

## Oblique Projections and Abstract Splines

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Given a closed subspace  $\mathcal{S}$  of a Hilbert space  $\mathcal{H}$  and a bounded linear operator  $A \in L(\mathcal{H})$  which is positive, consider the set of all  $A$ -self-adjoint projections onto  $\mathcal{S}$ :

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in L(\mathcal{H}) : Q^2 = Q, \quad Q(\mathcal{H}) = \mathcal{S}, \quad AQ = Q^*A\}.$$

In addition, if  $\mathcal{H}_1$  is another Hilbert space,  $T : \mathcal{H} \rightarrow \mathcal{H}_1$  is a bounded linear operator such that  $T^*T = A$  and  $\xi \in \mathcal{H}$ , consider the set of  $(T, \mathcal{S})$  spline interpolants to  $\xi$ :

$$sp(T, \mathcal{S}, \xi) = \left\{ \eta \in \xi + \mathcal{S} : \|T\eta\| = \min_{\sigma \in \mathcal{S}} \|T(\xi + \sigma)\| \right\}.$$

A strong relationship exists between  $\mathcal{P}(A, \mathcal{S})$  and  $sp(T, \mathcal{S}, \xi)$ . In fact,  $\mathcal{P}(A, \mathcal{S})$  is not empty if and only if  $sp(T, \mathcal{S}, \xi)$  is not empty for every  $\xi \in \mathcal{H}$ . In this case, for any  $\xi \in \mathcal{H} \setminus \mathcal{S}$  it holds

$$sp(T, \mathcal{S}, \xi) = \{(1 - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\}$$

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and for any  $\xi \in \mathcal{H}$ , the unique vector of  $sp(T, \mathcal{S}, \xi)$  with minimal norm is  $(1 - P_{A, \mathcal{S}})\xi$ , where  $P_{A, \mathcal{S}}$  is a distinguished element of  $\mathcal{P}(A, \mathcal{S})$ . These results offer a generalization to arbitrary operators of several theorems by de Boor, Atteia, Sard and others, which hold for closed range operators. © 2002 Elsevier Science (USA)

### 1. INTRODUCTION

Given two Hilbert spaces  $\mathcal{H}$  and  $\mathcal{H}_1$ ,  $T \in L(\mathcal{H}, \mathcal{H}_1)$ ,  $\mathcal{S} \subseteq \mathcal{H}$  a closed subspace and  $\xi \in \mathcal{H}$ , an abstract spline or a  $(T, \mathcal{S})$ -spline interpolant to  $\xi$  is any element of the set

$$sp(T, \mathcal{S}, \xi) = \left\{ \eta \in \xi + \mathcal{S} : \|T\eta\| = \min_{\sigma \in \mathcal{S}} \|T(\xi + \sigma)\| \right\}.$$

Observe that  $A = T^*T = |T|^2$ , as a positive bounded operator on  $\mathcal{H}$ , defines a semiinner product  $\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  by  $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$ ,  $\xi, \eta \in \mathcal{H}$  and a corresponding seminorm  $\| \cdot \|_A : \mathcal{H} \rightarrow \mathbb{R}^+$  given by  $\|\eta\|_A = \langle \eta, \eta \rangle_A^{1/2} = \langle A\eta, \eta \rangle^{1/2} = \|T\eta\|$ . Thus, if for any  $\eta \in \mathcal{H}$  we consider  $d_A(\eta, \mathcal{S}) = \inf_{\sigma \in \mathcal{S}} \|\eta + \sigma\|_A$ , then

$$sp(T, \mathcal{S}, \xi) = \{ \eta \in \xi + \mathcal{S} ; \|\eta\|_A = d_A(\xi, \mathcal{S}) \}.$$

If  $A$  is an invertible operator, then  $\langle \cdot, \cdot \rangle_A$  is a scalar product,  $(\mathcal{H}, \langle \cdot, \cdot \rangle_A)$  is a Hilbert space and, by the projection theorem,  $d_A(\xi, \mathcal{S}) = \|(I - P_{A, \mathcal{S}})\xi\|_A$  and  $sp(T, \mathcal{S}, \xi) = \{(I - P_{A, \mathcal{S}})\xi\}$ , where  $P_{A, \mathcal{S}}$  is unique orthogonal projection onto  $\mathcal{S}$  which is orthogonal to the inner product  $\langle \cdot, \cdot \rangle_A$ . However, if  $A$  is not invertible then  $\| \cdot \|_A$  is or a seminorm or an incomplete norm and we cannot use the projection theorem unless we complete the quotient  $\mathcal{H}/ker A$ . One of the main goals of this paper is to get a simpler way of describing the set  $sp(T, \mathcal{S}, \xi)$ .

We start with a positive bounded linear operator  $A$  on a Hilbert space  $\mathcal{H}$  and a closed subspace  $\mathcal{S}$  of  $\mathcal{H}$ . The subspace  $\mathcal{S}^{\perp A} = \{ \xi : \langle A\xi, \eta \rangle = 0 \ \forall \eta \in \mathcal{S} \}$  is called the *A-orthogonal companion* of  $\mathcal{S}$ . Note the identities

$$\mathcal{S}^{\perp A} = A^{-1}(\mathcal{S}^{\perp}) = A(\mathcal{S})^{\perp} = ker(PA). \tag{1}$$

Instead of defining adjoint operators with respect to  $\langle \cdot, \cdot \rangle_A$ , we restrict our discussion to *A-self-adjoint* operators, i.e.  $W \in L(\mathcal{H})$  such that  $AW = W^*A$ . Note that any such  $W$  satisfies  $\langle W\xi, \eta \rangle_A = \langle \xi, W\eta \rangle_A$ ,  $\xi, \eta \in \mathcal{H}$ .

The pair  $(A, \mathcal{S})$  is said to be *compatible* if there exists a projection  $Q \in L(\mathcal{H})$  such that  $Q(\mathcal{H}) = \mathcal{S}$  and  $AQ = Q^*A$ . The main result in this paper is the description of the relationship between the set

$$\mathcal{P}(A, \mathcal{S}) = \{ Q \in \mathcal{Q} : R(Q) = \mathcal{S}, AQ = Q^*A \}$$

and  $sp(T, \mathcal{S}, \xi)$ , where  $T: \mathcal{H} \rightarrow \mathcal{H}_1$  is any bounded linear operator such that  $T^*T = A$ . A relevant point here is that this method allows to tackle the case of operators with non-closed range. Thus, several results by Atteia [3], Sard [18], Golomb [11], Shekhtman [19], de Boor [4], Izumino [13], Delvos [9], Deutsch [8] are generalized to any bounded linear operators  $T$ .

If  $(A, \mathcal{S})$ , is compatible, there exists a distinguished element  $P_{A, \mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$ . The study of the map  $(A, \mathcal{S}) \rightarrow P_{A, \mathcal{S}}$  was initiated by Pasternak-Winiarski [15] at least for invertible  $A$ . A geometrical description of that map can be found in [2]. In [7, 12] the inversibility hypothesis on  $A$  was removed, opening, in that way, the possibility that  $\mathcal{P}(A, \mathcal{S})$  be empty or have many elements. This induces the notion of compatibility of a pair  $(A, \mathcal{S})$ . This paper is mainly devoted to explore the relationship of the compatibility of  $(A, \mathcal{S})$  with the existence of spline interpolants for every  $\xi \in \mathcal{H}$ . Section 2 contains a short study on compatibility of a pair  $(A, \mathcal{S})$ . If  $(A, \mathcal{S})$  is compatible, the properties of the distinguished element  $P_{A, \mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$  are described. In Section 3, we show that  $(A, \mathcal{S})$  is compatible if and only if  $sp(T, \mathcal{S}, \xi)$  is not empty for any  $\xi \in \mathcal{H}$  and that  $sp(T, \mathcal{S}, \xi) = \{(1 - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\}$  for any  $\xi \in \mathcal{H} \setminus \mathcal{S}$ . Moreover, the vector of  $sp(T, \mathcal{S}, \xi)$  with minimal norm is exactly  $(1 - P_{A, \mathcal{S}})\xi$ . In Section 4, we present some characterizations of  $P_{A, \mathcal{S}}$  which are useful for the study of the convergence of  $\{P_{A, \mathcal{S}_n} \xi\}$  if  $(A, \mathcal{S}_n)$  is compatible for every  $n \in \mathbb{N}$  and  $\mathcal{S}_n$  decreases to 0. This study is the goal of Section 5. Finally, Section 6 includes several examples of compatibility and spline projections.

