Limb-Darkened Radiation-Driven Winds from Massive Stars

M. Curé

Departamento de Física y Astronomía, Facultad de Ciencias, Universidad de Valparaíso

Av. Gran Bretaña 1111, Casilla 5030, Valparaíso, Chile

michel.cure@uv.cl

and

L. Cidale[1]

Departamento de Espectroscopía, Facultad de Ciencias Astronómicas y Geofísicas, Universidad Nacional de La Plata (UNLP), and

Instituto de Astrofísica La Plata, CCT La Plata, CONICET-UNLP

Paseo del Bosque S/N, 1900 La Plata, Argentina

lydia@fcaglp.unlp.edu.ar

and

D. F. Rial[1]

Departamento de Matemáticas, Facultad de Ciencias Exactas y Naturales,

Universidad de Buenos Aires, Argentina.

and

IMAS, CONICET

drial@dm.uba.ar

Received ______________________; accepted ______________________

[1]Member of the Carrera del Investigador Científico, CONICET, Argentina

ABSTRACT

We calculated the influence of the limb-darkened finite disk correction factor in the theory of radiation-driven winds from massive stars. We solved the 1-D m-CAK hydrodynamical equation of rotating radiation-driven winds for all three known solutions, i.e., fast, Ω-slow and δ-slow. We found that for the fast solution, the mass loss rate is increased by a factor $\sim$ 10%, while the terminal velocity is reduced about 10%, when compared with the solution using a finite disk correction factor from a uniformly bright star. For the other two slow solutions the changes are almost negligible. Although, we found that the limb darkening has no effects on the wind momentum luminosity relationship, it would affect the calculation of synthetic line profiles and the derivation of accurate wind parameters.

Subject headings: hydrodynamics — methods: analytical— stars: early-type — stars: mass-loss — stars: rotation — stars: winds, outflows

1. Introduction

The CAK theory (Castor et al. 1975) describes the mass loss due to radiation force in massive stars. This theory is based on a simple parameterization of the line force (α and k) which represents the contribution of the spectral lines to the radiative acceleration by a power law distribution function. Abbott (1982) improved this theory calculating the line force considering the contribution of the strengths of the hundreds of thousands of lines. He also included a third parameter (δ) that takes into account the change in ionization throughout the wind. Despite this immense effort to give a more realistic representation of the line force, evident discrepancies still remained. Further improvements to this theory done by Friend & Abbott (1986) and Pauldrach et al. (1986) (hereafter m-CAK model) relaxed the point star approximation with the introduction of the finite disk correction factor, assuming a uniform bright spherical source of radiation. From then on, this model has succeeded in describing both, wind terminal velocities (v_∞) and mass-loss rates ($\dot{M}$) from very massive stars. As a result of the radiation force, the properties of the stellar winds must somehow reflect the luminosities of the stars. This relationship can be obtained from the line driven wind theory (Kudritzki et al. 1995, 1999) and, nowadays, it is known as the Wind Momentum–Luminosity Relationship (WM–L). It predicts a strong dependence of wind momentum rate on the stellar luminosity with α (Puls et al. 1996).

The m-CAK hydrodynamical solution (hereafter the fast solution) is characterized by an exponential growth at the base of the wind that matches very quickly a β-law profile when the velocity reaches some few kilometers per second, with a β index in the range 0.8 to 1.0.

However, in the last decade, Curé (2004) and Curé et al. (2011) found two new physical solutions from the 1-D non-linear m-CAK hydrodynamics equation that describe the wind velocity profile and mass loss rates from rapidly rotating stars (the Ω-slow solution) and from slowly rotating A- and late B-type supergiants (the δ-slow solution). The Ω-slow solution only exists when the star's rotational speed is larger than $\sim 3/4$ of the breakup speed. This Ω-slow solution posses a larger mass loss rate (the higher the rotational speed, the higher the mass loss rate) and reaches a terminal velocity which is about 1/3 of the fast solution's terminal speed. On the other hand, the δ-slow solution is found when the line-force parameter δ is slightly larger than ~ 0.25. High values of δ are expected in hydrogen rich environments; for a pure hydrogen gas Puls et al. (2000) demonstrated that δ is $1/3$. This last solution, where the Abbott δ factor represents changes in the ionization of the wind with distance, reaches a slow terminal velocity, similar to the Ω-slow solution, but with a much lower mass loss rate.

