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Generalized Schur complements and P -complementable operators

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Abstract

Let A be a selfadjoint operator and P be an orthogonal projection both operating on a Hilbert space \mathcal{H} . We say that A is P -complementable if $A - \mu P \geq 0$ holds for some $\mu \in \mathbb{R}$. In this case we define $I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geq 0\}$. As a tool for computing $I_P(A)$ we introduce a natural generalization of the Schur complement or shorted operator of A to $\mathcal{S} = R(P)$, denoted by $\Sigma(A, P)$. We give expressions and a characterization for $I_P(A)$ that generalize some known results for particular choices of P . We also study some aspects of the shorted operator $\Sigma(A, P)$ for P -complementable A , under the hypothesis of *compatibility* of the pair (A, \mathcal{S}) . We give some applications in the finite dimensional context.

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1. Introduction

Let \mathcal{H} be a Hilbert space and $L(\mathcal{H})$ the algebra of bounded linear operators on \mathcal{H} . Given a closed subspace $\mathcal{S} \subseteq \mathcal{H}$ and $P = P_{\mathcal{S}} \in L(\mathcal{H})$ the orthogonal

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projection onto \mathcal{S} , we study the following two problems: for any selfadjoint operator $A \in L(\mathcal{H})$,

1. determine whether there exists some $\mu \in \mathbb{R}$ such that

$$A - \mu P \geq 0; \quad (1)$$

2. in case Eq. (1) holds for some μ , compute the optimum number

$$I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geq 0\}. \quad (2)$$

The general solution of problem (1) is well known, see for example [14] or Proposition 3.3. Also, if $A \geq 0$, problem (2) has a known answer (see, for example, [7]). We state this case in the preliminary Section 2 (Corollary 2.2). Therefore our main interest is to study problem (2) in the non-positive case. It should be mentioned that the general case seems not to be easily reduced to the positive case (see Remark 5.1).

If condition (1) is satisfied by A , we shall say that A is P -complementable, because in this case there exists the *shorted operator* (or Schur complement, see [1]) defined as follows:

$$\Sigma(A, P) = \max\{D \in L(\mathcal{H}) : D = D^*, D \leq A, D(\mathcal{H}) \subseteq \mathcal{S}\}.$$

Using the identity $\Sigma(A - \mu P, P) = \Sigma(A, P) - \mu P$, in Section 3 we extend several known properties of shorted operators of positive operators to our case. On the other hand, in Section 5 we show that, if $A \not\geq 0$ but it is P -complementable, then

$$I_P(A) = \lambda_{\min}(\Sigma(A, P)),$$

where $\lambda_{\min}(C)$ denotes the minimum of the spectrum $\sigma(C)$ of $C \in L(\mathcal{H})$.

Although most applications of the problems mentioned above appear in matrix theory, i.e., when $\dim \mathcal{H} < \infty$, an additional hypothesis of the operator A allows to extend all finite dimensional results to our setting. This hypothesis is the so called *compatibility* of the pair (A, \mathcal{S}) . This notion, defined by Corach, Maestripieri and the second author in [4–6], is the following: The pair (A, \mathcal{S}) is compatible if there exists a A -selfadjoint projection onto \mathcal{S}^\perp , i.e., $Q \in L(\mathcal{H})$ such that $Q^2 = Q$, $AQ = Q^*A$ and $R(Q) = \mathcal{S}^\perp$.

There are several characterization of the compatibility of (A, \mathcal{S}) and general properties of such pairs in the case $A \geq 0$ (see, for example, [4]); some of them are stated in Section 4 of this paper, where we also extend these properties to the non-positive case. We say that compatibility is an additional condition because, if $(1 - P)A(1 - P) \geq 0$ and (A, \mathcal{S}) is compatible, then A is P -complementable. The reverse implication is false in general, but it is true if $\dim \mathcal{S}^\perp < \infty$, in particular in the finite dimensional case.

Section 5 is devoted to the computation of the number $I_P(A)$ for A selfadjoint, not necessarily positive. We first obtain the formula

$$I_P(A) = \inf\{\langle A\xi, \xi \rangle : \xi \in \mathcal{H}, \|P\xi\| = 1\}$$

so that, if \mathcal{S} is the subspace generated by the unit vector $\xi \in \mathcal{H}$, then

$$I_P(A) = \inf\{\langle A\eta, \eta \rangle : \eta \in \mathcal{H}, \langle \eta, \xi \rangle = 1\}.$$

If (A, \mathcal{S}) is compatible, we show that the computation of $I_P(A)$ can be reduced to the case in which $\mathcal{S} \subseteq \overline{R(A)}$, by replacing \mathcal{S} by $\overline{\mathcal{S} \cap R(A)}$. We state the results of the rest of this section in the following theorem:

Theorem. *Let $A = A^* \in L(\mathcal{H})$, $A \not\equiv 0$, and $P = P^* = P^2 \in L(\mathcal{H})$ with $R(P) = \mathcal{S}$. Suppose that $(1 - P)A(1 - P) \geq 0$ and (A, \mathcal{S}) is compatible. Then*

1. $R(\Sigma(A, P)) = \mathcal{S} \cap R(A) \neq \{0\}$.
2. If $\mathcal{T} = \overline{\mathcal{S} \cap R(A)}$ and $Q = P_{\mathcal{T}}$, then the pair (A, \mathcal{T}) is compatible, $\Sigma(A, P) = \Sigma(A, Q)$ and $I_P(A) = I_Q(A)$.
3. If $R(A)$ is closed, then $\mathcal{T} = \mathcal{S} \cap R(A)$ and

$$I_P(A) = I_Q(A) = \lambda_{\min}((QA^\dagger Q)^\dagger), \tag{3}$$

where C^\dagger denotes the Moore–Penrose pseudoinverse of a closed range operator C .

Formula (3) is the natural generalization of $I_P(A) = \|PA^\dagger P\|^{-1}$, which holds if A is positive (semidefinite) with closed range (see Corollary 2.2). In Section 6 we study some applications of the mentioned results, particularly to problems posed by Fiedler–Markham [9] and Reams [15]. Given a completely positive map $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$, we also compute the number

$$I(\Phi) = \max\{\mu \in \mathbb{R} : \Phi - \mu \cdot \text{Id is completely positive}\},$$

which can be considered as a notion of index for such maps.

2. Preliminary results

In this paper \mathcal{H} denotes a Hilbert space, $L(\mathcal{H})$ is the algebra of all linear bounded operators on \mathcal{H} , $\text{Gl}(\mathcal{H})$ is the group of invertible operators in $L(\mathcal{H})$ and $L(\mathcal{H})^+$ is the subset of $L(\mathcal{H})$ of all positive (semidefinite) operators. If $\dim \mathcal{H} = n < \infty$ we shall identify \mathcal{H} with \mathbb{C}^n and $L(\mathcal{H})$ with the space of $n \times n$ complex matrices $M_n(\mathbb{C})$. The elements of \mathbb{C}^n are considered as column vectors. For simplicity we sometimes describe a column vector $\xi \in \mathbb{C}^n$ as $\xi = (\xi_1, \dots, \xi_n)$.

For every $C \in L(\mathcal{H})$ its range is denoted by $R(C)$, $\sigma(C)$ denotes the spectrum of C and $\rho(C)$ the spectral radius of C . If $C^* = C$, we denote $\lambda_{\min}(C)$ the minimum of $\sigma(C)$. If $R(C)$ is closed, then C^\dagger denotes the Moore–Penrose pseudoinverse of C . The orthogonal projection onto a closed subspace \mathcal{S} is denoted by $P_{\mathcal{S}}$. We use the notations $\mathbb{Q} = \{Q \in L(\mathcal{H}) : Q^2 = Q\}$ for the set of idempotents and $\mathbb{P} = \{P \in \mathbb{Q} : P = P^*\}$ for the set of orthogonal projections. For every $P \in \mathbb{P}$, the decomposition $\mathcal{H} = R(1 - P) \oplus R(P)$ induces a 2×2 representation of $A \in L(\mathcal{H})$:

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix},$$

which we call the matrix representation induced by P .

