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Generalized Borel transform technique in quantum mechanics ☐

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Abstract

We present the Generalized Borel Transform (GBT). This new approach allows one to obtain approximate solutions of Laplace/Mellin transform valid in both, perturbative and non-perturbative regimes. We compare the results provided by the GBT for a solvable model of quantum mechanics with those provided by standard techniques, as the conventional Borel sum, or its modified versions. We found that our approach is very efficient for obtaining both the low and the high energy behavior of the model.

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Recently, Penin and Pivovarov [1] presented a numerical analysis of renormalon techniques in quantum mechanics. They used a simple solvable model, namely a delta function scattering potential, to address the problem of resummation of perturbative series [2]. In fact, this quantum mechanics potential can be considered as a confining one, mimicking the long sought property of Quantum Chromodynamics (QCD). This is one of the many examples where the finite order perturbation theory predictions present uncertainties comparable with experimental errors [3]. In all these cases one is forced to take into account, in a way or another, non-perturbative contributions. In connection with these difficulties, it is certainly instructive to study exactly solvable models so that the efficiency and precision of the different proposals [1,4] can be quantitatively checked. This analysis can help us to define criteria for selecting the appropriate approach to be used in more realistic cases as QCD is.

In Ref. [1] the delta function model is tackled by means of standard Borel summation techniques combined with the renormalon approach [5]. On the other hand, we have recently introduced [6] a Generalized Borel Transform (GBT) that avoids the implementation of a perturbative expansion. This proposal was successfully applied [7] to obtain an analytic expression for the heavy quark–antiquark potential, valid for all distances.

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The purpose of this Letter is to discuss the use and advantages of the GBT in the case of quantum mechanics and in particular in connection with the model studied in Ref. [1]. We are able to obtain analytic expressions valid in all the range of the variables.

We start by briefly summarizing Ref. [1] in order to prepare the basic material of the delta potential that is further treated with our GBT approach.

The quantum mechanics model, analyzed first in the scattering region, is defined through the singular potential

$$V(r) = v\delta(r-a); \quad v, a = \text{const}, \tag{1}$$

and can be considered as a kind of confining (not completely) interaction. Usually, the dispersion relation related to this kind of singular potential is analyzed in analogy with dispersion relations in elementary particle theory [8]. Here, to simplify the presentation, we consider only the s-wave amplitude and the study of the wave function at the origin. To analyze the scattering of a wave packet, one considers an integral of the form

$$\Psi(Q) \equiv 1 + F(Q) = \int_0^\infty \psi(q) W(q, Q)\, dq, \tag{2}$$

where $W(q, Q) \equiv Q \exp(-Q/q)/q^2$ is a momentum distribution function normalized to one, and

$$\psi(q) = \left[1 + \frac{v}{2q}\big(1 - \exp(-2qa)\big)\right]^{-1} \tag{3}$$

is the exact solution for the scattering of a plane wave in the "Euclidean" region where the momentum $q > 0$ is obtained through a Wick rotation. This is possible whenever $m = 1/a > |v|$ and consequently there are no bound states present.

The perturbative solution in Eq. (2) suffers from the same difficulties present in perturbative QCD and it mimics some general features of renormalons.

We start the analysis with the perturbative treatment in order to make explicit the difficulties of the standard method. At high energies ($q \gg m$) one can write

$$\psi^{\text{as}}(q) = \sum_{n=0}^{\infty} \big(-\alpha(q)\big)^n, \tag{4}$$

where $\alpha(q) \equiv v/2q$ is the natural parameter of expansion in expression (3).

