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Peiffer elements in simplicial groups and algebras

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Abstract

The main objective of this paper is to prove in full generality the following two facts:

A. *For an operad* $\mathcal{O}$ *in* Ab, *let* A *be a simplicial* $\mathcal{O}$*-algebra such that* A_m *is generated as an* $\mathcal{O}$*-ideal by* $(\sum_{i=0}^{m-1} s_i(A_{m-1}))$, *for* $m > 1$, and *let* $\mathbf{N}A$ *be the Moore complex of* A. *Then*

$$d(\mathbf{N}_m A) = \sum_I \gamma \left(\mathcal{O}_p \otimes \bigcap_{i \in I_1} \ker d_i \otimes \cdots \otimes \bigcap_{i \in I_p} \ker d_i \right)$$

where the sum runs over those partitions of $[m-1]$, $I = (I_1, \ldots, I_p)$, $p \geq 1$, *and* γ *is the action of* $\mathcal{O}$ *on* A.

B. *Let* G *be a simplicial group with Moore complex* $\mathbf{N}G$ *in which* G_n *is generated as a normal subgroup by the degenerate elements in dimension* $n > 1$, *then* $d(\mathbf{N}_n G) = \prod_{I,J} [\bigcap_{i \in I} \ker d_i, \bigcap_{j \in J} \ker d_j]$, *for* $I, J \subseteq [n-1]$ *with* $I \cup J = [n-1]$.

In both cases, d_i is the i-th face of the corresponding simplicial object.

The former result completes and generalizes results from Akça and Arvasi [I. Akça, Z. Arvasi, Simplicial and crossed Lie algebras, Homology Homotopy Appl. 4 (1) (2002) 43–57], and Arvasi and Porter [Z. Arvasi, T. Porter, Higher dimensional Peiffer elements in simplicial commutative algebras, Theory Appl. Categ. 3 (1) (1997) 1–23]; the latter completes a result from Mutlu and Porter [A. Mutlu, T. Porter, Applications of Peiffer pairings in the Moore complex of a simplicial group, Theory Appl. Categ. 4 (7) (1998) 148–173]. Our approach to the problem is different from that of the cited works. We have first succeeded with a proof for the case of algebras over an operad by introducing a different description of the inverse of the normalization functor $\mathbf{N} : Ab^{\Delta^{\mathrm{op}}} \to Ch_{\geq 0}$. For the case of simplicial groups, we have then adapted the construction for the inverse equivalence used for algebras to get a simplicial group $\mathbf{N}G \boxtimes \Lambda$ from the Moore complex $\mathbf{N}G$ of a simplicial group G. This construction could be of interest in itself.

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1. Introduction

Brown and Loday have noted [4] that if in dimension two, the simplicial group (G_*, d_i, s_i) is normally generated by degenerate elements, then

$$d(\mathbf{N}_2 G) = [\ker d_0, \ker d_1].$$

Here $\mathbf{N}_2 G = \ker d_0 \cap \ker d_1$, d is here induced by d_2 and the square brackets denote the commutator subgroup. Thus, the subgroup $d(\mathbf{N}_2 G)$ of $\mathbf{N}_1 G$ is generated by elements of the form $s_0 d_1(x) y s_0 d_1(x^{-1})(xyx^{-1})^{-1}$, and it is just the Peiffer subgroup of $\mathbf{N}_1 G$, the vanishing of which is equivalent to $d_1 : \mathbf{N}_1 G \to \mathbf{N}_0 G$ being a crossed module.

Arvasi and Porter [3] have shown that if A is a simplicial commutative algebra with Moore complex $\mathbf{N}A$, and for $n > 0$ the ideal generated by the degenerate elements in dimension n is A_n, then

$$d(\mathbf{N}_n A) \supseteq \sum_{I,J} K_I K_J.$$

This sum runs over those $\emptyset \neq I, J \subset [n-1] = \{0, \dots, n-1\}$ with $I \cup J = [n-1]$, and $K_I := \bigcap_{i \in I} \ker d_i$. They have also shown the equality for $n = 2$, 3 and 4, and argued for its validity for all n. A similar result for simplicial Lie algebras was obtained by Arvasi and Akça in [1].

Mutlu [15] and Mutlu and Porter [16], have adapted Arvasi's method to the case of simplicial groups. They succeeded to prove that for $n = 2$, 3 and 4,

$$d(\mathbf{N}_n G) = \prod_{I,J} [K_I, K_J]$$

and that the inclusion $d(\mathbf{N}_n G) \supseteq \prod_{I,J} [K_I, K_J]$ holds for every n.

Simplicial groups are algebraic models for all connected homotopy types. Besides, the homotopy groups of a simplicial group coincide with the homology groups of its Moore complex; therefore, the study of the Moore complex of a simplicial group has important applications in the study of its homotopy type. The decomposition of the group of boundaries of the Moore complex of a simplicial group (algebra) as a product of commutator subgroups (sum of product ideals) is of interest in various topological and homological settings. For instance, in calculations of non-abelian homology of groups [12] and non-abelian homology of Lie algebras [13], and to explain the relations among several algebraic models of connected homotopy 3-types: braided regular crossed modules, 2-crossed modules, quadratic modules, crossed squares and simplicial groups with Moore complex of length 2 [1,3,15,16]. It is possible that such decomposition contributes to light complete descriptions of algebraic models of the n-types of specific families of spaces for low values of n, to calculate Samelson and Whitehead products [10] and analogues in homotopy theory of simplicial Lie algebras or to link simplicial groups and weak infinity categoric models. On the other hand, the vanishing of the higher-dimensional Peiffer elements is important in the construction of the simplicial version of the cotangent complex of André [2] and Quillen [18].

The objective of this paper is to give a general proof for the inclusions partially proved in [1,3,15] and [16].

The methods used by the previous authors to study higher-dimensional Peiffer elements are based on the ideas of Conduché [9] and the techniques developed by Carrasco and Cegarra [6,7]. Our approach to the problem is different from that of the cited papers. We have first succeeded with a proof for the case of algebras over an operad $\mathcal{O} \in \mathrm{Op}(Ab^{\Delta^{\mathrm{op}}})$, by introducing a new description of the inverse of the normalization functor $\mathbf{N} : Ab^{\Delta^{\mathrm{op}}} \to Ch_{\geq 0}$. We have then adapted this construction to get a simplicial group $G \boxtimes \Lambda$ from the Moore complex of a simplicial group G, which was used in the case of groups. This construction could be of interest in itself.

In Section 2 we give an alternative description of the Dold–Kan functor, which we shall use later. In Section 3 we take an operad $\mathcal{O} \in \mathrm{Op}(Ab^{\Delta^{\mathrm{op}}})$, an $\mathcal{O}$-simplicial algebra, and we study what happens when we apply the normalization functor to this algebra. We finally give a description of the kind of algebras that we get in $Ch_{\geq 0}$.

Section 4 is devoted to state and prove the first important result, Theorem 8. In Section 5 we introduce a simplicial group built up from a chain of groups (Definition 11) and prove some properties of this construction when applied to the Moore complex of a simplicial group (Proposition 18 and Remark 19).

Finally, in Section 6, we prove the other main result of this paper, Lemma 21, which completes the proof of Theorem 25.

2. The inverse of the normalization functor

In this section we give a description of the Dold–Kan functor $Ch_{\geq 0} \to Ab^{\Delta^{\mathrm{op}}}$, useful for our purposes. This description is in the spirit of that given in [8] for the inverse of the conormalization functor.

Let Δ be the simplicial category whose objects are the finite linearly ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for integers $n \geq 0$, and whose morphisms are nondecreasing monotone functions. Let Fin be the category with the same objects as Δ, but where the morphisms $[m] \to [n]$ are all maps. We associate to each n the free abelian group $\mathbb{Z}[n] := \mathbb{Z}e_0 \oplus \cdots \oplus \mathbb{Z}e_n \simeq \mathbb{Z}^{n+1}$, and to each $\alpha : [m] \to [n] \in$ Fin, $\alpha : \mathbb{Z}[m] \to \mathbb{Z}[n]$ given by

$$\alpha(e_i) := e_{\alpha(i)}.$$

In this way we have a cosimplicial abelian group, after restriction to Δ. Given a category $\mathcal{A}$, and a small category $\mathcal{C}$, we call a *$\mathcal{C}$-object* of $\mathcal{A}$ to any functor $\mathcal{C} \to \mathcal{A}$.