In the m-CAK model, the calculation of the radiation force is often carried out assuming a uniform bright finite-sized spherical star. The rapid rotation, however, changes the shape of the star to an oblate configuration (Cranmer & Owocki 1995; Pelupessy et al. 2000) and induces gravity darkening (von Zeipel 1924) as function of (co)-latitude. In both cases, a rotating and non-rotating star, the decrease of the temperature outwards the photosphere produces a limb darkening effect which also modifies the finite disk correction factor. The theoretical formalism for computing the self-consistent radiation force for non-spherical rotating stars, including the effects of stellar oblateness, limb darkening and gravity darkening, was developed by Cranmer & Owocki (1995). However, to disentangle the effects of each one of these competing processes upon the wind structure, these authors present a semi-quantitative analysis and estimated that the limb darkening effect could increase the mass loss rate ($\dot{M}$) in an amount of $\sim 11\%$ to $\sim 13\%$ over the uniformly bright models. However, that larger mass loss would imply a reduction in the wind terminal speed. Owocki & ud-Doula (2004) carried out (for the fast solution) a perturbation analysis of the effects of the gas pressure on the mass loss rate and wind terminal velocity in terms of the ratio of sound speed to escape speed ($a/v_{\rm esc}$). They showed that for finite-disk-corrected spherical wind, typical increases in mass-loss rate are 10%–20%, with comparable relative decreases in the wind terminal speed.

Then, considering that the radiative flux does not change significantly when the limb darkening is taken into account, an enhancement of $\sim$ 10% in the mass loss rate might lead not only to a lower terminal speed (v_∞) but also to a change in the theoretical WM–L. An accurate determination of the WM-L relationship for A and B supergiants (Asgs and Bsgs) is important because it would allow the use of these stars as extragalactic distance indicators (Bresolin & Kudritzki 2004).

In this work, we present an analytical expression for the limb darkening finite disk correction factor and solve the 1-D hydrodynamical equation for all three known solutions for radiation driven winds; i.e., fast, Ω-slow and δ-slow solutions. These results are compared with the wind solutions computed with the finite disk correction factor assuming a uniform bright star, finding that the effects of the limb darkening are only important for fast solution.

In §2 we briefly describe the 1-D momentum equation of the wind, in §3 we present an analytical expression for the limb-darkened finite disk correction factor and in §4 we solve numerically the hydrodynamics equations for model parameters corresponding to the fast, Ω-slow, δ-slow and Ωδ-slow solutions. Finally, in §5, we discuss the results, conclusions and future work.

2. The m-CAK hydrodynamic model

The m-CAK model for radiation driven winds considers one dimensional component isothermal fluid in a stationary regime with spherical symmetry. Neglecting the effects of viscosity, heat conduction and magnetic fields (Castor et al. 1975), the equations of mass conservation and radial momentum read:

$$4\pi r^2 \rho v = \dot{M}, \tag{1}$$

and

$$v\,\frac{dv}{dr} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM\,(1-\Gamma)}{r^2} + \frac{v_\phi^2(r)}{r} + g^{line}\,(\rho, dv/dr, n_E). \tag{2}$$

Here v is the fluid velocity and dv/dr its gradient. All other variables have their standard meaning (see Curé (2004) for a detailed derivation and definitions of variables, constants and functions). We adopted the standard parametrization for the line force term, given by Abbott (1982), Friend & Abbott (1986), Pauldrach et al. (1986):

$$g^{line} = \frac{C}{r^2}\, f_{\rm D}\,(r, v, dv/dr)\,\left(r^2\,v\,\frac{dv}{dr}\right)^{\alpha}\,\left(\frac{n_E}{W(r)}\right)^{\delta}, \tag{3}$$

where the coefficient C depends on $\dot{M}$, $W(r)$ is the dilution factor and $f_{\rm D}$ is the finite disk correction factor.

Introducing the following change of variables $u = -R_*/r$, $w = v/a$ and $w' = dw/du$, where a is the isothermal sound speed and $a_{\rm rot} = v_{\rm rot}/a$, where $v_{\rm rot}$ is the equatorial rotation speed at the stellar surface, the momentum equation becomes:

$$F(u, w, w') \equiv \left(1 - \frac{1}{w^2}\right) w\,\frac{dw}{du} + A + \frac{2}{u} + a_{\rm rot}^2\,u - C'\;f_{\rm D}\;g(u)\,(w)^{-\delta}\left(w\,\frac{dw}{du}\right)^{\alpha} = 0. \tag{4}$$

The standard method for solving this non-linear differential equation (4) together with the constant $C'(\dot{M})$ (eigenvalue of this problem) is imposing that the solution passes through a singular (or critical) point.
Critical points are defined at the roots of the singularity condition, namely:

$$\frac{\partial}{\partial w'}\,F(u, w, w') = 0. \tag{5}$$

At this specific point and in order to find a physical wind solution, a regularity condition must be also imposed, i.e.,

$$\frac{d}{du}\,F(u, w, w') = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial w}\,w' = 0. \tag{6}$$

In order to solve this equation we need to know the behaviour of the finite disk correction factor f_D. To disentangle limb darkening from rotational effects (gravity-darkening and oblateness) we will analize them independently. A discussion on the effects of the oblate finite disk correction factor on the velocity profile and mass loss rate was presented by Araya et al. (2011). Therefore, in this work we mainly discuss the importance of the limb darkening on radiation driven winds.

