Discrete Applied Mathematics 164 (2014) 500–511

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the correspondence between tree representations of chordal and dually chordal graphs

Pablo De Caria *, Marisa Gutierrez

CONICET/Departamento de Matemática, Universidad Nacional de La Plata, Calle 50 y 115, C.C. 172, (1900), La Plata, Argentina

ARTICLE INFO

Article history:
Received 31 October 2011
Received in revised form 8 July 2013
Accepted 8 July 2013
Available online 12 August 2013

Keywords:
Chordal graph
Dually chordal graph
Basic chordal graph
Clique tree
Compatible tree

ABSTRACT

Chordal graphs and their clique graphs (called dually chordal graphs) possess characteristic tree representations, namely, the clique tree and the compatible tree, respectively. The following problem is studied: given a chordal graph G, determine if the clique trees of G are exactly the compatible trees of the clique graph of G. This leads to a new subclass of chordal graphs, basic chordal graphs, which is here characterized. The question is also approached backwards: given a dually chordal graph G, we find all the basic chordal graphs with clique graph equal to G. This approach leads to the possibility of considering several properties of clique trees of chordal graphs and finding their counterparts in compatible trees of dually chordal graphs.

1. Introduction

Chordal graphs have been vastly studied and form a class of both theoretical and practical interest.

Chordal graphs have an associated characterizing tree representation, the clique tree. A clique tree T of a chordal graph G has vertex set equal to the family of cliques of G and, for each vertex of G, the set of cliques to which that vertex belongs induces a subtree of T.

Another class that has been studied for some decades is that of dually chordal graphs, which are the clique graphs of chordal graphs. If G is a chordal graph with clique tree T, then it is possible to verify that every clique of $K(G)$ induces a subtree of T.

A spanning tree T of a graph G such that every clique of G induces a subtree of T receives the name of compatible tree. Compatible trees are characteristic to dually chordal graphs and the previous paragraph implies that every clique tree of a chordal graph is compatible with its clique graph. The converse is not always true. We define a graph to be basic chordal if the converse is true. In other words, we say that a chordal graph G is basic chordal if its clique trees are exactly the compatible trees of $K(G)$. Basic chordal graphs will be the major focus of our attention.

The structure of the main part of this paper is as follows.

In Section 3, we review some classical properties of chordal graphs and clique trees that are fundamental for the development of our work.

In Section 4, we introduce dually chordal graphs and the compatible tree, and we start to study the relationship between the clique trees of a chordal graph and the compatible trees of its clique graph. Thus, basic chordal graphs arise and their first characterization is given (Theorem 4.6).

* Corresponding author. Tel.: +54 0221 4245875; fax: +54 0221 4229850.
E-mail addresses: pdecaria@mate.unlp.edu.ar (P. De Caria), marisa@mate.unlp.edu.ar (M. Gutierrez).
URL: http://www.mate.unlp.edu.ar/~pdecaria (P. De Caria).

0166-218X/$ – see front matter
http://dx.doi.org/10.1016/j.dam.2013.07.011

In Section 5, we study the sets that induce subtrees in every clique tree or in every compatible tree of a given graph. We show that these families of sets are characterized by a special subfamily, called the basis, and we find how bases can be computed for both the case of chordal graphs and of dually chordal graphs. This knowledge enables a better understanding of basic chordal graphs and of dually chordal graphs, and many results about these classes are stated. For example, we describe, for a dually chordal graph G, all the basic chordal graphs with clique graph equal to G (Theorem 5.7), and we give a more general characterization of compatible trees (Theorem 5.13). Some of the results are new and others approach the known facts about dually chordal graphs under a new perspective.

Finally, in Section 6 we use the information gained from Section 5 to give a new characterization of basic chordal graphs in terms of minimal vertex separators (Theorem 6.4) and we apply it to find additional properties of basic chordal graphs.

2. Definitions

For a simple graph G, the set of vertices of G is denoted by $V(G)$ and $E(G)$ denotes the set of its edges. A subset of $V(G)$ is *complete* when its elements are pairwise adjacent in G. A *clique* is defined to be a maximal complete set, and the family of cliques of G is denoted by $\mathcal{C}(G)$. The subgraph *induced* by a subset A of $V(G)$, denoted by $G[A]$, has A as vertex set, and two vertices are adjacent in $G[A]$ if and only if they are adjacent in G.