Put $v_i := e_i - e_n$ in $\mathbb{Z}[n]$ and write $\mathbb{Z}[[n]]$ for the $\mathbb{Z}$-module freely generated by $\{v_0, \ldots, v_{n-1}\}$. Since $v_n = e_n - e_n = 0$ and for $\alpha : \mathbb{Z}[m] \to \mathbb{Z}[n]$ we have that

$$\alpha(v_i) = \alpha(e_i - e_m) = e_{\alpha(i)} - e_{\alpha(m)} = e_{\alpha(i)} - e_{\alpha(n)} + e_{\alpha(n)} - e_{\alpha(m)} = v_{\alpha(i)} - v_{\alpha(m)},$$

we conclude that $\mathbb{Z}[[n]]$ is a Fin-subgroup of $\mathbb{Z}[n]$; i.e., we can associate to $\mathbb{Z}[[n]]$ a functor from the category Fin to the category of abelian groups.

Write $\mathbb{Z}(n)$ for the abelian group $\hom_{\mathbb{Z}}(\mathbb{Z}[[n]], \mathbb{Z})$. For $\varphi \in \mathbb{Z}(n)$ and $\alpha : [m] \to [n] \in$ Fin we take $\alpha(\varphi) := \varphi\alpha \in \mathbb{Z}(m)$.

In this way, $\mathbb{Z}(*)$ is a functor from the category $\mathrm{Fin}^{\mathrm{op}}$ to that of abelian groups. Observe that $\hom_{\mathbb{Z}}(\mathbb{Z}[[n]], \mathbb{Z})$ is freely generated by the morphisms φ_j, $0 \leq j \leq n-1$, of the form

$$\varphi_j(v_i) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus, we shall identify $\mathbb{Z}(n)$ with $\mathbb{Z}\varphi_0 \oplus \cdots \oplus \mathbb{Z}\varphi_{n-1}$.

In particular, if we restrict to those arrows in Δ, then $n \mapsto \mathbb{Z}(n)$ is a simplicial abelian group. Faces and degeneracies with source $\mathbb{Z}(n)$ are explicitly given by

$$s_j(\varphi_i) = \begin{cases} \varphi_i & \text{if } i < j \\ \varphi_i + \varphi_{i+1} & \text{if } i = j \\ \varphi_{i+1} & \text{if } i > j \end{cases} \quad \text{and} \quad d_j(\varphi_i) = \begin{cases} \varphi_i & \text{if } i < j \\ 0 & \text{if } i = j \\ \varphi_{i-1} & \text{if } i > j \end{cases} \tag{1}$$

for $j \neq n$, and

$$s_n(\varphi_i) = \varphi_i \quad \text{and} \quad d_n(\varphi_i) = \begin{cases} \varphi_i & \text{if } i < n-1 \\ 0 & \text{if } i = n-1. \end{cases} \tag{2}$$

Let us write Ass for the category of associative algebras. We can now apply to $\mathbb{Z}(*)$ the exterior algebra functor, $\Lambda : Ab \to Ass$ and then the forgetful functor $Ass \to Ab$ by considering the exterior algebra just as an abelian group. We write $\Lambda\mathbb{Z}(*)$ for the simplicial abelian group so obtained.

Recall that if $\alpha : A \to B$ is a morphism of rings then $\delta : A \to B$ is called an *α-derivation* if whenever $x, y \in A$ it holds that $\delta(xy) = \alpha(x)\delta(y) + \delta(x)\alpha(y)$.

Definition 1. Let (A_*, d) be a connected chain complex of abelian groups, and B_*^* a sequence of $\mathbb{Z}^+$-graded abelian groups. We write $A \boxtimes B$ for the sequence of groups $n \mapsto \bigoplus_{i \geq 0}(A_i \otimes B_n^i)$.

Write $\mathbf{K}_*A := A \boxtimes \Lambda\mathbb{Z}(*) = \bigoplus_{i=0}^{*}(A_i \otimes \Lambda^i\mathbb{Z}(*))$. We associate to each $\alpha \in \mathrm{Fin}(m, n)$, the morphism $\mathbf{K}(\alpha) : \mathbf{K}_nA \to \mathbf{K}_mA$ by the formula

$$\mathbf{K}(\alpha)(a \otimes \varphi) := a \otimes \alpha(\varphi) + dg \otimes \delta_\alpha(\varphi).$$

Here δ_α is the α-derivation $\Lambda\mathbb{Z}(n) \to \Lambda\mathbb{Z}(m)$ completely characterized by

$$\delta_\alpha(\varphi_i) := \begin{cases} 0 & \text{if } i \neq \alpha(m) \\ 1 & \text{if } i = \alpha(m). \end{cases}$$

Proposition 2. *Let* $[m] \xrightarrow{\alpha} [n] \xrightarrow{\beta} [p] \in Fin$, *then* $\mathbf{K}(\beta\alpha) = \mathbf{K}(\alpha)\mathbf{K}(\beta)$. *In consequence,* $\mathbf{K}_*A$ *is a* Fin^{op}-*group. In particular, restricting to* Δ, *it defines a simplicial abelian group, which we will denote by the same letter.*

Proof. Take $a \otimes \varphi \in \mathbf{K}_p A$. By evaluating both $\mathbf{K}(\beta\alpha)$ and $\mathbf{K}(\alpha)\mathbf{K}(\beta)$ at $a \otimes \varphi$ and comparing, we get that in order to prove the identity, it suffices to verify if $\delta_{\beta\alpha} = \alpha\delta_\beta + \delta_\alpha\beta$.

Let us observe that both $\delta_{\beta\alpha}$ and $\alpha\delta_\beta + \delta_\alpha\beta$ are $\alpha\beta$-derivations. Hence they coincide if they agree on the generators of $\Lambda\mathbb{Z}(p)$. Take $\varphi_i \in \mathbb{Z}(p)$ as above,

$$(\alpha\delta_\beta + \delta_\alpha\beta)(\varphi_i) = \alpha\delta_\beta(\varphi_i) + \delta_\alpha(\beta(\varphi_i)) \quad \text{and} \tag{3}$$

$$\delta_\alpha(\beta(\varphi_i)) = \begin{cases} \displaystyle\sum_{\beta(j)=i} \delta_\alpha(\varphi_j) & \text{if } i \neq \beta(n) \\ \displaystyle -\sum_{\beta(j)\neq\beta(n)} \delta_\alpha(\varphi_j) & \text{if } i = \beta(n). \end{cases} \tag{4}$$

We have to analyze the following possible cases:

i. If $i = \beta(n)$, then $i \neq \beta\alpha(m)$ or $i = \beta\alpha(m)$. If $i \neq \beta\alpha(m)$, (4) $= -1$ and $\delta_\beta(\varphi_i) = 1$. Hence (3) is 0. If $i = \beta\alpha(m)$, $\alpha(\delta_\beta(\varphi_i)) = 1$ and (4) $= 0$; so (3) $= 1$.
ii. If $i \neq \beta(n)$, we have two possibilities, $i = \beta\alpha(m)$ or $i \neq \beta\alpha(m)$. If $i = \beta\alpha(m)$, then (4) is 1, $\delta_\beta(\varphi_i) = 0$, and in consequence (3) is 1. If $i \neq \beta\alpha(m)$, then (4) is 0, $\delta_\beta(\varphi_i) = 0$ and (3) is 0.

Thus (3) coincides with $\delta_{\beta\alpha}(\varphi_i)$. □

Let us take a closer view on faces and degeneracies in $\mathbf{K}_*A$. Take $a \otimes \varphi \in \mathbf{K}_n A$, and write simply s_i and d_i for either $\mathbf{K}(s_i)$ and $\mathbf{K}(d_i)$. For $0 \leq i \leq n$, we have that $s_i(a \otimes \varphi) = a \otimes s_i(\varphi)$, for $0 \leq i \leq n-1$, $d_i(a \otimes \varphi) = a \otimes d_i(\varphi)$, and $d_n(a \otimes \varphi) = a \otimes d_n(\varphi) + da \otimes \delta_{d_n}(\varphi)$. So, for $i \neq n$, we can immediately say that $a \otimes \varphi \in \ker d_i$ if $i \in \sharp\varphi$. Here we write for a monomial φ, $\sharp\varphi := \{i_1, \ldots, i_r\}$ if and only if $\varphi \in \mathbb{Z}\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_r}$.