In this paper,  $L(\mathcal{H})$  is the algebra of all linear bounded operators on the Hilbert space  $\mathcal{H}$  and  $L(\mathcal{H})^+$  is the subset of  $L(\mathcal{H})$  of all self-adjoint positive (i.e., non-negative definite) operators. For every  $C \in L(\mathcal{H})$  its range is denoted by  $R(C)$ . If  $R(C)$  is closed, then  $C^\dagger$  denotes the Moore–Penrose pseudoinverse of  $C$ . The orthogonal projections onto a closed subspace  $\mathcal{S}$  is denoted by  $P_{\mathcal{S}}$ . The direct sum of subspaces  $\mathcal{S}$  and  $\mathcal{T}$  is denoted  $\mathcal{S} \dot{+} \mathcal{T}$ . Finally,  $\mathcal{S} \ominus \mathcal{T}$  denotes  $\mathcal{S} \cap \mathcal{T}^\perp$ .

## 2. A-SELF-ADJOINT PROJECTIONS

Throughout this paper  $\mathcal{S}$  denotes a closed subspace of  $\mathcal{H}$  and  $A$  is a fixed operator in  $L(\mathcal{H})^+$ . Recall that  $\mathcal{S}^{\perp A} = A^{-1}(\mathcal{S}^\perp)$ . It is easy to see that a projection  $Q$  belongs to  $\mathcal{P}(A, \mathcal{S})$  if and only if  $R(Q) = \mathcal{S}$  and  $ker Q \subseteq A^{-1}(\mathcal{S}^\perp)$ . Then

$$\text{the pair } (A, \mathcal{S}) \text{ is compatible if and only if } \mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{H}. \quad (2)$$

In this case,  $\mathcal{P}(A, \mathcal{S})$  has a single element if and only if  $ker A \cap \mathcal{S} = \{0\}$  because

$$\mathcal{S} \cap A^{-1}(\mathcal{S}^\perp) = ker A \cap \mathcal{S}. \quad (3)$$

If  $(A, \mathcal{S})$  is compatible, then there is a distinguished element in  $\mathcal{P}(A, \mathcal{S})$ , namely the unique projection  $P_{A, \mathcal{S}}$  onto  $\mathcal{S}$  with kernel  $A^{-1}(\mathcal{S}^\perp) \ominus (\ker A \cap \mathcal{S})$ . The elements of  $\mathcal{P}(A, \mathcal{S})$  can be parametrized by the set of relative supplements of  $\ker A \cap \mathcal{S}$  into  $A^{-1}(\mathcal{S}^\perp)$ .

The set  $\mathcal{P}(A, \mathcal{S})$  can also be characterized using the matrix operator decomposition induced by the orthogonal projection  $P = P_{\mathcal{S}}$ . Under this representation,  $A$  has a matrix form

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \tag{4}$$

where  $a \in L(\mathcal{S})^+$ ,  $b \in L(\mathcal{S}^\perp, \mathcal{S})$  and  $c \in L(\mathcal{S}^\perp)^+$ . Observe that  $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $PA = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$  and  $PAP = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ . Every projection  $Q$  with range  $\mathcal{S}$  has the matrix form  $Q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$  for some  $x \in L(\mathcal{S}^\perp, \mathcal{S})$ . It is easy to see that  $Q \in \mathcal{P}(A, \mathcal{S})$  if and only if  $x$  satisfies the equation  $ax = b$ . Then

$$\mathcal{P}(A, \mathcal{S}) = \left\{ Q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} : x \in L(\mathcal{S}^\perp, \mathcal{S}) \text{ and } ax = b \right\}. \tag{5}$$

Note that Eq. (5) implies that if  $(A, \mathcal{S})$  is compatible, then  $R(b) \subseteq R(a)$ . As a corollary of a well-known theorem of R.G. Douglas, it can be shown that these two conditions are, indeed, equivalent. First, we recall Douglas' theorem [10]:

**THEOREM 2.1.** *Let  $B, C \in L(\mathcal{H})$ . Then the following conditions are equivalent:*

1.  $R(B) \subseteq R(C)$ .
2. There exists a positive number  $\lambda$  such that  $BB^* \leq \lambda CC^*$ .
3. There exists  $D \in L(\mathcal{H})$  such that  $B = CD$ . Moreover, there exists a unique operator  $D$  which satisfies the conditions

$$B = CD, \quad \ker D = \ker B \quad \text{and} \quad R(D) \subseteq \overline{R(C^*)}.$$

In this case,  $\|D\|^2 = \inf\{\lambda : BB^* \leq \lambda CC^*\}$ ;  $D$  is called the reduced solution of the equation  $CX = B$ . If  $R(C)$  is closed, then  $D = C^\dagger B$ .

**COROLLARY 2.2.** *Let  $A \in L(\mathcal{H})^+$  and  $\mathcal{S} \subseteq \mathcal{H}$  a closed subspace. If  $A$  has matrix form as in (4), then  $(A, \mathcal{S})$  is compatible if and only if  $R(b) \subseteq R(a)$ .*

The next theorem describes some properties of  $\mathcal{P}(A, \mathcal{S})$  and  $P_{A, \mathcal{S}}$ . The norm of  $P_{A, \mathcal{S}}$  will be computed in Section 5.

**THEOREM 2.3.** *Let  $A \in L(\mathcal{H})^+$  with matrix form (4), such that the pair  $(A, \mathcal{S})$  is compatible.*

1. *The distinguished projection  $P_{A, \mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$  has the matrix form*

$$P_{A, \mathcal{S}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix},$$

where  $d \in L(\mathcal{S}^\perp, \mathcal{S})$  is the reduced solution of the equation  $ax = b$ .

2.  *$\mathcal{P}(A, \mathcal{S})$  is an affine manifold which can be parametrized as*

$$\mathcal{P}(A, \mathcal{S}) = P_{A, \mathcal{S}} + L(\mathcal{S}^\perp, \mathcal{N}),$$

where  $\mathcal{N} = A^{-1}(S^\perp) \cap \mathcal{S} = \ker A \cap \mathcal{S}$  and  $L(\mathcal{S}^\perp, \mathcal{N})$  is viewed as a subspace of  $L(\mathcal{H})$ . A matrix representation of this parametrization is

$$\mathcal{P}(A, \mathcal{S})Q = P_{A, \mathcal{S}} + z = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \ominus \mathcal{N} \\ \mathcal{N} \\ \mathcal{S}^\perp \end{matrix}. \quad (6)$$

3.  *$P_{A, \mathcal{S}}$  has minimal norm in  $\mathcal{P}(A, \mathcal{S})$ , i.e.  $\|P_{A, \mathcal{S}}\| = \min \{\|Q\| : Q \in \mathcal{P}(A, \mathcal{S})\}$ .*

*Proof.*

(1) If  $Q = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}$ , then  $Q \in \mathcal{P}(A, \mathcal{S})$  and  $\ker Q \subseteq A^{-1}(\mathcal{S}^\perp)$ . Since  $P_{A, \mathcal{S}}$  is characterized by the properties  $R(P_{A, \mathcal{S}}) = \mathcal{S}$  and  $\ker P_{A, \mathcal{S}} = A^{-1}(\mathcal{S}^\perp) \ominus \mathcal{N}$  then, in order to show that  $Q = P_{A, \mathcal{S}}$  it suffices to prove that  $\ker Q \subseteq \mathcal{N}^\perp$ . Let  $\xi \in \ker Q$  and write  $\xi = \xi_1 + \xi_2$  with  $\xi_1 \in \mathcal{S}$  and  $\xi_2 \in \mathcal{S}^\perp$ . Then  $0 = Q\xi = \xi_1 + d\xi_2$ . If  $\eta \in \mathcal{N}$ , then  $\langle \xi, \eta \rangle = \langle \xi_1, \eta \rangle = -\langle d\xi_2, \eta \rangle = 0$  because, by Theorem 2.1,  $R(d) \subseteq \overline{R(a)}$  and, as an operator in  $L(\mathcal{S})$ ,  $\ker a = \mathcal{S} \cap \ker PAP = \mathcal{S} \cap \ker A = \mathcal{N}$ .