Using now the series (4) in (2) one obtains

$$\Psi^{\text{as}}(Q) = \sum_{n=0}^{\infty} n! \big(-\alpha(Q)\big)^n. \tag{5}$$

For $v > 0$, the theory is Borel summable [5] and the conventional Borel transform is

$$B(s) = \sum_{n=0}^{\infty} (-s)^n = \frac{1}{s+1} \to \Psi_{\text{S}}^{\text{as}}(Q) = \frac{1}{\alpha(q)} \int_0^\infty ds\, B(s) \exp\big(-s/\alpha(Q)\big). \tag{6}$$

Consequently, the approximate solution (5) results

$$\Psi_{\text{S}}^{\text{as}}(Q) = \frac{1}{\alpha(Q)} \exp\big(1/\alpha(Q)\big)\, \text{Ei}\big(1, 1/\alpha(Q)\big), \tag{7}$$

where [9]

$$\mathrm{Ei}(n,x) \equiv \int_1^\infty \frac{\exp(-tx)\,dt}{t^n}, \quad n = 1,2,\ldots,\ x > 0. \tag{8}$$

A numerical comparison of the approximate expression (7) with the exact solution (2), presented in Fig. 1, clearly shows that it is not a good approximation for intermediate values of $q \sim m$.

An alternative approach [10] uses a modified perturbation theory and provides results of better precision at very small momenta. This technique starts by choosing, instead of $\alpha(q)$, a modified expansion parameter of the form

$$\alpha_\mu(q) \equiv \frac{v}{2q+\mu}\left(1 + \frac{\mu - v}{2q+\mu}\right),$$

where $\mu = \mu(m, v)$ is fitted using experimental information. In this model, the exact solution $\Psi(Q)$ given in Eq. (2), plays the role of experimental data.

The accuracy of the approximation and the optimal value of μ clearly depend on the range where data have to be fitted. Unfortunately, the value that optimizes the result at very small Q, does not fulfill this requirement at very large Q as it is evident in Fig. 1.

Hence, the method and its modification work reasonably well in providing an approximation to the exact solution but each one in different regimes of Q. They are not able to reproduce simultaneously, with sufficient precision, the exact result in both IR and UV regimes. We have found that the Generalized Borel Transform [6,7] copes with that difficulty. This GBT approach was designed to obtain the approximate solution of any Laplace/Mellin transform in the wide range of the parameter involved.

In fact, when using our proposal, one is performing a whole class of transformations, depending on a parameter that we call λ. As it should be, the result does not explicitly depend on λ. Consequently, one can choose for this parameter the best adapted value in each particular problem under consideration. In practice, as one is performing an approximate calculation, one can get rid of λ by means of a suitable saddle-point like technique.

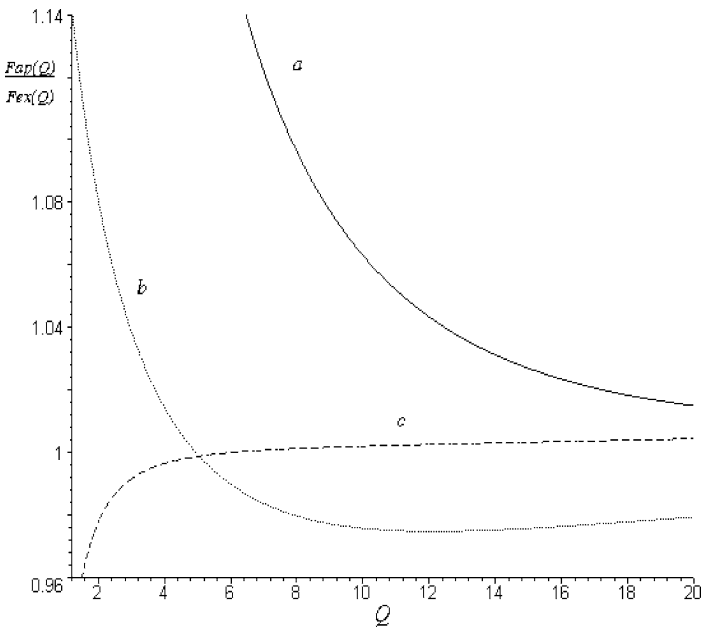

Fig. 1. Numerical comparation for Borel-summable solutions.