3. Limb-darkened finite disk correction factor

Cranmer & Owocki (1995) derived an integral expression for the limb-darkened finite disk correction factor ($f_{\rm LD}$), based in a simple linear gray atmosphere, namely:

$$f_{\rm LD}\,(r, v, dv/dr) = \frac{r^2}{R_*^2\,(1+\sigma)^\alpha} \int_{\mu_*}^{1} (1+\sigma\mu'^2)^\alpha \times \left(1 + \frac{3}{2}\sqrt{\frac{\mu'^2-\mu_*^2}{1-\mu_*^2}}\right) \mu' d\mu', \tag{7}$$

where $\sigma \equiv (d\ln v/d\ln r) - 1$ and $\mu_* = \sqrt{1 - R_*^2/r^2}$.
The integration of eq. 7 gives the following analytical expression :

$$\begin{aligned} f_{\rm LD}\,(r, v, dv/dr) \;\; = \;\; & \frac{1}{2\,\sigma\,(\alpha+1)} \times \left(\frac{\sigma+1}{-\frac{\sigma}{r^2}+\sigma+1}\right)^\alpha \times \\ & \left[\sigma\,(\alpha+1)\;{}_2F_1\left(\frac{3}{2}, -\alpha, \frac{5}{2}, -\frac{\sigma}{r^2\,(\sigma+1)-\sigma}\right)\right. \\ & \left. + r^2\,(\sigma+1)\left(\left(\frac{\sigma+1}{-\frac{\sigma}{r^2}+\sigma+1}\right)^\alpha - 1\right) + \sigma\right] \end{aligned} \tag{8}$$

where ${}_2F_1$ is the Gauss Hypergeometric function.

Figure 1 compares the run of both uniformly bright and limb-darkened finite disk correction factors as function u, using two different β-law velocity profiles ($\beta = 0.8$ and 2.5) and a typical value of the α line-force parameter equals 0.6. This figure clearly shows that at the base of the wind the factor $f_{\rm LD}$ (shown in gray-dashed line) is about $\sim$ 10% larger than the one obtained for a uniform bright stellar disk, f_D (continuous line), increasing, the value of the mass loss rate. Instead, at larger distances from the stellar surface, both correction factors have the same behaviour as a function of u.

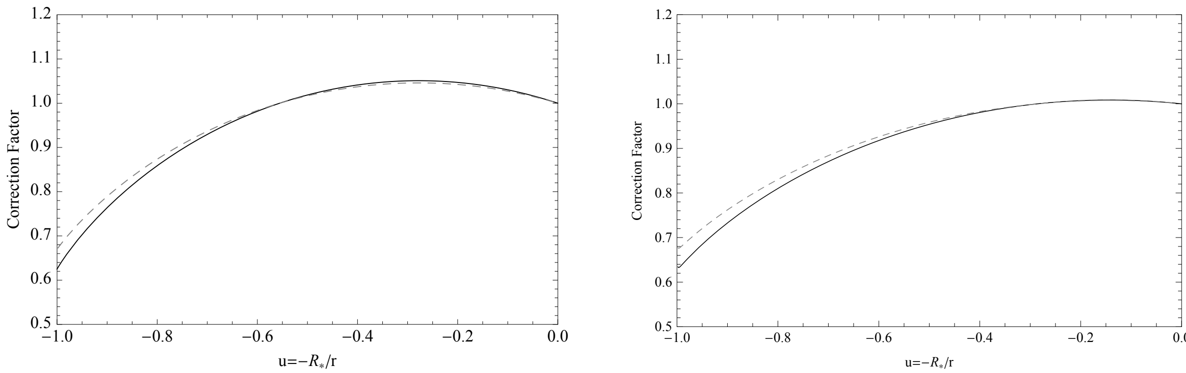

Fig. 1.— Uniformly bright f_D (continuous line) and limb-darkened f_{LD} (gray-dashed line) finite disk correction factors. Calculations where performed using a β-law velocity field with $\beta = 0.8$ (left panel) and $\beta = 2.5$ (right panel). In both cases the value for α was 0.6

Mathematically, $f_{\rm LD}$ can be considered as a small perturbation of $f_{\rm D}$, as it is shown in Figure 1, even for different β-law index. Thus, based on the standard theory of dynamical system (see, e.g., J. Palis & de Melo 1982), we expect no large differences when considering velocity profiles from the equation of motion (eq. 4), for the cases where uniformly bright or limb-darkened finite disk correction factors are used. This is a consequence of the theorem of the continuous dependence of the solutions of the ordinary differential equations (ODE) on their parameters (Hirsch & Smale 1974).

Therefore, the scope of this paper is limited to the study of the numerical 1-D stationary solutions of eq. 4, leaving a theoretical topological analysis (and also a time–dependent one) for a future work.

