Alfvén waves and wings in Hall magnetohydrodynamics

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[1] The problem of a conducting body moving in a magnetized plasma when the electronic pressure and Hall terms in Ohm's law cannot be neglected is analyzed in the magnetohydrodynamic approximation. Since Alfvén wings are closely related to Alfvén waves, the influence of these terms in the propagation of Alfvénic perturbations of large amplitude is studied. Instead of linearizing the magnetohydrodynamic equations and searching monochromatic waves, the conditions that the group velocity be parallel to the background magnetic induction field, in the reference system in which the plasma is locally at rest, that the perturbation be incompressible, that the perturbations in velocity and the magnetic induction field be related, and that a magnitude connected to the pressure remain constant are imposed. It is shown that large-amplitude Alfvén waves can propagate in homogeneous plasmas if a "polarization condition" on the current density is fulfilled. The value of their group velocity is different from the value that it takes when simple Ohm's law is used. On the other hand, the methodology of stream functions is used for the analysis of Alfvén wings. Their existence, when the Hall term in Ohm's law is relevant, is proved, and the relations among the plasma pressure, induction magnetic field, velocity, and electric current density in the wing are found. The present results can be applied, as an approximation, to spacecraft or space tethers moving in a circular orbit if one can consider that the density and the magnetic induction field do not change as the source is orbiting and if the influence of partial ionization can be neglected. *INDEX TERMS:* 0654 Electromagnetics: Plasmas; 7827 Space Plasma Physics: Kinetic and MHD theory; 7855 Space Plasma Physics: Spacecraft sheaths, wakes, charging; 7839 Space Plasma Physics: Nonlinear phenomena; *KEYWORDS:* magnetohydrodynamics, Hall, Alfvén waves, Alfvén wings, space plasmas

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1. Introduction

[2] Alfvén waves have been detected in a variety of space plasmas like the solar wind [*Denskat and Neubauer*, 1982] and the Earth's magnetosphere [*Drell et al.*, 1965; *Mallinckrodt and Carlson*, 1978]. The density and magnetic induction field in some of these systems are such that the electronic pressure and Hall terms in Ohm's law cannot be neglected [*Ovenden et al.*, 1983; *Wolf-Gladrow et al.*, 1987]. The main characteristic of Alfvén waves in the magnetohydrodynamic approximation (MHD) and in uniform plasmas, when the Ohm's law reduces to $\mathbf{E} + (\mathbf{V} \times \mathbf{B})/c = 0$, is that they propagate without distortion with a group velocity parallel to the background magnetic induction field. Moreover, Alfvén waves are incompressible perturbations, there exists a relation between the velocity and magnetic induction field disturbances, and the total pressure (plasma plus magnetic) is constant [*Priest*, 1982]. Another interesting characteristic of Alfvén waves is that they can build up structures in the plasma, called Alfvén wings. These are produced when a conducting source moves uniformly in a magnetized plasma. From the beginning of the space age, Alfvén wings were a center of interest for space plasma physics researchers in the quest for understanding interactive plasma-source systems. They have analyzed the cases of the Io-Jupiter system [*Goertz*, 1980; *Acuña et al.*, 1981; *Wright*, 1987; *Wolf-Gladrow et al.*, 1987; *Hastings et al.*, 1988; *Wright and Schwartz*, 1990], the Europa and Callisto-Jupiter system [*Neubauer*, 1999], satellites moving through the Earth's ionosphere [*Drell et al.*, 1965; *Dobrowolny and Veltri*, 1986], and tethered probes [*Sanmartín and Estes*, 1997]. The Alfvén wings problem, supposing simple Ohm's law, was studied by *Neubauer* [1980] using a nonlinear analytic model, by *McKenzie* [1991] using Green functions, and for nonuniform plasmas by *Sallago and Platzeck* [2000, 2002] using the methodology of stream functions. The aim of this paper is to analyze the influence of electronic pressure and Hall terms on the construction of Alfvén wings in a perfectly conducting plasma in the magnetohydrodynamic approximation. Since Alfvén wings are closely related to Alfvén waves, we start by searching the existence, in this case, of Alfvénic perturbations. Alfvén waves with a Hall term have been studied by other authors [*Mattei*, 1969; *Ovenden et al.*,

