\mathcal{S}_j = R(Q_j)$. For every $i \in \mathbb{I}_m$ put $\mathcal{T}_i = V_i(\mathcal{S}_j) \subseteq \mathcal{K}_i$, $t_i = \dim \mathcal{T}_i$ and $W_i = V_iQ_j \in L(\mathcal{H}_j , \mathcal{T}_i)$. Since $Q_j \in C_{\mathcal{V}}$ then each matrix $v_i^{-1} W_i^*$ is an isometry, so that the compression of $\mathcal{V}$ given by $\mathcal{W} = \{W_i\}_{i\in \mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}(m , \mathbf{t} , s_j)$, where $\mathbf{t} = (t_1 , \ldots , t_m)$ and $s_j = \dim \mathcal{S}_j$. Recall that $S_{\mathcal{V}}$ commutes with each Q_j, so that also $S_{\mathcal{V}}^{-1}$ commutes with them. This implies that $\mathcal{W}^{\#}$ is the same type of compression to $\mathcal{RS}_{\mathbf{v}}(m , \mathbf{t} , s_j)$ of the system $\mathcal{V}^{\#}$.

A straightforward computation shows that the pair $(\mathcal{W}, \mathcal{W}^{\#}) \in \mathcal{DP}_{\mathbf{v}}(m , \mathbf{t} , s_j)$ is still a d_P-local minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}(m , \mathbf{t} , s_j)$. Indeed, the key argument is that one can "complete" other systems in $\mathcal{PRS}_{\mathbf{v}}(m , \mathbf{t} , s_j)$ near $\mathcal{W}$ (and acting in $\mathcal{S}_j$) with the fixed orthogonal complement $\{V_i(I_d - Q_j)\}_{i\in\mathbb{I}_m}$, getting systems in $\mathcal{PRS}_{\mathbf{v}}(m , \mathbf{k} , d)$ near $\mathcal{V}$. It is easy to see that all the computations involved in the joint potential work independently on each orthogonal subsystem. This shows the minimality of $(\mathcal{W}, \mathcal{W}^{\#})$.

Observe that $W_i^*W_i = Q_jV_i^*V_iQ_j = V_i^*V_iQ_j$ for every $i \in \mathbb{I}_m$. Therefore, the minimality of Q_j in $C_{\mathcal{V}}$ shows that the system $\mathcal{W}$ satisfies that $C_{\mathcal{W}} = \mathbb{C}\, I_{\mathcal{S}_j}$. Hence, we can apply Lemma 4.6 on $\mathcal{S}_j$, and get that $S_{\mathcal{W}} = \alpha_j I_{\mathcal{S}_j}$ for some $\alpha_j > 0$. But when we return to $L(\mathcal{H})$, we get that $S_{\mathcal{V}} Q_j = \sum_{i\in\mathbb{I}_m} V_i^*V_iQ_j = \sum_{i\in\mathbb{I}_m} W_i^*W_i = S_{\mathcal{W}} = \alpha_j Q_j$. In particular, $\alpha_j \in \sigma(S_{\mathcal{V}})$.

We have proved that for every $j \in \mathbb{I}_p$ there exists $\alpha_j \in \sigma(S_{\mathcal{V}})$ such that $S_{\mathcal{V}} Q_j = \alpha_j Q_j$ and hence each projector $Q_j \leqslant P_{\alpha_j} = P_{\alpha_j}(S_{\mathcal{V}})$. Using that $\sum_{j\in\mathbb{I}_p} Q_j = I_d$ we see that each

$$P_{\sigma_k} = \sum_{j\in J_k} Q_j \in C_{\mathcal{V}}, \quad \text{where} \quad J_k = \{j \in \mathbb{I}_p : \alpha_j = \sigma_k\}. \tag{26}$$

Therefore also $S_{\mathcal{V}} = \sum_{k\in\mathbb{I}_r} \sigma_k P_{\sigma_k} \in C_{\mathcal{V}}$. □

4.3. Concluding remarks

Theorem 4.8 assures that if $(\mathcal{V} , \mathcal{V}^{\#})$ is a d_P-local minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$, then $\mathcal{V}$ is an orthogonal sum of **tight** systems in the following sense.