4. Results

We are now in conditions to solve the non-linear differential equation (eq. 4) considering the factor $f_{\rm LD}$ given by eq. 8. The calculation of all partial derivatives of $f_{\rm LD}\,(u, w, w')$ are given in Appendix A. These derivatives are needed in order to evaluate the singularity and regularity conditions (Eqs. 5 and 6, respectively). Depending on the selected parameter-space, each type of solution explains the wind of a different kind of massive object, i.e., fast solution describes the wind of hot stars, the Ω-slow solution explains the wind of rapid rotators such as Be stars, and the δ-slow solution characterizes the wind of A-type supergiants. In the following subsections we will adopt a prototype star for each one of these three known physical solutions, in order to analyse the effects of the limb darkening on the m–CAK hydrodynamical model.

4.1. Fast Solution

For the standard fast solution we selected, as in Curé (2004), a typical O5 V star with the following stellar and line-force parameters: $T_{\rm eff} = 45\,000$ K, $\log g = 4.0$, $R/R_\odot = 12$, $v_{\rm rot} = 0$, $k = 0.124$, $\alpha = 0.64$ and $\delta = 0.07$ (Lamers & Cassinelli 1999). The numerical code we used to solve the momentum equation is described in Curé (2004). Figure 2 (left panel) shows the velocity profile for the standard case, where a uniform bright star disk (continuous line) and the limb-darkened one (gray-dashed line) are used. Figure 2 (right panel) displays the difference in the velocity, $\Delta v = v_{\rm un} - v_{\rm LD}$ (where $v_{\rm un}$ is the velocity profile when the uniform finite disk correction factor is taken into account, while $v_{\rm LD}$ is the wind solution obtained using $f_{\rm LD}$). The $v_{\rm LD}(u)$ profile is always smaller than the $v_{\rm un}(u)$ one, with a monotonically increasing difference. The effect of the limb-darkened finite disk correction factor changes the behaviour of the velocity field in the most external layers, it reaches a smaller terminal velocity by about 10% of the v_∞ value of the standard m-CAK case. There is no significant change of the velocity field at the base of the wind. Therefore, the location of the singular point is almost the same in both cases. Our calculations confirmed the predictions of Cranmer & Owocki (1995); Owocki & ud-Doula (2004) based on the behaviour of the $f_{\rm LD}$ at the base of the wind, i.e., that the mass loss rate is increased a factor of about $\sim$ 10%. Concerning to the WM-L relationship, the value of $D_{\rm mom} = (\dot{M}\, v_\infty \sqrt{R_*/R_\odot})$ shows almost no change due to a compensation of the increase in the mass loss rate and the decrease of the terminal velocity, as shown in Table 1. Although the value of $D_{\rm mom}$ seems to remain unaltered, we would expect minor differences in the synthetic spectra when they are computed with the two different velocity profiles.

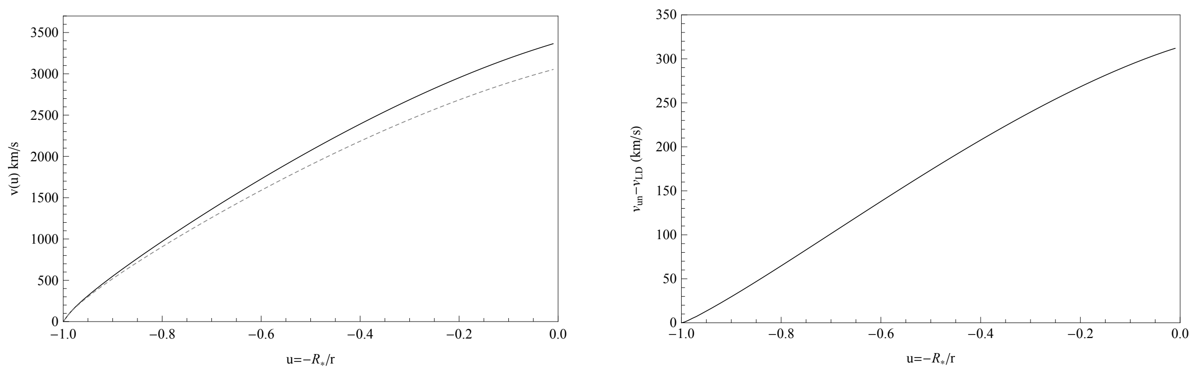

Fig. 2.— Left panel: velocity profile as function of the inverse radial coordinate u. The standard m-CAK model is shown in continuous line and the solution with the limb-darkened finite disk correction factor is in gray dashed line. The effect of the $f_{\rm LD}$ in the velocity profile is very significant reducing the terminal velocity approximately in 10% with respect to the standard m-CAK model. Right panel: Velocity difference between the standard solutions with a uniformly bright, $f_{\rm D}$, and the limb-darkened, $f_{\rm LD}$, correction factors.

These results show that the correction to the line radiation force due to the limb darkening effect leads to lower mass loss rates and higher wind terminal velocities, both in approximately 10%, when compare with the contribution of a uniformly bright star disk radiation source.