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1983; *Woodward and McKenzie*, 1994a, 1994b; *Pokhotelov et al.*, 1996], but they linearize the magnetohydrodynamic equations or impose particular dependencies for the perturbations. In a similar way as it is done by *Sallago and Platzeck* [2000], instead of linearizing the magnetohydrodynamic equations and searching monochromatic waves, we impose the conditions that the perturbation propagate without distortion with a group velocity parallel to the background magnetic induction field, in the reference system in which the plasma is locally at rest, the perturbation be incompressible, there exist a relation between the velocity and induction magnetic field perturbations, and a magnitude called total generalized pressure remain constant in the disturbed region. It is proved in section 2 that large-amplitude Alfvén waves can propagate, when the electronic pressure and Hall terms are taken into account, if a condition on the spatial dependence of the current density is fulfilled. This condition, called the "polarization condition," relates the current density and its curl. In the linearized case for small perturbations and monochromatic waves, this condition means that the perturbation on magnetic induction field must be circularly polarized [*Mattei*, 1969; *Pokhotelov et al.*, 1996]. Owing to this condition it is not possible to impose the adiabaticity condition because the system would result overdetermined; as a result, the plasma pressure is not constant in the perturbed region. It is a known fact that linearized incompressible waves with Hall term are dispersive. Our result is discussed in connection with this fact finding that there is not any contradiction: one can construct wave packets with monochromatic waves in such a way that their group velocity be independent of the wave vector direction.

[3] For the analysis of Alfvén wings we consider in section 3 that the problem of a source moving with constant velocity in a uniformly magnetized plasma is a stationary one in the source rest frame. Dividing the space in two regions, one containing the source and one containing the wing, an invariant direction can be defined in the latter region. This direction coincides with the wing's axis. Stationary problems in MHD when an ignorable coordinate is present can be treated with a methodology based on stream functions [*Tsinganos*, 1982; *Agim and Tataronis*, 1985; *Palumbo and Platzeck*, 1998]. The stream function methodology when the electronic pressure and Hall terms in Ohm's law cannot be neglected was developed by *Palumbo* [1993]. In the present paper we apply this methodology when analyzing the Alfvén wings. We find the relations among the different fields in the wing. These results can be applied to spacecraft or space tethers moving in a circular orbit, if one can consider that the density and the magnetic induction field do not change as the source is orbiting and if the influence of partial ionization can be neglected.

[4] Finally, we want to remark that other perturbations produced by a conducting source moving uniformly in a magnetized plasma cannot build up wings, in the sense given above, or may not be studied in the magnetohydrodynamic approximation.

2. Alfvén Waves in Hall Magnetohydrodynamics (HMHD)

[5] If Hall term in Ohm's law cannot be neglected, the magnetohydrodynamic equations system is often called HMHD [*Turner*, 1986]. The generalized Ohm's law for an electrically neutral fully ionized plasma results in [*Boyd and Sanderson*, 1969; *Rossi and Olbert*, 1970]

$$\frac{\mathbf{J}}{\sigma} = \mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B} - \frac{1}{en^+c}\mathbf{J} \times \mathbf{B} + \frac{1}{en^+}\nabla p_e, \tag{1}$$

where e is the proton charge, n^+ is the proton density number, and p_e is the electronic pressure. The Hall term is relevant if $\Omega_e \tau_{ei} \gg 1$, where Ω_e is the electron girofrequency, and τ_{ei} the electron-ion collision time [*Priest*, 1982].

[6] The HMHD equations for a perfectly conducting plasma, in the absence of gravitational and viscous forces, are

$$\frac{\partial \rho}{\partial t} + (\mathbf{V} \cdot \nabla)\rho + \rho \nabla \cdot \mathbf{V} = 0, \tag{2}$$

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}\right] = -\nabla p + \frac{\mathbf{J}}{c} \times \mathbf{B}, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\left(\mathbf{V} - \frac{m^+}{e\rho}\mathbf{J}\right) \times \mathbf{B} + \frac{m^+ c}{e\rho}\nabla p_e\right], \tag{5}$$

the energy and state equations.

[7] Let us consider a perfectly conducting plasma with uniform and constant density ρ_0, plasma pressure p_0, velocity $\mathbf{V}_0$, and magnetic induction field $\mathbf{B}_0$. We search solutions of nonlinearized HMHD equations under the Alfvénic conditions imposed to perturbations:

[8] 1. The perturbations ρ_1, p_1, $\mathbf{V}_1$, $\mathbf{B}_1$ are functions of $(\mathbf{r} - \mathbf{V}_A'^H t)$, where $\mathbf{V}_A'^H$ is the group velocity

$$\mathbf{V}_A'^H = \mathbf{V}_0 - a\mathbf{B}_0, \tag{6}$$

where a is a constant to be determined in order to show the influence of the Hall term. $\mathbf{V}_A'^H$ is parallel to the background magnetic induction field in the reference system in which the plasma is locally at rest.