For a vertex $v \in V(G)$, the *open neighborhood* of v, denoted by $N(v)$ or $N_G(v)$, is the set of all the vertices adjacent to v in G. The *degree* $\deg(v)$ of v is the number $|N(v)|$. The *closed neighborhood* of v, denoted by $N[v]$ or $N_G[v]$, is the set $N(v) \cup \{v\}$. Vertex v is said to be *simplicial* if $N[v]$ is complete. This is equivalent to $N[v]$ being a clique. Any clique equaling the closed neighborhood of a vertex is called *simplicial clique*.

Given two nonadjacent vertices u and v in the same connected component of G, a *uv-separator* is a set S contained in $V(G)$ such that u and v are in different connected components of $G - S$, where $G - S$ denotes the induced subgraph $G[V(G) \setminus S]$. This separator S is *minimal* if no proper subset of S is also a uv-separator. We will just say *minimal vertex separator* to refer to a set S that is a uv-minimal separator for some pair of nonadjacent vertices u and v in G. The family of all minimal vertex separators of G will be denoted by $\mathbb{S}(G)$.

Let T be a tree. For $v, w \in V(T)$, the notation $T[v, w]$ denotes the path in T from v to w and $T(v, w)$ denotes the inner vertices of that path.

Let $\mathcal{F}$ be a family of nonempty sets of vertices of G. If $F \in \mathcal{F}$, then F is called a *member* of $\mathcal{F}$. If $v \in \bigcup_{F\in\mathcal{F}} F$, then we say that v is a *vertex* of $\mathcal{F}$. The family $\mathcal{F}$ is *Helly* if the intersection of all the members of every subfamily of pairwise intersecting sets is not empty. If $\mathcal{C}(G)$ is a Helly family, then we say that G is a *clique-Helly graph*. We say that $\mathcal{F}$ is *separating* if, for every ordered pair (v, w) of vertices of $\mathcal{F}$, there exists $F \in \mathcal{F}$ such that $v \in F$ and $w \notin F$. The *intersection graph* of $\mathcal{F}$, denoted $L(\mathcal{F})$, has the members of $\mathcal{F}$ as vertices, two of them being adjacent if and only if they are not disjoint. The *clique graph* $K(G)$ of G is the intersection graph of $\mathcal{C}(G)$. The *two-section graph* $S(\mathcal{F})$ of $\mathcal{F}$ is another graph whose vertices are the vertices of $\mathcal{F}$, in such a way that two vertices v and w are adjacent in $S(\mathcal{F})$ if and only if there exists $F \in \mathcal{F}$ such that $\{v, w\} \subseteq F$.

For every vertex v of $\mathcal{F}$, let $D_v = \{F \in \mathcal{F} : v \in F\}$. The *dual family* $D\mathcal{F}$ of $\mathcal{F}$ consists of all the sets D_v. For the particular case of $\mathcal{C}(G)$, the notation $\mathcal{C}_v$ will be used instead of D_v. An even more general notation will also be used: given a set A of vertices, $\mathcal{C}_A$ is defined to be equal to $\{C \in \mathcal{C}(G) : A \subseteq C\}$.

3. Properties of chordal graphs and clique trees

Given a cycle C of a graph G, a *chord* is defined as an edge joining two nonconsecutive vertices of C. *Chordal graphs* are defined as those graphs for which every cycle of length greater than or equal to four has a chord. That definition is not the only possible way to introduce chordal graphs because they have many characterizations. Some of them are:

(i) [3] a graph is chordal if and only if every minimal separator of two nonadjacent vertices of the graph is a complete set.
(ii) [5] an ordering $v_1 \dots v_n$ of the vertices of G is called a *perfect elimination ordering* if v_i is simplicial in $G[\{v_i, \dots, v_n\}]$ for $1 \le i \le n$. A graph is chordal if and only if it has a perfect elimination ordering.
(iii) [14] this characterization is the most important given the purpose of this paper. A *clique tree* of G is a tree T whose vertex set is $\mathcal{C}(G)$ and such that every member of $D\mathcal{C}(G)$ induces a subtree of T, that is, $T[\mathcal{C}_v]$ is a subtree of T for every $v \in V(G)$. A graph is chordal if and only if it has a clique tree.