If $\sigma(S_{\mathcal{V}}) = \{\sigma_1, \ldots, \sigma_r\}$, and we denote $\mathcal{H}_j = R(P_{\sigma_j}) = \ker(S - \sigma_j I_d)$ for every $j \in \mathbb{I}_r$, then $\mathcal{H} = \bigoplus_{j\in\mathbb{I}_r} \mathcal{H}_j$. By Theorem 4.8 each $P_{\sigma_j} \in C_{\mathcal{V}}$. Then, putting $d_j = \dim \mathcal{H}_j$,

$$\mathcal{K}_{i,j} = V_i(\mathcal{H}_j) \subseteq \mathcal{K}_i \quad , \quad k_{i,j} = \dim \mathcal{K}_{i,j} \quad \text{and} \quad \mathbf{k}^j = (k_{1,j} , \ldots , k_{m,j}) ,$$

for every $i \in \mathbb{I}_m$ and $j \in \mathbb{I}_r$, we can define the compression of $\mathcal{V}$ to $\mathcal{H}_j$:

$$\mathcal{V}^j = \{V_i P_{\sigma_j}\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}(m , \mathbf{k}^j , d_j) \quad \text{for} \quad j \in \mathbb{I}_r.$$

Each of these compressions forms a tight projective RS (with the same corresponding parameters) for $\mathcal{H}_j$. Indeed, since $P_{\sigma_j} \in C_{\mathcal{V}}$ then $\mathcal{V}^j$ is projective. Also $S_{\mathcal{V}^j} = S_{\mathcal{V}} P_{\sigma_j} = \sigma_j P_{\sigma_j}$, which means that $\mathcal{V}^j$ is σ_j – tight. Observe that the decomposition of each $\mathcal{V}^j$ into irreducible tight systems (as in Remark 4.7) follows from the orthogonal decomposition of $\mathcal{H}_j$ given in Eq. (26). On the other hand, the decomposition into tight components obtained by compression to the $\mathcal{H}_j$'s is optimal in the sense that the sizes of the blocks in the decomposition (i.e., the dimensions $d_j = \dim \mathcal{H}_j$) are maximal.

In particular, every $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ such that $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$ (the unique vector of Theorem 3.9) must have this structure, because in this case $(\mathcal{V} , \mathcal{V}^{\#})$ is a d_S (hence also d_P) local minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$. Observe that the structures of all global minimizers $\mathcal{V}$ share many features: Since $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$, the number r of tight components, the sizes d_j and the tight constants σ_j for each space $\mathcal{H}_j$ coincide for every such minimizer $\mathcal{V}$.

A similar decomposition can be obtained for the canonical dual $\mathcal{V}^{\#}$ of a d_P-local minimizer $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ of the joint potential. Indeed, using the definitions and notations in the beginning of this subsection, we can consider the compression of $\mathcal{V}^{\#}$ to each $\mathcal{H}_j$ given by

$$(\mathcal{V}^{\#})^j = \{\sigma_j^{-1} V_i P_{\sigma_j}\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}(m\,,\,\mathbf{k}^j\,,\,d_j) \quad \text{for} \quad j \in \mathbb{I}_r .$$

As in the case of $\mathcal{V}$, $\mathcal{V}^{\#}$ is the orthogonal sum of the systems $(\mathcal{V}^{\#})^j$ for $j \in \mathbb{I}_r$. Therefore, although the canonical dual $\mathcal{V}^{\#}$ may be not projective - indeed $\mathcal{V}^{\#}$ is projective if $V_i P_{\sigma_j} = 0$ or V_i for every $i \in \mathbb{I}_m$ and $j \in \mathbb{I}_r$ – it can be decomposed as the orthogonal sum of projective tight subsystems. $\square$

5. Examples

The following two examples are about irreducible systems.

Example 5.1. Let $d = k_1 + k_2$ and $\mathbf{k} = (k_1\,,\,k_2)$. Assume that $k_1 > k_2$. We shall see that, in this case, there is no irreducible (Riesz) systems in $\mathcal{PRS}(2\,,\,\mathbf{k}\,,\,d)$. Observe that the situation is the same whatever the weights $(v_1\,,\,v_2)$ are.