(2) Let  $Q = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}$  with  $y \in L(\mathcal{S}^\perp, \mathcal{S})$  and let  $d \in L(\mathcal{S}^\perp, \mathcal{S})$  be the reduced solution of the equation  $ax = b$ . Then  $Q \in \mathcal{P}(A, \mathcal{S})$  if and only if  $ay = b$ . Therefore, if  $z = y - d$ , then  $Q \in \mathcal{P}(A, \mathcal{S})$  if and only if  $Q = P_{A, \mathcal{S}} + z$  and  $R(z) \subseteq \ker a = \mathcal{N}$ . Concerning the matrix representation (6), recall that  $R(d) \subseteq \overline{R(a)} = (\ker a)^\perp = \mathcal{S} \ominus \mathcal{N}$ . Therefore,

$$Q = P_{A, \mathcal{S}} + z = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \ominus \mathcal{N} \\ \mathcal{N} \\ \mathcal{S}^\perp \end{matrix}.$$

(3) If  $Q \in \mathcal{P}(A, \mathcal{S})$  has the matrix form given in Eq. (6), then

$$\|Q\|^2 = \|QQ^*\| = 1 + \left\| \begin{pmatrix} 0 & 0 & d \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right\|^2 \geq 1 + \|d\|^2 = \|P_{A, \mathcal{S}}\|^2. \quad \blacksquare$$

*Remark 2.4.* Under additional hypothesis on  $A$ , other characterizations of compatibility can be used. We mention a sample of these, taken from [6, 7]:

1. If  $A$  is injective then the following conditions are equivalent: (a) The pair  $(A, \mathcal{S})$  is compatible. (b)  $\mathcal{S}^\perp \subseteq R(A + \lambda(1 - P))$  for some (and then for any)  $\lambda > 0$ . (c)  $P(\overline{A(\mathcal{S})}) = \mathcal{S}$  and  $\overline{A(\mathcal{S})} \cap \mathcal{S}^\perp = \{0\}$ .

2. If  $A$  has closed range then the following conditions are equivalent: (a) The pair  $(A, \mathcal{S})$  is compatible. (b)  $R(PAP)$  is closed. (c)  $\mathcal{S} + \ker A$  is closed.

3. If  $R(PAP)$  is closed (or, equivalently, if  $R(PA^{1/2})$  or  $A^{1/2}(\mathcal{S})$  are closed), then  $(A, \mathcal{S})$  is compatible. Indeed, using the matrix form (4), the positivity of  $A$  implies that  $R(b) \subseteq R(a^{1/2})$  (see, e.g., [1]). If  $R(PAP) = R(a)$  is closed, then  $R(b) \subseteq R(a^{1/2}) = R(a)$  so that  $(A, \mathcal{S})$  is compatible by Corollary 2.2.

### 3. SPLINES AND $A$ -SELF-ADJOINT PROJECTIONS

In this section, we characterize the existence of splines in terms of the existence of  $A$ -self-adjoint projections. The first result extends a theorem of Izumino [13] to operators whose ranges are not necessarily closed.

**PROPOSITION 3.1.** *Let  $T \in L(\mathcal{H}, \mathcal{H}_1)$ ,  $A = T^*T \in L(\mathcal{H})$  and  $\mathcal{S} \subseteq \mathcal{H}$  a closed subspace. Then, for any  $\xi \in \mathcal{H}$ ,*

$$sp(T, \mathcal{S}, \xi) = (\xi + \mathcal{S}) \cap \mathcal{S}^{\perp A}.$$

*In particular,  $sp(T, \mathcal{S}, \xi)$  is an affine manifold of  $L(\mathcal{H})$  and, if  $\eta \in sp(T, \mathcal{S}, \xi)$ , then  $sp(T, \mathcal{S}, \xi) = \eta + \ker T \cap \mathcal{S}$ .*

*Proof.* Suppose that  $\eta \in (\xi + \mathcal{S}) \cap A^{-1}(\mathcal{S}^\perp)$  and  $\sigma \in \mathcal{S}$ . Then  $\langle A\eta, \sigma \rangle = \langle A\sigma, \eta \rangle = 0$  and

$$\|T(\eta + \sigma)\|^2 = \langle A(\eta + \sigma), \eta + \sigma \rangle = \langle A\eta, \eta \rangle + \langle A\sigma, \sigma \rangle \geq \langle A\eta, \eta \rangle = \|T\eta\|^2.$$

Therefore,  $\eta \in sp(T, \mathcal{S}, \xi)$ . Conversely, if  $\eta \in sp(T, \mathcal{S}, \xi)$  and  $\sigma \in \mathcal{S}$ , then, for any  $t \in \mathbb{R}$ ,

$$\begin{aligned} \|T\eta\|^2 &\leq \|T(\eta + t\sigma)\|^2 = \langle A(\eta + t\sigma), \eta + t\sigma \rangle \\ &= \langle A\eta, \eta \rangle + t^2 \langle A\sigma, \sigma \rangle + 2t \operatorname{Re} \langle A\eta, \sigma \rangle \\ &= \|T\eta\|^2 + t^2 \langle A\sigma, \sigma \rangle + 2t \operatorname{Re} \langle A\eta, \sigma \rangle, \end{aligned}$$

therefore  $t^2 \langle A\sigma, \sigma \rangle + 2t \operatorname{Re} \langle A\eta, \sigma \rangle \geq 0$  for all  $t \in \mathbb{R}$  and a standard argument shows that  $\langle A\eta, \sigma \rangle = 0$  and then  $\eta \in (\xi + \mathcal{S}) \cap A^{-1}(\mathcal{S}^\perp)$ . ■

**THEOREM 3.2.** *Let  $T \in L(\mathcal{H}, \mathcal{H}_1)$ ,  $A = T^*T \in L(\mathcal{H})$  and  $\mathcal{S} \subseteq \mathcal{H}$  a closed subspace.*

1. *If  $\xi \in \mathcal{H}$ ,  $sp(T, \mathcal{S}, \xi)$  is not empty  $\xi \in \mathcal{S} + A^{-1}(\mathcal{S}^\perp)$ .*
2. *The following conditions are equivalent: (a)  $sp(T, \mathcal{S}, \xi)$  is not empty for every  $\xi \in \mathcal{H}$ . (b)  $\mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{H}$ . (c) The pair  $(A, \mathcal{S})$  is compatible.*
3. *If  $(A, \mathcal{S})$  is compatible and  $\xi \in \mathcal{H} \setminus \mathcal{S}$ , it holds  $sp(T, \mathcal{S}, \xi) = \{(I - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\}$ .*
4. *If  $(A, \mathcal{S})$  is compatible, then for every  $\xi \in \mathcal{H}$ ,  $(I - P_{A, \mathcal{S}})\xi$  is the unique vector in  $sp(T, \mathcal{S}, \xi)$  with minimal norm.*

*Proof.* The first assertion follows directly from Proposition 3.1. Indeed, if  $\eta \in sp(T, \mathcal{S}, \xi)$  and  $\eta = \xi + \sigma$  with  $\sigma \in \mathcal{S}$ , then  $\xi = -\sigma + \eta \in \mathcal{S} + A^{-1}(\mathcal{S}^\perp)$ ; the converse implication is similar. The second assertion follows from the first one and Eq. (2). In order to prove the third item, let  $\xi \in \mathcal{H}$  and  $Q \in \mathcal{P}(A, \mathcal{S})$ . Then, by Proposition 3.1 and Eq. (2),

$$(I - Q)\xi = \xi - Q\xi \in (\xi + \mathcal{S}) \cap \ker Q \subseteq (\xi + \mathcal{S}) \cap A^{-1}(\mathcal{S}^\perp) = sp(T, \mathcal{S}, \xi).$$

Conversely, let  $\eta \in sp(T, \mathcal{S}, \xi)$  and  $\sigma \in \mathcal{S}$  such that  $\xi = \sigma + \eta$ . We are looking for some  $Q \in \mathcal{P}(A, \mathcal{S})$  such that  $Q\xi = \sigma$ . Let  $\eta_1 = (I - P_{A, \mathcal{S}})\xi$  and  $\sigma_1 = \xi - \eta_1 = P_{A, \mathcal{S}}\xi \in \mathcal{S}$ . Then, by Proposition 3.1,

$$\sigma - \sigma_1 = \eta_1 - \eta \in \mathcal{S} \cap A^{-1}(\mathcal{S}^\perp) = \ker A \cap \mathcal{S}.$$

If  $\xi = \sigma_2 + \rho$  with  $\sigma_2 \in \mathcal{S}$  and  $0 \neq \rho \in \mathcal{S}^\perp$ , choose  $z \in L(\mathcal{S}^\perp, \ker A \cap \mathcal{S})$  ( $\subseteq L(\mathcal{H})$ ) such that  $z(\rho) = \sigma - \sigma_1$ . By Theorem 2.3,  $Q = P_{A, \mathcal{S}} + z \in \mathcal{P}(A, \mathcal{S})$  and clearly  $Q\xi = \sigma$ .