Now we state the well known criterion due to Douglas [8] (see also [10]) about ranges and factorization of operators:

Theorem 2.1. *Let $A, B \in L(\mathcal{H})$. Then the following conditions are equivalent:*

1. $R(B) \subseteq R(A)$.
2. There exists a positive number μ such that $BB^* \leq \mu AA^*$.
3. There exists $D \in L(\mathcal{H})$ such that $B = AD$.

Moreover, in this case there exists a unique solution D of the equation $AX = B$ such that $R(D) \subseteq \overline{R(A)}$. The operator D is called the reduced solution of the equation $AX = B$ and $\|D\|^2 = \min\{\mu : BB^* \leq \mu AA^*\}$. If $R(A)$ is closed, then $D = A^\dagger B$.

Corollary 2.2. *Let $A, B \in L(\mathcal{H})^+$. Then there exists $\mu > 0$ such that $A - \mu B \geq 0$ if and only if $R(B^{1/2}) \subseteq R(A^{1/2})$. In this case, if $B \neq 0$ and D is the reduced solution of the equation $A^{1/2}X = B^{1/2}$, we have*

$$\max\{\mu \geq 0 : A - \mu B \geq 0\} = \|D\|^{-2}.$$

If $R(A)$ is closed, this number coincides with $\rho(A^\dagger B)^{-1} = \|B^{1/2}A^\dagger B^{1/2}\|^{-1}$.

Proof. If $R(A)$ is closed, then $R(A) = R(A^{1/2})$ and $D = (A^{1/2})^\dagger B^{1/2}$. Hence

$$\|D\|^2 = \|D^*D\| = \|B^{1/2}A^\dagger B^{1/2}\| = \rho(B^{1/2}A^\dagger B^{1/2}) = \rho(A^\dagger B). \quad \square$$

Corollary 2.3. *Let $A \in L(\mathcal{H})^+$ and $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Consider the rank one projection $P = \xi \otimes \xi = P_\xi$ onto the subspace generated by ξ . If*

$$I_\xi(A) = \max\{\mu \geq 0 : A - \mu P \geq 0\},$$

then $I_\xi(A) \neq 0 \iff \xi \in R(A^{1/2})$. In this case, if $\eta \in \ker A^\perp$ satisfies $A^{1/2}\eta = \xi$, we get $I_\xi(A) = \|\eta\|^{-2}$. If $\xi \in R(A)$, then for every $\zeta \in \mathcal{H}$ such that $A\zeta = \xi$ it holds $I_\xi(A) = \langle A\zeta, \zeta \rangle^{-1}$. If $R(A)$ is closed, then $I_\xi(A) = \langle A^\dagger \xi, \xi \rangle^{-1}$.

Proof. The first part follows from Corollary 2.2. Let $\eta \in \ker A^\perp$ such that $A^{1/2}\eta = \xi$. Then the reduced solution of the equation $A^{1/2}X = P$ is $\eta \otimes \xi$, the one rank operator defined by

$$\eta \otimes \xi(\gamma) = \langle \gamma, \xi \rangle \eta, \quad \gamma \in \mathcal{H}.$$

It is easy to see that $\|\eta \otimes \xi\| = \|\xi\| \|\eta\| = \|\eta\|$. If there exists $\zeta \in \mathcal{H}$ such that $A\zeta = \xi$, then $A^{1/2}\zeta = \eta$ and $\langle A\zeta, \zeta \rangle = \|\eta\|^2$. If $R(A)$ is closed, then $\eta = (A^{1/2})^\dagger \xi$, so that $\|\eta\|^2 = \langle \eta, \eta \rangle = \langle A^\dagger \xi, \xi \rangle$. \square

2.4. Suppose that $\dim \mathcal{H} = n < \infty$. We identify $L(\mathcal{H})$ with $M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices. Let $A \in L(\mathcal{H})^+$. In [16] the notion of minimal index for A was defined as

$$\begin{aligned}
 I(A) &= \max \{ \mu \geq 0 : A \circ B \geq \mu \cdot B \quad \forall B \in L(\mathcal{H})^+ \} \\
 &= \max \{ \mu \geq 0 : \Phi_A - \mu \cdot \text{Id} \geq 0 \quad \text{on } L(\mathcal{H})^+ \} \\
 &= \max \{ \mu \geq 0 : A - \mu \cdot ee^t \geq 0 \},
 \end{aligned}
 \tag{4}$$

where $e = (1, \dots, 1)$, the symbol \circ denotes the Hadamard product of matrices and $\Phi_A(C) = A \circ C$, $C \in L(\mathcal{H})$. The last equality follows from the fact that for $C \in L(\mathcal{H})$, $\Phi_C \geq 0 \Leftrightarrow C \geq 0$ (see [13]).

Note that, if $\xi = n^{-1/2}e$, then $\|\xi\| = 1$ and $I(A) = n^{-1}I_\xi(A)$. By Corollary 2.3, $I(A) > 0$ if and only if e belongs to the range of A . In [7,16] it is shown that, in this case, for any vector y such that $A(y) = e$,

$$I(A) = \langle Ay, y \rangle^{-1} = \langle A^\dagger e, e \rangle^{-1} = \min \{ \langle Az, z \rangle : \langle z, e \rangle = 1 \}.
 \tag{5}$$

Note that the first two equalities are particular cases of Corollary 2.3.

3. The shorted operator for selfadjoint operators

Let $A = A^* \in L(\mathcal{H})$ and $P \in \mathbb{P}$. We first need a characterization of those pairs (A, P) such that, for some $\mu \in \mathbb{R}$, it holds

$$A - \mu P \geq 0.
 \tag{6}$$

The solution of this problem is well known, see for example [14]. We shall give a brief survey of the characterization of pairs (A, P) satisfying Eq. (6), for the sake of completeness.

Note that if $Px = 0$ then $\langle (A - \mu P)x, x \rangle = \langle Ax, x \rangle$. Thus, a necessary condition for A and P to satisfy condition (6) is that $(1 - P)A(1 - P) \geq 0$.

Definition 3.1. Let $A \in L(\mathcal{H})$ such that $A = A^*$ and let $P \in \mathbb{P}$. We shall say that A is P -positive if $(1 - P)A(1 - P) \geq 0$.

Remark 3.2. Let $e = (1, \dots, 1) \in \mathbb{C}^n$ and let $P_e \in M_n(\mathbb{C})$ denote the orthogonal projection onto the subspace generated by e . A real symmetric matrix $A \in M_n(\mathbb{R})$ is called *almost positive* if $\langle A\xi, \xi \rangle \geq 0$ for all $\xi \in \mathbb{R}^n$ such that $\langle \xi, e \rangle = 0$. Therefore a real selfadjoint matrix $A \in M_n(\mathbb{C})$ is almost positive if and only if it is P_e -positive.

Proposition 3.3. Let $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$ and $A \in L(\mathcal{H})$ be hermitian and P -positive. Let $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ be the representation induced by P . Then the following conditions are equivalent:

1. There exists $\mu \in \mathbb{R}$ such that $A - \mu P \geq 0$.
2. The partial matrix $\begin{pmatrix} a & b \\ b^* & ? \end{pmatrix}$ admits a positive completion.

3. The set $M(A, \mathcal{S}) = \{D \in L(\mathcal{H}) : D = D^*, D \leq A, R(D) \subseteq \mathcal{S}\}$ is not empty.
4. There exists $x \in L(\mathcal{S}, \mathcal{S}^\perp)$ such that $b = a^{1/2}x$.
5. $R(b) \subseteq R(a^{1/2})$.
and, if $R(a)$ is closed, also
6. $\ker a = \ker A \cap \mathcal{S}^\perp$.