Indeed, if $\mathcal{V} = (V_1\,,\,V_2) \in \mathcal{PRS}_{\mathbb{1}}(2\,,\,\mathbf{k}\,,\,d)$, let $\mathcal{S}_i = R(V_i^*)$ and $P_i = P_{\mathcal{S}_i} = V_i^* V_i$ for $i = 1, 2$. Then $\mathbb{C}^d = \mathcal{S}_1 \oplus \mathcal{S}_2$ (not necessarily orthogonal). Observe that $\dim S_1 = \dim S_2^{\perp} = k_1$ and $2\,k_1 > d$. Hence $\mathcal{T} = \mathcal{S}_1 \cap \mathcal{S}_2^{\perp} \neq \{0\}$. Since $P = P_{\mathcal{T}} \leq P_1$ and $P \leq I_d - P_2$, then $P \in C_{\mathcal{V}}$ and $0 \neq P \neq I_d$. Therefore $C_{\mathcal{V}} \neq \mathbb{C}\, I_d$.

In particular, if the decomposition $\mathbb{C}^d = \mathcal{S}_1 \oplus \mathcal{S}_2$ is orthogonal, then $S_{\mathcal{V}} = P_1 + P_2 = I_d$. So, in this case $\mathcal{V}$ is tight and reducible. $\square$

Example 5.2. If $m \geq d$ and $\mathbf{k} = \mathbb{1}_m$, then $\mathcal{PRS}(m\,,\,\mathbf{k}\,,\,d)$ is the set of m-vector frames for the space $\mathbb{C}^d$. In this case $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}$ is reducible $\iff$ there exists $J \subseteq \mathbb{I}_m$ such that $\emptyset \neq J \neq \mathbb{I}_m$ and the subspaces $\operatorname{span}\{f_i : i \in J\}$ and $\operatorname{span}\{f_j : j \notin J\}$ are orthogonal.

Indeed, if $A = A^*$, then $A \in C_{\mathcal{F}} \iff$ every f_i is an eigenvector of A. But different eigenvalues of A must have orthogonal subspaces of eigenvectors. Observe that in this case the set of irreducible systems is an open and dense subset of $\mathcal{PRS}_{\mathbf{v}}$, since it is the intersection of $2^m - 2$ open dense sets (one for each fixed nontrivial $J \subseteq \mathbb{I}_m$). $\square$

5.3. Minimizers and majorization: Theorem 3.9 states that there exists a vector $\lambda_{\mathbf{v}} = \lambda_{\mathbf{v}}(m, \mathbf{k}, d) \in (\mathbb{R}^d_{>0})^{\downarrow}$ such that a system $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}(m, \mathbf{k}, d)$ satisfies that $(\mathcal{V},\, \mathcal{V}^{\#})$ is a global minimizer of the joint potential in $\mathcal{DP}_{\mathbf{v}}$ if and only if $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$. This vector is found as the unique minimizer of the map $F(\lambda) = \sum_{i=1}^{d} \lambda_i^2 + \lambda_i^{-2}$ on the convex set $\Lambda(\mathcal{OP}_{\mathbf{v}})$.

In all the examples where $\lambda_{\mathbf{v}}$ could be explicitly computed, it satisfied a stronger condition, in terms of majorization (see [4, Chapter II] for definitions and basic properties). We shall see that in these examples there is a vector $\lambda \in \Lambda(\mathcal{OP}_{\mathbf{v}})$ such that

$$\lambda \prec \lambda(S_{\mathcal{V}}) \;\; \text{for every} \;\; \mathcal{V} \in \mathcal{PRS}_{\mathbf{v}} \quad (\text{the symbol } \prec \text{ means majorization}) . \tag{27}$$

Observe that such a vector $\lambda \in \Lambda(\mathcal{OP}_{\mathbf{v}})$ must be the unique minimizer for F on $\Lambda(\mathcal{OP}_{\mathbf{v}})$, since the map F is permutation invariant and convex. Hence $\lambda = \lambda_{\mathbf{v}}$. Moreover, those cases where $\lambda_{\mathbf{v}}$ satisfies Eq. (27) have some interesting properties regarding the structure of minimizers of the joint potential. For example, that $\lambda_{t\mathbf{v}}(m, \mathbf{k}, d) = t^2 \lambda_{\mathbf{v}}(m, \mathbf{k}, d)$ for $t > 0$, a fact that is not evident at all from the properties of these vectors.