[9] 2. The disturbance is incompressible:

$$\nabla \cdot \mathbf{V}_1 = 0. \tag{7}$$

[10] 3. There exists a relation between the perturbations on velocity and magnetic induction fields:

$$\mathbf{V}_1 = a\mathbf{B}_1 + \boldsymbol{\alpha}, \tag{8}$$

where $\boldsymbol{\alpha}$ must vanish if the Hall term has not influence or if the disturbance is null.

[11] Under these conditions, from the continuity equation (2), the perturbation on density is null:

$$\rho_1 = 0. \tag{9}$$

The induction equation (5) takes the form

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\left(\mathbf{V} - \frac{\epsilon}{c\rho_0}\mathbf{J}\right) \times \mathbf{B}\right], \tag{10}$$

where

$$\epsilon = \frac{m^+ c}{e}. \tag{11}$$

The last term in equation (5) is zero, because the background density is uniform and the perturbation on density is null. Notice that for Alfvénic perturbations the electronic pressure does not modify the induction equation, whereas Hall's effect appears explicitly; the terms including ϵ will denote its influence in what follows.

[12] Replacing the relation between the perturbations on velocity and magnetic induction field (condition 3), the induction equation (10) results in

$$\nabla \times \left[\left(\boldsymbol{\alpha} - \frac{\epsilon}{c\rho_0} \mathbf{J} \right) \times \mathbf{B} \right] = 0; \tag{12}$$

the only value of $\boldsymbol{\alpha}$ satisfying equation (12) that is independent of the background magnetic induction field $\mathbf{B}_0$ is

$$\boldsymbol{\alpha} = \frac{\epsilon}{c\rho_0} \mathbf{J}. \tag{13}$$

Moreover, since the perturbation is incompressible, from equations (8) and (13) the equation $\nabla \cdot \mathbf{B} = 0$ is fulfilled.

[13] Finally, after replacing equations (6), (8), and (13) in the equation of motion, one gets

$$\rho_0 \left[\left(\frac{\epsilon}{c\rho_0} \mathbf{J}_1 + a\mathbf{B} \right) \cdot \nabla \right] \left(\frac{\epsilon}{c\rho_0} \mathbf{J}_1 + a\mathbf{B} \right) = -\nabla p + \frac{\mathbf{J}_1}{c} \times \mathbf{B} \tag{14}$$

or, using vectorial identities,

$$\rho_0 \left[\nabla \times \left(\frac{\epsilon}{c\rho_0} \mathbf{J}_1 + a\mathbf{B} \right) \right] \times \left(\frac{\epsilon}{c\rho_0} \mathbf{J}_1 + a\mathbf{B} \right) = -\nabla \left(p + \frac{\rho_0}{2} \left| \frac{\epsilon}{c\rho_0} \mathbf{J}_1 + a\mathbf{B} \right|^2 \right) + \frac{\mathbf{J}_1}{c} \times \mathbf{B}. \tag{15}$$

[14] After taking the curl of equation (15), one gets

$$\nabla \times \left\{ \left[\left(a^2 4\pi - \frac{1}{\rho_0} \right) \mathbf{J}_1 + \frac{\epsilon a}{\rho_0} (\nabla \times \mathbf{J}_1) \right] \times \left(\frac{\epsilon}{ca\rho_0} \mathbf{J}_1 + \mathbf{B} \right) \right\} = 0; \tag{16}$$

the only possible solution for $\mathbf{J}_1$ satisfying equation (16) that is independent of the background magnetic induction field $\mathbf{B}_0$ is

$$\left(a^2 4\pi - \frac{1}{\rho_0} \right) \mathbf{J}_1 + \frac{\epsilon a}{\rho_0} (\nabla \times \mathbf{J}_1) = 0. \tag{17}$$