Proof. 1 \rightarrow 2: Take $d = c - \mu P$. Then $\begin{pmatrix} a & b \\ b^* & d \end{pmatrix} = A - \mu P \geq 0$.

2 \rightarrow 3: Let $d \in L(\mathcal{S})$ such that $\begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \geq 0$. Then $D = \begin{pmatrix} 0 & 0 \\ 0 & c - d \end{pmatrix} \in M(A, \mathcal{S})$.

3 \rightarrow 1: If $D \in M(A, \mathcal{S})$, take $\mu \in \mathbb{R}$ such that $-D \leq -\mu P$.

4 \leftrightarrow 5: It is a consequence of Douglas Theorem 2.1.

2 \leftrightarrow 5: It is well known (see [1] or [14]). For example, if $b = a^{1/2}x$ with $x \in L(\mathcal{S}, \mathcal{S}^\perp)$, then

$$\begin{pmatrix} a & b \\ b^* & x^*x \end{pmatrix} = \begin{pmatrix} a^{1/2} & 0 \\ x^* & 0 \end{pmatrix} \begin{pmatrix} a^{1/2} & x \\ 0 & 0 \end{pmatrix} \geq 0.$$

If $R(a)$ is closed, then $R(a^{1/2}) = R(a) = (\ker a)^\perp$. In this case

$$\begin{aligned} R(b) \subseteq R(a) &\Leftrightarrow \ker a \subseteq \ker b^* \\ &\Leftrightarrow (\forall \xi \in \mathcal{S}^\perp, a\xi = 0 \Rightarrow a\xi + b^*\xi = A\xi = 0), \end{aligned}$$

i.e., condition 5 is equivalent to $\ker a \subseteq \ker A \cap \mathcal{S}^\perp$. Note that the reverse inclusion always holds. \square

Remark 3.4. With the notations of Proposition 3.3, if $R(a)$ is not closed, then conditions 1–5 still imply, with the same proof, that $\ker a = \ker A \cap \mathcal{S}^\perp$.

Definition 3.5. Let $A \in L(\mathcal{H})$ be hermitian and $P \in \mathbb{P}$ such that A is P -positive.

1. A is called P -complementable if any of the conditions of Proposition 3.3 holds.
2. In this case we define: $I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geq 0\}$.

If A is P -complementable, the shorted operator can be defined for the pair (A, P) , and several results for shorted operators of positive operators (see [1]) remain true in this case. We show these properties in the rest of this section.

Definition 3.6. Let $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$ and let $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in L(\mathcal{H})$ be hermitian P -complementable. Let $d \in L(\mathcal{S}, \mathcal{S}^\perp)$ be the reduced solution of the equation $b = a^{1/2}x$. Then we define the Schur complement (or shorted operator) of A with respect to \mathcal{S} as

$$\Sigma(A, P) = \begin{pmatrix} 0 & 0 \\ 0 & c - d^*d \end{pmatrix}.$$

Proposition 3.7. *Let $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$ and let A be P -complementable.*

1. *If $A \geq 0$, then $\Sigma(A, P)$ is the usual shorted operator for A and \mathcal{S} .*
2. *Let $\mu \in \mathbb{R}$. Then $\Sigma(A - \mu P, P) = \Sigma(A, P) - \mu P$.*
3. *$\Sigma(A, P) = \max\{D \in L(\mathcal{H}) : D = D^*, D \leq A, R(D) \subseteq \mathcal{S}\}$.*
4. *$\Sigma(A, P) = \inf\{QAQ^* : Q = Q^2, R(Q) = \mathcal{S}\}$.*
5. *Let $\xi \in \mathcal{S}$. Then*

$$\langle \Sigma(A, P)\xi, \xi \rangle = \inf \{ \langle A(\xi + \eta), \xi + \eta \rangle, \eta \in \mathcal{S}^\perp \}.$$

6. *If $a = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ and $R(a)$ is closed, then*

$$\Sigma(A, P) = \begin{pmatrix} 0 & 0 \\ 0 & c - b^*a^\dagger b \end{pmatrix}.$$

where a^\dagger is the Moore–Penrose pseudoinverse of a in $L(\mathcal{S}^\perp)$.

Proof

1. It is shown in [1].
2. It is clear by definition.
3. If $A \geq 0$, then $\Sigma(A, P) = \max\{D \in L(\mathcal{H}) : D \geq 0, D \leq A \text{ and } R(D) \subseteq \mathcal{S}\}$ (see [1]). The general case can be easily deduced from the positive case using item 2.
4. If $A \geq 0$, then $\Sigma(A, P) = \inf\{QAQ^* : Q = Q^2, R(Q) = \mathcal{S}\}$ (see [1]). The general case can be easily deduced from the positive case using item 2 and the fact that, if $Q \in \mathbb{Q}$ has $R(Q) = \mathcal{S}$, then $QPQ^* = P$.
5. The positive was shown in [1]. If $A \not\geq 0$, denote by $B = A - I_P(A)P \geq 0$. By item 2, $\Sigma(A, P) = \Sigma(B, P) + I_P(A)P$. Thus,

$$\begin{aligned} \langle \Sigma(B, P)\xi, \xi \rangle &= \inf \{ \langle B(\xi + \eta), \xi + \eta \rangle, \eta \in \mathcal{S}^\perp \} \\ &= \inf \{ \langle A(\xi + \eta), \xi + \eta \rangle, \eta \in \mathcal{S}^\perp \} - I_P(A)\|\xi\|^2. \end{aligned}$$

6. If $R(a)$ is closed, then $R(a^{1/2})$ is also closed, $(a^{1/2})^\dagger = (a^\dagger)^{1/2}$ and $d = (a^{1/2})^\dagger b$ is the reduced solution of the equation $a^{1/2}x = b$. \square

Remark 3.8. The following properties are easy consequences of Proposition 3.7 and the corresponding results for the positive case (see [1,11]):

1. Let $P, Q \in \mathbb{P}$ such that $P \leq Q$, let $A \in L(\mathcal{H})$ P -complementable and $B \in L(\mathcal{H})$ such that $A \leq B$. Then B is P -complementable, $\Sigma(A, P) \leq \Sigma(B, P)$, A is Q -complementable and $\Sigma(A, P) \leq \Sigma(A, Q)$.

2. Let $\{E_n\} \in L(H)$ be a monotone decreasing sequence of positive operators strongly convergent to 0 and let $A \in L(H)$ be P -complementable. Then $\Sigma(A + E_n, P)$ converges strongly to $\Sigma(A, P)$.
3. Let $A \in L(H)$ be an invertible P -complementable operator. Then $\|\Sigma(A + \epsilon, P) - \Sigma(A, P)\| \rightarrow 0$ as $\epsilon \rightarrow 0^+$.
4. Let $A \in L(H)$ be P -complementable. Then there exist unique operators F and G such that $A = F + G$ with $R(F) \subseteq \mathcal{S}$, $G \geq 0$ and $R(G^{1/2}) \cap \mathcal{S} = \{0\}$.
5. Let $A \in L(\mathcal{H})$ and $P \in \mathbb{P}$ such that A is P -complementable. Let f be a operator monotone map defined on $\sigma(A) \cup \sigma(\Sigma(A, P))$ such that $f(0) \geq 0$. Then $\Sigma(f(A), P) \geq f(\Sigma(A, P))$.
6. Let $\{P_n\} \in \mathbb{P}$ be a decreasing sequence of projections such that $P_n \xrightarrow{\text{S.O.T}} P$ and let $A \in L(\mathcal{H})$ be P -complementable. Then $\{\Sigma(A, P_n)\}$ decreases to $\Sigma(A, P)$ (see [14] or [2]).