Table 1: Wind parameters for the fast solution with uniformly bright (f_D) and limb-darkened (f_{LD}) finite disk correction factors

	f_D	f_{LD}
$\dot{M}$ (10^{-6} $M_{\odot}$ yr^{-1})	2.206	2.449
v_{∞} (km s^{-1})	3384	3071
$r_{\rm singular}$ (R_*)	1.027	1.031
EigenValue (C')	40.89	38.53
log $D_{\rm mom}$ (cgs)	29.21	29.22

4.2. Ω-Slow Solution

The Ω-slow solution is present when the star is rotating at velocities near the breakup rotational speed. Therefore, to study the effects of the limb darkening in the radiation force we select, the case of a typical B1 V star with high rotational speed ($\Omega = v_{\rm rot}/v_{\rm breakup} = 0.9$) and the following stellar parameters: $T_{\rm eff} = 25\,000$ K, log $g = 4.03$, $R/R_{\odot} = 5.3$. The corresponding line force parameters: $k = 0.3$, $\alpha = 0.5$ and $\delta = 0.07$ were taken from Abbott (1982).

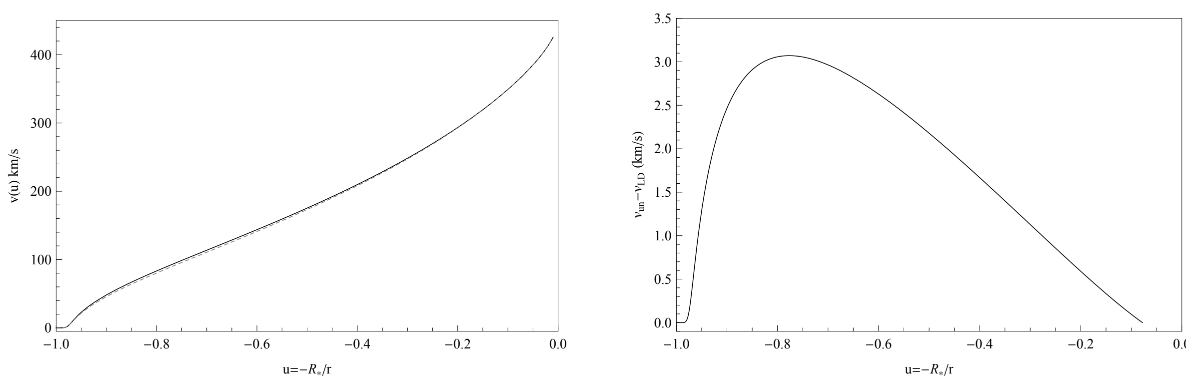

Fig. 3.— Left panel: Same as Figure 1, $v(u)$ versus u. In this case the effect of the f_{LD} in the velocity profile is minimal. Right panel: Velocity difference.

The resulting velocity profiles with uniform and limb-darkened correction factors and the corresponding differences in the velocities are shown in Figure 3. These plots show clearly that the effect of f_{LD} in the velocity profile is minima. The influence of the f_{LD} on the mass loss rate

Table 2: Wind parameters for the Ω-slow solution with uniformly bright ($f_{\rm D}$) and limb-darkened ($f_{\rm LD}$) finite disk correction factors

	$f_{\rm D}$	$f_{\rm LD}$
$\dot{M}$ (10^{-6} $M_{\odot}\,yr^{-1}$)	4.22 10^{-3}	4.22 10^{-3}
v_{∞} ($km\,s^{-1}$)	446.8	446.5
$r_{\rm singular}$ (R_{*})	26.14	26.14
EigenValue (C')	78.31	78.27
$\log D_{\rm mom}$ (cgs)	25.44	25.44

and other wind quantities are shown in Table 2, together with the comparison of the velocity profile using the uniform correction factor. All the changes in these quantities are minimal or even negligible. There is an important dominance of the centrifugal force term in the momentum equation (4).

4.3. δ-Slow Solution

For the calculation of the $f_{\rm LD}$ correction factor in the parameter-space of the δ-slow solution, we select an A-type supergiant star with the following fundamental parameters: $T_{\rm eff}$ = 10 000 K, $\log g = 2.0$, $R/R_{\odot} = 60$, $v_{\rm rot} = 0$, and line-force parameters: $k = 0.37$, $\alpha = 0.49$ and δ =0.3 (model W03 from Curé et al. 2011).

Similar to the Ω-slow wind solution, the effect of the limb darkening is negligible in both, the velocity profile and mass loss rate (see Figure 4 and Table 3).

Concerning the influence of the limb darkening on the WM-L relationship, there is no substantial effect.