It means that the electric current density and its curl are parallel:

$$\nabla \times \mathbf{J}_1 = b\, \mathbf{J}_1. \tag{18}$$

This is the "polarization condition." After replacing it in equation (17), one gets the following relation between the constants a and b:

$$a = -\frac{b\epsilon}{8\pi\rho_0} \mp \sqrt{\frac{1}{4\pi\rho_0} + \left(\frac{b\epsilon}{8\pi\rho_0} \right)^2}. \tag{19}$$

Then, from equation (6), the group velocity results in

$$\mathbf{V}_A'^H = \mathbf{V}_0 + \frac{\epsilon b\, \mathbf{B}_0}{8\pi\rho_0} \pm \frac{\mathbf{B}_0}{|\mathbf{B}_0|} \sqrt{\frac{|\mathbf{B}_0|^2}{4\pi\rho_0} + \left(\frac{\epsilon b |\mathbf{B}_0|}{8\pi\rho_0} \right)^2}. \tag{20}$$

[15] Returning to the motion equation (15), after replacing the "polarization condition" and the value of a, it results that there is a magnitude P^*, called the generalized total pressure, that remains uniform and constant in the perturbed region:

$$P^* = p + \frac{\rho_0 a^2}{2} \left| \mathbf{B} + \frac{\epsilon}{ca\rho_0} \mathbf{J}_1 \right|^2. \tag{21}$$

A relation between the vorticity and the current density,

$$\mathbf{w}_1 = \frac{\mathbf{J}_1}{ca\rho_0}, \tag{22}$$

is obtained immediately by taking the curl on equation (8) and using equation (18); a similar relation also exists for Alfvén waves in MHD. Furthermore, due to the fact that the plasma pressure must be always positive, the amplitude of the perturbation cannot be arbitrary.

[16] Notice that for finite wave packets the "polarization condition" implies a similar relation for $\mathbf{B}_1$; however, it is not a force free solution since there is a nonnull Lorentz's force $\mathbf{J}_1 \times \mathbf{B}_0/c$. The meaning of the "polarization condition" for the current density is analyzed in the linearized limit when the perturbation is developed in monochromatic waves. If the perturbation on magnetic induction field can be written

$$\mathbf{B}_1 = \mathbf{b}_1 \; \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)], \tag{23}$$

the current density fulfills the condition given by equation (18) if

$$\mathbf{b}_1 = \frac{ib}{|\mathbf{k}|^2} \mathbf{k} \times \mathbf{b}_1. \tag{24}$$

Thus defining two versors $\breve{\mathbf{e}}_1$ and $\breve{\mathbf{e}}_2$ perpendicular to $\mathbf{k}$, the phase between the two components of the perturbation on the magnetic induction field must be $\pi/2$:

$$\mathbf{B}_1 = \frac{b_1}{\sqrt{2}} (\breve{\mathbf{e}}_1 \pm i\breve{\mathbf{e}}_2) \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)], \tag{25}$$

being, in this case $b = \pm|\mathbf{k}|$. The fact that linearized Alfvén waves in HMHD are circularly polarized was pointed out by *Mattei* [1969]. Moreover, the dispersion relation obtained, if $\mathbf{V}_0$ is null and for right polarization, is

$$\omega = \frac{\epsilon |\mathbf{k}| (\mathbf{k} \cdot \mathbf{B}_0)}{8\pi\rho_0} \pm \sqrt{\frac{(\mathbf{k} \cdot \mathbf{B}_0)^2}{4\pi\rho_0} + \left(\frac{\epsilon |\mathbf{k}| (\mathbf{k} \cdot \mathbf{B}_0)}{8\pi\rho_0} \right)^2}. \tag{26}$$

In the limit for short wavelengths ($k \gg \omega_{pi}/c$, where ω_{pi} is the ion plasma frequency) one obtains the whistler mode. Its dispersion relation is $\omega = \epsilon|\mathbf{k}|(\mathbf{k} \cdot \mathbf{B}_0)/4\pi\rho_0$ [*Goldston and Rutherford*, 1995; *Biskamp*, 2000].