4. A -selfadjoint projections

Given $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$ and $A \in L(\mathcal{H})$ P -positive, we shall consider a condition stronger than being P -complementable which is the existence of A -selfadjoint projections onto \mathcal{S}^\perp , i.e., $Q \in \mathbb{Q}$ such that $AQ = Q^*A$ and $R(Q) = \mathcal{S}^\perp$.

Definition 4.1. Let $A = A^* \in L(\mathcal{H})$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. We denote by

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in \mathbb{Q} : R(Q) = \mathcal{S}^\perp, AQ = Q^*A\}.$$

The pair (A, \mathcal{S}) is said to be *compatible* if $\mathcal{P}(A, \mathcal{S})$ is not empty.

The notion of a compatible pair was introduced in [4], where a characterization of compatible pairs (A, \mathcal{S}) in terms of the Schur complements $\Sigma(A, P)$ is given, in case that $A \geq 0$. The following two results are taken from [4]:

Lemma 4.2. Let $A = A^* \in L(\mathcal{H})$ and $Q \in \mathbb{Q}$. Then the following conditions are equivalent:

1. Q satisfies that $AQ = Q^*A$, i.e., Q is A -selfadjoint.
2. $\ker Q \subseteq A^{-1}(R(Q)^\perp)$,
and, if $A \geq 0$,
3. $Q^*AQ \leq A$.

Proposition 4.3. Given $A = A^* \in L(\mathcal{H})$ and $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$, the following conditions are equivalent:

1. The pair (A, \mathcal{S}) is compatible (i.e., $\mathcal{P}(A, \mathcal{S})$ is not empty).

2. If $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ then $R(b) \subseteq R(a)$.
3. $\mathcal{S}^\perp + A^{-1}(\mathcal{S}) = \mathcal{H}$.

In this case, for every $E \in \mathcal{P}(A, \mathcal{S})$, $\ker E \subseteq A^{-1}(\mathcal{S})$.

Corollary 4.4. *If (A, \mathcal{S}) is compatible and A is P -positive, then A is P -complementable.*

Proof. Just note that, if $a = (1 - P)A(1 - P) \geq 0$, then $R(a) \subseteq R(a^{1/2})$. \square

Remark 4.5. Let $A \in L(\mathcal{H})$ be hermitian and $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$ such that A is P -positive and suppose that $R((1 - P)A(1 - P))$ is closed. Then (A, \mathcal{S}) is compatible if and only if A is P -complementable. This last condition holds whenever $\dim \mathcal{S}^\perp < \infty$. Therefore if \mathcal{H} is a finite dimensional space and A is P -positive, the conditions (A, \mathcal{S}) is compatible and A is P -complementable are equivalent.

Proposition 4.6. *Let $A = A^* \in L(\mathcal{H})$ such that A is P -positive and the pair (A, \mathcal{S}) is compatible. Let $E \in \mathcal{P}(A, \mathcal{S})$ and $Q = I - E$. Then*

1. $\Sigma(A, P) = AQ = Q^*A = Q^*AQ$.
2. $\Sigma(A, P) = \min\{FAF^* : F \in \mathbb{Q}, R(F) = \mathcal{S}\}$.
3. $R(\Sigma(A, P)) \subseteq R(A) \cap S$.

Proof. The case $A \geq 0$ was shown in [4] (with equality in item 3). The general case follows from the fact that if $F \in \mathbb{Q}$ and $R(F) = \mathcal{S}$, then $FP = PF^* = FPF^* = P$. Recall that if $B = A - I_P(A)P$, then $\Sigma(A, P) = \Sigma(B, P) + I_P(A)P$; and $R((I - E)^*) = \ker(I - E)^\perp = \mathcal{S}$. Item 3 is clear because $R(AQ) \subseteq R(A)$. \square

Lemma 4.7. *Let $A = A^* \in L(\mathcal{H})$ and $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$. Suppose that A is P -positive and (A, \mathcal{S}) is compatible. Let $E \in \mathcal{P}(A, \mathcal{S})$ and $Q = I - E$. Consider the operator $T = (1 - P) + Q$. Then*

1. $T \in \text{Gl}(\mathcal{H})$ with $T^{-1} = E + P$.
2. If $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ in terms of P , then

$$T^*AT = \begin{pmatrix} a & 0 \\ 0 & \Sigma(A, P) \end{pmatrix}. \tag{7}$$

3. If $A \in \text{Gl}(\mathcal{H})$ then $a \in \text{Gl}(\mathcal{S}^\perp)$ and $\Sigma(A, P) \in \text{Gl}(\mathcal{S})$. Moreover, if we view $\Sigma(A, P) \in L(\mathcal{S})$, then $\Sigma(A, P)^{-1} = PA^{-1}P$ or, in other words,

$$\Sigma(A, P) = (PA^{-1}P)^\dagger. \tag{8}$$

Proof

1. Since $R(1 - P) = R(E) = \ker P = \ker Q = \mathcal{S}^\perp$, then $(1 - P)E = E$ and $QP = Q$. Thus $T(E + P) = E + Q = 1$. The other case is similar.
2. The fact that $R(Q) = \ker E \subseteq A^{-1}(\mathcal{S})$ implies that $Q^*A(1 - P) = (1 - P)AQ = 0$. By Proposition 4.6, $Q^*AQ = \Sigma(A, P)$.
3. Note that $(T^*AT)^{-1} = T^{-1}A^{-1}(T^*)^{-1} = (E + P)A^{-1}(E^* + P)$. But $PE = E^*P = 0$, so that $\Sigma(A, P)^{-1} = P(T^*AT)^{-1}P = PA^{-1}P$. \square

Proposition 4.8. Let $A = A^* \in L(\mathcal{H})$ and $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$. Suppose that A is P -positive and (A, \mathcal{S}) is compatible. Then $R(\Sigma(A, P)) = R(A) \cap \mathcal{S}$.

Proof. We use the notations of Lemma 4.7. By formula (7), $R(T^*AT) \cap \mathcal{S} = R(\Sigma(A, P))$. On the other hand, if $\xi \in \mathcal{S}$, then $T^*\xi = Q^*\xi = \xi$, because $R(Q^*) = \ker Q^\perp = \mathcal{S}$ and $Q^* \in \mathbb{Q}$. Hence $R(A) \cap \mathcal{S} = R(AT) \cap \mathcal{S} \subseteq R(T^*AT) \cap \mathcal{S} = R(\Sigma(A, P))$. The reverse inclusion was shown in Proposition 4.6. \square

5. Computation of $I_P(A)$

Let $P \in \mathbb{P}$ and $A = A^* \in L(\mathcal{H})$. Recall that, if A is P -complementable, we have defined

$$I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geq 0\}.$$

Remark 5.1. If $A \geq 0$ then, by Corollary 2.2, $I_P(A) \neq 0$ if and only if $R(P) \subseteq R(A^{1/2})$ and, in this case, $I_P(A) = \|D\|^{-2}$, where D is the reduced solution of the equation $A^{1/2}X = P$. Thus, if $R(A)$ is closed, then $I_P(A) = \rho(A^\dagger P)$.

Suppose now that $A \not\geq 0$. It is easy to see that if $B = A + \mu P$, then $I_P(B) = I_P(A) + \mu$. Therefore a way to compute $I_P(A)$ would be to find a lower bound $\mu \leq I_P(A)$ in order to compute firstly $I_P(B)$ for $B = A - \mu P \geq 0$, reducing the general case to the positive case. Nevertheless this way seems to be not applicable. For example, it is easy to get, for any $M > 0$, selfadjoint matrices $A \in M_2(\mathbb{C})$ with $\|A\| \leq 2$ such that $I_P(A) < -M$, where P is a fixed projection of rank one. Indeed, take $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 0 \end{pmatrix}$, for $\varepsilon < M^{-1}$.