The minimality of  $\|(I - P_{A, \mathcal{S}})\xi\|$  is proved as follows. If  $\xi \in \mathcal{S}$ , then  $(I - P_{A, \mathcal{S}})\xi = 0$ , which must be minimal. If  $\xi \notin \mathcal{S}$ , let  $\xi = \sigma_2 + \rho$  with  $\sigma_2 \in \mathcal{S}$  and  $0 \neq \rho \in \mathcal{S}^\perp$ . By Theorem 2.3, any  $Q \in \mathcal{P}(A, \mathcal{S})$  has the form  $Q = P_{A, \mathcal{S}} + z$ , with  $z \in L(\mathcal{S}^\perp, \ker A \cap \mathcal{S})$  ( $\subseteq L(\mathcal{H})$ ). Recall that  $R(P_{A, \mathcal{S}}) = \mathcal{S} \ominus (\ker A \cap$

$\mathcal{S}$ ). Therefore,

$$\begin{aligned} \|(I - Q)\xi\|^2 &= \|(I - Q)\rho\|^2 = \|\rho - P_{A,\mathcal{S}}(\rho) - z(\rho)\|^2 = \|\rho\|^2 + \|P_{A,\mathcal{S}}(\rho)\|^2 + \|z(\rho)\|^2 \\ &\geq \|\rho\|^2 + \|P_{A,\mathcal{S}}(\rho)\|^2 = \|\rho - P_{A,\mathcal{S}}(\rho)\|^2 = \|(I - P_{A,\mathcal{S}})\xi\|^2. \quad \blacksquare \end{aligned}$$

**COROLLARY 3.3.** *Let  $T \in L(\mathcal{H}, \mathcal{H}_1)$ ,  $A = T^*T \in L(\mathcal{H})$  and  $\mathcal{S} \subseteq \mathcal{H}$  a closed subspace. Then the following are equivalent:*

1.  $sp(T, \mathcal{S}, \xi)$  has a unique element for every  $\xi \in \mathcal{H}$ .
2. The pair  $(A, \mathcal{S})$  is compatible and  $\ker T \cap \mathcal{S} = \{0\}$ .

**Remark 3.4.** Let  $T \in L(\mathcal{H}, \mathcal{H}_1)$ ,  $A = T^*T \in L(\mathcal{H})$  and  $\mathcal{S} \subseteq \mathcal{H}$  a closed subspace.

1. If  $(A, \mathcal{S})$  is compatible then, by item 4 of Theorem 3.2, the projection  $1 - P_{A,\mathcal{S}}$  coincides with the so-called *spline projection* for  $T$  and  $\mathcal{S}$  when  $T$  has a closed range.
2. If  $R(T)$  is closed, then, by Remark 2.4 and Theorem 3.2,  $sp(T, \mathcal{S}, \xi) \neq \emptyset$  for every  $\xi \in \mathcal{H}$  if and only if  $\ker T + \mathcal{S}$  is closed. In case that  $\ker T \cap \mathcal{S} = \{0\}$ , then it is equivalent to the condition that the inclination between  $\ker T$  and  $\mathcal{S}$  is less than one (see [4, 8]).
3. If  $\xi \in \mathcal{S}$ , then  $sp(T, \mathcal{S}, \xi) = \ker T \cap \mathcal{S}$ . On the other hand,  $(I - Q)\xi = 0$  for every  $Q \in \mathcal{P}(A, \mathcal{S})$ . So the equality of item 3 of Theorem 3.2 may be false in this case.

#### 4. CHARACTERIZATIONS OF THE SPLINE PROJECTION $P_{A,\mathcal{S}}$

Fix  $A \in L(\mathcal{H})^+$  and a closed subspace  $\mathcal{S} \subseteq \mathcal{H}$ . As before, we denote  $P = P_{\mathcal{S}}$ . In this section, two different descriptions of the spline projection  $P_{A,\mathcal{S}}$  are given and, as a consequence, we relate  $P_{A,\mathcal{S}}$  with the shorted operator (see [1] and Remark 4.4 below).

By Corollary 2.2, it holds that the pair  $(A, \mathcal{S})$  is compatible if and only if  $R(PA) \subseteq R(PAP)$ . In case that  $A$  is invertible, it is known (see [2]) that, in the matrix form (4),  $a$  is invertible in  $L(\mathcal{S})$  and

$$P_{A,\mathcal{S}} = \begin{pmatrix} a^{-1} & 0 \\ 0 & 0 \end{pmatrix} \quad PA = \begin{pmatrix} 1 & a^{-1}b \\ 0 & 0 \end{pmatrix} \tag{7}$$

because  $a^{-1}b$  is the reduced solution of  $ax = b$  (see Theorem 2.3). Rewriting (7), we get  $(PAP)P_{A,\mathcal{S}} = PA$ . Thus, if  $A$  is invertible,  $P_{A,\mathcal{S}}$  is the reduced solution of the equation  $(PAP)X = PA$ . Let us consider the general case, in other words, if the pair  $(A, \mathcal{S})$  is compatible, let us relate  $P_{A,\mathcal{S}}$  with the



reduced solution  $Q$  of the equation

$$(PAP)X = PA. \tag{8}$$

Observe that, in general,  $\overline{R(PAP)}$  is strictly contained in  $\mathcal{S}$ . Therefore,  $R(Q)$  may be smaller than  $\mathcal{S} = R(P_{A,\mathcal{S}})$ .

**PROPOSITION 4.1.** *If the pair  $(A, \mathcal{S})$  is compatible,  $Q$  is the reduced solution of Eq. (8) and  $\mathcal{N} = \ker A \cap \mathcal{S}$ , then*

$$P_{A,\mathcal{S}} = P_{\mathcal{N}} + Q.$$

Moreover,  $Q$  verifies the following properties:

1.  $Q^2 = Q$ ,  $\ker Q = A^{-1}(\mathcal{S}^\perp)$  and  $R(Q) = \mathcal{S} \ominus \mathcal{N}$ .
2.  $Q$  is  $A$ -self-adjoint.
3.  $Q = P_{A,\mathcal{S} \ominus \mathcal{N}}$ .

*Proof.* Using the matrix form (4) of  $A$ , observe that, in  $L(\mathcal{S})$ ,  $\ker a = \mathcal{N}$  and  $\overline{R(a)} = \overline{R(a^{1/2})} = \mathcal{S} \ominus \mathcal{N}$ . Note that  $R(Q) \subseteq \overline{R(a)}$ . Also  $\ker Q = \ker(PA) = A^{-1}(\mathcal{S}^\perp)$ . If  $\xi \in \mathcal{S} \ominus \mathcal{N}$ , then

$$a(Q\xi) = (PAP)Q\xi = PA\xi = PAP\xi = a(\xi).$$

Since  $a$  is injective in  $\mathcal{S} \ominus \mathcal{N}$ , we can deduce that  $Q\xi = \xi$  for all  $\xi \in \mathcal{S} \ominus \mathcal{N}$ . Now, the compatibility of  $(A, \mathcal{S})$  implies that  $\mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{H}$ . Also  $A^{-1}(\mathcal{S}^\perp) \cap \mathcal{S} = \ker A \cap \mathcal{S} = \mathcal{N}$ . Therefore  $A^{-1}(\mathcal{S}^\perp) \dot{+} (\mathcal{S} \ominus \mathcal{N}) = \mathcal{H}$ . Then  $Q^2 = Q$  and  $R(Q) = \mathcal{S} \ominus \mathcal{N}$ . Note that

$$\ker Q = A^{-1}(\mathcal{S}^\perp) \subseteq A^{-1}((\mathcal{S} \ominus \mathcal{N})^\perp) = R(Q)^\perp,$$

so that  $Q$  is  $A$ -self-adjoint by Eq. (2). On the other hand,  $(\mathcal{S} \ominus \mathcal{N}) \cap \ker A = \{0\}$ , so that  $Q$  is the unique element of  $P(A, \mathcal{S} \ominus \mathcal{N})$ , by Theorem 2.3. Observe that  $R(Q) \subseteq \mathcal{N}^\perp$  and  $\mathcal{N} \subseteq \ker A \subseteq A^{-1}(\mathcal{S}^\perp) = \ker Q$ . Therefore,  $(P_{\mathcal{N}} + Q)^2 = P_{\mathcal{N}} + Q$ ,  $R(P_{\mathcal{N}} + Q) = \mathcal{S}$  and  $\ker(P_{\mathcal{N}} + Q) = (A^{-1}(\mathcal{S}^\perp)) \ominus \mathcal{N}$ . These formulae clearly imply that  $P_{\mathcal{N}} + Q = P_{A,\mathcal{S}}$  (see Theorem 2.3). ■