Proposition 3. *Write* $\mathbf{N} : Ab^{\Delta^{\mathrm{op}}} \to Ch_{\geq 0}$ *the normalization complex functor and let* $\mathbf{K} : Ch_{\geq 0} \to Ab^{\Delta^{\mathrm{op}}}$ *be as before. Then* $\mathbf{KN} \simeq 1_{Ab^{\Delta^{\mathrm{op}}}}$ *and* $\mathbf{NK} \simeq 1_{Ch_{\geq 0}}$. *Thus* $\mathbf{K}$ *is (isomorphic to) the classical inverse equivalence of the normalization functor.*

Proof. Recall that $\mathbf{N}_m A = \bigcap_{i=0}^{m-1} \ker d_i$ for $A \in Ab^{\Delta^{\mathrm{op}}}$. Observe that when $A = \mathbf{K}C$ for some $C \in Ch_{\geq 0}$, then $a \otimes \varphi \in \ker d_i$ iff $i \in \sharp\varphi$. On the other hand, it can be seen by a simple computation that, if $i \notin \sharp\varphi$, $i \notin \sharp\psi$ and $i \leq n$, then $\sharp\varphi \neq \sharp\psi$ implies $\sharp d_i\varphi \neq \sharp d_i\psi$. Hence we have that $\sum_{\varphi\in\Lambda_m} a_\varphi \otimes \varphi$, with Λ_m the set of monic monomials in $\Lambda\mathbb{Z}(n)$, is in $\ker d_i$ iff each $a_\varphi \otimes \varphi \in \ker d_i$. Then $x \in \mathbf{N}_m\mathbf{K}C$ if and only if $x = a \otimes \varphi_0 \wedge \cdots \wedge \varphi_{m-1}$ for some $a \in C_m$. In this case,

$$d_m(a \otimes \varphi_0 \wedge \cdots \wedge \varphi_{m-1}) = da \otimes \varphi_0 \wedge \cdots \wedge \varphi_{m-2} \in \mathbf{N}_{m-1}\mathbf{K}C.$$

Since $\mathbf{N}_m\mathbf{K}C \simeq C_m$, as $\mathbb{Z}$-modules, and d_m induces d, we get that $\mathbf{NK}C \simeq C$.

Let A_* a simplicial abelian group. We now want to see that $\mathbf{KN}A \cong A$.

Take, for $m \geq 0$, $\psi_m : \bigoplus_{i\geq 0} \mathbf{N}_i A \otimes \Lambda^i\mathbb{Z}(m) \to A_m$ the homomorphisms given by $\psi_m(a \otimes \varphi_{j_1} \wedge \cdots \wedge \varphi_{j_p}) = s_{j_p} \ldots s_{j_1}(a)$. Here we identify $a \in \mathbf{N}_p A$ with its inclusion in A_p. The map $\psi : \mathbf{KN}_*A \to A_*$ defined degreewise by ψ_* is clearly a morphism of simplicial groups. We claim that it is in fact an isomorphism of simplicial abelian groups. We shall show this by induction on m:

For $m = 0$, we have that $A_0 = \mathbf{N}_0 A$, and ψ_0 is bijective.

Suppose now that for $k \leq m$, $\psi_k : \mathbf{KN}_k A \to A_k$ is bijective.

Observe that $\mathbf{N}_*(\psi) : \mathbf{N}_*(\mathbf{KN}A) \to \mathbf{N}_*A$ is an isomorphism of chain complexes and, since ψ_{m-1} is bijective, for every $x \in A_{m-1}$ and every $j = 0, \ldots, m-1$, $s_j x$ is in the image of ψ_m. Hence, ψ_m is surjective.

On the other hand, suppose that $\psi(a \otimes \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p}) = 0$ and $p \leq m-1$. There is a map $\sigma : [m] \to [p] \in \Delta$ such that

$$\begin{array}{ccc} [m] & \xrightarrow{\sigma} & [p] \\ {\scriptstyle\epsilon}\uparrow & \overset{id}{\nearrow} & \\ [p] & & \end{array}$$

commutes, $\mathbf{K}(\sigma)(a \otimes \varphi_0 \wedge \cdots \wedge \varphi_{p-1}) = a \otimes \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p}$ and $\mathbf{K}(\epsilon) = s_{j_p} \ldots s_{j_1}$. Since $0 = \psi_m(\mathbf{K}(\sigma)(a \otimes \varphi_0 \wedge \cdots \wedge \varphi_{p-1})) = \mathbf{K}(\sigma\epsilon)(a \otimes \varphi_0 \wedge \cdots \wedge \varphi_{p-1}) = a \otimes \varphi_0 \wedge \cdots \wedge \varphi_{p-1}$, we conclude that $a \otimes \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p} = 0$. Thus, ψ_m is also injective.

Finally, by the uniqueness of the inverse of the normalization functor, $\mathbf{K}$ and the classical inverse equivalence of $\mathbf{N}$ [11] must agree up to a natural isomorphism. □

3. Algebras in $Ab^{\Delta^{\mathrm{op}}}$ and $Ch_{\geq 0}$

Let us recall the following definitions and notation from [14].

Let k be a commutative and unital ring. An *operad* $\mathcal{O}$ in the category of k-modules consists of a sequence of k-modules, $\mathcal{O}(j)$, $j \geq 0$, together with a map $\eta : k \to \mathcal{O}(1)$, a right action on $\mathcal{O}(j)$ by the symmetric group Σ_j for each j, and maps

$$\gamma : \mathcal{O}(p) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_p) \to \mathcal{O}(j_1 + \cdots + j_p).$$

Tensor products are always taken in the respective categories, in this case, in that of k-modules. The maps γ are required to satisfy suitable associativity, unitality and equivariance conditions.

Thinking of elements of $\mathcal{O}(n)$ as n-ary operations, we think of $\gamma(c \otimes b_1 \otimes \cdots \otimes b_p)$ as the composite of the operation c with the tensor product of the operations b_i. By convention, the 0-th tensor power of a k-module is interpreted to be k. The module $\mathcal{O}(0)$ parametrizes the 0-ary operations. If $\mathcal{O}(0) = k$, we say that $\mathcal{O}$ is a unital operad. For classes of algebras without units, such as Lie algebras, it is natural to set $\mathcal{O}(0) = 0$.

Let A be a k-module, and let $A^{\otimes j}$ represent its j-fold tensor power, with Σ_j acting on the left. An $\mathcal{O}$-*algebra* is a k-module together with maps

$$\theta : \mathcal{O}(j) \otimes A^{\otimes j} \to A$$

for $j \geq 0$ that are associative, unital and equivariant in suitable senses.

An $\mathcal{O}$-*ideal* or *normal subobject* of an $\mathcal{O}$-algebra A, is a k-submodule I of A such that $\theta(\mathcal{O}(p+q+1) \otimes A^{\otimes p} \otimes I \otimes A^{\otimes q}) \subseteq I$, for every p and q.

Let now $\mathcal{O} \in \mathrm{Op}(Ab)$, the category of operads on Ab. $\mathcal{O}$ induces an operad, also written $\mathcal{O} \in \mathrm{Op}(Ab^{\Delta^{\mathrm{op}}})$, obtained by applying $\mathcal{O}$ dimensionwise. Let $\mathbf{F}$ be the monad associated to $\mathcal{O}$ (see for example [14, 1.3]). For any $A \in Ab$ we have

$$\mathbf{F}(A) := \bigoplus_{n \geq 0} \mathcal{O}(n) \otimes_{\Sigma_n} A^{\otimes n}$$

and, for any $A \in Ab^{\Delta^{\mathrm{op}}}$,

$$\mathbf{F}_m(A) := \bigoplus_{n \geq 0} \mathcal{O}(n) \otimes_{\Sigma_n} A_m^{\otimes n}.$$

We associate to $\alpha \in \Delta(m, n)$, $\mathbf{F}(\alpha) : \mathbf{F}_n A \to \mathbf{F}_m A$ by taking α degreewise.

Write $\mathbf{N}_m A = \tilde{A}_m$ and $\Lambda_m^j = \Lambda^j \mathbb{Z}(m)$. Using that $A_* \simeq \mathbf{K}_* \mathbf{N} A$, we also write

$$A_m \simeq \bigoplus_{j=0}^{m} \tilde{A}_j \otimes \Lambda_m^j. \tag{5}$$

Then

$$\begin{aligned}\mathbf{F}_m A &\simeq \mathbf{F}\left(\bigoplus_{j=0}^{m} \tilde{A}_j \otimes \Lambda_m^j\right) = \bigoplus_{p \geq 0} \mathcal{O}(p) \otimes_{\Sigma_p} \left(\bigoplus_{j=0}^{m} \tilde{A}_j \otimes \Lambda_m^j\right)^{\otimes p} \\ &= \bigoplus_{p \geq 0} \bigoplus_{0 \leq r \leq mp} \bigoplus_{i_1 + \cdots + i_p = r} \mathcal{O}(p) \otimes_{\Sigma_p} (\tilde{A}_{i_1} \otimes \Lambda_m^{i_1}) \otimes \cdots \otimes (\tilde{A}_{i_p} \otimes \Lambda_m^{i_p}) \\ &\simeq \bigoplus_{p \geq 0} \bigoplus_{0 \leq r \leq mp} \bigoplus_{i_1 + \cdots + i_p = r} \mathcal{O}(p) \otimes_{\Sigma_p} (\tilde{A}_{i_1} \otimes \cdots \otimes \tilde{A}_{i_p}) \otimes (\Lambda_m^{i_1} \otimes \cdots \otimes \Lambda_m^{i_p}).\end{aligned} \tag{6}$$

Observe from (5) that, as we have already done in (6), we can identify $A_m^{\otimes p}$ with

$$\bigoplus_{0\le r\le mp}\bigoplus_{i_1+\cdots+i_p=r}(\tilde{A}_{i_1}\otimes\cdots\otimes\tilde{A}_{i_p})\otimes(\Lambda_m^{i_1}\otimes\cdots\otimes\Lambda_m^{i_p})\simeq\bigoplus_{I\in\wp(m)^{\times p}}\tilde{A}_I.$$

In this last expression we have written $I := (I_1,\ldots,I_p)\in\wp(m)^{\times p}$ for $\varphi_{I_1}\otimes\cdots\otimes\varphi_{I_p}$. Here $\varphi_J := \varphi_{j_1}\wedge\cdots\wedge\varphi_{j_s}$ whenever $J=\{j_1<\cdots<j_s\}\subseteq[m-1]$. $\wp(m)^{\times p}$ is the p-th cartesian power of the powerset of $\{0,\ldots,m-1\}$.