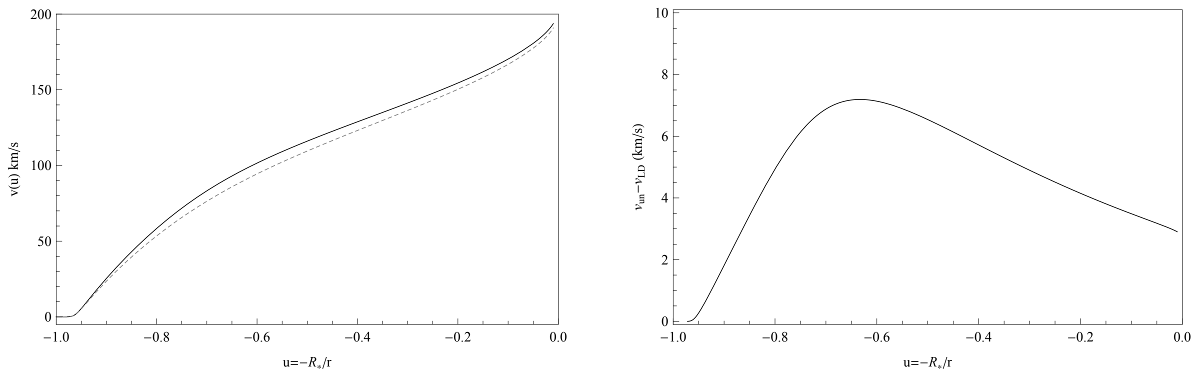

Fig. 4.— Left panel: Same as Figure 1, $v(u)$ versus u. In this case the effect of the $f_{\rm LD}$ in the velocity profile is minimal. Right panel: Velocity difference.

Table 3: Wind parameters for the δ-slow solution with uniformly bright ($f_{\rm D}$) and limb-darkened ($f_{\rm LD}$) finite disk correction factors

	$f_{\rm D}$	$f_{\rm LD}$
$\dot{M}$ (10^{-6} $M_{\odot}\,yr^{-1}$)	7.22 10^{-4}	7.36 10^{-4}
v_{∞} ($km\,s^{-1}$)	203	200
$r_{\rm singular}$ (R_{*})	11.06	11.06
EigenValue (C')	63.78	63.54
log $D_{\rm mom}$ (cgs)	24.85	24.86

4.4. Ωδ-Slow Solution

Here we investigate the particular case when Ω and δ take higher values. We selected the same test star as in §4.2 but with a different value of the δ parameter (δ = 0.25). The computed hydrodynamic solutions for uniformly bright and limb-darkened correction factors are almost the same, as it shown in Figure 5 and Table 4.

When we compare the velocity profiles between the Ω-slow solution computed with $\delta = 0.07$ (see figure 4 left panel) and $\delta = 0.25$ (see figure 5 left panel), we find that both profiles have the same behaviour as a function of r. Therefore, the centrifugal term due to the high dominates over the δ-factor in g^{line}. Nevertheless, the influence of the δ-factor is not negligible, it reduces the mass

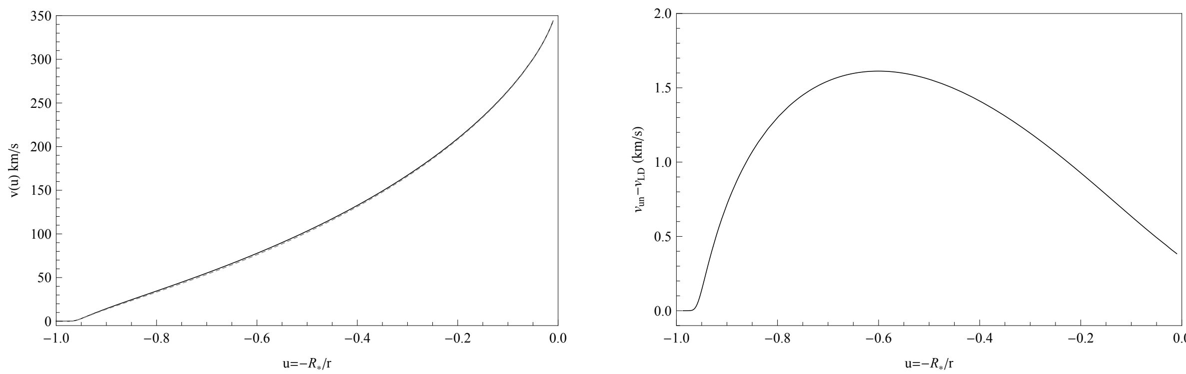

Fig. 5.— Left panel: Same as Figure 1, $v(u)$ versus u. In this case the effect of the $f_{\rm LD}$ in the velocity profile is minimal. Right panel: Velocity difference.

Table 4: Wind parameters for the Ωδ-slow solution with uniformly bright ($f_{\rm D}$) and limb-darkened ($f_{\rm LD}$) finite disk correction factors

	$f_{\rm D}$	$f_{\rm LD}$
$\dot{M}$ (10^{-6} $M_{\odot}\,yr^{-1}$)	8.63 10^{-4}	8.64 10^{-4}
v_{∞} ($km\,s^{-1}$)	367.8	367.5
$r_{\rm singular}$ (R_*)	38.17	38.17
EigenValue (C')	113.9	113.9
log $D_{\rm mom}$ (cgs)	24.66	24.66

loss rate in ∼ 80% and the terminal velocity in ∼ 20%.