**PROPOSITION 4.2.** *If  $(A, \mathcal{S})$  is compatible and  $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$ , then  $R(P_{\mathcal{M}}A^{1/2}) \subseteq R(A^{1/2}P)$ . Moreover, Eq. (8) and*

$$(A^{1/2}P)X = P_{\mathcal{M}}A^{1/2} \tag{9}$$

have the same reduced solution. In particular, if  $A^{1/2}(\mathcal{S})$  is closed and  $\ker A \cap \mathcal{S} = \{0\}$ , then

$$P_{A,\mathcal{S}} = (A^{1/2}P)^\dagger P_{\mathcal{M}} A^{1/2} = (A^{1/2}P)^\dagger A^{1/2} = (TP)^\dagger T \tag{10}$$

for every  $T \in L(\mathcal{H}, \mathcal{H}_1)$  such that  $T^*T = A$ .

*Proof.* Denote  $B = A^{1/2}$ . Recall that  $\mathcal{M} = \overline{B(\mathcal{S})} = B^{-1}(\mathcal{S}^\perp)^\perp$ . Observe that

$$BP_{\mathcal{M}}B = AP_{A,\mathcal{S}} = APP_{A,\mathcal{S}} : \tag{11}$$

in fact, for  $\zeta \in \mathcal{H}$ , let  $\eta = P_{A,\mathcal{S}}\zeta$  and  $\rho = \zeta - \eta \in A^{-1}(\mathcal{S}^\perp)$ ; then  $B\eta \in \mathcal{M}$  and  $B\rho \in B^{-1}(\mathcal{S}^\perp) = \mathcal{M}^\perp$ . Hence,  $BP_{\mathcal{M}}B\zeta = A\eta = AP_{A,\mathcal{S}}\zeta$ . By Proposition 4.1, the projection  $Q = P_{A,\mathcal{S}} - P_{\mathcal{N}}$  is the reduced solution of the equation  $PAPX = PA$ . We shall see that  $Q$  is the reduced solution of Eq. (9). First note that, by Eq. (11),  $BP_{\mathcal{M}}B = (AP)P_{A,\mathcal{S}} = (AP)Q$ , so  $B(P_{\mathcal{M}}B - BPQ) = 0$ . But  $R(P_{\mathcal{M}}B - BPQ) \subseteq \overline{R(B)} = (\ker B)^\perp$ . Hence,  $Q$  is a solution of (9). Note that  $\ker P_{\mathcal{M}}B = B^{-1}(B^{-1}(\mathcal{S}^\perp)) = A^{-1}(\mathcal{S}^\perp) = \ker Q$  by Proposition 4.1. Finally,

$$\overline{R((BP)^*)} = \overline{R(PB)} = \overline{R(PAP)} = \mathcal{S} \ominus \mathcal{N} = R(Q).$$

The first equality of Eq. (10) follows directly. The second, from the fact that  $(A^{1/2}P)^\dagger P_{\mathcal{M}} = (A^{1/2}P)^\dagger$ . The last equality follows easily using the polar decomposition of  $T$  because  $A^{1/2} = |T|$ . ■

Formula (10), for operators with closed range, is due to Golomb [11].

**COROLLARY 4.3.** *Under the notations of Proposition 4.2, the pair  $(A, \mathcal{S})$  is compatible if and only if  $R(P_{\mathcal{M}}A^{1/2}) \subseteq R(A^{1/2}P)$ .*

*Proof.* Suppose that  $R(P_{\mathcal{M}}A^{1/2}) \subseteq R(A^{1/2}P)$ . Then, given  $\xi \in \mathcal{H}$ , there must exist  $\sigma \in \mathcal{S}$  such that  $P_{\mathcal{M}}A^{1/2}\xi = A^{1/2}\sigma$ . Therefore,  $A^{1/2}(\xi - \sigma) = (1 - P_{\mathcal{M}})A^{1/2}\xi$  and

$$\begin{aligned} \|A^{1/2}(\xi - \sigma)\| &= \|(1 - P_{\mathcal{M}})A^{1/2}\xi\| = d(A^{1/2}\xi, A^{1/2}(\mathcal{S})) \\ &= \inf \{ \|A^{1/2}(\xi + \tau)\| : \tau \in \mathcal{S} \}. \end{aligned} \tag{12}$$

Hence,  $\xi - \sigma \in sp(T, \mathcal{S}, \xi)$  and  $sp(T, \mathcal{S}, \xi) \neq \emptyset$  for every  $\xi \in \mathcal{H}$ . This implies compatibility by Theorem 3.2. The converse implication was shown in Proposition 4.2. ■

*Remark 4.4.* If  $A \in L(\mathcal{H})^+$  and  $\mathcal{S} \in \mathcal{H}$  is a closed subspace, then the set

$$\{X \in L(\mathcal{H})^+ : X \leq A \text{ and } R(X) \subseteq \mathcal{S}^\perp\}$$

has a maximum (for the natural order relation in  $L(\mathcal{H})^+$ ), which is called the *shorted operator* of  $A$  to  $\mathcal{S}^\perp$ . We denote it by  $\Sigma(P, A)$ . This notion, due to Krein [14] and Anderson–Trapp [1], has many applications to electrical engineering. It is well known (see [16]) that

$$\Sigma(P, A) = A^{1/2}P_{\mathcal{T}}A^{1/2},$$

where  $\mathcal{T} = A^{-1/2}(\mathcal{S}^\perp) = A^{1/2}(\mathcal{S})^\perp$ . From the proof of Proposition 4.2, it follows that, if  $(A, \mathcal{S})$  is compatible, then  $A^{1/2}(1 - P_{\mathcal{T}})A^{1/2} = AP_{A, \mathcal{S}}$ . Therefore, in this case,  $\Sigma(P, A) = A(1 - P_{A, \mathcal{S}})$ . More generally, it can be shown that  $\Sigma(P, A) = A(1 - Q)$  for every  $Q \in \mathcal{P}(A, \mathcal{S})$  (see [7]).

### 5. CONVERGENCE OF SPLINE PROJECTIONS

This section is devoted to the study of the convergence of abstract splines in the general (i.e. not necessarily closed range) case. Given  $A \in L(\mathcal{H})^+$ , let us consider a sequence of closed subspaces  $\mathcal{S}_n$  such that all pairs  $(A, \mathcal{S}_n)$  are compatible. Following de Boor [4] and Izumino [13], it is natural to look for conditions which are equivalent to the fact that  $P_{A, \mathcal{S}_n} \xrightarrow{SOT} 0$  (i.e. the spline projections converge to  $I$ ), where  $\xrightarrow{SOT}$  means convergence in the strong operator topology. This problem has a well-known solution under the assumption that  $R(A)$  is closed (see [4] or [13]). However, in our more general setting, it is possible that the sequence  $\{\mathcal{S}_n\}$  decreases to  $\{0\}$ , while  $\|P_{A, \mathcal{S}_n}\|$  tends to infinity (see Example 5.7). This induces us to consider the following weaker convergence:

**DEFINITION 5.1.** Let  $A \in L(\mathcal{H})^+$  and  $T_n, T \in L(\mathcal{H}), n \in \mathbb{N}$ . We shall say that the sequence  $T_n$  converges  $A$ -SOT to  $T$ :  $T_n \xrightarrow{A-SOT} T$  if

$$\|(T_n - T)\xi\|_A \rightarrow 0 \text{ for every } \xi \in \mathcal{H}.$$

Note that  $T_n \xrightarrow{A-SOT} T$  if and only if  $A^{1/2}T_n \xrightarrow{SOT} A^{1/2}T$ .