For any two modules $A := \bigoplus_{I\in\wp(m)^{\times p}} A_I$ and $B := \bigoplus_{I\in\wp(m)^{\times p}} B_I$, indexed by the same set $\wp(m)^{\times p}$, we take

$$A\hat{\otimes}B := \bigoplus_{I\in\wp(m)^{\times p}} A_I\otimes_{\Sigma_p} B_I.$$

Denoting

$$\tilde{\mathcal{O}}_m(p) := \bigoplus_{0\le r\le mp}\bigoplus_{i_1+\cdots+i_p=r}\mathcal{O}(p)\otimes_{\Sigma_p}(\Lambda_m^{i_1}\otimes\cdots\otimes\Lambda_m^{i_p})\simeq\bigoplus_{I\in\wp(m)^{\times p}}\tilde{\mathcal{O}}_I(p) \tag{7}$$

Eq. (6) can also be written as

$$\mathbf{F}_m A\simeq\bigoplus_{p\ge 0}\tilde{\mathcal{O}}_m(p)\hat{\otimes}\tilde{A}_m^{\otimes p}. \tag{8}$$

We can now look at the operad structure inherited by $\tilde{\mathcal{O}}$. Take $p\ge 0$ and $l_1+\cdots+l_p=l$. We have to define the operad action $\gamma:\tilde{\mathcal{O}}_m(p)\otimes\tilde{\mathcal{O}}_m(l_1)\otimes\cdots\otimes\tilde{\mathcal{O}}_m(l_p)\to\tilde{\mathcal{O}}_m(l)$. We do so in the following way,

$$\gamma(\tilde{\mathcal{O}}_{I_0}(p)\otimes\tilde{\mathcal{O}}_{I_1}(l_1)\otimes\cdots\otimes\tilde{\mathcal{O}}_{I_p}(l_p))$$
$$:=\begin{cases}\gamma(\mathcal{O}(p)\otimes\mathcal{O}(l_1)\otimes\cdots\otimes\mathcal{O}(l_p))\otimes J & \text{if } I_0=\left(\bigcup_{j=1}^{q_1}I_{1j},\ldots,\bigcup_{j=1}^{q_p}I_{pj}\right)\\ 0 & \text{in any other case}\end{cases}$$

where we have written $I_i=(I_{i1},\ldots,I_{iq_i})\in\wp(m)^{\times l_i}$, for $i=1,\ldots,p$, and $J := (I_{11},\ldots,I_{1q_1},\ldots,I_{p1},\ldots,I_{pq_p})\in\wp(m)^{\times l}$. This formula together with multilinearity completely determines γ.

Since $m\mapsto\mathbf{F}_m A$ is actually a simplicial abelian group, we can apply the normalized chain complex functor to pass from a simplicial $\mathbb{Z}$-module to a chain complex. This is the functor assigning to a simplicial abelian group the chain complex made of those elements from the latter which lie in the kernel of all the face operators except the last one, with the differential induced by the last face operator. The passage from simplicial $\mathbb{Z}$-modules to $\mathbb{Z}$-complexes carries an operad of simplicial $\mathbb{Z}$-modules to an operad of $\mathbb{Z}$-complexes [14, pp. 36].

We know that a basic element $o\otimes(a_1\otimes\cdots\otimes a_p)\otimes(x_1\otimes\cdots\otimes x_p)\in\mathcal{O}(p)\otimes(\tilde{A}_{i_1}\otimes\cdots\otimes\tilde{A}_{i_p})\otimes(\Lambda_m^{i_1}\otimes\cdots\otimes\Lambda_m^{i_p})$ is in $\bigcap_{i=0}^{m-1}\ker d_i$ if and only if $\sharp x_1\cup\cdots\cup\sharp x_p=\{0,\ldots,m-1\}$. We can write

$$\mathbf{N}_m\mathbf{F}A\simeq\bigoplus_{p\ge 0}\bigoplus_{0\le r\le mp}\bigoplus_{i_1+\cdots+i_p=r}\mathcal{O}(p)\otimes(\tilde{A}_{i_1}\otimes\cdots\otimes\tilde{A}_{i_p})\otimes\mathbf{N}(\Lambda_m^{i_1}\otimes\cdots\otimes\Lambda_m^{i_p})$$

where $\mathbf{N}(\Lambda_m^{i_1}\otimes\cdots\otimes\Lambda_m^{i_p})$ is a shorthand for the $\mathbb{Z}$-submodule of $(\Lambda_m^{i_1}\otimes\cdots\otimes\Lambda_m^{i_p})$ generated by the elements $(x_1\otimes\cdots\otimes x_p)$ with $\sharp x_1\cup\cdots\cup\sharp x_p=\{0,\ldots,m-1\}$. If we associate $(x_1\otimes\cdots\otimes x_p)\in(\Lambda_m^{i_1}\otimes\cdots\otimes\Lambda_m^{i_p})$ with $(\sharp x_1,\ldots,\sharp x_p)\in\wp(m)^{\times p}$, we can put a basis of $\mathbf{N}(\Lambda_m^*)^{\otimes p}$ in a one-to-one correspondence with the subset $\wp_m(m)^{\times p}$ of $\wp(m)^{\times p}$ whose elements $I := (I_1,\ldots,I_p)$ are such that $\bigcup_{i=1}^p I_i=\{0,\ldots,m-1\}$. We shall use $\wp_m(m)^{\times p}$ as index set, and write

$$\mathbf{N}\left(\bigoplus_{0\le r\le mp}\bigoplus_{i_1+\cdots+i_p=r}\mathcal{O}(p)\otimes_{\Sigma_p}(\Lambda_m^{i_1}\otimes\cdots\otimes\Lambda_m^{i_p})\right)=\bigoplus_{I\in\wp_m(m)^{\times p}}\mathcal{O}[I]. \tag{9}$$

Here we make use of the identification between $\mathbf{N}(\Lambda_m^*)^{\otimes p}$ and $\wp_m(m)^{\times p}$ and write $\mathcal{O}[I]$ for $\mathcal{O}(p)\otimes_{\Sigma_n}(x_1\otimes\cdots\otimes x_p)$, with $I=(I_1,\ldots,I_p)$ and $I_q=\sharp x_q$.

Remark 4. Suppose that $o \otimes (a_1 \otimes x_1) \otimes \cdots \otimes (a_p \otimes x_p) \in \mathcal{O}(p) \otimes (\tilde{A}_{i_1} \otimes \Lambda_m^{i_1}) \otimes \cdots \otimes (\tilde{A}_{i_p} \otimes \Lambda_m^{i_p}) \simeq \mathcal{O}(p) \otimes (\tilde{A}_{i_1} \otimes \cdots \otimes \tilde{A}_{i_p}) \otimes (\Lambda_m^{i_1} \otimes \cdots \otimes \Lambda_m^{i_p})$. Then,

$$\begin{aligned} d_m(o \otimes (a_1 \otimes x_1) \otimes \cdots \otimes (a_p \otimes x_p)) &= o \otimes d_m(a_1 \otimes x_1) \otimes \cdots \otimes d_m(a_p \otimes x_p) \\ &= o \otimes (a_1 \otimes d_m(x_1) + da_1 \otimes \delta_{d_m}(x_1)) \otimes \cdots \otimes (a_p \otimes d_m(x_p) + da_p \otimes \delta_{d_m}(x_p)) \\ &= o \otimes (a_1 \otimes d_m(x_1)) \otimes \cdots \otimes (a_p \otimes d_m(x_p)) + \cdots + o \otimes (da_1 \otimes \delta_{d_m}(x_1)) \otimes \cdots \otimes (da_p \otimes \delta_{d_m}(x_p)). \end{aligned}$$

This corresponds to the sum of all the elements of the form

$$o \otimes (\varepsilon'_1(x_1) \otimes \cdots \otimes \varepsilon'_p(x_p)) \otimes (\varepsilon''_1(a_1) \otimes \cdots \otimes \varepsilon''_p(a_p))$$

where ε'_i is either d_m or δ_{d_m} and ε''_i is either 1 or d, in accordance with the value of ε'_i. Since $\sharp x_1 \cup \cdots \cup \sharp x_p = \{0, \ldots, m-1\}$, the term with all $\varepsilon'_i = d_m$ is zero.