We first show the key relation between $I_P(A)$ and the shorted operator $\Sigma(A, P)$:

Proposition 5.2. Let $A \in L(\mathcal{H})$ be hermitian, $A \not\geq 0$, and $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$ such that A is P -complementable. Then

$$I_P(A) = \lambda_{\min}(\Sigma(A, P)) = \min\{\langle \Sigma(A, P)\xi, \xi \rangle : \xi \in \mathcal{S}, \|\xi\| = 1\}. \quad (9)$$

Proof. Denote by $\mu = \lambda_{\min}(\Sigma(A, P))$. Since $A \not\geq 0$, it is easy to see that $\mu < 0$. In particular this shows the last equality in Eq. (9). Note that $\mu P \leq \Sigma(A, P)$, so that

$$A - \mu P \geq A - \Sigma(A, P) \geq 0 \quad \text{and} \quad \mu \leq I_P(A).$$

On the other hand, since $A - I_P(A)P \geq 0$, then $I_P(A)P \in M(A, \mathcal{S})$ and $I_P(A)P \leq \Sigma(A, P)$ (see Propositions 3.3 and 3.7), which implies that $I_P(A) \leq \mu$. \square

Remark 5.3. With the notations of Proposition 5.2, if $A \geq 0$, then the identity $I_P(A) = \min\{\langle \Sigma(A, P)\xi, \xi \rangle : \xi \in \mathcal{S}, \|\xi\| = 1\}$ remains true; and this number coincides with $\lambda_{\min}(\Sigma(A, P))$ if we consider the spectrum of $\Sigma(A, P)$ as an operator of $L(\mathcal{S})$ (in order to remove the number 0 if necessary).

The following properties of $I_P(A)$ follow immediately from Remark 3.8 and Proposition 5.2.

Corollary 5.4. *Let $A \in L(\mathcal{H})$ be hermitian and $P \in \mathbb{P}$ such that A is P -complementable:*

1. *Let $Q \in \mathbb{P}$ such that $P \leq Q$ and suppose that $A \not\geq 0$. Then $I_P(A) \leq I_Q(A)$. If $A \geq 0$ this property may fail because of the fact observed in Remark 5.3.*
2. *Let $B \in L(\mathcal{H})$ such that $A \leq B$. Then $I_P(A) \leq I_P(B)$.*
3. *Let $\{E_n\} \in L(\mathcal{H})$ be a monotone (not necessary strictly) decreasing sequence of positive operators strongly convergent to 0. Then the sequence $\{I_P(A + E_n)\}$ decreases to $I_P(A)$.*
4. *Let $\{A_n\} \in L(\mathcal{H})$ be a sequence of P -complementable operators which is norm convergent to an invertible P -complementable operator A . Then $\{I_P(A_n)\}$ converges to $I_P(A)$.*
5. *Let f be a operator monotone map defined on $\sigma(A) \cup \sigma(\Sigma(A, P))$ such that $f(0) \geq 0$. Then $I_P(f(A)) \geq f(I_P(A))$.*
6. *Let $\{P_n\}$ be a decreasing sequence of orthogonal projections such that $P_n \xrightarrow{\text{S.O.T}} P$. Then $\{I_{P_n}(A)\}$ decreases to $I_P(A)$.*

Remark 5.5. It was pointed out in [16] that the hypothesis in item 3 can not be relaxed, i.e the map $A \mapsto I_P(A)$ is not norm continuous in general, as we see in the following example.

Example 5.6. Let $a \neq 1$ and $\{b_n\} \subseteq \mathbb{R}_{>a}$ such that $\lim_{n \rightarrow \infty} b_n = a$. Then the sequence of positive matrices

$$A_n = \begin{pmatrix} a^2 + a^{-2} & ab_n + (ab_n)^{-1} \\ ab_n + (ab_n)^{-1} & b_n^2 + b_n^{-2} \end{pmatrix}$$

converges in norm to $A = (a^2 + a^{-2})ee^t$, where $e = (1, 1)$. Let $x_n = (a, b_n)$ and $y_n = (a^{-1}, b_n^{-1})$. Note that $A_n = x_n x_n^* + y_n y_n^*$ and $e = \lambda_n x_n + \mu_n y_n$, with $\lambda_n = (a + b_n)^{-1}$ and $\mu_n = ab_n(a + b_n)^{-1}$. If a vector z satisfies that $A_n z = e$, then

$$e = A_n z = (x_n x_n^* + y_n y_n^*)z = \langle z, x_n \rangle x_n + \langle z, y_n \rangle y_n$$

and $\langle z, e \rangle^{-1} = (\langle z, x_n \rangle^2 + \langle z, y_n \rangle^2)^{-1} = \frac{(a+b_n)^2}{1+a^2 b_n^2}$. Since $I_{P_e}(A_n) = 2\langle A_n^{-1}e, e \rangle^{-1} = \frac{2(a+b_n)^2}{1+a^2 b_n^2}$, we get

$$\lim_{n \rightarrow \infty} I_{P_e}(A_n) = \lim_{n \rightarrow \infty} \frac{2(a+b_n)^2}{a^2 b_n^2 + 1} = \frac{8}{a^2 + a^{-2}} \neq 2(a^2 + a^{-2}) = I_{P_e}(A).$$

The following results are the natural generalizations of formula (5) to our setting.

Corollary 5.7. *Let $A \in L(\mathcal{H})$ be hermitian and $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$ such that A is P -complementable. Then*

$$I_P(A) = \inf\{\langle A\xi, \xi \rangle : \xi \in \mathcal{H}, \|P\xi\| = 1\}. \quad (10)$$

Proof. It is a consequence of Eq. (9) in Proposition 5.2 (or Remark 5.3 in case that $A \geq 0$) and item 5 of Proposition 3.7. \square

Corollary 5.8. *Let A and P be as above and suppose that $P = \xi \otimes \xi$ for some unit vector $\xi \in \mathcal{H}$. Then*

$$I_P(A) = \inf\{\langle A\eta, \eta \rangle : \eta \in \mathcal{H}, \langle \eta, \xi \rangle = 1\}. \quad (11)$$

Proof. Note that $P\eta = \langle \eta, \xi \rangle \xi$ and $\|P\eta\| = |\langle \eta, \xi \rangle|$. Also, if $\omega \in \mathbb{C}$ has $|\omega| = 1$, then $\langle A\omega\eta, \omega\eta \rangle = \langle A\eta, \eta \rangle$. \square

Throughout, we shall consider $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$ and $A \in L(\mathcal{H})$ P -positive such that (A, \mathcal{S}) is compatible. In this case almost all results which can be shown for matrices can be extended to the infinite dimensional case.

Remark 5.9. Let $A \in L(\mathcal{H})$ be hermitian and $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$ such that (A, \mathcal{S}) is compatible. Suppose that $I_P(A) \neq 0$. Then

$$R(A) \cap \mathcal{S} = R(\Sigma(A, P)) \neq \{0\}.$$

Indeed, since (A, \mathcal{S}) is compatible, $R(\Sigma(A, P)) = R(A) \cap \mathcal{S}$ by Proposition 4.8. On the other hand, $0 \neq I_P(A) = \lambda_{\min}(\Sigma(A, P))$, by Proposition 5.2. Hence $\Sigma(A, P) \neq 0$.

Theorem 5.10. *Let $A \in L(\mathcal{H})$ be hermitian and $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$, such that A is P -positive and (A, \mathcal{S}) is compatible with $I_P(A) \neq 0$. Denote by $\mathcal{T} = \mathcal{S} \cap R(A)$ and $Q = P_{\mathcal{T}}$. Then*

1. A is Q -complementable. Moreover, the pair (A, \mathcal{T}) is compatible.
2. $\Sigma(A, P) = \Sigma(A, Q)$.
3. $I_P(A) = I_Q(A)$.