Let us briefly present our technology in connection with a Laplace–Mellin transform related to the quantum mechanical problem under analysis. In this case, we typically face a transformation like

$$S(g) = \int_0^\infty x^n H(x) \exp(-gx)\, dx, \quad g > 0, \tag{9}$$

where we have explicitly extracted a factor x^n ($n > 0$) from the function to be transformed, because, at it will become clear below, this operation improves the implementation of the saddle point method that is part of our proposal.

The Generalized Borel Transform (GBT) of S is defined as

$$B_\lambda(s) \equiv -\int_0^\infty \exp\bigl[s/\eta(g)\bigr]\left[\frac{1}{\lambda\eta(g)} + 1\right]^{-\lambda s} \frac{S(g)}{[\eta(g)]^2} \frac{\partial\eta(g)}{\partial g}\, dg, \quad \mathrm{Re}(s) < 0, \tag{10}$$

λ being any real positive non zero value and where we have defined

$$\frac{1}{\eta} \equiv \lambda\bigl[\exp(g/\lambda) - 1\bigr]. \tag{11}$$

This particular election of η allows one to define the function $u_\lambda(g)$ monotonically increasing in the interval $0 < u_\lambda(g) < \infty$, namely,

$$u_\lambda(g) \equiv \frac{1}{\eta} - \lambda \ln\left[\frac{1}{\lambda\eta} + 1\right]. \tag{12}$$

Consequently, $B_\lambda(s)$ results in

$$B_\lambda(s) = \int_0^\infty \exp(su) S\bigl[g_\lambda(u)\bigr]\bigl[1 + \lambda\eta(u)\bigr]\, du, \quad \mathrm{Re}(s) < 0.$$

The last expression can be written as a Laplace transform

$$B_\lambda(s) = \int_0^\infty \exp(su) L_\lambda\bigl[g(u)\bigr]\, du, \quad \mathrm{Re}(s) < 0,$$

of the function $L_\lambda[g(u)]$ there implicitly defined.

Replacing (9) and (11) in Eq. (10) we can easily test that the GBT is an analytic function on the negative Borel half-plane, such that its extension to the other half-plane also exists and is also analytic with a cut on the real positive axis. From this observation, $S(g)$ can be expressed in terms of the inverse Laplace transform integrated on the above mentioned cut [7]

$$S(g) = \frac{1}{\lambda\eta + 1} \int_0^\infty \exp\bigl[-su_\lambda(g)\bigr] \Delta B_\lambda(s)\, ds. \tag{13}$$

The main advantages of this proposal come from the fact that it allows one to perform the calculations in terms of the parameter λ. Moreover, as it will become clear below, this approach avoids the implementation of a perturbative expansion. Each value of the parameter λ defines a particular Borel Transform. This can be summarized by writing $S(g) = T_\lambda^{-1}[T_\lambda(S(g))]$, where $T_\lambda(S(g)) \equiv B_\lambda(s)$.

The discontinuity of the $B_\lambda(s)$ can be expressed as

$$\Delta B_\lambda(s) = 2\lambda \int_{-\infty}^{\infty} dw\, \exp\bigl[R_\lambda(w,s)\bigr],$$

where

$$R_\lambda(w,s) \equiv -\ln\bigl\{\Gamma\bigl[\lambda(s+x(w))\bigr]\bigr\} + \bigl\{\lambda\bigl[s+x(w)\bigr]-1\bigr\}\ln(\lambda s) - \lambda s + w + \ln\bigl[x^n H(x)\bigr]$$

with $x = \exp(w)$ and Γ representing the Euler gamma function. After the change of variables $s = \lambda\exp(t)$ one gets

$$S(g) = 2\lambda^2\bigl(1-\exp(-g/\lambda)\bigr)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\bigl[C_\lambda(w,t,g)\bigr]\,dw\,dt \tag{14}$$

with

$$C_\lambda(w,t,g) = -s(t)u_\lambda(g) + t + R_\lambda\bigl(w,s(t)\bigr). \tag{15}$$