4. Peiffer pairings in $\mathcal{O}$-simplicial modules

Let us suppose that for all $m > 1$,

$$A_m = \mathrm{Ideal}_{\mathcal{O}}\left(\sum_{i=0}^{m-1} s_i(A_{m-1})\right). \tag{10}$$

Here $\mathrm{Ideal}_{\mathcal{O}}(X)$ means the $\mathcal{O}$-ideal generated by X. Since the degeneracies are injective $\mathcal{O}$-morphisms, we have that $s_i\mathcal{O}[I_1, \ldots, I_p] \simeq \mathcal{O}[s_i^* I_1, \ldots, s_i^* I_p]$, where $s_i^* I := s_i(\varphi_I)$. Hence, condition (10) can also be stated as

$$\begin{aligned} \tilde{A}_m &= \sum_{\bigcup I=[m-1]} \gamma(\mathcal{O}[I] \otimes \tilde{A}_{i_1} \otimes \cdots \otimes \tilde{A}_{i_{|I|}}) \\ &= \sum_{\bigcup I=[m-1]} \gamma(\mathcal{O}_{|I|} \otimes (\tilde{A}_{i_1} \otimes I_1) \otimes \cdots \otimes (\tilde{A}_{i_{|I|}} \otimes I_{|I|})) \end{aligned} \tag{11}$$

or equivalently, as $\gamma : \sum_{\bigcup I=[m-1]} \gamma(\mathcal{O} \otimes (\tilde{A}_{i_1} \otimes I_1) \otimes \cdots \otimes (\tilde{A}_{i_{|I|}} \otimes I_{|I|})) \to \tilde{A}_m$ being surjective. We have written I for $(I_1, \ldots, I_p)$. If γ is not surjective, we can still consider the $\mathcal{O}$-ideal $\tilde{D}_* = \operatorname{im}\gamma$, as in [1,3].

Let us write K_I for the ideal $\bigcap_{i \in I} \ker d_i \subseteq A_m$. Observe that $(\tilde{A}_{i_j} \otimes I_j) \subseteq K_{I_j}$.

Lemma 5. *Suppose (11) holds for the simplicial $\mathcal{O}$-algebra A_*. Then, for each $m \geq 0$, the following inclusion also holds,*

$$d\tilde{A}_m \subseteq \sum_{\bigcup I=[m-2]} \gamma(\mathcal{O}_{|I|} \otimes K_{I_1} \otimes \cdots \otimes K_{I_{|I|}}).$$

Proof. Apply d_m to both sides of (11). We get that

$$\begin{aligned} d_m(\tilde{A}_m) &= d_m \sum_{\bigcup I=[m-1]} \gamma(\mathcal{O}_{|I|} \otimes (\tilde{A}_{i_1} \otimes I_1) \otimes \cdots \otimes (\tilde{A}_{i_{|I|}} \otimes I_{|I|})) \\ &= \sum_{\bigcup I=[m-1]} \gamma(\mathcal{O}_{|I|} \otimes d_m(\tilde{A}_{i_1} \otimes I_1) \otimes \cdots \otimes d_m(\tilde{A}_{i_{|I|}} \otimes I_{|I|})). \end{aligned} \tag{12}$$

The simplicial identity $d_k d_m = d_{m-1} d_k$ if $k < m$, implies $d_m(\tilde{A}_{i_j} \otimes I_j) \subseteq K_{I_j}$. Hence, from (12) follows that

$$d_m(\tilde{A}_m) \subseteq \sum_{\bigcup I=[m-2]} \gamma(\mathcal{O}_{|I|} \otimes K_{I_1} \otimes \cdots \otimes K_{I_{|I|}}). \quad \square$$

The other inclusion was shown in [3] for the case $\mathcal{O} = Comm$ and in [1] for the case $\mathcal{O} = Lie$. Essentially the same proof can be adapted for a general $\mathcal{O}$. We do this in the following

Proposition 6. *Let A_* be a simplicial $\mathcal{O}$-algebra. Let $I = (I_1, \ldots, I_p)$, with nonempty I_i's and $\bigcup_{i=1}^{p} I_i = [m-1]$. Then,*

$$\gamma(\mathcal{O}_p \otimes K_{I_1} \otimes \cdots \otimes K_{I_p}) \subseteq d\tilde{A}_m.$$

To prove this proposition, we shall use the following lemma, whose proof can be found in [6,3] or [1].

Lemma 7. *For a simplicial algebra A_*, if $0 \le r \le n$ let $\overline{\mathbf{N}A}_n^{(r)} = \bigcap_{i \neq r} \ker d_i$. Then the map $\psi : \mathbf{N}A_n \to \overline{\mathbf{N}A}_n^{(r)}$, given by*

$$\psi(a) := a - \sum_{k=0}^{n-r-1} s_{r+k} d_n a$$

is a bijection.

In consequence, $d_n(A_n) = d_r(\overline{\mathbf{N}A}_n^{(r)})$ for each n, r.

Proof (*of* Proposition 6). Let $o \in \mathcal{O}_p$ and $x_i \in K_{|I_i|}$, $i = 1, \ldots, p$. Suppose that $\bigcup_i I_i = [m-1]$ and $I_i \neq \emptyset$ for all i. Let r be the smallest nonzero element not in $\bigcap_k I_k$, and i_0 the first i such that $r \in I_i$. Take $x = \gamma(o \otimes s_r x_1 \otimes \cdots \otimes s_{r-1} x_{i_0} \otimes \cdots \otimes s_r x_p)$. One obtains that $d_j x = 0$, for $j \neq r$ and $\gamma(o \otimes x_1 \otimes \cdots \otimes x_{i_0} \otimes \cdots \otimes x_p) = d_r x \in d_r(\overline{\mathbf{N}A}_n^{(r)}) = d_n(A_n)$. Thus,

$$\gamma(\mathcal{O}_p \otimes K_{I_1} \otimes \cdots \otimes K_{I_p}) \subseteq d_n \tilde{A}_n. \quad \square$$

We can join both Lemma 5 and Proposition 6 in

Theorem 8. *Let A be a simplicial $\mathcal{O}$-algebra such that $A_m = \mathrm{Ideal}_{\mathcal{O}}(\sum_{i=0}^{m-1} s_i(A_{m-1}))$ for every $m > 1$. Then*

$$d\tilde{A}_m = \sum_{\bigcup I = [m-1]} \gamma(\mathcal{O}_{|I|} \otimes K_{I_1} \otimes \cdots \otimes K_{I_{|I|}}).$$

In fact, the previous theorem is still true if we replace $\tilde{A}_m$ by $\tilde{D}_m$.

Remark 9. Suppose that $\mathcal{O} = Comm$, the operad whose algebras are the commutative ones, and $I = (I_1, \ldots, I_p)$ with $\bigcup_{i=1}^{p} I_i = [m-1]$. Recall that $\mathcal{O}_m \simeq \mathbb{Z}$ for all m. Composing and using the surjectivity of the product, we get that

$$\sum_{\bigcup I = [m-1]} \gamma(\mathbb{Z} \otimes K_{I_1} \otimes \cdots \otimes K_{I_p}) = \sum_{\bigcup I = [m-1]} \gamma(\mathbb{Z} \otimes K_{I'} \otimes K_{I''})$$

with $I' = \bigcup_{i=1}^{q} I_i$, $I'' = \bigcup_{i=q}^{p} I_i$, $1 < q < p$. Hence

$$\sum_{\bigcup I = [m-1]} \gamma(\mathbb{Z} \otimes K_{I_1} \otimes \cdots \otimes K_{I_p}) = \sum_{I' \cup I'' = [m-1]} K_{I'} K_{I''}.$$

Compare this last expression with that of [3]. Something similar happens with any quadratic operad; i.e., the expression for $d\tilde{A}_m$ takes the form

$$\sum_{I' \cup I'' = [m-1]} \gamma(\mathcal{O}_2 \otimes K_{I'} \otimes K_{I''})$$

with $K_{I'} = \bigcap_{i \in I'} \ker d_i$ and $K_{I''} = \bigcap_{i \in I''} \ker d_i$.