Conjecture 5.4. For every set of parameters $(m, \mathbf{k}, d)$ and $\mathbf{v} \in \mathbb{R}^d_{>0}$, the vector $\lambda_{\mathbf{v}}(m, \mathbf{k}, d)$ of Theorem 3.9 satisfies the majorization minimality of Eq. (27) on $\Lambda(\mathcal{OP}_{\mathbf{v}})$. $\square$

Example 5.5. Given $\mathbf{v} = \mathbf{v}^{\downarrow} \in \mathbb{R}^m_{>0}$ and $d \leq m$, the d-irregularity of $\mathbf{v}$ is the index

$$r = r_d(\mathbf{v}) \overset{\text{def}}{=} \max\left\{ j \in \mathbb{I}_{d-1} \;:\; (d-j)\, v_j^2 > \sum_{i=j+1}^{m} v_i^2 \right\},$$

or $r = 0$ if this set is empty. In [24, Proposition 2.3] (see also [2, Proposition 4.5]) it is shown that for any set of parameters $(m\,,\ \mathbb{1}_m\,,\ d)$ and every $\mathbf{v} = \mathbf{v}^\downarrow \in \mathbb{R}^m_{>0}$, there is $c \in \mathbb{R}$ such that

$$\lambda_{\mathbf{v}}(m,d) \stackrel{\text{def}}{=} (v_1^2\,,\ \dots\,,\ v_r^2\,,\ c\,\mathbb{1}_{d-r}) \in \Lambda(\mathcal{OP}_{\mathbf{v}}(m,\mathbb{1}_m\,,\ d))$$

and it satisfies Eq. (27). Therefore $\lambda_{\mathbf{v}}(m,d) = \lambda_{\mathbf{v}}(m,\mathbb{1}_m\,,\ d)$ by 5.3. Thus, in the case of vector frames, Conjecture 5.4 is known to be true. $\square$

In the following examples we shall compute explicitly the vector $\lambda_{\mathbf{v}}$ and the global minimizers of the joint potential in $\mathcal{PRS}_{\mathbf{v}}$. Since we shall use Eq. (27) as our main tool (showing Conjecture 5.4 in these cases), we need a technical result about majorization, similar to [23, Lemma 2.2]. Recall that the symbol $\prec_w$ means weak majorization.

Lemma 5.6. Let $\alpha\,,\ \gamma \in \mathbb{R}^n$, $\beta \in \mathbb{R}^m$ and $b \in \mathbb{R}$ such that $b \le \min_{k\in\mathbb{I}_n} \gamma_k$. Then, if

$$\operatorname{tr}(\gamma\,,\ b\,\mathbb{1}_m) \le \operatorname{tr}(\alpha\,,\ \beta) \quad \text{and} \quad \gamma \prec_w \alpha \implies (\gamma\,,\ b\,\mathbb{1}_m) \prec_w (\alpha\,,\ \beta).$$

Observe that we are not assuming that $(\alpha\,,\ \beta) = (\alpha\,,\ \beta)^\downarrow$.

Proof. Let $h = \operatorname{tr}\ \beta$ and $\rho = \frac{h}{m}\,\mathbb{1}_m$. Then it is easy to see that

$$\sum_{i\in\mathbb{I}_k} (\gamma^\downarrow\,,\ b\,\mathbb{1}_m)_i \le \sum_{i\in\mathbb{I}_k} (\alpha^\downarrow\,,\ \rho)_i \le \sum_{i\in\mathbb{I}_k} (\alpha^\downarrow\,,\ \beta^\downarrow)_i \quad \text{for every} \quad k \in \mathbb{I}_{n+m}.$$

Since $(\gamma^\downarrow\,,\ b\,\mathbb{1}_m) = (\gamma\,,\ b\,\mathbb{1}_m)^\downarrow$, we can conclude that $(\gamma\,,\ b\,\mathbb{1}_m) \prec_w (\alpha\,,\ \beta)$. $\square$