In this way we can obtain the dominant contribution of the double integral using the steepest descent technique in the combined variables $[t,w]$. In so doing, one first computes the saddle point $t_o(g)$ and $w_o(g)$ in the limit $\lambda \gg 1$. In this case, the saddle point is

$$t_o(g) = \ln\left\{\frac{x_o^2(g)}{f[x_o(g)]}\right\}, \quad w_o(g) = \ln\bigl[x_o(g)\bigr], \tag{16}$$

where $x_o(g)$ is the real and positive solution of the implicit equation coming from the extremes of the function C in the asymptotic limit in λ, namely,

$$x_o^2 g^2 = f(x_o)\bigl[f(x_o)+1\bigr], \tag{17}$$

and

$$f(x_o) \equiv 1 + n + x_o\frac{d\ln[H(x_o)]}{dx_o}. \tag{18}$$

In the range of the parameters where $f[x_o(g)] \gg 1$, we have retained the first order in the expansion of C around the saddle point. In so doing, it remains

$$S^{\text{Approx}}(g) \simeq 4\pi\lambda^2\bigl[1-\exp(-g/\lambda)\bigr]\frac{\exp[C_\lambda(w_o,t_o,g)]}{\sqrt{D(t_o,w_o)}}, \tag{19}$$

where

$$D(t_o,w_o) \equiv \frac{\partial^2 C}{\partial w^2}\left|\frac{\partial^2 C}{\partial t^2}\right| - \left[\frac{\partial^2 C}{\partial w\,dt}\right]^2.$$

Then one checks the positivity condition [11], in particular, when the discriminant $D(t_o,w_o)$ of the second derivatives of C at this point is positive. We can now obtain the approximate expression for the starting amplitude $S(g)$ (9):

$$S^{\text{Approx}}(g) = \sqrt{2\pi}\,e^{-1/2}\left[\frac{f(x_o)+1}{D(x_o)}\right]^{1/2} x_o^{n+1} H(x_o)\exp\bigl[-f(x_o)\bigr], \tag{20}$$

where explicitly

$$D(x_o) = -x_o\frac{df(x_o)}{dx_o}\left[\frac{1}{2}+f(x_o)\right] + f(x_o)\bigl[1+f(x_o)\bigr] \tag{21}$$

with $x_o(g)$, and f defined above in Eqs. (17) and (18), respectively.

Notice that the expression (20) is valid for functions $H(x)$ that fulfill the following general conditions:

(1) The relation (17) must be biunivocal.
(2) $D(x_o)$ should be positive and $[x_o \frac{df(x_o)}{dx_o} - 2 f(x_o)]$ should be negative in x_o.
(3) $f(x_o) \gg 1$.

These conditions provide the range of values of the involved parameters where the approximate solution (20) is valid. Then, the calculation of the GBT simply consists in solving the implicit equation (17) to obtain the saddle point expression. In general, the functional complexity of H can add constraints on the range of the parameters.

We are now prepared to study the amplitude coming from the delta potential, namely, the wave packet

$$\Phi(Q) \equiv \frac{\Psi(Q)}{Q} = \frac{F(Q)+1}{Q} = \int_0^\infty \frac{\exp(-Qx)\,dx}{1+\frac{ux}{2}\left[1-\exp\left(-\frac{2}{xm}\right)\right]} \equiv \int_0^\infty H(x)\exp(-Qx)\,dx. \tag{22}$$

On the other hand, the GBT is able to deal better with

$$\Phi_n(Q) = \int_0^\infty x^n H(x)\exp(-Qx)\,dx \tag{23}$$

because the acuteness of the saddle point increases rapidly with n. In fact, in all the many examples we have worked out, the approximation to the exact result improves dramatically with increasing n. Notice that the relation between (22) and (23)

$$\Phi_n(Q) = (-)^n \frac{\partial^n}{\partial Q^n}\Phi(Q)$$

can be inverted obtaining

$$\Phi(Q) = (-)^n \underbrace{\int dQ \cdots \int dQ}_{n}\ \Phi_n(Q) + \sum_{p=0}^{n-1} c_p\, Q^p, \tag{24}$$

where the finite sum come from the indefinite integrations. Moreover, the expression (24) is valid for any value of n, in particular when $n \gg 1$. Consequently, the approximate solution reads