[17] We want to remark that, in spite of being dispersive, one can build up wave packets with linearized Alfvén waves in such a way that the group velocity results independent of the direction of **k**, this can be attained by considering only wave vectors with the same modulus. Under these conditions, the group velocity results in

$$\mathbf{V}_g = \nabla_{\mathbf{k}}\, \omega\big|_{|\mathbf{k}|=\text{const.}} = \frac{\epsilon|\mathbf{k}|\mathbf{B}_0}{8\pi\rho_0} \pm \frac{\mathbf{B}_0}{|\mathbf{B}_0|}\sqrt{\frac{|\mathbf{B}_0|^2}{4\pi\rho_0} + \left(\frac{\epsilon|\mathbf{k}|\,|\mathbf{B}_0|}{8\pi\rho_0}\right)^2}.$$

This group velocity coincides with $\mathbf{V}_A'^H$, equation (20), for $\mathbf{V}_0 = 0$ and $b = |\mathbf{k}|$. As a result these wave packets propagate without distortion in the direction of the background magnetic induction field. Notice that by considering only monochromatic waves with the same value of $|\mathbf{k}|$, each one satisfying equation (18) with $b = |\mathbf{k}|$, the superposition of these waves also satisfies equation (18).

[18] Summarizing, one of the effects of Hall term on the propagation of Alfvén waves is to modify the group velocity; it depends on the proportionality constant between the current density and its curl. We want to remark that if the influence of Hall term is important, the direction of the group velocity for Alfvén waves (compare equation (20)) can be too different from the direction that results if the Hall term is not taken into account. On the other hand, $\mathbf{V}_A'^H$ tends to $\mathbf{V}_0 \pm \mathbf{B}_0/\sqrt{4\pi\rho_0}$ if Hall term is negligible. The relation between the perturbations on velocity and induction magnetic field also changes, including the perturbation on current density, (compare equation (8)). Finally, if Hall term is not taken into account, the generalized total pressure P^* (compare equation (21)) tends to the total pressure P, the sum of the plasma pressure plus the magnetic pressure.

[19] It is interesting to remark that if the Hall term can be neglected, the Alfvén waves in HMHD tend to the circularly polarized Alfvén waves in MHD. The energy conservation law shows the existence of a heat flux when circularly polarized Alfvén waves propagate.

3. Alfvén Wings in HMHD

[20] Suppose now that in the plasma previously described there exists a conductor which is moving with constant velocity. It generates Alfvén waves that build up Alfvén wings. The wings are cylindrical regions which section depends on the source's shape, characterized by intense electric currents where the disturbed fields are different from zero. In the conductor's rest frame the problem is a stationary one. Let us consider the plus sign in the expression of $\mathbf{V}_A'^H$.

[21] Defining a separation surface in such a way that one can divide the space into two regions, one containing the source and one containing the wing, in the latter the direction of $\mathbf{V}_A'^H$ is invariant. Choosing orthogonal Cartesian coordinates (ξ, η, ζ) with ζ in the direction of $\mathbf{V}_A'^H$, ζ becomes an ignorable variable and

$$V_{A\xi}'^H = V_{A\eta}'^H = 0. \tag{27}$$

[22] In these conditions, we can define stream functions for all the magnitudes of null divergence. The methodology of stream functions for stationary problems with a symmetry in magnetohydrodynamics, when the electronic pressure and Hall terms are considered, was developed by *Palumbo* [1993]. There exists two independent stream functions in this case: ψ, the current function of **B**, often called the magnetic flux, and χ, the current function of $\rho\mathbf{V}$. These functions are defined in such a way that **B** and $\rho\mathbf{V}$ can be written

$$\mathbf{B} = \nabla\psi \times \breve{e}_\zeta + B_\zeta \breve{e}_\zeta \tag{28}$$

$$\rho\mathbf{V} = \nabla\chi \times \breve{e}_\zeta + \rho V_\zeta \breve{e}_\zeta. \tag{29}$$

[23] From the HMHD equations (equations (2), (4), and (10)), and taking the curl of equation (3), *Palumbo* [1993] obtained an equivalent system of equations written as relations among the jacobians of different physical magnitudes. In the particular case of cartesian or cylindrical coordinates and constant density, these equations can be integrated giving

$$\chi - \frac{\epsilon B_\zeta}{4\pi} = G_1(\psi), \tag{30}$$

$$\psi + \epsilon V_\zeta = G_2(\chi), \tag{31}$$

$$-V_\zeta + \frac{G_1' B_\zeta}{\rho_0} + \frac{\epsilon J_\zeta}{c\rho_0} = G_3(\psi), \tag{32}$$

$$-G_2' V_\zeta + \frac{B_\zeta}{\rho_0} + \frac{\epsilon w_\zeta}{\rho_0} = G_4(\chi). \tag{33}$$

Here G_1, G_2, G_3, and G_4 are arbitrary functions. Different functional relations correspond to different physical situations [*Palumbo*, 1993]. In what follows we will analyze the case of Alfvén wings.