Proof. If $A \geq 0$, by Remark 5.3, we know that $\Sigma(A, P)$ is invertible in $L(\mathcal{S})$. On the other hand, since (A, \mathcal{S}) is compatible, $\mathcal{S} = R(\Sigma(A, P)) = R(A) \cap \mathcal{S} \subseteq R(A)$.

Suppose now that $A \not\geq 0$. By Remark 5.9, $R(\Sigma(A, P)) = R(A) \cap \mathcal{S} \subseteq \mathcal{T}$. Hence

$$\Sigma(A, P) \in M(A, \mathcal{T}) = \{D \in L(\mathcal{H}) : D = D^*, D \leq A, R(D) \subseteq \mathcal{T}\} \neq \emptyset.$$

Therefore, by Proposition 3.3, A is Q -complementable and, by Proposition 3.7, $\Sigma(A, P) \leq \Sigma(A, Q)$. The inequality $\Sigma(A, Q) \leq \Sigma(A, P)$ follows by Remark 3.8. Then,

$$I_P(A) = \lambda_{\min}(\Sigma(A, P)) = \lambda_{\min}(\Sigma(A, Q)) = I_Q(A).$$

Using Proposition 4.3 item 3, in order to show that the pair (A, \mathcal{T}) is compatible, it suffices to verify that $\mathcal{T}^\perp + A^{-1}(\mathcal{T}) = \mathcal{H}$, which follows from the following facts: $\mathcal{S}^\perp + A^{-1}(\mathcal{S}) = \mathcal{H}$ (since (A, \mathcal{S}) is compatible), $\mathcal{S}^\perp \subseteq \mathcal{T}^\perp$ and $A^{-1}(\mathcal{S}) = A^{-1}(\mathcal{S} \cap R(A)) \subseteq A^{-1}(\mathcal{T})$. \square

Remark 5.11. When $\dim \mathcal{S} = 1$, if A is P -positive and P -compatible we can deduce that $\mathcal{S} \subseteq R(A)$. More generally, if $\dim \mathcal{S} < \infty$, A is injective and (A, \mathcal{S}) is compatible, then $\mathcal{S} \subseteq R(A)$. Indeed, note that $\dim A^{-1}(\mathcal{S}) = \dim \mathcal{S} \cap R(A)$, and $A^{-1}(\mathcal{S})$ must be a supplement of \mathcal{S}^\perp . Nevertheless, if we remove the condition (A, \mathcal{S}) is compatible, this is not true, even if $\dim \mathcal{S} = 1$ and A is injective and P -complementable, as the following example shows.

Example 5.12. Let $A \in L(\mathcal{H})^+$ be injective non-invertible. Let $\xi \in \mathcal{H} \setminus R(A)$ be a unit vector. Denote by \mathcal{S} the subspace generated by ξ , $P = P_{\mathcal{S}}$. If

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

in terms of P and $A\xi = \lambda\xi + \eta$ with $\eta \in \mathcal{S}^\perp$, then $\lambda = \langle A\xi, \xi \rangle \neq 0$ and $\eta \neq 0$ (otherwise $\xi \in R(A)$). Therefore $c = \lambda P$ and $b(\mu\xi) = \mu\eta$, $\mu \in \mathbb{C}$.

Suppose that $\eta \in R(a)$, i.e., there exists $v \in \mathcal{S}^\perp$ which verifies $av = b\xi$. Then $(1 - P)A(v - \xi) = av - b\xi = 0$, so $A(v - \xi)$ is a multiple of ξ , which must be 0 ($\xi \notin R(A)$). So $v = \xi$, a contradiction. Therefore $R(b) \not\subseteq R(a)$ and the pair (A, \mathcal{S}) is incompatible.

Now consider $B = A + \mu P$, for any $\mu \in \mathbb{R}$. It is clear that B must be P -complementable ($B - \mu P = A \geq 0$). But the facts that A is injective and $\xi \notin R(A)$, clearly imply that B is injective and $\xi \notin R(B)$.

5.13. Fix $E \in \mathbb{P}$ with range \mathcal{M} . Denote by $L(\mathcal{H})_{\mathcal{M}} = \{C \in L(\mathcal{H}) : ECE = C\}$. For $C \in L(\mathcal{H})_{\mathcal{M}}$, denote by $C_0 \in L(\mathcal{M})$ the compression of C to \mathcal{M} . With respect to the matrix representation induced by E

$$C = \begin{pmatrix} 0 & 0 \\ 0 & C_0 \end{pmatrix} \begin{matrix} \mathcal{M}^\perp \\ \mathcal{M} \end{matrix}.$$

The following properties of this compression are easy to see:

1. The map $L(\mathcal{H})_{\mathcal{M}} \ni C \mapsto C_0 \in L(\mathcal{M})$ is a $*$ -isomorphism of C^* -algebras, i.e., it is isometric and compatible with sums, products and adjoints.
2. If $C = C^* \in L(\mathcal{H})_{\mathcal{M}}$ and $R(C) = \mathcal{M}$, then $C_0 \in \text{Gl}(\mathcal{M})$ and $(C_0)^{-1} = (C^\dagger)_0$. If $R(C)$ is closed, then $(C_0)^\dagger = (C^\dagger)_0$.

Theorem 5.14. Let $A \in L(\mathcal{H})$ be hermitian, $A \not\geq 0$, and $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$ such that A is P -positive and (A, \mathcal{S}) is compatible. Suppose that $R(A)$ is closed. Denote by $\mathcal{F} = \mathcal{S} \cap R(A)$ and $Q = P_{\mathcal{F}}$. Then

$$I_P(A) = I_Q(A) = \lambda_{\min}(QA^\dagger Q)^\dagger. \quad (12)$$

Proof. Since we only need to prove the equality $I_Q(A) = \lambda_{\min}(QA^\dagger Q)^\dagger$, we shall directly suppose that $R(P) \subseteq R(A)$. Denote $\mathcal{M} = R(A)$ and $E = P_{\mathcal{M}}$. Using the notations of 5.13, we have that A, P and $\Sigma(A, P) \in L(\mathcal{H})_{\mathcal{M}}$. It is clear that $\Sigma(A, P)_0 = \Sigma(A_0, P_0)$, $I_P(A) = I_{P_0}(A_0)$ and A_0 is invertible. Therefore, by Lemma 4.7,

$$\Sigma(A_0, P_0) = (P_0(A_0)^{-1}P_0)^\dagger = (PA^\dagger P)_0^\dagger = ((PA^\dagger P)^\dagger)_0$$

and

$$\begin{aligned} I_P(A) &= I_{P_0}(A_0) = \lambda_{\min} \Sigma(A_0, P_0) = \lambda_{\min}((PA^\dagger P)^\dagger)_0 \\ &= \lambda_{\min}(PA^\dagger P)^\dagger. \quad \square \end{aligned}$$

6. Some applications

The problem of calculating $I_P(A)$ of a P -complementable operator A with respect to a projection P has already been considered for certain projections P , mainly in the finite dimensional case. Reams [15] showed that if $A \in M_n(\mathbb{R})$ is invertible and almost positive (see Remark 3.2), then A is P_e -complementable and $I_{P_e}(A) = n \cdot \langle A^{-1}e, e \rangle^{-1}$, where $e = (1, \dots, 1) \in \mathbb{C}^n$ and P_e denotes the orthogonal projection onto the subspace generated by e . We obtain a generalization of this result in the non-positive case. The general positive case was already considered in [7] and Corollary 2.3 (for every unit vector $\xi \in \mathbb{C}^n$).