$$\Phi_{\mathrm{App}}(Q) \simeq (-)^n \underbrace{\int dQ \cdots \int dQ}_{n}\ \Phi_n^{\mathrm{App}}(Q) + \sum_{p=0}^{n-1} c_p\, Q^p \tag{25}$$

where, for $n \gg 1$

$$\Phi_n(Q) \simeq \Phi_n^{\mathrm{App}}(Q) = \sqrt{\frac{2\pi/e[f(x_o)+1]}{D(x_o)}}\,(x_o)^{n+1} H(x_o)\exp\left[-f(x_o)\right] \tag{26}$$

is the approximate solution provided by GBT, with the D and f functions defined above (18) and (21).

Notice that in Eq. (25) all the coefficients c_p vanish whenever the Laplace transform (22) fulfills the following asymptotic behavior

$$\lim_{Q\to\infty} \Phi(Q) = 0.$$

For n sufficiently large and $Q > 0$, the explicit expression of the saddle point becomes $x_o = (n+3/2)/Q$. Then, we can substitute this expression into (26), obtaining the following approximate solution for the expression (23)

$$\Phi_n^{\mathrm{App}}(Q) \simeq \frac{\sqrt{2\pi}\,(n+1)^{n/2}(2+n)^{n/2+1/2}\exp(-n-3/2)}{Q^{n+1}\{1+v\frac{n+3/2}{2Q}[1-\exp(-\frac{2Q}{(n+3/2)m})]\}} \simeq \frac{\Gamma(n+1)}{Q^{n+1}}\frac{1}{1+P(n,Q)}.$$

If $0 < Q < \infty$, then $0 < P(n,Q) < v/m < 1$ and we can expand to obtain

$$\Phi_n^{\mathrm{App}}(Q) \simeq \Gamma(n+1)\sum_{p=0}^{\infty}\frac{[-v\frac{n+3/2}{2}]^p}{Q^{p+n+1}}\sum_{k=0}^{p}\binom{p}{k}(-)^k\exp\left[-\frac{2kQ}{(n+3/2)m}\right]. \tag{27}$$

It is illustrative to separate the perturbative and non-perturbative contributions from this expression. To this end we analyze the first contribution, e.g., the term $k = 0$. This correspond to the asymptotic solution on Q. After, we add the terms with $k \neq 0$, corresponding to nonperturbative corrections.

For $k = 0$, by solving the n-integrations, we obtain

$$\Phi_o^{\mathrm{GBT}}(Q) \simeq \sum_{p=0}^{\infty}\frac{[-v\frac{n+3/2}{2}]^p}{Q^{p+1}}\frac{\Gamma(p+1)\Gamma(n+1)}{\Gamma(n+p+1)} \simeq \frac{1}{Q}\sum_{p=0}^{\infty}\frac{[-v/2]^p}{Q^p}\Gamma(p+1),$$

where in the last step, we have explicitly taken the limit $n \to \infty$. Then, from Eq. (22)

$$\Psi_o^{\mathrm{GBT}}(Q) \simeq \sum_{p=0}^{\infty}\big(-\alpha(Q)\big)^p\Gamma(p+1) = \Psi_{\mathrm{S}}^{\mathrm{as}}(Q) = \frac{1}{\alpha(Q)}\exp\big(1/\alpha(Q)\big)\,\mathrm{Ei}\big(1,1/\alpha(Q)\big).$$

Consequently, we have recovered the approximate solution (7) provided by the conventional Borel transform in the region where this one provides sensible results.