[24] Let us suppose that the background fields are uniform, $\mathbf{B}_0 = (B_{0\xi}, B_{0\eta}, B_{0\zeta})$, $\mathbf{V}_0 = (V_{0\xi}, V_{0\eta}, V_{0\zeta})$, the ξ and η components can be derived from the stream functions $\psi_0(\xi, \eta)$ and $\chi_0(\xi, \eta)$, according to equations (28) and (29). Taking the derivatives with respect to η and ξ of equations (30) and (31), and considering equations (28) and (29), one obtains

$$\rho_0 V_\xi - \frac{\epsilon J_\xi}{c} = G_1'(\psi) B_\xi, \tag{34}$$

$$\rho_0 V_\eta - \frac{\epsilon J_\eta}{c} = G_1'(\psi) B_\eta, \tag{35}$$

$$B_\xi + \epsilon w_\xi = G_2'(\chi)\rho_0 V_\xi, \tag{36}$$

$$B_\eta + \epsilon w_\eta = G_2'(\chi)\rho_0 V_\eta. \tag{37}$$

Since the functional dependence for $G_1(\psi)$, $G_2(\chi)$, $G_3(\psi)$ and $G_4(\chi)$ must be the same for perturbed and unperturbed fields, first are analyzed the equations (34)–(37) for the background fields:

$$\rho_0 V_{0\xi} = G_1'(\psi_0) B_{0\xi}, \tag{38}$$

$$\rho_0 V_{0\eta} = G_1'(\psi_0) B_{0\eta}, \tag{39}$$

$$B_{0\xi} = G_2'(\chi_0)\rho_0 V_{0\xi}, \tag{40}$$

$$B_{0\eta} = G_2'(\chi_0)\rho_0 V_{0\eta}. \tag{41}$$

By replacing the first equation in the third or the second in the fourth, it results in

$$G_1'(\psi_0) G_2'(\chi_0) = 1. \tag{42}$$

The invariant direction coincides with the direction of the group velocity, so the only nonnull component of the group velocity is the ζ component. After considering equation (6), from equations (38) or (39), one gets

$$G_1' = a\rho_0. \tag{43}$$

[25] Using these results, equations (34)–(37) are rewritten for the perturbations:

$$V_{1\xi} = aB_{1\xi} + \frac{\epsilon J_{1\xi}}{c\rho_0}, \tag{44}$$

$$V_{1\eta} = aB_{1\eta} + \frac{\epsilon J_{1\eta}}{c\rho_0}, \tag{45}$$

$$V_{1\xi} = aB_{1\xi} + a\epsilon w_{1\xi}, \tag{46}$$

$$V_{1\eta} = aB_{1\eta} + a\epsilon w_{1\eta}. \tag{47}$$

From these relations one obtains that

$$w_{1\xi} = \frac{J_{1\xi}}{ca\rho_0} \tag{48}$$

$$w_{1\eta} = \frac{J_{1\eta}}{ca\rho_0}. \tag{49}$$

[26] Let us now analyze equations (32) and (33); for the background fields one obtains

$$G_3(\psi_0) = -V_{0\zeta} + aB_{0\zeta} = -V_A'^H \tag{50}$$

$$G_4(\chi_0) = -\frac{V_{0\zeta}}{a\rho_0} + \frac{B_{0\zeta}}{\rho_0} = -\frac{V_A'^H}{a\rho_0}, \tag{51}$$

while for the perturbations one gets

$$V_{1\zeta} = aB_{1\zeta} + \frac{\epsilon J_{1\zeta}}{c\rho_0} \tag{52}$$

$$w_{1\zeta} = \frac{J_{1\zeta}}{ca\rho_0}. \tag{53}$$

Equations (44), (45), and (52) and equations (48), (49), and (53) can be written in vectorial form:

$$\mathbf{V}_1 = a\mathbf{B}_1 + \frac{\epsilon \mathbf{J}_1}{c\rho_0} \tag{54}$$

$$\mathbf{w}_1 = \frac{\mathbf{J}_1}{ca\rho_0}. \tag{55}$$

The first expression states a relation between the perturbations on velocity and magnetic induction field, and the second a relation between the vorticity and the current density; they coincide with the corresponding relations for Alfvén waves with Hall term (compare equations (8) and (22)). Taking the curl of the first of these equations and using the second, it is evident that $\nabla \times \mathbf{J}_1$ is proportional to $\mathbf{J}_1$, in agreement to what happens for waves (compare equation (18)). Owing to the fact that for Alfvén wings ζ is ignorable, from this proportionality, one obtains