We start with the computation of the norm of  $P_{A, \mathcal{S}}$  for any compatible pair  $(A, \mathcal{S})$ . Before that, recall the following formula, due to Ptak [17] (see also [5, 7]): if  $Q_1$  and  $Q_2$  are orthogonal projections such that  $R(Q_1) + R(Q_2) = \mathcal{H}$ , then the norm of the unique projection  $Q_3$  with  $\ker Q_3 = R(Q_1)$

and  $R(Q_3) = R(Q_2)$  is

$$\|Q_3\| = (1 - \|Q_1 Q_2\|^2)^{-1/2}. \quad (13)$$

**PROPOSITION 5.2.** *Let  $A \in L(\mathcal{H})^+$  such that the pair  $(A, \mathcal{S})$  is compatible. Then,*

$$\|P_{A, \mathcal{S}}\|^2 = \inf \{ \lambda > 0 : PA^2P \leq \lambda(PAP)^2 \}. \quad (14)$$

*If, in addition,  $\ker A \cap \mathcal{S} = \{0\}$ , then*

$$\|P_{A, \mathcal{S}}\| = (1 - \|QP\|^2)^{-1/2}, \quad (15)$$

*where  $Q$  denotes the orthogonal projection onto  $A^{-1}(\mathcal{S}^\perp)$ .*

*Proof.* Let  $Q$  be the reduced solution of the equation  $(PAP)X = PA$ . Then  $\|Q\|^2$  equals the infimum of Eq. (14) by Douglas Theorem. On the other hand, by Proposition 4.1,  $\|Q\| = \|P_{A, \mathcal{S}}\|$ , showing formula (14). If  $\ker A \cap \mathcal{S} = \{0\}$ , then Theorem 2.3 assures that  $R(P_{A, \mathcal{S}}) = \mathcal{S}$  and  $\ker P_{A, \mathcal{S}} = A^{-1}(\mathcal{S}^\perp)$ . Therefore, (15) follows from Ptak formula (13). ■

**Remark 5.3.** Let  $A \in L(\mathcal{H})^+$  such that the pair  $(A, \mathcal{S})$  is compatible and  $\ker A \cap \mathcal{S} = \{0\}$ . Then, if  $P_{\ker A}$  is the orthogonal projection onto  $\ker A$ , then

$$\|P_{A, \mathcal{S}}\| \geq (1 - \|P_{\ker A} P\|^2)^{-1/2}.$$

Indeed, if  $Q$  is the projection of Eq. (15), then  $P_{\ker A} \leq Q$  because  $\ker A \subseteq A^{-1}(\mathcal{S}^\perp)$ . Then  $\|P_{\ker A} P\|^2 = \|PP_{\ker A} P\| \leq \|PQP\| = \|QP\|^2$ . This inequality, shown by de Boor [4] in the closed range case, relates the norm of  $P_{A, \mathcal{S}}$  with the angle between  $\ker A$  and  $\mathcal{S}$ .

**PROPOSITION 5.4.** *Let  $A \in L(\mathcal{H})^+$  and let  $\mathcal{S}_n$  ( $n \in \mathbb{N}$ ) be closed subspaces such that all pairs  $(A, \mathcal{S}_n)$  are compatible. Denote  $\mathcal{M}_n = A^{1/2}(\mathcal{S}_n)$ ,  $n \in \mathbb{N}$ .*

1. *The following conditions are equivalent: (a)  $P_{A, \mathcal{S}_n} \xrightarrow{A-SOT} 0$ . (b)  $\langle AP_{A, \mathcal{S}_n} \xi, \xi \rangle \rightarrow 0$ , for every  $\xi \in \mathcal{H}$  (i.e.  $AP_{A, \mathcal{S}_n} \xrightarrow{WOT} 0$  by polarization). (c)  $AP_{A, \mathcal{S}_n} \xrightarrow{SOT} 0$ . (d)  $\Sigma(P_{\mathcal{S}_n}, A) \xrightarrow{SOT} A$ . (e)  $P_{\mathcal{M}_n} A^{1/2} \xrightarrow{SOT} 0$ .*
2. *If there exists  $C \geq 0$  such that  $\|P_{A, \mathcal{S}_n}\| \leq C$  for all  $n \in \mathbb{N}$  and  $P_{\mathcal{S}_n} A \xrightarrow{SOT} 0$ , then  $P_{A, \mathcal{S}_n} \xrightarrow{A-SOT} 0$ .*
3. *If  $P_{A, \mathcal{S}_n} \xrightarrow{A-SOT} 0$ , then  $P_{\mathcal{S}_n} A \xrightarrow{SOT} 0$ .*

*Proof.*

1. Because  $P_{A, \mathcal{S}_n}^* A = AP_{A, \mathcal{S}_n}$ , it is clear that conditions (a)–(c) are equivalent. By Remark 4.4,  $\Sigma(P_{\mathcal{S}_n}, A) = A(1 - P_{A, \mathcal{S}_n})$  so that (c) is equivalent

to (d). Finally, by Proposition 4.2, we know that  $A^{1/2}P_{A,\mathcal{S}_n} = P_{\mathcal{M}_n}A^{1/2}$  and this shows that (a) is equivalent to (e).

2. Suppose that there exists  $C \geq 0$  such that  $\|P_{A,\mathcal{S}_n}\| \leq C$  for all  $n \in \mathbb{N}$  and that  $P_{\mathcal{S}_n}A \xrightarrow{SOT} 0$ . Denote  $P_n = P_{\mathcal{S}_n}$ . The fact that  $R(P_{A,\mathcal{S}_n}) = R(P_n)$  implies that  $P_nP_{A,\mathcal{S}_n} = P_{A,\mathcal{S}_n}$ . Therefore, for every  $\xi \in \mathcal{H}$ ,

$$\|P_{A,\mathcal{S}_n}^*A\xi\| = \|P_{A,\mathcal{S}_n}^*P_nA\xi\| \rightarrow 0,$$

since  $\|P_{A,\mathcal{S}_n}\|$  is bounded. Hence  $P_{A,\mathcal{S}_n}^*A = AP_{A,\mathcal{S}_n} \xrightarrow{SOT} 0$  so that  $P_{A,\mathcal{S}_n} \xrightarrow{A-SOT} 0$  by item 1.

3. Suppose that  $P_{A,\mathcal{S}_n} \xrightarrow{A-SOT} 0$ . Then, by item 1,  $AP_{A,\mathcal{S}_n} \xrightarrow{SOT} 0$ . Note that  $P_{A,\mathcal{S}_n}P_n = P_n$ , so that  $P_nP_{A,\mathcal{S}_n}^* = P_n$ . Given  $\xi \in \mathcal{H}$ , we have that

$$\|P_nA\xi\| = \|P_nP_{A,\mathcal{S}_n}^*A\xi\| = \|P_nAP_{A,\mathcal{S}_n}\xi\| \leq \|AP_{A,\mathcal{S}_n}\xi\| \rightarrow 0. \quad \blacksquare$$

*Remark 5.5.* With the notations of Proposition 5.4, it follows that  $P_{A,\mathcal{S}_n} \xrightarrow{A-SOT} 0$  if and only if  $A^{1/2}(1 - P_{A,\mathcal{S}_n})\xi \rightarrow A^{1/2}\xi$  for every  $\xi \in \mathcal{H}$  or, equivalently, the spline interpolants  $\xi_n = (1 - P_{A,\mathcal{S}_n})\xi$  satisfy that  $T\xi_n \rightarrow T\xi$  in  $\mathcal{H}_1$ , if  $T \in L(\mathcal{H}, \mathcal{H}_1)$  and  $T^*T = A$ . In particular, if  $P_{A,\mathcal{S}_n} \xrightarrow{A-SOT} 0$ , then  $\min\{\|T(\xi + \tau)\| : \tau \in \mathcal{S}_n\} = \|T(1 - P_{A,\mathcal{S}_n})\xi\| \rightarrow \|T\xi\|$ .