Corollary 6.1. *Let $\xi \in \mathbb{C}^n$ be a unit vector. Let $A \in M_n(\mathbb{C})$ be non-positive but P_ξ -positive. Then A is P_ξ -complementable if and only if*

$$\forall \eta \in \mathbb{C}^n, \langle \eta, \xi \rangle = 0 \quad \text{and} \quad \langle A\eta, \eta \rangle = 0 \Rightarrow A\eta = 0. \quad (13)$$

In this case $\xi \in R(A)$ and

$$I_{P_\xi}(A) = \langle A^\dagger \xi, \xi \rangle^{-1} = \min\{\langle Az, z \rangle : \langle z, \xi \rangle = 1\}. \quad (14)$$

Proof. Condition (13) is equivalent to $\text{Ker}((1 - P_\xi)A(1 - P_\xi)) \cap \{\xi\}^\perp = \text{Ker}(A) \cap \{\xi\}^\perp$. By Proposition 3.3, this is equivalent to the fact that A is P_ξ -complementable, since $R(A)$ is closed. Note that $I_{P_\xi}(A) < 0$, since $A \not\geq 0$. By Remarks 4.5 and 5.9 we get $R(A) \cap R(P_\xi) \neq \{0\}$. Therefore $\xi \in R(A)$ and $\langle A^\dagger \xi, \xi \rangle \neq 0$. By Eq. (12) in Theorem 5.14,

$$I_{P_\xi}(A) = \lambda_{\min}(P_\xi A^\dagger P_\xi)^\dagger = \lambda_{\min}(\langle A^\dagger \xi, \xi \rangle P_\xi)^\dagger = \langle A^\dagger \xi, \xi \rangle^{-1}.$$

In order to prove Eq. (14), it only remains to show that the infimum in Eq. (11) is actually a minimum. Let $\zeta = A^\dagger \xi$ and $\eta = \langle A^\dagger \xi, \xi \rangle^{-1} \zeta$. Then

$$\langle A\eta, \eta \rangle = \langle A^\dagger \xi, \xi \rangle^{-2} \langle A\zeta, \zeta \rangle = \langle A^\dagger \xi, \xi \rangle^{-2} \langle \xi, A^\dagger \xi \rangle = \langle A^\dagger \xi, \xi \rangle^{-1},$$

and the minimum is attained at η . \square

It was also noted in [15] that the problem of calculating $I_P(A)$ with respect to $P = P_e$ is equivalent to a problem posed by Fiedler and Markham in [9], that is to calculate

$$\max\{\lambda_{\min}((A \circ C)C^{-1}), C > 0\}$$

for a positive matrix $A \in M_n(\mathbb{C})$, where $A \circ B$ denotes the Hadamard product of A and B . The corollary above complements the results obtained in [9] in the non-positive, non-invertible case.

Recall that given a positive matrix $A \in M_n(\mathbb{C})$, the minimal index was introduced in [16] as

$$I_A = \max\{\mu \geq 0 : A \circ B \geq \mu B, B \geq 0\}.$$

Given $P \in M_n(\mathbb{C})$ an orthogonal projection and a P -complementable matrix A , there is a relation between $I_P(A)$ and the Schur multiplier induced by A .

Corollary 6.2. *Let $M = \{x_1, \dots, x_k\} \subseteq \mathbb{C}^n$ be an orthonormal set and let P be the orthogonal projection onto the subspace spanned by M . Suppose that $A \in M_n(\mathbb{C})$ is P -complementable. Then*

$$I_P(A) = \max \left\{ \mu \in \mathbb{R} : A \circ B \geq \mu \sum_{i=1}^k D_{x_i} B D_{x_i}^*, B \geq 0 \right\}, \quad (15)$$

where D_x denotes the diagonal matrix with main diagonal $x \in \mathbb{C}^n$.

Proof. First note that $P = \sum_{i=1}^k x_i x_i^*$. Thus $A - \mu P \geq 0$ if and only if every $B \geq 0$ satisfies $(A - \mu \sum_{i=1}^k x_i x_i^*) \circ B \geq 0$, which is equivalent to $A \circ B \geq \mu \sum_{i=1}^k D_{x_i} B D_{x_i}^*$

$BD_{x_i}^*$, since a simple calculation shows that $C \circ xx^* = D_x C D_x^*$ for every $C \in M_n(\mathbb{C})$ and $x \in \mathbb{C}^n$. This shows formula (15). \square

6.1. Completely positive maps on $M_n(\mathbb{C})$

Definition 6.3. Let $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a linear map. Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. Φ is selfadjoint if $\Phi(A^*) = \Phi(A)^*$ or equivalently if $\Phi(A)$ is selfadjoint whenever A is selfadjoint.

Let $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a linear map. If $m \in \mathbb{N}$, we denote $\Phi^{(m)} : M_m(M_n(\mathbb{C})) \rightarrow M_m(M_n(\mathbb{C}))$ the map given by

$$\Phi^{(m)}((a_{ij})_{ij}) = (\Phi(a_{ij}))_{ij}, \quad (a_{ij})_{ij} \in M_m(M_n(\mathbb{C})),$$

and call it the *inflation* of order m of Φ .

Definition 6.4. The linear map $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is called *completely positive* if $\Phi^{(m)}$ is positive for every $m \in \mathbb{N}$.

In the following, $\{e_{ij}\} \subseteq M_n(\mathbb{C})$ denotes the canonical basis for $M_n(\mathbb{C})$. Now we state a result due to Choi [3].

Theorem 6.5. Let $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a linear map. Then Φ is completely positive if and only if $\Phi^{(n)}((e_{ij})_{ij}) = (\Phi(e_{ij}))_{ij} \in M_n(M_n(\mathbb{C}))$ is positive.

Remark 6.6. Note that the matrix $E = ((e_{ij})_{ij}) \in M_n(M_n(\mathbb{C})) \simeq M_{n^2}(\mathbb{C})$ is a scalar multiple of a rank one projection. Indeed, if $\{e_i\}$ denotes the canonical basis of \mathbb{C}^n and $v \in \mathbb{C}^{n^2}$ is the vector $v = (e_1, \dots, e_n)$, then $(e_{ij})_{ij} = vv^*$. Thus $E = \frac{1}{n} P_v$, where P_v is the projection onto the subspace generated by v .

Remark 6.7. Let $A \in M_n(\mathbb{C})$. Then the linear map $\Phi_A : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ given by $\Phi_A(B) = A \circ B$ is selfadjoint (resp. positive) if and only if A is selfadjoint (resp. positive). Moreover, if $A \geq 0$, then Φ_A is completely positive, since the inflated matrix $A^{(n)} \geq 0$ and $\Phi_A^{(n)} = \Phi_{A^{(n)}}$ (see [13]). Therefore $\Phi_A - \mu \text{Id}$ is completely positive if and only if $A - \mu e e^* \geq 0$, where $e \in \mathbb{C}^n$ is given by $e = (1, \dots, 1)$. Note that $e e^* = n P_e$, since $\|e\| = n^{1/2}$. Therefore we conclude that for every P_e -complementable matrix A ,

$$I(A) = \max\{\mu \in \mathbb{R} : \Phi_A - \mu \text{Id} \text{ is completely positive}\} = \frac{1}{n} I_{P_e}(A),$$

where $I(A)$ is the minimal index of A defined in 2.4 (in fact, its natural generalization for A not necessarily positive, but P_e -complementable).

Definition 6.8. Let $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a selfadjoint map. We say that Φ is complementable if there exists $\mu \in \mathbb{R}$ such that $\Phi - \mu\text{Id}$ is completely positive. In this case we define:

$$I(\Phi) = \max\{\mu \in \mathbb{R} : \Phi - \mu\text{Id} \text{ is completely positive}\}.$$

Note that all completely positive maps Φ are complementable and $I(\Phi) \geq 0$. But in general not all selfadjoint maps are complementable. For example, if $A \in M_n(\mathbb{C})$ is selfadjoint, then Φ_A is complementable if and only if A is P_e -complementable.