$$J_{1\zeta} = \frac{bcB_{1\zeta}}{4\pi}. \tag{56}$$

Notice that, given b, all the perturbed magnitudes in the wing can be obtained from the value of $J_{1\zeta}$. This comes from the above relations and from the fact that

$$\nabla^2\psi_1 = -\frac{4\pi J_{1\zeta}}{c} \tag{57}$$

$$\nabla^2\chi_1 = -\rho_0 w_{1\zeta}. \tag{58}$$

However, $J_{1\zeta}$ cannot be arbitrarily taken, the proportionality of $\mathbf{J}_1$ and its curl gives the following equation:

$$\nabla^2 J_{1\zeta} + b^2 J_{1\zeta} = 0. \tag{59}$$

The constant b is related to the source's size. As the convective derivative of χ is zero the value of χ for a given plasma element is the same before and after entering the wing. As a consequence, the intensity of $J_{1\zeta}$ is bounded.

[27] The equation of motion remains to be analyzed. Using the stream functions, it can be written [*Palumbo*, 1993]

$$\rho\frac{\nabla V^2}{2} - \rho V_\zeta \nabla V_\zeta + w_\zeta \nabla\chi = -\nabla p - \frac{B_\zeta}{4\pi}\nabla B_\zeta + \frac{J_\zeta}{c}\nabla\psi. \tag{60}$$

After replacing the values of $w_{1\zeta}$, $J_{1\zeta}$, $\nabla\psi$, and $\nabla\chi$ for Alfvén wings it results in

$$\nabla\left[p + \frac{a^2\rho_0}{2}\left|\mathbf{B} + \frac{\epsilon}{ca\rho_0}\mathbf{J}_1\right|^2\right] = 0; \tag{61}$$

this means that the generalized total pressure P^* (compare equation (21)) remains constant in the wing. In order to the boundary conditions on the edge of the wing be fulfilled, a current density appears on the surface. This also occurs on the surface of the wave packets.

[28] Finally, since we have supposed a perfectly conducting plasma, the electric field can be determined from equation (1). After replacing the pressure from equation (21), if $p_e = p/2$, the electric field takes the form

$$\mathbf{E} = \frac{G_3}{c}\nabla\psi - \frac{\rho_0}{2}\nabla\left|a\mathbf{B}_0 + \frac{1}{4\pi a\rho_0}\mathbf{B}_1\right|^2;$$

the second term is due to the electronic pressure.

[29] In the remainder of this section we give a simple example, in order to show the construction strategy for Alfvén wings when Hall term is taken into account. We propose the following background fields:

$$\mathbf{B}_0 = \left(B_{0\xi}, 0, B_{0\zeta}\right) \qquad \mathbf{V}_0 = \left(V_{0\xi}, 0, V_{0\zeta}\right),$$

where $B_{0\zeta}$ and $V_{0\zeta}$ are positive constants, and $B_{0\xi}$ and $V_{0\xi}$ are taken in such a way that $V'^{H}_{A\xi} = 0$. For these fields the nonnull component of the group velocity $\mathbf{V}'^{H}_{A}$ is

$$\mathbf{V}'^{H}_{A\zeta} = V_{0\zeta} + \left[\frac{b\epsilon}{8\pi\rho_0} + \sqrt{\frac{1}{4\pi\rho_0} + \left(\frac{b\epsilon}{8\pi\rho_0}\right)^2}\,\right]B_{0\zeta}.$$

From the values of $B_{0\xi}$ and $V_{0\xi}$ it is possible to obtain the values of $\psi_0(\eta)$ and $\chi_0(\eta)$. Let us suppose that the wing is a circular cylinder and that the ζ component of the current density in the wing due to the perturbations, a solution of equation (59), can be written

$$J_{1\zeta} = a_0 \mathrm{J}_0(br) + a_1 \mathrm{J}_1(br)\cos(\varphi + \alpha_1),$$

where r, φ, ζ are cylindrical coordinates, and J_0 and J_1 are Bessel functions. If the perturbed region lies in $r < R$, in order that the electric current across the wing be null, bR must be one zero of J_1.