**PROPOSITION 5.6.** *Let  $A \in L(\mathcal{H})^+$  and  $\mathcal{S}_2 \subseteq \mathcal{S}_1 \subseteq \mathcal{H}$  be closed subspaces. Suppose that  $(A, \mathcal{S}_1)$  is compatible. Denote by  $P_i = P_{\mathcal{S}_i}$ ,  $i = 1, 2$  and  $a_1 = P_1AP_1 \in L(\mathcal{S}_1)^+$ . Then*

*$(A, \mathcal{S}_2)$  is compatible if and only if  $(a_1, \mathcal{S}_2)$  is compatible in  $L(\mathcal{S}_1)$ .*

*Proof.* We know that, if  $A = \begin{pmatrix} a_1 & b_1 \\ b_1^* & c_1 \end{pmatrix}$ , in the matrix decomposition induced by  $P_1$ , then  $R(b_1) \subseteq R(a_1)$ . Hence also  $R(P_2b_1) \subseteq R(P_2a_1)$ . If  $a_1 = \begin{pmatrix} a_2 & b_2 \\ b_2^* & c_2 \end{pmatrix}$ , using now the matrix decomposition induced by  $P_2$ , then  $P_2a_1 = a_2 + b_2$  and  $P_2A(1 - P_2) = b_2 + P_2b_1$ . Hence,

$$R(P_2b_1) \subseteq R(P_2a_1) = R(a_2) + R(b_2) \text{ and } R(P_2A(1 - P_2)) = R(b_2) + R(P_2b_1).$$

Therefore, the pair  $(A, \mathcal{S}_2)$  is compatible if and only if  $R(P_2A(1 - P_2)) \subseteq R(P_2AP_2) = R(a_2)$  if and only if  $R(b_2) \subseteq R(a_2)$  if and only if the pair  $(a_1, \mathcal{S}_2)$  is compatible.  $\blacksquare$

**EXAMPLE 5.7.** Let  $A \in L(\mathcal{H})^+$  injective but not invertible. With the notations of Proposition 5.6 it is easy to see that  $P_1P_{A,\mathcal{S}_2}P_1 = P_{A,\mathcal{S}_2}P_1 \in \mathcal{P}(a_1, \mathcal{S}_2)$ . Note that  $a_1$  is injective, so that  $\mathcal{P}(a_1, \mathcal{S}_2)$  has a unique

element and

$$P_{a_1, \mathcal{S}_2} = P_{A, \mathcal{S}_2} P_1 \Rightarrow \|P_{A, \mathcal{S}_2}\| \geq \|P_{a_1, \mathcal{S}_2}\|. \quad (16)$$

We shall see that there exists a sequence  $\mathcal{S}_n$ ,  $n \in \mathbb{N}$ , of closed subspaces of  $\mathcal{H}$  such that

1. the pair  $(A, \mathcal{S}_n)$  is compatible for every  $n \in \mathbb{N}$ ,
2.  $\mathcal{S}_{n+1} \subseteq \mathcal{S}_n$  for every  $n \in \mathbb{N}$ ,
3.  $\bigcap_{n \geq 1} \mathcal{S}_n = \{0\}$ , so that  $P_{\mathcal{S}_n} \xrightarrow{SOT} 0$ ,
4.  $\|P_{A, \mathcal{S}_n}\| \rightarrow \infty$ .

In order to prove this fact, we need the following lemma:

LEMMA 5.8. *Let  $B \in L(\mathcal{H})^+$  be injective non-invertible. Then, for every  $\varepsilon > 0$ , there exists a closed subspace  $\mathcal{S} \subseteq \mathcal{H}$  such that the pair  $(B, \mathcal{S})$  is compatible,  $P_{\mathcal{S}} B P_{\mathcal{S}}$  is not invertible in  $L(\mathcal{S})$  and  $\|P_{B, \mathcal{S}}\| \geq \varepsilon^{-1}$ .*

*Proof.* Let  $\eta \in \mathcal{H}$  be a unit vector. Denote by  $\xi = B\eta$  and consider the subspace  $\mathcal{S} = \{\xi\}^\perp$  and  $P = P_{\mathcal{S}}$ . It is clear that  $\eta \in B^{-1}(\mathcal{S}^\perp)$ . First note that  $\langle \xi, \eta \rangle = \langle B\eta, \eta \rangle > 0$ , so that  $\eta \notin \mathcal{S}$ . Since  $\mathcal{S}$  is an hyperplane, this implies that  $\mathcal{S} + B^{-1}(\mathcal{S}^\perp) = \mathcal{H}$  and the pair  $(B, \mathcal{S})$  is compatible. Also  $PBP$  is not invertible because  $\dim \mathcal{S}^\perp = 1 < \infty$ . Note that  $B^{-1}(\mathcal{S}^\perp)$  is the subspace generated by  $\eta$ . Hence, if  $Q = P_{B^{-1}(\mathcal{S}^\perp)}$ , it is easy to see that  $\|PQ\| = \|P\eta\|$ . Then, by Eq. (15),

$$\|P_{B, \mathcal{S}}\| = (1 - \|PQ\|^2)^{-1/2} = (1 - \|P\eta\|^2)^{-1/2} = \|(1 - P)\eta\|^{-1}$$

and

$$\|(1 - P)\eta\| = \left| \left\langle \eta, \frac{\xi}{\|\xi\|} \right\rangle \right| = \frac{\langle \eta, B\eta \rangle}{\|B\eta\|}.$$

So, it suffices to show that there exists a unit vector  $\eta$  such that  $\langle \eta, B\eta \rangle \leq \varepsilon \|B\eta\|$ . Consider  $\rho \in \mathcal{H} \setminus R(B^{1/2})$  a unit vector. Let  $\rho_n$  be a sequence of unit vectors in  $R(B^{1/2})$  such that  $\rho_n \rightarrow \rho$ . Let  $\mu_n \in \mathcal{H}$  such that  $B^{1/2}\mu_n = \rho_n$ ,  $n \in \mathbb{N}$ , and denote by  $\xi_n = B^{1/2}\rho_n = B\mu_n$ , and  $\xi = B^{1/2}\rho$ . It is easy to see, using that  $B(\mu_n) = \xi_n \rightarrow \xi \notin R(B)$ , that  $\|\mu_n\| \rightarrow \infty$ . Denote by  $\eta_n = \mu_n / \|\mu_n\|^{-1}$ . Then

$$\frac{\langle \eta_n, B\eta_n \rangle}{\|B\eta_n\|} = \frac{\langle \mu_n, B\mu_n \rangle}{\|\mu_n\|^2 \|B\eta_n\|} = \frac{\|B^{1/2}\mu_n\|^2}{\|\mu_n\| \|B\mu_n\|} = \frac{1}{\|\mu_n\| \|\xi_n\|} \rightarrow 0$$

because  $\xi_n \rightarrow \xi \neq 0$ . ■

By an inductive argument, using Lemma 5.8, Proposition 5.6 and Eq. (16), we can construct a sequence of compatible subspaces  $\mathcal{S}_n, n \in \mathbb{N}$ , such that  $\mathcal{S}_{n+1} \subseteq \mathcal{S}_n$  and  $\|P_{A, \mathcal{S}_n}\| \rightarrow \infty$ . We can also get that  $P_{\mathcal{S}_n} \xrightarrow{SOT} 0$  by interlacing, before constructing the subspace  $\mathcal{S}_{n+1}$ , a spectral subspace  $\mathcal{T}_n$  of  $P_{\mathcal{S}_n} A P_{\mathcal{S}_n}$  (as an operator in  $L(\mathcal{S}_n)$ ), in such a way that  $P_{\mathcal{T}_n} A P_{\mathcal{T}_n}$  is not invertible and the projections  $P_{\mathcal{T}_n} \xrightarrow{SOT} 0$  (this can be done recursively by testing the projections  $P_{\mathcal{T}_n}$  in the first  $n$  elements of a countable dense subset of  $\mathcal{H}$ ), and taking  $\mathcal{S}_{n+1}$  as a subspace of  $\mathcal{T}_n$ . Note that the pairs  $(P_{\mathcal{S}_n} A P_{\mathcal{S}_n}, \mathcal{T}_n)$  are clearly compatible, so that also the pairs  $(A, \mathcal{T}_n)$  are compatible by Proposition 5.6.

*Remark 5.9.* Recall from Remark 4.4 that if  $(A, \mathcal{S})$  is compatible, then  $A(1 - P_{A, \mathcal{S}}) = \Sigma(P, A)$ . Then

$$0 \leq AP_{A, \mathcal{S}} = A - \Sigma(P, A) \leq A.$$

This implies that  $\|AP_{A, \mathcal{S}}\| \leq \|A\|$ , while  $\|P_{A, \mathcal{S}}\|$  can be arbitrarily large.

### 6. SOME EXAMPLES

In this section, we present several examples of pairs  $(A, \mathcal{S})$  which are not compatible and pairs  $(A, \mathcal{S})$  which are compatible and such that the spline projector  $P_{A, \mathcal{S}}$  can be explicitly computed. Observe that Example 6.4 cannot be studied under the closed range hypothesis, considered by Atteia, de Boor and Izumino.