The rest of this section is dedicated to stating some relevant properties of clique trees that are required for the next section. All graphs considered will be assumed to be connected.

We first express a slightly different way to characterize a clique tree. The following result is widely known and can be found in many papers on acyclic hypergraphs and on tree-width of graphs:

Proposition 3.1. *Let G be a graph and T be a tree such that $V(T) = \mathcal{C}(G)$. The following are equivalent:*

(a) *T is a clique tree of G.*
(b) $\forall C_1, C_2, C_3 \in \mathcal{C}(G), C_3 \in T[C_1, C_2] \Longrightarrow C_1 \cap C_2 \subseteq C_3$.

Proof. (a) $\Rightarrow$ (b). Let $C_1, C_2, C_3 \in \mathcal{C}(G)$ be such that $C_3 \in T[C_1, C_2]$ and v be a vertex of $C_1 \cap C_2$. Then, C_1 and C_2 are in $\mathcal{C}_v$. Since $\mathcal{C}_v$ induces a subtree of T and $C_3 \in T[C_1, C_2]$, we have that $C_3 \in \mathcal{C}_v$, that is, $v \in C_3$. Therefore, every element of $C_1 \cap C_2$ is an element of C_3 and the inclusion $C_1 \cap C_2 \subseteq C_3$ follows.

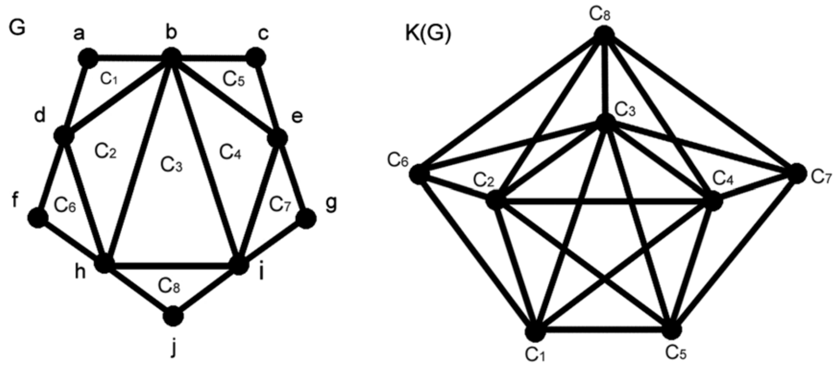

Fig. 1. The graph at right is dually chordal because it is the clique graph of the chordal graph G at left.

(b) ⇒ (a). Let v be any vertex of G and C_1, C_2 be cliques in $\mathcal{C}_v$. Thus $v \in C_1 \cap C_2$. Suppose that $C_3 \in T[C_1, C_2]$. By the hypothesis, it follows that $C_1 \cap C_2 \subseteq C_3$. Hence $v \in C_3$, which is equivalent to $C_3 \in \mathcal{C}_v$. Therefore, $T[\mathcal{C}_v]$ is a connected subgraph of T, that is, $\mathcal{C}_v$ induces a subtree of T. □

A clique tree of a given chordal graph can be found in polynomial time by using numerical algorithms. The most classical one relies on the following well known characterization of clique trees that is a reformulation of a result from acyclic hypergraph theory:

Theorem 3.2 ([11]). *Let G be a graph and let $K(G)^w$ be the graph obtained from $K(G)$ by assigning each edge CC' the weight $|C \cap C'|$. Then, T is a clique tree of G if and only if T is a maximum weight spanning tree of $K(G)^w$ and the total weight of T equals $\sum_{C \in \mathcal{C}(G)} |C| - |V(G)|$.*

Although we will not use here the characterization of chordal graphs mentioned in (i), minimal vertex separators are going to play an important role in this paper and are useful to gain insight into the edges of clique trees. We need the following three properties.

As minimal vertex separators of chordal graphs are complete, it is a priori possible that some of them are cliques. The property that appears below shows that it is not the case.