5. Simplicial groups

The use of constructions involving near-rings in the study of simplicial groups is not new [5], even if the use of near-rings we do in this section seems not to appear before in the literature.

We begin by recalling some definitions from [17].

Definition 10. A *right distributive near-ring* is a set N together with two binary operations "+" and " · " such that,

a. $(N, +, 0)$ is a (not necessarily abelian) group,
b. $(N, \cdot)$ is a semigroup,
c. $(l + m) \cdot n = l \cdot n + m \cdot n$, for all $l, m, n \in N$.

N is said to be *zero-symmetric* if $n \cdot 0 = 0$ for all n in N. N is *unital* if the semigroup $(N, \cdot)$ has a neutral element 1. An element $d \in N$ is said to be *distributive* if for any $m, n \in N$, $d \cdot (m + n) = d \cdot m + d \cdot n$. A distributive unital zero-symmetric near-ring is a ring.

Write N_d for $\{d \in N \mid d \text{ is distributive}\}$. $(N_d, \cdot)$ is a sub-semigroup of N. We say that N is *distributively generated* if $(N, +, 0)$ is generated by some subset $D \subseteq N_d$.

Let $X_m := \{\varphi_0, \ldots, \varphi_{m-1}\}$. Put $(F_m, \cdot, 1)$ for the free monoid generated by X_m. Following [17, Definition 6.20], we take $(N_m, +, 0)$ for the free group on F_m, and endow it with the product

$$\left(\sum_i \sigma_i \varphi_i\right) \cdot \left(\sum_j \sigma_j \varphi_j\right) := \sum_i \sigma_i \left(\sum_j \sigma_j \varphi_i \cdot \varphi_j\right)$$

where the σ_i's are integers. We call $(N_m, +, \cdot, 0, 1)$ the *free distributively generated unital near-ring generated by the set* X_m. Since $(\sum_i \sigma_i\varphi_i) \cdot 0 = (\sum_i \sigma_i\varphi_i) \cdot (1 - 1) = \sum_i \sigma_i\varphi_i \cdot (1 - 1) = \sum_i \sigma_i((\varphi_i \cdot 1) - (\varphi_i \cdot 1)) = 0$, N_m is also zero-symmetric. Let us write $(\Lambda(m), +, \cdot, 0, 1)$ for the free distributively generated unital zero-symmetric near-ring generated by the set X_m, and which also satisfies the relations

$$\varphi_i \cdot \varphi_j = -\varphi_j \cdot \varphi_i.$$

We can endow $\Lambda(*)$ with a simplicial near-ring structure by formulas (1) and (2), where + is now the not necessarily abelian group operation in $\Lambda(*)$. Note that this group is graded by the length of the words in the φ's.

By forgetting the operation $\cdot$ in $\Lambda(*)$, we get a simplicial group $(\Lambda(*), +, 0)$, also written $\Lambda(*)$. In what follows we simply write $\varphi_i\varphi_j$ for $\varphi_i \cdot \varphi_j$.

Definition 11. Let (G_*, d) be a connected chain complex of (not necessarily abelian) groups, and A_* a family of graded groups. We write $G \boxtimes A$ for the sequence of groups $n \mapsto \coprod_{i\geq 0}(G_i \otimes A_n^i)$; where $G_i \otimes A_n^i$ is the group generated by the symbols $g \otimes a$ with $g \in G_i$, $a \in A_n^i$ and subject to the relations

$$g \otimes 0 \approx 1 \otimes a \approx 1 \otimes 0$$
$$g \otimes (a + b) \approx (g \otimes a)(g \otimes b)$$

and $\coprod$ is the coproduct in the category of groups.

We can endow $G \boxtimes \Lambda(*)$ with a simplicial group structure. We associate to each face or degeneracy $\alpha \in \Delta$, the unique group morphism $\daleth(\alpha) : G \boxtimes \Lambda(n) \to G \boxtimes \Lambda(m)$, $m = n \pm 1$ given by the formula

$$\daleth(\alpha)\,(g \otimes x) := (dg \otimes \bar{\alpha}(x))(g \otimes \alpha(x)),$$

where $\alpha(\varphi_{i_1} \ldots \varphi_{i_p}) = \alpha(\varphi_{i_1}) \ldots \alpha(\varphi_{i_p})$, and $\bar{\alpha} = 0$, except for $\bar{\alpha} = \bar{d_n}$, where we take $\bar{d_n}(\varphi_{i_1} \ldots \varphi_{i_p}) = d_n\varphi_{i_1} \ldots d_n\varphi_{i_{p-1}}$, if $i_p = n - 1$, and 0 otherwise. We have assumed that $i_1 < i_2 < \cdots < i_p$.

Take $g \otimes x \in G \boxtimes \Lambda(*)$, and write simply s_i and d_i for $\daleth(s_i)$ and $\daleth(d_i)$. Write for a monomial $x \in \Lambda(n)$, $\sharp x := \{i_1, \ldots, i_r\}$ iff $x \in \mathbb{Z}\,\varphi_{i_1} \ldots \varphi_{i_r}$. For $0 \leq i \leq n$, we have that $s_i(g \otimes x) = g \otimes s_i(x)$, for $0 \leq i \leq n - 1$, $d_i(g \otimes x) = g \otimes d_i(x)$, and $d_n(g \otimes x) = (dg \otimes \bar{d_n}(x))(g \otimes d_n(x))$. Well definiteness of this maps follows from the fact that $\Lambda(*)$ is a distributively generated near-ring, and d a homomorphism. For $i \neq n$, we can immediately say that

$g \otimes x \in \ker d_i$ if $i \in \sharp x$, just as it is the case for abelian groups. Observe that not all elements of $\ker d_i$ has to be of this form; for example, $[g \otimes \varphi_i, h \otimes \varphi_j]$, is not of this form, although it is in $\ker d_i$ (and in $\ker d_j$).

Furthermore, we have that,

Proposition 12. *$G \boxtimes \Lambda(*)$ is a simplicial group.*

Proof. We just have to verify that the maps $s_i : G \boxtimes \Lambda(n) \to G \boxtimes \Lambda(n+1)$ and $d_i : G \boxtimes \Lambda(n) \to G \boxtimes \Lambda(n-1)$, $0 \le i \le n$, which are group morphisms, satisfy the simplicial identities. This is a straightforward computation. □

Let us now recall from [6] the following notation. Let $I = \{i_1, \ldots, i_r\}$, with $0 \le i_1 < \cdots < i_r \le m$, or $I = \emptyset$. We shall write $s_I := s_{i_r} \ldots s_{i_1}$ or 1, respectively, and call them the canonical inclusions. Similarly, we define $d_I := d_{i_1} \ldots d_{i_r}$ and $d_\emptyset := 1$.

Since the group is not necessarily commutative, we write $\widetilde{\sum}_I s_I(x_I)$ for the ordered sum of the $s_I(x_I)$, according to the inverse lexicographical order.

A central result for us is,

Proposition 13 (*[6, 3.1.10]*). *Let G be a simplicial group, and* **N***G its Moore complex. For every $n > 1$ each element $x \in G_n$ admits a unique expression of the form*

$$x = \widetilde{\sum}_{I \in \wp(n)} s_I(x_I) \quad \text{for } x_I \in \mathbf{N}_{|I|}G$$

such that the map

$$\prod_{I \in \wp(n)} \mathbf{N}_{|I|}G \to G_n$$

given by $(x_I)_{I \in \wp(n)} \mapsto \widetilde{\sum}_{I \in \wp(n)} s_I(x_I)$ is a bijection.