Example 5.7. Assume that $\operatorname{tr}\mathbf{k} = d$. Then the elements of $\mathcal{PRS}_{\mathbf{v}}(m\,,\ \mathbf{k}\,,\ d)$ are **Riesz** systems. Assume that the weights are ordered in such a way that $\mathbf{v} = \mathbf{v}^\downarrow$. We shall see that the vector $\lambda = (v_1^2\,\mathbb{1}_{k_1}\,,\ \dots\,,\ v_m^2\,\mathbb{1}_{k_m}) \prec \lambda(S_{\mathcal{V}})$ for every $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}(m\,,\ \mathbf{k}\,,\ d)$. Hence λ satisfies Eq. (27), and $\lambda_{\mathbf{v}} = \lambda$ by 5.3.

Indeed, given $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{PRS}_{\mathbf{v}}$, consider the projections $P_i = v_i^{-2}\,V_i^*V_i$ and denote by $\mathcal{S}_i = R(P_i)$ for every $i \in \mathbb{I}_m$. Then $S_{\mathcal{V}} = \sum_{i\in\mathbb{I}_m} v_i^2\,P_i$ and $\mathbb{C}^d = \bigoplus_{i\in\mathbb{I}_m}\mathcal{S}_i$ where the direct sum is not necessarily orthogonal. Let

$$\mathcal{S} = \bigoplus_{i\in\mathbb{I}_{m-1}}\mathcal{S}_i \subseteq \mathbb{C}^d\ ,\quad P = P_{\mathcal{S}} \quad \text{and} \quad Q = I_d - P = P_{\mathcal{S}^\perp}\ .$$

Consider the restriction $A = \sum_{i=1}^{m-1} v_i^2\,P_i \in L(\mathcal{S})^+$. It is well known that the pinching matrix

$$M = P\,S_{\mathcal{V}}\,P + Q\,S_{\mathcal{V}}\,Q = \begin{array}{c} \begin{bmatrix} A + v_m^2\,PP_2P & 0 \\ 0 & v_m^2\,QP_mQ \end{bmatrix} \end{array} \begin{array}{l} \mathcal{S} \\ \mathcal{S}^\perp \end{array}$$

satisfies that $\lambda(M) \prec \lambda(S_{\mathcal{V}})$. Using an inductive argument on m (the case $m = 1$ is trivial), for the Riesz system $\mathcal{V}_0 = \{V_i|_{\mathcal{S}}\}_{i\in\mathbb{I}_{m-1}}$ (for $\mathcal{S}$) such that $S_{\mathcal{V}_0} = A$, we can assure that

$$\gamma = (v_1^2\,\mathbb{1}_{k_1}\,,\ \dots\,,\ v_{m-1}^2\,\mathbb{1}_{k_{m-1}}) \prec \lambda(A) \prec_w \lambda\big(A + v_m^2\,PP_2P\big) = \alpha \quad \text{in} \quad \mathbb{R}^{d-k_m}.$$

Since $v_m \le v_{m-1}$, Lemma 5.6 assures that $\lambda = (\gamma\,,\ v_m^2\,\mathbb{1}_{k_m}) \prec (\alpha\,,\ \beta) = \lambda(M)$, where $\beta = \lambda(v_m^2\,QP_mQ) \in \mathbb{R}^{k_m}$. Hence, we have proved that $\lambda \prec \lambda(S_{\mathcal{V}})$.

Recall a system $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ is a minimizer if and only if $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}} = \lambda$. Now, it is easy to see that $\lambda(S_{\mathcal{V}}) = \lambda_{\mathbf{v}}$ if and only if the projections P_i are mutually orthogonal. $\square$

Example 5.8. Assume that the parameters $(m\,,\ \mathbf{k}\,,\ d)$ satisfy that

$$m = 2 \quad \text{and} \quad \operatorname{tr}\mathbf{k} = k_1 + k_2 > d\,, \quad \text{but} \quad k_1 \neq d \neq k_2\,.$$