To determine the nonperturbative corrections, we must solve the n-integrals

$$I_{n+p+1}(a,Q) = \underbrace{\int dQ\cdots\int dQ}_{n}\frac{\exp(-aQ)}{Q^{n+p+1}}, \quad a \neq 0,$$

to get

$$I_{n+p+1}(a,Q) = \frac{(-)^{n+p}(-)^n}{\Gamma(n+p+1)a^{n-1}}\frac{d^{n+p}}{db^{n+p}}\exp(ab)\,\mathrm{Ei}\big(n,a(b+Q)\big)\big|_{b=0}$$

where $\mathrm{Ei}(n,x)$ is defined in Eq. (8). Then, one can carry out the $(n+p)$ derivations valued at $b = 0$

$$I_{n+p+1}(a,Q) = a^{p+1}\exp(-aQ)(-)^n G(n+p+1,p+2,aQ) \tag{28}$$

where G is the second class confluent hypergeometric function.

Hence, using the expression (27) in (25) and using also Eq. (28) we can write

$$\Psi_{\mathrm{S}}^{\mathrm{GBT}}(Q) \simeq \Psi_o^{\mathrm{GBT}}(Q) + \lim_{n\to\infty} Q\Gamma(n+1)\sum_{p=1}^{\infty}\sum_{k=1}^{p}\binom{p}{k}(-)^k\left[-v\frac{n+3/2}{2}\right]^p\left[\frac{2k}{(n+3/2)m}\right]^{p+1}$$
$$\times\exp\left[-\frac{2k}{(n+3/2)m}Q\right]G\left[n+p+1,p+2,\frac{2k}{(n+3/2)m}Q\right].$$

To perform the limit $n \to \infty$, we use here the integral representation of G to conclude that the approximate solution reads

$$\Psi_{\mathrm{S}}^{\mathrm{GBT}}(Q) \simeq \Psi_{\mathrm{S}}^{\mathrm{as}}(Q) - 4\frac{Q}{v}\sum_{p=1}^{\infty}\sum_{k=1}^{p}\binom{p}{k}(-)^{k+p}\left[\frac{v}{\sqrt{\frac{2m}{k}Q}}\right]^{p+1}K_{p+1}\left(2\sqrt{\frac{2k}{m}Q}\right), \tag{29}$$

where K_p is the Bessel function of second class.

The approximate solution (29) converges rapidly to the exact solution (22). The nonperturbative corrections recover the corresponding dependence on m. The deviation with respect to the exact solution obtained is certainly small within all the range of Q. In particular, the second order approximation suffers a deviation lower than 4% for $Q \gtrsim 2$ (see Fig. 1).

The test of the accuracy provided by the GBT is presented in Fig. 1 where we have compared the ratio $F_{\rm S}^{\rm GBT}(Q)/F(Q)$ (line c) between our approximate analytical solution (29) up to second order ($k = 2$) and the numerical integration of the exact expression (22), with the corresponding to the Borel resummation technique $F_{\rm S}^{\rm as}(Q)/F(Q)$ (line a) and optimization of the PT $F_{\mu=4}(Q)/F(Q)$ (line b). This comparison has been performed for the particular values of the parameters $m = 3$ and $v = 0.1$.

Let us now turn to the case of an attractive potential. This corresponds to $v < 0$, and can be studied by changing $v \to -|v|$ in the previous definition of the potential. In this case the expression for the wave packet (5) is not Borel summable because now its conventional Borel transform (6) has a pole that does not allow the integration. This situation is similar to the one present in infrared perturbative QCD [12–14]. In this case, the renormalon technique usually extends the result of Borel summation (7) by using the principal value (PV) prescription. Then adds a term, coming from the ambiguity generated by the pole, that represents the non-perturbative contribution. Consequently, the series (5) results in the following approximate solution

$$\Psi_{\rm NS}^{\rm as}(Q) = \Psi_{\rm PV}^{\rm as}(Q) + C\exp(1/\alpha), \tag{30}$$

where

$$\Psi_{\rm PV}^{\rm as}(Q) = -\frac{1}{\alpha(Q)}\exp\left(-\frac{1}{\alpha(Q)}\right)\overline{\rm Ei}\left(1, \frac{1}{\alpha(Q)}\right)$$

and

$$\overline{\rm Ei}\left(1, \frac{1}{\alpha(Q)}\right) \equiv {\rm P.V.}\int_1^{-\infty} \frac{\exp(t/\alpha)\,dt}{t}. \tag{31}$$

The first term in (30) has the same perturbative asymptotic expansion as the exact function (2) and the constant C is obtained by using purely non-perturbative methods or is extracted from experiment. The optimal value obtained is $C = -0.06$ (see [1]). However, the results (30) diverge strongly at small Q (see Fig. 2, line a).