Since $\mathbf{N}G \boxtimes \Lambda(*)$, as defined in Definition 11, is itself a simplicial group, the results just stated apply to it. Observe that $g \in \mathbf{N}_n G$ if and only if $g \otimes \varphi_0 \ldots \varphi_{n-1} \in \mathbf{N}_n(\mathbf{N}G \boxtimes \Lambda(*))$, although not all the elements of $\mathbf{N}_n(\mathbf{N}G \boxtimes \Lambda(*))$ are of this form. Take $s_I(g_I) := s_I(g \otimes \varphi_0 \ldots \varphi_{n-1}) = g \otimes s_I(\varphi_0 \ldots \varphi_{n-1}) = \widetilde{\sum}_i g \otimes \varphi^{(i)}$. The i-th term of this ordered sum is in $\bigcap_{j \in \sharp \varphi^{(i)}} \ker d_i \in G \boxtimes \Lambda(n + |I|)$. On the other hand, any φ_J, with $J \subseteq [m-1]$, can be written as $\varphi_J = \widetilde{\sum}_i \varepsilon_i s_{j_i}(\varphi_{J_i})$ for some $0 \le j_i \le m-1$, $J_i \subseteq [m-2]$ and $\varepsilon_i = \pm 1$. Indeed, the following proposition holds,

Proposition 14. *Any φ_J, with $J \subseteq [m-1]$, can be written as $\varphi_J = \widetilde{\sum}_{I \in \mathcal{I}} \varepsilon_I s_I(\varphi_{[r]})$, with $r = |J| - 1$, $\varepsilon_i = \pm 1$ and $\mathcal{I} \subseteq \wp(m-1)$. The order in $\mathcal{I}$ shall become clear after the proof of this proposition.*

Proof. We do induction on $t = m - r$. For $t = 0$ there is nothing to do, so suppose $r = m - 1$. Then $\varphi_J = \varphi_0 \ldots \hat{\varphi}_j \ldots \varphi_{m-1}$, where the hat over φ_j points out that $j \notin J$. We shall now show how we can write φ_J as $\widetilde{\sum}_{i \in \mathcal{I}} \varepsilon_i s_i(\varphi_{[m-2]})$. In this case we can identify $\mathcal{I}$ with a subset of $[m-2]$, with certain order. The construction of $\mathcal{I}$ is based on the following observations. If $j = m-1$ then $\varphi_J = s_{m-1}\varphi_{[m-2]}$, if $j = m-1-1$ then $\varphi_J = -s_{m-1}\varphi_{[m-2]} + s_{m-1-1}\varphi_{[m-2]}$, and in general, if $j = m-1-q$ then $\varphi_J = -\varphi_{J'} + s_{m-1-q}\varphi_{[m-2]}$, where $J' = [m-1] - \{m-1-q+1\}$. In this way we get an effective recursive procedure to find the appropriate $\mathcal{I}$. Observe that this procedure does not affect those φ_k with $k < j$.

Now, take $t > 1$ and suppose that $j_1 < \cdots < j_t$ are all the elements in the complement of J. Suppose that we have already built up $\mathcal{I}'$ such that $\varphi_{J'} = \widetilde{\sum}_{I \in \mathcal{I}'} \varepsilon_I s_I(\varphi_{[r-1]})$, with $J' = [m-2] - \{j_1, \ldots, j_{t-1}\}$. Now we do apply the procedure firstly described to get

$$\varphi_J = \widetilde{\sum}_{i \in \mathcal{I}} \varepsilon_i s_i(\varphi_{J'}).$$

We can do so, since this procedure is blind to the $j \in J$ with $j < j_t$. Finally, we get

$$\varphi_J = \widetilde{\sum}_{i \in \mathcal{I}} \varepsilon_i s_i \left(\widetilde{\sum}_{I \in \mathcal{I}'} \varepsilon_I s_I(\varphi_{[r-1]}) \right) = \widetilde{\sum}_{i \in \mathcal{I}} \widetilde{\sum}_{I \in \mathcal{I}'} \varepsilon_i \varepsilon_I s_i s_I(\varphi_{[r-1]})$$
$$= \widetilde{\sum}_{I \in \mathcal{I}''} \varepsilon_I s_I(\varphi_{[r-1]}). \quad \square$$

Remark 15. The following observations, although trivial, may be useful.

Let $I, J \subseteq [m]$, and G a simplicial group. Suppose that $x, y \in G_m$ are such that $x \in \bigcap_{i\in I} \ker d_i$ and $y \in \bigcap_{j\in J} \ker d_j$. Then $[x, y] \in \bigcap_{i\in I\cup J} \ker d_i$.

The second observation is that,

$$s_I(\varphi_{[r-1]}) = \sum_{l\in s_I^{-1}(0)\times\cdots\times s_I^{-1}(r-1)} \varphi_{\sharp l}$$

with $s_I^{-1}(0) \times \cdots \times s_I^{-1}(r-1)$ lexicographically ordered, and $\sharp l := \{l_0, \ldots, l_{r-1}\}$, whenever $l = (l_0, \ldots, l_{r-1})$.

Remark 16 (*From [15]*). Let $x \in \mathbf{N}_n G$ and $y \in G_{n-1}$. Take $\theta_y(x) := s_{n-1}(y)xs_{n-1}(y^{-1}) : \mathbf{N}_n G \to G_n$. Since $d_i\theta_y(x) = 1$ for $0 \le i \le n-1$, $\theta_y(x) \in \mathbf{N}_n G$. Furthermore, $d_n\theta_y(x) = yd_n(x)y^{-1}$, and in consequence, $yd(x)y^{-1} \in d(\mathbf{N}_n G)$. Hence, $d(\mathbf{N}_n G)$ is a normal subgroup of G_{n-1}.

We are now ready to relate the construction in [6] with ours. We construct a morphism of groups $\Phi : \mathbf{N}G \boxtimes \Lambda \to G$. We do it degreewise. The map $\Phi_0 : \mathbf{N}_0 G \to G_0$ is simply the identity. Let us denote by $\mathbb{Z}\varphi_I$ the subgroup generated by φ_I. The restriction of Φ_m to $\mathbf{N}_m G \otimes \mathbb{Z}\varphi_{[m-1]}$ is the obvious isomorphism with $\mathbf{N}_m G \subseteq G_m$. On the other hand, for $J \subset [m-1]$, we have seen in Proposition 14 that $\varphi_J = \widetilde{\sum}_{I\in\mathcal{I}} \varepsilon_I s_I(\varphi_{[r]})$; then we define

$$\Phi_m(g \otimes \varphi_J) := \widetilde{\sum}_{I\in\mathcal{I}} \varepsilon_I s_I(g).$$

Since Φ_m is defined on each $\mathbf{N}G \boxtimes \mathbb{Z}\varphi_I$, Definition 11 and the universal property of the coproduct, allow us to extend it in a unique way to all of $\mathbf{N}G \boxtimes \Lambda(m)$.

Lemma 17. *The homomorphism Φ_m defined above is onto.*

Proof. Immediate from Proposition 13. □

In fact, we have that

Proposition 18. *The map $\Phi : \mathbf{N}G \boxtimes \Lambda \to G$, defined above, is a surjective morphism of simplicial groups.*

Proof. The same definition of Φ guarantees that it commutes with the degeneracies. So we must just verify it also commutes with the faces; that is to say, that for $0 \le i \le m$,

$$d_i \Phi_m = \Phi_{m-1} d_i. \tag{13}$$

Since the elements of the form $g \otimes s_I(\varphi_{[r-1]})$, with $g \in N_r G$ generate $G \boxtimes \Lambda(m)$, it will suffice to see that (13) holds when evaluating on these elements. Suppose $i \ne m$. Then,

$$d_i \Phi_m(g \otimes s_I(\varphi_{[r-1]})) = d_i s_I(g).$$

On the other hand,

$$\Phi_{m-1} d_i(g \otimes s_I(\varphi_{[r-1]})) = \Phi_{m-1}(g \otimes d_i s_I(\varphi_{[r-1]})) = d_i s_I(g).$$

Hence they agree.

Suppose now that $i = m$. On the one hand, we have that

$$d_m \Phi_m(g \otimes s_I(\varphi_{[r-1]})) = d_m s_I(g) = s_I s_{r-1}(dg) s_I(g)$$

(see for example [6, pp. 123] or compute it). On the other hand,

$$\begin{aligned}\Phi_{m-1} d_m(g \otimes s_I(\varphi_{[r-1]})) &= \Phi_{m-1}(dg \otimes \bar{s}_I(\varphi_{[r-1]}))\, \Phi_{m-1}(g \otimes d_m s_I(\varphi_{[r-1]})) \\ &= s_I s_{r-1}(dg) s_I(g).\end{aligned}$$

This finishes the proof. □

Remark 19. Let us consider the construction of Definition 11, for the case $A = \Lambda$. We would like to get back the construction of Section 2 in the abelian case, even though there is nothing like a "distributivity on the left" for $\otimes$ in Definition 11. This situation can be amended by asking for new identities involving elements of the form $gh \otimes a$; at least when $A = \Lambda$. The problem with this approach is that we did not find a small nice set of such identities implying them all.

In the rest of this remark, we use the notation of Definition 11. It holds, in each $G \boxtimes \Lambda(n)$, that $gh \otimes \varphi_{[n-1]} = (g\otimes\varphi_{[n-1]})(h\otimes\varphi_{[n-1]})$. Once we know this identity to hold, we have a procedure to express $gh\otimes\varphi_I$ when $I < [n-1]$ by using Proposition 14. We shall illustrate this by an example.