**EXAMPLE 6.1.** Let  $A \in L(\mathcal{H})^+$  and

$$M = \begin{pmatrix} A & A^{1/2} \\ A^{1/2} & I \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix} \in L(\mathcal{H} \oplus \mathcal{H})^+.$$

Denote by  $\mathcal{S} = \mathcal{H} \oplus \{0\}$  and by  $N = \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix}$ . Since  $M = N^*N$ , then  $\ker M = \ker N = \{\xi \oplus -A^{1/2}\xi: \xi \in \mathcal{H}\}$  which is the graph of  $-A^{1/2}$ . Note that  $R(N) = (R(A^{1/2}) + R(I)) \oplus \{0\} = \mathcal{S}$ , so that  $R(M)$  is also closed. If  $A$  is injective with non-closed range, then  $(M, \mathcal{S})$  is not compatible (because  $R(A)$  is properly included in  $R(A^{1/2})$ ). Observe that this implies that the inclination between  $\mathcal{S}$  and  $\ker M$  is one, cf. [4].

*Remark 6.2.* Let  $P \in \mathcal{P}$ ,  $R(P) = \mathcal{S}$  and  $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in L(\mathcal{H})^+$ . It is well known that the positivity of  $A$  implies that  $R(b) \subseteq R(a^{1/2})$ . Therefore, if  $\dim \mathcal{S} < \infty$  then the pair  $(A, \mathcal{S})$  is compatible : in fact in this case  $R(a) = R(PAP)$  must be closed, so  $R(b) \subseteq R(a^{1/2}) = R(a)$  and Corollary 2.2, can be

applied. On the other hand, if  $\dim \mathcal{S}^\perp < \infty$  and  $R(A)$  is closed then, by Remark 2.4,  $(A, \mathcal{S})$  is compatible. However, if  $R(A)$  is not closed, then the pair  $(A, \mathcal{S})$  can be non-compatible:

**PROPOSITION 6.3.** *Let  $P \in \mathcal{P}$ ,  $R(P) = \mathcal{S}$  and  $A \in L(\mathcal{H})^+$ . Suppose that  $A$  is injective non-invertible and  $\dim \mathcal{S}^\perp < \infty$ . Then  $(A, \mathcal{S})$  is compatible if and only if  $\mathcal{S}^\perp \subseteq R(A)$ .*

*Proof.* By Eq. (2),  $(A, \mathcal{S})$  is compatible if and only if  $A^{-1}(\mathcal{S}^\perp) + \mathcal{S} = \mathcal{H}$ . Since  $A$  is injective, Eq. (3) says that  $A^{-1}(\mathcal{S}^\perp) \cap \mathcal{S} = \{0\}$ . Now the result becomes clear because  $\dim A^{-1}(\mathcal{S}^\perp) = \dim (\mathcal{S}^\perp \cap R(A))$ . ■

**EXAMPLE 6.4.** Let  $T \in L(\mathcal{H}, L^2)$  given by  $Te_m = \frac{e^{i(m+1)t}}{m}$ , where  $e_m$  ( $m \in \mathbb{N}$ ) is an orthonormal basis of  $\mathcal{H}$ . Then  $A = T^*T$  is given by  $Ae_m = \frac{e_m}{m^2}$ , which is injective non-invertible. Let  $\xi_1, \dots, \xi_n \in R(A)$ , denote by  $\mathcal{S} = \{\xi_1, \dots, \xi_n\}^\perp$  and  $P = P_{\mathcal{S}}$ . If  $\xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \dots, \xi_i^{(m)}, \dots)$ ,  $1 \leq i \leq n$ , denote by

$$\eta_i = (\xi_i^{(1)}, 4\xi_i^{(2)}, \dots, m^2\xi_i^{(m)}, \dots) \in \mathcal{H}, \quad 1 \leq i \leq n,$$

and  $Q$  the orthogonal projection onto the subspace  $\mathcal{T}$  generated by  $\eta_1, \dots, \eta_n$ . It is clear that  $\mathcal{T} = A^{-1}(\mathcal{S}^\perp)$ . Then  $(A, \mathcal{S})$  is compatible and  $P_{A, \mathcal{S}}$  is the projection onto  $\mathcal{S}$  with kernel  $\mathcal{T}$ . Therefore (cf. [5] or [17]),  $\|PQ\| < 1$ ,

$$P_{A, \mathcal{S}} = (1 - QP)^{-1}(1 - Q) = \sum_{k=0}^{\infty} (QP)^k(1 - Q)$$

and  $\|P_{A, \mathcal{S}}\| = \|1 - P_{A, \mathcal{S}}\| = (1 - \|PQ\|^2)^{-1/2}$ .

*Remark 6.5.* Let  $B \in L(\mathcal{H})^+$  be injective and non-invertible. Let  $\xi \in \mathcal{H}$  be a unit vector,  $\mathcal{S} = \{\xi\}^\perp$ ,  $P = P_{\mathcal{S}}$  and  $P_\xi = 1 - P$ . Let  $B = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$  in terms of  $P$ . By Proposition 6.3,  $(B, \mathcal{S})$  is compatible if and only if  $\xi \in R(B)$ . Note that the sequence  $\xi_n$  (in  $R(B)$ ) of Lemma 5.8 converges to  $\xi \notin R(B)$ . This is, precisely, the fact which implies that  $\|P_{B, \{\xi_n\}^\perp}\|$  converges to infinity.

**EXAMPLE 6.6.** Fix  $\mathcal{S}$  a closed subspace of  $\mathcal{H}$  and consider the set

$$\mathcal{A}_{\mathcal{S}} = \{A \in L(\mathcal{H})^+ : \text{the pair } (A, \mathcal{S}) \text{ is compatible}\}$$

and the map  $\alpha: \mathcal{A}_{\mathcal{S}} \rightarrow \mathcal{Q}$  given by  $\alpha(A) = P_{A, \mathcal{S}}$ . We shall see that  $\alpha$  is not continuous. Indeed, let  $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ , and suppose that  $R(b) = R(a)$  is a closed subspace  $\mathcal{M}$  properly included in  $\mathcal{S}$ . Denote by  $\mathcal{N} = \mathcal{S} \ominus \mathcal{M}$  and consider the projection  $P_{\mathcal{N}}$  and some element  $u \in L(\mathcal{S}^\perp, \mathcal{N}) \subseteq L(\mathcal{H})$ ,  $u \neq 0$ . Consider,



for every  $n \in \mathbb{N}$ ,

$$\begin{aligned}
 A_n &= A + \frac{1}{n}(P_{\mathcal{N}} + u)^*(P_{\mathcal{N}} + u) = A + \frac{1}{n} = \begin{pmatrix} 1 & 0 & u \\ 0 & 0 & 0 \\ u^* & 0 & u^*u \end{pmatrix} \begin{matrix} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^\perp \end{matrix} \\
 &= \begin{pmatrix} \frac{1}{n} & 0 & \frac{1}{n}u \\ 0 & a & b \\ \frac{1}{n}u^* & b^* & c + \frac{1}{n}u^*u \end{pmatrix} \geq A \geq 0.
 \end{aligned}$$

It is clear that  $A_n \rightarrow A$ . Note that  $a$  is invertible in  $L(\mathcal{M})$ . Then, by Theorem 2.3,

$$P_{A,\mathcal{S}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a^{-1}b \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^\perp \end{matrix},$$

Also  $a + \frac{1}{n}P_{\mathcal{N}}$  is invertible in  $L(\mathcal{S})$  for every  $n \in \mathbb{N}$ . Then,

$$\begin{aligned}
 P_{A_n,\mathcal{S}} &= \begin{pmatrix} n & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{n} & 0 & \frac{1}{n}u \\ 0 & a & b \\ \frac{1}{n}u^* & b^* & c + \frac{1}{n}u^*u \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & a^{-1}b \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{N} \\ \mathcal{M} \\ \mathcal{S}^\perp \end{matrix}
 \end{aligned}$$

for all  $n \in \mathbb{N}$ . Therefore,  $\alpha(A_n) = P_{A_n,\mathcal{S}}P_{A,\mathcal{S}} = \alpha(A)$ . Note that the sequence  $\alpha(A_n)$  converges (actually, it is constant) to  $P_{A,\mathcal{S}} + u$ , which belongs to  $\mathcal{P}(A, \mathcal{S})$  by Theorem 2.3.

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