On the other hand, the alternative approach of the modified perturbation theory obtain the best approximate results for the value of the parameter $\mu = 3.6$. The obtained result presents an improvement with respect to the previous approximate solution reducing the deviation at small Q to approximately 20% but loses the asymptotic exact behavior (see Fig. 2, line b).

The GBT follows the same steps presented in the case of the repulsive potential. Notice that previously, the term $k = 0$ generates the asymptotic solution series (5) but now this series is non-alternative. In this regime the approximate value for this sum is given by expression (31). The different behavior is related to the non-perturbative contribution. As was mentioned above, the renormalon technique shows that this contribution can be estimated through the ambiguity generated by the non-analyticity of the conventional Borel transform. However, as we have shown in the case of a repulsive potential, the origin is in the remaining terms with $k \neq 0$ in the expansion of the binomial expression (27). In fact, those terms cannot be taken into account in the renormalon technique. Thus, our approximate solution reads

$$\Psi_{\rm NS}^{\rm GBT}(Q) \simeq \Psi_{\rm PV}^{\rm as}(Q) - 4\frac{Q}{|v|}\sum_{p=1}^{\infty}\sum_{k=1}^{p}\binom{p}{k}(-)^k\left[\frac{|v|}{\sqrt{\frac{2m}{k}Q}}\right]^{p+1} K_{p+1}\left(2\sqrt{\frac{2k}{m}Q}\right).$$

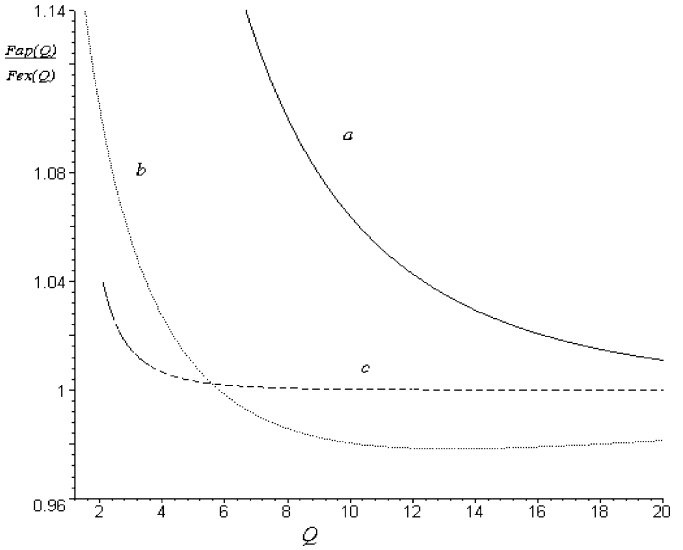

Fig. 2. Numerical comparation for no-Borel-summable solutions.

As it was the case for the repulsive potential, our approximate solution has the correct behavior for all the range of the momentum Q. In fact, the deviation, when one considers up to second order ($k = 2$), is lower than 3.5% for $Q \gtrsim 2$ (see Fig. 2, line c).

In Fig. 2 we present the corresponding comparative plots of the ratio between the approximate solutions $F_{\mathrm{NS}}^{\mathrm{as}}$, $F_{\mu=3.6}$, $F_{\mathrm{NS}}^{\mathrm{GBT}}$ and the exact one F for the attractive potential.

In summary, we have presented an analytic expression for the wave packet valid for all the range of Q, based on the GBT. In the case of the summable Borel theory, our results show explicitly how the GBT completes the perturbative solution provided by conventional Borel transform by means of the adequate incorporation of non-perturbative contributions. In the non-summable Borel theory case, besides obtaining sensible results, we have shown that the real origin of the non-perturbative contributions is not the Borel ambiguity.

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