Let $g, h \in G_1$, and consider $g \otimes \varphi_i, h \otimes \varphi_i$, with $i = 0, 1$, in $G \boxtimes \Lambda(2)$. We want to calculate $gh \otimes \varphi_0$ and $gh \otimes \varphi_1$. First, observe that in $G \boxtimes \Lambda(1)$ we have $gh \otimes \varphi_0 = (g \otimes \varphi_0)(h \otimes \varphi_0)$. Then, we also have in $G \boxtimes \Lambda(2)$,

$$gh \otimes \varphi_0 = s_1(gh \otimes \varphi_0) = s_1((g \otimes \varphi_0)(h \otimes \varphi_0)) = (g \otimes \varphi_0)(h \otimes \varphi_0).$$

On the other hand,

$$s_0(gh \otimes \varphi_0) = gh \otimes (\varphi_0 + \varphi_1) = (gh \otimes \varphi_0)(gh \otimes \varphi_1) = (h \otimes \varphi_0)(g \otimes \varphi_0)(gh \otimes \varphi_1)$$

and

$$s_0((g \otimes \varphi_0)(h \otimes \varphi_0)) = (g \otimes \varphi_0 + \varphi_1)(h \otimes \varphi_0 + \varphi_1) = (g \otimes \varphi_0)(g \otimes \varphi_1)(h \otimes \varphi_0)(h \otimes \varphi_1).$$

Comparing the last expressions we deduce that

$$\begin{aligned}(gh \otimes \varphi_1) &= (h \otimes -\varphi_0)(g \otimes -\varphi_0)(g \otimes \varphi_0)(g \otimes \varphi_1)(h \otimes \varphi_0)(h \otimes \varphi_1)\\ &= (h \otimes -\varphi_0)(g \otimes \varphi_1)(h \otimes \varphi_0)(h \otimes \varphi_1)\\ &= (g \otimes \varphi_1)^{(h\otimes\varphi_0)} \quad (h \otimes \varphi_1).\end{aligned}$$

Unfortunately, although relations for $n > 2$ can be found in essentially the same way, they are much more complicated than those just obtained for $n \leq 2$. Despite this fact, all relations reduce to "left distributivity" up to commutators.

6. Peiffer pairings in simplicial groups

It was shown in [15, Prop. 2.3.7] (see also [16]) that,

Lemma 20. *Let G be a simplicial group. If $n \geq 2$ and $I, J \subseteq [n-1]$ with $I \cup J = [n-1]$, we have that,*

$$\left[\bigcap_{i\in I} \ker d_i, \bigcap_{j\in J} \ker d_j\right] \subseteq d(\mathbf{N}_n G).$$

We refer the interested reader to [15] for a proof of this lemma. We will be concerned in this section in proving the following

Lemma 21. *Let G be a simplicial group. Let D_n be the normal subgroup of G_n generated by the degenerate elements. If $G_n = D_n$ for $n \geq 2$, then we have that*

$$d(\mathbf{N}_n G) \subseteq \prod_{I\cup J=[n-1]} \left[\bigcap_{i\in I} \ker d_i, \bigcap_{j\in J} \ker d_j\right].$$

In fact, more can be said. If we call $N_n = \mathbf{N}_n G \cap D_n$, it holds that

$$d(N_n) \subseteq \prod_{I\cup J=[n-1]} \left[\bigcap_{i\in I} \ker d_i, \bigcap_{j\in J} \ker d_j\right].$$

Compare with [16,15].

Before proving this lemma, we shall make a couple of remarks that make more clear the relationship between $\mathbf{N}G \boxtimes \Lambda$ and G.

Proposition 22. *Let G be the simplicial group $\mathbf{N}H \boxtimes \Lambda$ for some $H \in Grp^{\Delta^{op}}$. We have the equality $\mathbf{N}_n G = \mathcal{N}_n \, \mathrm{C}_n$, where $\mathcal{N}_n$ is the normal subgroup of G_n generated by $\mathbf{N}_n H \otimes \mathbb{Z}\varphi_{[n-1]}$, $\mathrm{C}_n = \tilde{\mathrm{C}}_n \cap \mathbf{N}_n G$ and $\tilde{\mathrm{C}}_n$ is the normal subgroup of G_n generated by $[\mathbf{N}_{|I|}H \otimes \varphi_I, \mathbf{N}_{|J|}H \otimes \varphi_J]$ with $I, J \subsetneq [n-1]$.*

Proof. Let us take $x \in \mathbf{N}_n G$. Each element of G_n is a product of the form $x = x_1 \cdots x_r$, with $x_i = g_i \otimes \varphi_{I_i}$ and $I_i \subseteq [n-1]$. Since $\mathcal{N}_n$ is normal in $\mathbf{N}_n G$, $\mathbf{N}_n G/\mathcal{N}_n$ is a group. Then $\bar{x} = \bar{x}'_1 \cdots \bar{x}'_t$ with each $\bar{x}'_i \in \mathbf{N}_n G/\mathcal{N}_n$, the image of some x_i not in $\mathcal{N}_n$. For any $0 \le j \le n-1$, we have that $d_j(\bar{x}'_1 \cdots \bar{x}'_t) = 1$ and as $\bar{x}'_i \notin \mathcal{N}_n$, there exists $0 \le k \le n-1$ such that $d_k(\bar{x}'_i) \neq 1$.

Take i such that $d_i(\bar{x}'_1) \neq 1$, and call y_i the elements of $\{\bar{x}'_1, \dots, \bar{x}'_t\}$ not in the kernel of d_i. This set is not void because we have, for example, $y_1 = \bar{x}'_1$. Modulo commutators, we have that $\bar{x} = y_1 \cdots y_q$. Since $d_i(\bar{x}) = 1$, we deduce that $y_1 \cdots y_q = 1$, and hence, $\bar{x}_1 \cdots \bar{x}_{i_q} = 1$ modulo commutators. Then $\bar{x} \in \mathrm{C}_n$. □

Proposition 23. *Using notation from Proposition 22, we have that $\mathcal{N}_n \cap D_n = 1$.*

Proof. Write G_n as $G_n = \coprod_{I \subseteq [n-1]} \mathbf{N}_{|I|}H \otimes \mathbb{Z}\varphi_I = (\mathbf{N}_n H \otimes \mathbb{Z}\varphi_{[n-1]}) \amalg \coprod_{I \subsetneq [n-1]} \mathbf{N}_{|I|}H \otimes \mathbb{Z}\varphi_I$. The proposition follows from the freeness of the coproduct and the fact that $D_n = \coprod_{I \subsetneq [n-1]} \mathbf{N}_{|I|}H \otimes \mathbb{Z}\varphi_I$. □

Remark 24. Observe that the condition $G_n = D_n$ may be written as a condition on $\mathbf{N}G \boxtimes \Lambda$. Indeed, $G_n = D_n$ if and only if for every $x \in \mathcal{N}_n$ there exists $y \in \mathrm{C}_n$ such that $\Phi_n(x) = \Phi_n(y)$, where Φ is the morphism of Proposition 18.

Proof (*of* Lemma 21). Suppose that $g \in \mathbf{N}_n G$. Then $g = \Phi_n(g \otimes \varphi_{[n-1]})$. By Remark 24, there is an $x \in \mathrm{C}_n$ such that $\Phi_n(x) = \Phi_n(g \otimes \varphi_{[n-1]}) = g$. Since Φ is a morphism of simplicial groups we have that $d_n(g) = d_n(\Phi_n(x)) = \Phi_{n-1}(d_n(x))$. Since $x \in \mathrm{C}_n$, $x = x_1 \cdots x_p$ with $x_i = [y_i, z_i]$ for $1 \le i \le p$, where $y_i \in K_{I_i}$, $z_i \in K_{J_i}$, $I_i \cup J_i = [n-1]$ and $I_i, J_i \neq [n-1]$. Then

$$\begin{aligned} d_n(g) &= \Phi_{n-1}(d_n x) = \Phi_{n-1}(d_n x_1) \cdots \Phi_{n-1}(d_n x_p) \\ &= \Phi_{n-1}(d_n x[y_1, z_1]) \cdots \Phi_{n-1}(d_n[y_p, z_p]) \\ &= [\Phi_{n-1}(d_n y_1), \Phi_{n-1}(d_n z_1)] \cdots [\Phi_{n-1}(d_n y_p), \Phi_{n-1}(d_n z_p)]. \end{aligned}$$

Since $d_j d_n = d_{n-1} d_j$ if $j < n$, we conclude that $\Phi_{n-1}(d_n y_i) \in K_{I_i}$ and $\Phi_{n-1}(d_n z_i) \in K_{J_i}$ for every i. Hence $d_n(g) \in \prod_{I \cup J = [n-1]} [K_I, K_J]$. □

We can collect previous results in the following

Theorem 25. *Let G be a simplicial group with Moore complex $\mathbf{N}G$ in which $G_n = D_n$, is the normal subgroup of G_n generated by the degenerate elements in dimension n, then*

$$d(\mathbf{N}_n G) = \prod_{I,J} \left[\bigcap_{i \in I} \ker d_i, \bigcap_{j \in J} \ker d_j \right]$$

for $I, J \subseteq [n-1]$ with $I \cup J = [n-1]$.

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