Theorem 6.9. Let $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a selfadjoint map. Then, with the notations of Remark 6.6,

1. Suppose that Φ is not completely positive. In this case Φ is complementable if and only if for all $\eta_1, \dots, \eta_n \in \mathbb{C}^n$

$$\sum_{i=1}^n (\eta_i)_i = 0 \Rightarrow \sum_{i,j=1}^n \langle \Phi(e_{ij})\eta_j, \eta_i \rangle \geq 0 \tag{16}$$

and

$$\sum_{i=1}^n (\eta_i)_i = 0 \quad \text{and} \quad \sum_{i,j=1}^n \langle \Phi(e_{ij})\eta_j, \eta_i \rangle = 0 \Rightarrow \sum_{j=1}^n \Phi(e_{ij})\eta_j = 0, \tag{17}$$

$i = 1, \dots, n$

or, equivalently, if $A_\Phi = \Phi^{(n)}E = (\Phi(e_{ij}))_{ij} \in M_{n^2}(\mathbb{C})$ is P_v -complementable.

2. In this case $I(\Phi) = n \cdot I_{P_v}(A_\Phi)$ and we have

$$I(\Phi) = \min \left\{ \sum_{i,j=1}^n \langle \Phi(e_{ij})\eta_j, \eta_i \rangle : \eta_1, \dots, \eta_n \in \mathbb{C}^n \text{ and } \sum_{i=1}^n (\eta_i)_i = 1 \right\}. \tag{18}$$

3. If conditions (16) and (17) hold, there exist $\eta_1, \dots, \eta_n \in \mathbb{C}^n$ such that

$$\sum_{j=1}^n \Phi(e_{ij})\eta_j = e_i, \quad i = 1, \dots, n \tag{19}$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of \mathbb{C}^n . For any such vectors,

$$I(\Phi)^{-1} = \sum_{i,j=1}^n \langle \Phi(e_{ij})\eta_j, \eta_i \rangle. \tag{20}$$

4. If Φ is completely positive then it is complementable and $I(\Phi) \geq 0$. Moreover, $I(\Phi) > 0$ if and only if there exist $\eta_1, \dots, \eta_n \in \mathbb{C}^n$ such that Eq. (19) holds. For any such vectors, Eq. (20) holds. Also Eq. (18) is true in this case.

Proof. From Theorem 6.5 we conclude that the map Φ is complementable if and only if the matrix $A_\Phi = (\Phi(e_{ij}))_{ij}$ is P_v complementable. It is easy to see that in fact $I(\Phi) = n \cdot I_{P_v}(A_\Phi)$. Thus we can apply Corollary 6.1 to the matrix $A_\Phi \in M_{n^2}(\mathbb{C})$ and the projection P_v . Note that Eq. (16) holds if and only if A_Φ is P_v -positive and condition (13) is equivalent to condition (17). Indeed, if $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{C}^{n^2}$ with $\eta_i \in \mathbb{C}^n$ ($i = 1, \dots, n$), then $\langle \eta, v \rangle = \sum_{i=1}^n (\eta_i)_i$ and $\langle A_\Phi \eta, \eta \rangle = \sum_{i,j=1}^n \langle \Phi(e_{ij}) \eta_j, \eta_i \rangle$.

Note that condition (19) is equivalent to the fact that $v \in R(A_\Phi)$, so this condition and Eq. (18) follow from Eq. (14). Similarly, $I(\Phi) = n \cdot I_{P_v}(A_\Phi) = \langle v, A_\Phi^\dagger v \rangle$.

Let $\zeta = (\zeta_1, \dots, \zeta_n) = A_\Phi^\dagger v$. If $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{C}^{n^2}$ satisfy condition (19) (i.e., $A_\Phi \eta = v$), then $P_{R(A_\Phi)} \eta = \zeta$. Therefore

$$\sum_{i,j=1}^n \langle \Phi(e_{ij}) \eta_j, \eta_i \rangle = \langle A_\Phi \eta, \eta \rangle = \langle A_\Phi \zeta, \zeta \rangle = \langle v, A_\Phi^\dagger v \rangle = I(\Phi)^{-1}.$$

Suppose now that Φ is completely positive. It is clear that Φ is complementable. By Corollary 2.3, it follows that $I(\Phi) = n \cdot I_{P_v}(A_\Phi) > 0$ if and only if $v \in R(A_\Phi)$, since $R(A_\Phi)$ is closed. This is equivalent to condition (19), and using Corollary 2.3, we can also deduce Eqs. (20) and (18) in this case. \square

Example 6.10. Consider the map $T : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ given by

$$T(A) = \frac{1}{n} \text{Tr}(A) I_n = \frac{1}{n} \sum_{i,j=i}^n e_{ij}^* A e_{ij},$$

where $\text{Tr}(A) = \sum A_{ii}$ is the usual trace. Then T is completely positive; moreover it is a conditional expectation. Note that the matrix

$$A_T = (T(e_{ij}))_{ij} = \frac{1}{n} I_{n^2}.$$

Then $I(T) > 0$, since $A_T(n e_i) = e_i$, $1 \leq i \leq n$, and T satisfies condition (19). Therefore, since $T(e_{ij}) = 0$ if $i \neq j$ and $T(e_{ii}) = \frac{1}{n} I_n$, using Eq. (20),

$$I(T)^{-1} = \sum_{i=1}^n \langle T(e_{ii}) n e_i, n e_i \rangle = \sum_{i=1}^n n = n^2.$$

This result is actually known in index theory of conditional expectations (using that $T^{(n)}(P_v) = n^{-1} A_T = n^{-2} I_{n^2}$, see [12]). Note that the number

$$\begin{aligned} J(T) &= \max\{\lambda \in \mathbb{R} : T - \lambda \text{Id is positive (not completely)}\} \\ &= n^{-1} \neq n^{-2} = I(T). \end{aligned}$$

Indeed, it is easy to see that $A \geq 0$ implies that $\text{Tr}(A) \geq \rho(A) = \|A\|$, so that

$$T(A) = \frac{1}{n} \text{Tr}(A) I_n \geq \frac{1}{n} A.$$

Taking $A = e_{11}$ we get $T(A) \not\geq \lambda A$ if $\lambda > \frac{1}{n}$; so that $J(T) = n^{-1}$.

Remark 6.11. Let $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a selfadjoint map. The formulation of Theorem 6.9 intends to characterize complementability and to compute $I(\Phi)$ in terms of Φ itself instead of doing it in terms of the “inflated” matrix A_Φ . Another way would be to recall the identity $I(\Phi) = n \cdot I_{P_v}(A_\Phi)$ and use all the previous results of the paper. For example, let $U_1, \dots, U_m \in M_n(\mathbb{C})$, and suppose that Φ is given by

$$\Phi(A) = \sum_{k=1}^m U_k^* A U_k, \quad A \in M_n(\mathbb{C}),$$

a prototypical completely positive map (see [3]). Denote by $V_k \in M_{n^2}(\mathbb{C})$ the block diagonal matrix with copies of U_k in its diagonal. Denote by $v = (e_1, \dots, e_n) \in \mathbb{C}^{n^2}$ and $E = (e_{ij})_{ij} = vv^*$. Note that $\|V_k v\| = \|U_k\|_2$ and $V_k^* E V_k = (V_k v)(V_k v)^*$. Therefore

$$A_\Phi = (\Phi(e_{ij}))_{ij} = \sum_{k=1}^m V_k^* E V_k = \sum_{k=1}^m \|U_k\|_2^2 P_{V_k v}.$$

Thus $I(\Phi)$ can be computed using this expression and Corollaries 6.1 and 2.3.

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