Fix $\mathbf{v} = (v_1\,,\ v_2)$ with $v_1 \geq v_2$. For the space $\mathcal{PRS}_{\mathbf{v}}(2\,,\ \mathbf{k}\,,\ d)$ the vector $\lambda_{\mathbf{v}}$ of Theorem 3.9 and all the global minimizers of the joint potential can be computed: Denote by

$$r_0 = k_1 + k_2 - d, \quad r_1 = k_1 - r_0 \quad \text{and} \quad r_2 = k_2 - r_0.$$

We shall see that the vector $\mu = (\,(v_1^2+v_2^2)\,\mathbb{1}_{r_0}\,,\ v_1^2\,\mathbb{1}_{r_1}\,,\ v_2^2\,\mathbb{1}_{r_2})$ satisfies Eq. (27), so that $\lambda_{\mathbf{v}}(2\,,\ \mathbf{k}\,,\ d) = \mu$ by 5.3. Moreover, the minimizers are those systems $\mathcal{V} = (V_1\,,\ V_2) \in \mathcal{PRS}_{\mathbf{v}}$ such that the two projections $P_i = v_i^{-2}\,V_i^*V_i$ (for $i = 1, 2$) commute.

Indeed, if $\mathcal{S}_i = R(P_i) = R(V_i^*)$ for $i = 1, 2$, then $\mathcal{M}_0 = \mathcal{S}_1 \cap \mathcal{S}_2$ has $\dim \mathcal{M}_0 = r_0$. Also $\mathcal{M}_i = \mathcal{S}_i \ominus \mathcal{M}_0$ have $\dim \mathcal{M}_i = r_i$ for $i = 1, 2$. Hence $\mathbb{C}^d = \mathcal{M}_0 \perp (\mathcal{M}_1 \oplus \mathcal{M}_2)$ and

$$S_{\mathcal{V}} = v_1^2\,P_1 + v_1^2\,P_2 \;=\; (v_1^2 + v_2^2)\,P_{\mathcal{M}_0} + v_1^2\,P_{\mathcal{M}_1} + v_1^2\,P_{\mathcal{M}_2}.$$

Note that $\mathcal{M}_1 \perp \mathcal{M}_2 \iff P_1\,P_2 = P_2\,P_1 = P_{\mathcal{M}_0}$. In this case $\lambda(S_{\mathcal{V}}) = \mu$. Otherwise, still $S_{\mathcal{V}}\big|_{\mathcal{M}_0} = (v_1^2 + v_2^2)\,I_{\mathcal{M}_0}$ and $S_{\mathcal{V}}(\mathcal{M}_1 \oplus \mathcal{M}_2) = \mathcal{M}_1 \oplus \mathcal{M}_2$. Hence, if we denote by $T = S_{\mathcal{V}}\big|_{\mathcal{M}_1 \oplus \mathcal{M}_2} = (v_1^2\,P_{\mathcal{M}_1} + v_1^2\,P_{\mathcal{M}_2})\big|_{\mathcal{M}_1 \oplus \mathcal{M}_2} \in \mathcal{G}l(\mathcal{M}_1 \oplus \mathcal{M}_2)^+$, then $\|T\|_{sp} \leq v_1^2 + v_2^2$ and

$$S_{\mathcal{V}} = \begin{array}{c} \begin{bmatrix} (v_1^2 + v_2^2)\,I_{r_0} & 0 \\ 0 & T \end{bmatrix} \end{array} \begin{array}{l} \mathcal{M}_0 \\ \mathcal{M}_0^{\perp} \end{array} \quad \text{with} \quad \lambda(S_{\mathcal{V}}) = \Big((v_1^2 + v_2^2)\,\mathbb{1}_{r_0}\,,\ \lambda(T)\Big) \in (\mathbb{R}^d_{>0})^{\downarrow}.$$

Using Example 5.7 for the space $\mathcal{M}_1 \oplus \mathcal{M}_2$, we can deduce that $(v_1^2\,\mathbb{1}_{r_1}\,,\ v_2^2\,\mathbb{1}_{r_2}) \prec \lambda(T)$. Therefore also $\mu = (\,(v_1^2 + v_2^2)\,\mathbb{1}_{r_0}\,,\ v_1^2\,\mathbb{1}_{r_1}\,,\ v_2^2\,\mathbb{1}_{r_2}) \prec (\,(v_1^2 + v_2^2)\,\mathbb{1}_{r_0}\,,\ \lambda(T)\,) = \lambda(S_{\mathcal{V}})$.