5. Discussion and Conclusions

In this work we improved the description of the radiation force taking into account the correction factor due to a limb-darkened disk. In particular, we derived an analytical formula to compute this contribution. Then, we solved the 1-D non-linear momentum equation for radiation driven winds and analized the influence of $f_{\rm LD}$ for all the three known solutions, namely: the fast, the Ω-slow and the δ-slow solutions, as well as, the case of a high Ω and δ parameter, the Ω δ-slow solution.

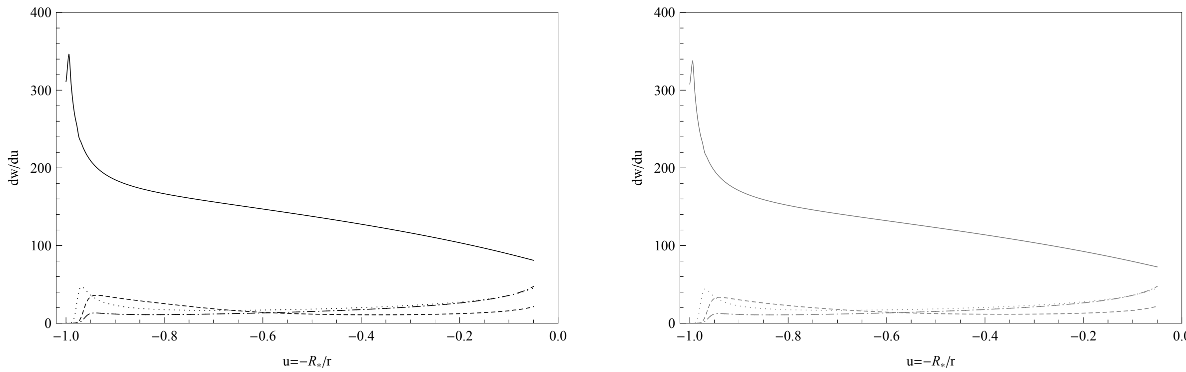

Fig. 6.— Left panel: Normalized velocity gradient dw/du versus u from the solutions of the equation of motion using the uniform finite disk f_D. In continuous-line is plotted the gradient of the fast solution; dashed–line correspond to the δ-slow solution; dotted–line to the Ω-slow solution and dashed–dotted–line to the $\Omega\delta$-slow solution. Right panel: id, for the cases where the f_{LD} us used in the equation of motion (eq. 4)

We selected the appropriate stellar parameters of massive stars that are representative of each possible hydrodynamical solution and evaluated the velocity profile as function of the radial coordinate.

We found a significant impact of $f_{\rm LD}$ in the radiation driven-wind of massive stars that are described by the fast solution. Due to the effect of a limb-darkened disk the mass loss rate increased in an amount of $\sim$ 10% while the terminal velocity is reduced about the same factor. Therefore, the limb darkening effect should be considered always in the calculation of the hydrodynamics fast solution.

On the other hand, the influence of $f_{\rm LD}$ on the Ω-slow and δ-slow solutions is minimal. The maximum difference obtained in the velocity profile computed with uniformly bright and limb-darkened disk radiation sources is less than 3 $km\ s^{-1}$ for the Ω-slow solution, 7 $km\ s^{-1}$ for the δ-slow solution and 1.5 $km\ s^{-1}$ for the case when both parameter δ and Ω are high (the $\Omega\delta$-slow solution). Therefore, the limb darkening effect is negligible when computing the wind parameters.

However, rotational effects like the star's oblateness should be considered, since it modifies the wind in the polar direction (see Araya et al. 2011) being much faster than the spherical one. Moreover, the slow solutions predict even slower and denser flows than the spherical ones.

The influence of $f_{\rm LD}$ on radiation driven winds can be interpreted in terms of the resulting velocity profile. The mayor differences between the uniformly bright and limb-darkened finite disk correction factors are in the region just above the stellar photosphere, as Figure 1 shows. In this region the velocity from all the models described in section 4 are small, however the value of the velocity gradient from the fast-solution is 5 to 10 times larger than the values from any slow-solution. Figure 6 shows the normalized velocity gradient dw/du as function of u for the different types of solutions. Is this dependence on the velocity gradient, specifically in the finite disk correction factor, that makes a significant difference in the terminal velocity and the mass loss rate *only* for the fast solution and not for the slow ones.

Concerning the WM-L relationship, the limb-darkened correction factor has no effects. The increase produced by the fast solution on $\dot{M}$ is compensated by a similar decrease of v_∞. Considering the importance of having a theoretical WM-L relationship for B- and A- type supergiants the effect of the star's oblateness and gravity-darkening should be explored, together with the calculation of the synthetic line spectrum in order to derive accurate wind parameters.