Given a graph G, two cliques C_1 and C_2 are a *separating pair* if $C_1 \cap C_2$ is a separator of every couple of vertices such that one is in $C_1 \setminus C_2$ and the other is in $C_2 \setminus C_1$. Since cliques are complete sets, this definition implies that $C_1 \cap C_2$ is a minimal vertex separator. It is also true that every minimal vertex separator of a chordal graph can be expressed in that way:

Theorem 3.3 ([9]). *Let G be a chordal graph and $S \in \mathbb{S}(G)$. Then, there exists a separating pair C_1, C_2 such that $S = C_1 \cap C_2$.*

Much of the importance of separating pairs lies on how they are related to clique trees. If we wonder what edges can be found in at least one clique tree, then the following theorem gives the answer:

Theorem 3.4 ([9]). *Let C_1 and C_2 be two distinct cliques of a chordal graph G. Then, there exists a clique tree T of G such that $C_1C_2 \in E(T)$ if and only if C_1 and C_2 form a separating pair.*

Finally, it is interesting to note that, when just one clique tree of a graph is known, it is possible to determine what the edges of the other clique trees (if they are more than one) can be:

Theorem 3.5 ([9]). *Let G be a chordal graph, T be a clique tree of G and $C_1, C_2 \in \mathcal{C}(G)$, with $C_1 \neq C_2$. Then, there exists a clique tree of G having C_1C_2 as an edge if and only if there are two cliques of G that are adjacent in $T[C_1, C_2]$ and whose intersection equals $C_1 \cap C_2$.*

4. Dually chordal graphs and an introduction on basic chordal graphs

As it was said in the introduction, a graph is *dually chordal* if it is the clique graph of some chordal graph (see example at Fig. 1). If T is a spanning tree of a graph G such that each clique of G induces a subtree of T, then we say that T is *compatible* with G (see example at Fig. 2). The three basic properties about dually chordal graphs and compatible trees that are necessary as a starting point for this section are written below:

Theorem 4.1 ([1,7]). *Let G be a graph and T be a tree such that $V(T) = V(G)$. The following statements are equivalent:*

(a) *T is compatible with G.*
(b) *$N[v]$ induces a subtree of T for every $v \in V(G)$.*
(c) *for all $u, v, w \in V(G)$, if $uv \in E(G)$ and $w \in T(u, v)$ then w is adjacent to both u and v.*

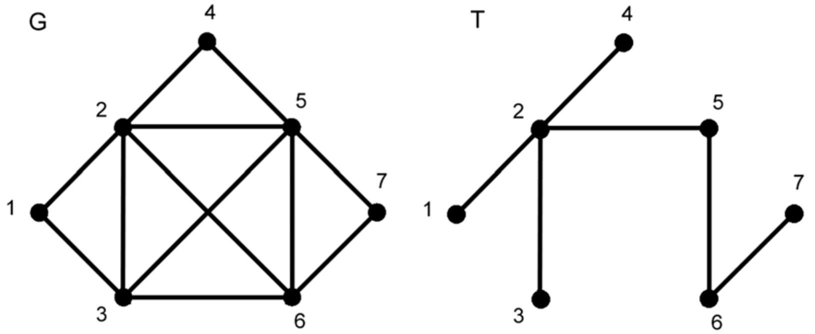

Fig. 2. A dually chordal graph and a tree compatible with it. The cliques of G are $\{1, 2, 3\}$, $\{2, 3, 5, 6\}$, $\{2, 4, 5\}$ and $\{5, 6, 7\}$, each inducing a subtree of T.

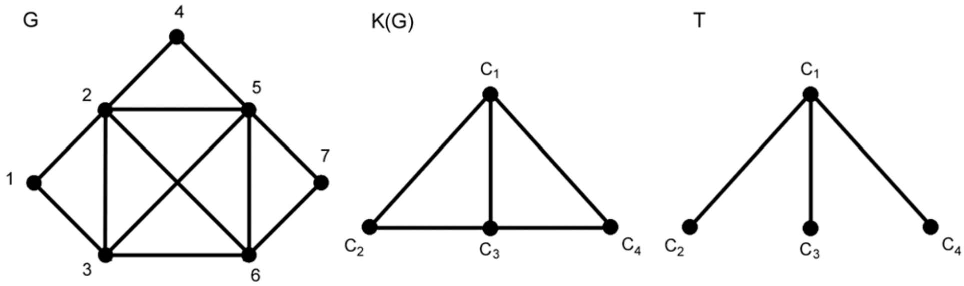

Fig. 3. A chordal graph G, its clique graph and a tree T that is compatible with $K(G)$ but is not a clique tree of G.