Example 5.9. Let $m = 3$, $d = 4$, $\mathbf{k} = (3\,,\ 2\,,\ 2)$ and $\mathbf{v} = \mathbb{1}_3$. Denote by $\mathcal{E} = \{e_i\,:\, i \in \mathbb{I}_4\}$ the canonical basis of $\mathbb{C}^4$. Then $\lambda_{\mathbb{1}}(3\,,\ \mathbf{k}\,,\ 4) = \left(2\,,\ 2\,,\ \frac{3}{2}\,,\ \frac{3}{2}\right)$ and a minimizer is given by any system $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_3} \in \mathcal{PRS}_{\mathbb{1}}$ such that the subspaces $\mathcal{S}_i = R(V_i^*)$ for $i \in \mathbb{I}_3$ are

$$\mathcal{S}_1 = \operatorname{span}\{e_1\,,\ e_2\,,\ e_3\} \quad , \quad \mathcal{S}_2 = \operatorname{span}\{\,e_1\,,\ w_2\,\} \quad \text{and} \quad \mathcal{S}_3 = \operatorname{span}\{\,e_2\,,\ w_3\,\},$$

where $w_2 = \frac{-e_3}{2} + \frac{\sqrt{3}\,e_4}{2}$ and $w_3 = \frac{-e_3}{2} - \frac{\sqrt{3}\,e_4}{2}$. The fact that $\lambda(\mathcal{S}_{\mathcal{V}}) = \left(2\,,\ 2\,,\ \frac{3}{2}\,,\ \frac{3}{2}\right)$ for such a system $\mathcal{V}$ is a direct computation. On the other hand, if $\mathcal{W} = \{W_i\}_{i\in\mathbb{I}_3} \in \mathcal{PRS}_{\mathbb{1}}(3\,,\ \mathbf{k}\,,\ 4)$, then there exist unit vectors $x_2 \in R(W_1^*) \cap R(W_2^*)$ and $x_3 \in R(W_1^*) \cap R(W_3^*)$.

Denote by $\mathcal{T} = \operatorname{span}\{x_2\,,\ x_3\}$. If $\dim \mathcal{T} = 1$ then $\lambda_1(S_{\mathcal{W}}) \geq \langle S_{\mathcal{W}}\,x_2\,,\ x_2\rangle = 3$ and $\lambda_2(S_{\mathcal{W}}) \geq 1$. If $\dim \mathcal{T} = 2$, using that $\mathcal{T} \subseteq R(W_1^*)$ and $x_i \in R(W_i^*)$ for $i = 2,\ 3$, we get

$$\lambda_1(S_{\mathcal{W}}) + \lambda_2(S_{\mathcal{W}}) \;\geq\; \sum_{i\in\mathbb{I}_3} \operatorname{tr}\left(P_{\mathcal{T}}\,W_i^*W_i\,P_{\mathcal{T}}\right) \geq \operatorname{tr} P_{\mathcal{T}} + \operatorname{tr} P_{\operatorname{span}\{x_2\}} + \operatorname{tr} P_{\operatorname{span}\{x_3\}} = 4.$$

In any case, we have shown that $(2\,,\ 2) \prec_w \alpha = (\lambda_1(S_{\mathcal{W}})\,,\ \lambda_2(S_{\mathcal{W}})\,)$. Therefore, using Lemma 5.6 we get that $\left(2\,,\ 2\,,\ \frac{3}{2}\,,\ \frac{3}{2}\right) \prec \lambda(S_{\mathcal{W}})$. Now, apply 5.3.

The minimizers $\mathcal{V} \in \mathcal{PRS}_{\mathbf{v}}$ such that $\lambda(S_{\mathcal{V}}) = (2\,,\ 2\,,\ \frac{3}{2}\,,\ \frac{3}{2}\,)$ have some interestig properties. For example, they are the sum of two tight systems, $\mathcal{V}^{\#}$ is not projective, and the involved projections do not commute. More precisely, the cosine of the Friedrich angles of their images are $c(\mathcal{S}_i\,,\ \mathcal{S}_j) = \frac{1}{2}$ for every $i \neq j$.

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