[30] From $J_{1\zeta}$, one can get the perturbation on magnetic flux ψ_1 that must satisfy equation (57). After imposing the condition $B_{1r}(R, \varphi) = 0$, one gets for ψ_1

$$\psi_1 = c_0 + \frac{4\pi a_0}{cb^2}\mathrm{J}_0(br) + \frac{4\pi a_1}{cb^2}\mathrm{J}_1(br)\cos(\varphi + \alpha_1).$$

The constants c_0, a_0, and a_1 are not arbitrary taken, they are bounded by the fact that the convective derivative of χ is zero.

[31] The components B_{1r}, and $B_{1\varphi}$ can be obtained from ψ_1:

$$B_{1r} = -\frac{4\pi a_1}{cb^2}\frac{\mathrm{J}_1(br)}{r}\mathrm{sen}(\varphi + \alpha_1)$$

$$B_{1\varphi} = \frac{4\pi a_0}{cb}\mathrm{J}_1(br) - \frac{4\pi a_1}{cb}\left(\mathrm{J}_0(br) - \frac{\mathrm{J}_1(br)}{br}\right)\cos(\varphi + \alpha_1).$$

Moreover $B_{1\zeta}$ gets its value from its relation with $J_{1\zeta}$ (compare equation (56)):

$$B_{1\zeta} = \frac{4\pi a_0}{cb}\mathrm{J}_0(br) + \frac{4\pi a_1}{cb}\mathrm{J}_1(br)\cos(\varphi + \alpha_1),$$

resulting then in

$$J_{1r} = -\frac{a_1}{b}\frac{\mathrm{J}_1(br)}{r}\mathrm{sen}(\varphi + \alpha_1)$$

$$J_{1\varphi} = a_0\mathrm{J}_1(br) - a_1\left(\mathrm{J}_0(br) - \frac{\mathrm{J}_1(br)}{br}\right)\cos(\varphi + \alpha_1).$$

From equation (54), one can get immediately $\mathbf{V}_1$.

[32] Finally, the plasma pressure in the wing can be obtained from the fact that the generalized total pressure P^* is constant in the wing, and that the total pressure P is continuous on the edge of the wing:

$$p = P^* - \frac{\rho_0 a^2}{2}\left|\mathbf{B}_0 + \frac{1}{4\pi\rho_0 a^2}\mathbf{B}_1\right|^2,$$

where P^* is given by

$$P^* = p_0 + \frac{|\mathbf{B}_0|^2}{8\pi} - \frac{|\mathbf{B}|^2}{8\pi}\bigg|_R + \frac{\rho_0 a^2}{2}\left|\mathbf{B}_0 + \frac{1}{4\pi\rho_0 a^2}\mathbf{B}_1\right|^2\bigg|_R.$$

4. Conclusions

[33] In this paper we have proved that if the electronic pressure and the Hall terms in Ohm's law cannot be neglected, Alfvén waves of large amplitude and Alfvén wings in homogeneous magnetized plasmas also exist in the magnetohydrodynamic approximation.

[34] It is found that the electronic pressure disappears from the induction equation for Alfvénic perturbations. The group velocity $\mathbf{V}'^{H}_{A}$ depends on the background plasma velocity, magnetic induction field, and density, and on a parameter b that enhances the importance of Hall term; if this term is negligible one recovers the MHD Alfvén velocity.

[35] In order to satisfy the full set of HMHD equations the electric current density and its curl must be proportional. This relation is called "polarization condition" since for monochromatic waves the perturbation on magnetic induction field must be circularly polarized. Owing to this condition, it is not possible to impose the adiabaticity condition because the system would result overdetermined. The relation between the perturbations in velocity and magnetic induction field has to be modified if Hall term cannot be neglected. Additionally, a magnitude $\mathcal{P}^*$ called generalized total pressure is uniform in the region perturbed by Alfvén waves.

[36] Taking advantage of the fact that one can define an ignorable coordinate, we apply the methodology of stream functions in HMHD when analyzing Alfvén wings. The Alfvén wing's axis coincides with the direction of the group velocity $\mathbf{V}'^{H}_{A}$. The "polarization condition" implies that the component of the current density due to perturbations in the direction of the wing axis, $J_{1\zeta}$, has to satisfy a differential equation. All the other physical magnitudes in the wing can be obtained from $J_{1\zeta}$. These results can be applied to tethered satellites moving in the Earth's ionosphere as an approximation, if the influence of partial ionization can be neglected.

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