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A. Partial derivatives of f_{LD}

In order to find the location of the singular point, we need to evaluate the singularity condition given by eq. 5 and, then, impose the regularity condition given by eq. 6. To perfom this calculation we need to know all the partial derivatives of $f_{\mathrm{LD}}(u, w, w')$; i.e., $\partial f_{\mathrm{LD}}/\partial u$, $\partial f_{\mathrm{LD}}/\partial w$ and $\partial f_{\mathrm{LD}}/\partial w'$.

Defining the following auxiliary variables:

$$Z = w/w' \tag{A1}$$

$$\lambda = u\,(u+Z) \tag{A2}$$

Thus, in terms of λ, the finite disk correction factor for a uniformly bright spherical star f_{D} reads,

$$f_{\mathrm{D}}(\lambda) = \frac{1}{(1+\alpha)}\,\frac{1}{\lambda}\left[1-(1-\lambda)^{(1+\alpha)}\right], \tag{A3}$$

while the limb-darkened finite disk correction factor f_{LD} is:

$$f_{\mathrm{LD}}(\lambda) = \frac{(1-\lambda)^{\alpha}\left[(1-\lambda)^{-\alpha}+\lambda\,(\alpha+1)\;{}_2F_1\left(\frac{3}{2},-\alpha,\frac{5}{2},\frac{\lambda}{\lambda-1}\right)+\lambda-1\right]}{2\,\lambda\,(\alpha+1)} \tag{A4}$$

Defining now $e(\lambda) = \partial f_{\mathrm{LD}}(\lambda)/\partial\lambda$, we obtain:

$$\begin{aligned} e(\lambda) \;=\; & \frac{(1-\lambda)^{\alpha}}{4\,(\alpha+1)\,(\lambda-1)\,\lambda^{2}} \times \left[(2-(3\alpha+5)\,\lambda)\,(1-\lambda)^{-\alpha} + \right. \\ & \left. +2\,(\lambda-1)\,(\alpha\lambda+1)+(\alpha+1)\,\lambda\,(2\alpha\lambda+3)\,{}_2F_1\left(\frac{3}{2},-\alpha,\frac{5}{2},\frac{\lambda}{\lambda-1}\right)\right] \end{aligned} \tag{A5}$$

Therefore, all the partial derivatives can be calculated using the chain rule, getting,

$$e(\lambda) = \frac{1}{2\,u+w/w'}\,\frac{\partial f_{\mathrm{LD}}}{\partial u} = \frac{w'}{u}\,\frac{\partial f_{\mathrm{LD}}}{\partial w} = -\frac{w'^{\,2}}{u\,w}\,\frac{\partial f_{\mathrm{LD}}}{\partial w'}. \tag{A6}$$

REFERENCES

Abbott, D. C. 1982, ApJ, 259, 282

Araya, I., Curé, M., Granada, A., & Cidale, L. S. 2011, in IAU Symposium, Vol. 272, IAU Symposium, ed. C. Neiner, G. Wade, G. Meynet, & G. Peters, 83–84

Bresolin, F., & Kudritzki, R. P. 2004, Origin and Evolution of the Elements, 283

Castor, J. I., Abbott, D. C., & Klein, R. I. 1975, ApJ, 195, 157

Cranmer, S. R., & Owocki, S. P. 1995, ApJ, 440, 308

Curé, M. 2004, ApJ, 614, 929

Curé, M., Cidale, L., & Granada, A. 2011, ApJ, 737, 18

Friend, D. B., & Abbott, D. C. 1986, ApJ, 311, 701

Hirsch, M. W., & Smale, S. 1974, Differential Equations, Dynamical Systems and Linear Algebra (Academic Press)

J. Palis, J., & de Melo, W. 1982, Geometric theory of dynamical systems: An introduction (Springer-Verlag, New York)

Kudritzki, R., Lennon, D. J., & Puls, J. 1995, in Science with the VLT, ed. J. R. Walsh & I. J. Danziger, 246

Kudritzki, R. P., Puls, J., Lennon, D. J., Venn, K. A., Reetz, J., Najarro, F., McCarthy, J. K., & Herrero, A. 1999, A&A, 350, 970

Lamers, H. J. G. L. M., & Cassinelli, J. P. 1999, in Introduction to Stellar Winds, 219

Owocki, S. P., & ud-Doula, A. 2004, ApJ, 600, 1004

Pauldrach, A., Puls, J., & Kudritzki, R. P. 1986, A&A, 164, 86

Pelupessy, I., Lamers, H. J. G. L. M., & Vink, J. S. 2000, A&A, 359, 695

Puls, J., Springmann, U., & Lennon, M. 2000, A&AS, 141, 23

Puls, J., et al. 1996, A&A, 305, 171

von Zeipel, H. 1924, MNRAS, 84, 684