Theorem 4.2 ([1]). *A graph is dually chordal if and only if there exists a tree compatible with it.*

Proposition 4.3. *Let G be a chordal graph. Then, every clique tree of G is compatible with $K(G)$.*

Proof. Let T be a clique tree of G and C be any clique of G. The closed neighborhood of C in $K(G)$ is equal to $\bigcup_{v\in C}\mathcal{C}_v$, which induces a subtree of T because T is a clique tree. Therefore, T is compatible with $K(G)$. □

The converse of Proposition 4.3 is not necessarily true. For example, consider again the graph G of Fig. 2. This graph is also chordal. Set $C_1 = \{2, 4, 5\}$, $C_2 = \{1, 2, 3\}$, $C_3 = \{2, 3, 5, 6\}$ and $C_4 = \{5, 6, 7\}$. The tree T in Fig. 3 is compatible with $K(G)$. However, T is not a clique tree of G. This can be easily verified by noting that the cliques of G that contain vertex 3 are C_2 and C_3, which do not induce a subtree in T. We are interested in the case that the converse is true, which prompts us to define a new graph class:

Definition. A graph G is *basic chordal* if it is chordal and the clique trees of G are exactly the compatible trees of $K(G)$.

One of the main goals set for this paper is to develop tools to answer as easily as possible whether a given chordal graph G is basic chordal or not. In view of Proposition 4.3, the problem reduces to determining if each tree compatible with $K(G)$ is a clique tree of G. Theorem 4.6 will establish the first necessary and sufficient condition for it to happen. We need to introduce two results before proving that theorem. The first one shows that compatible trees can also be characterized as a special type of maximum weight spanning trees. The second one can be proved by using the arguments similar to those that demonstrate the effectivity of some algorithms such as Kruskal's [10].

Theorem 4.4 ([4]). *Let G be a graph and G^w be the same graph after assigning each edge uv the weight $p(u, v) = |N[u] \cap N[v]|$. Then, a tree T is compatible with G if and only if T is a maximum weight spanning tree of G^w and the total weight of T equals $2|E(G)|$.*

Proof. Let T be a spanning tree of G^w. Then,

$$\begin{aligned}\sum_{uv\in E(T)} p(u,v) &= \sum_{uv\in E(T)} |N[u]\cap N[v]| = \sum_{uv\in E(T)}\sum_{w\in V(G)} |\{w\}\cap N[u]\cap N[v]| \\ &= \sum_{w\in V(G)}\sum_{uv\in E(T)} |\{w\}\cap N[u]\cap N[v]| = \sum_{w\in V(G)} |E(T[N[w]])| \le \sum_{w\in V(G)} (|N[w]|-1) \\ &= \sum_{w\in V(G)} (1+\deg(w)-1) = \sum_{w\in V(G)} \deg(w) = 2|E(G)|.\end{aligned}$$

The equality holds if and only if $|E(T[N[w]])| = |N[w]|-1$ for all $w \in V(G)$, that is, if and only if the closed neighborhood of every vertex of G induces a subtree of T. □

Proposition 4.5. *Let G be an edge-weighted graph and T, T' be two maximum weight spanning trees of G. Then, there exists a sequence $T = T_1, \ldots, T_k = T'$ such that T_i is a maximum weight spanning tree of G for $1 \le i \le k$, and, for $2 \le i \le k$, tree T_i can be obtained from T_{i-1} by adding one edge of G to it and removing another.*

Theorem 4.6. *Let G be a chordal graph. Then, there exists a tree compatible with $K(G)$ that is not a clique tree of G if and only if there exist $S \in \mathtt{S}(G)$ and $C_1, C_2 \in \mathcal{C}(G)$ such that $C_1 \cap C_2 \subsetneq S$ and, for every $C \in \mathcal{C}(G)$ with $C \cap S \neq \emptyset$, the intersections $C \cap C_1$ and $C \cap C_2$ are not empty.*

Proof. Suppose that, there exists a tree T that is compatible with $K(G)$ but that is not a clique tree of G. Let T' be a clique tree of G. Then, T' is compatible with $K(G)$ by Proposition 4.3 and, as a consequence of Theorem 4.4 and Proposition 4.5, there exists a sequence $T' = T_1, \ldots, T_k = T$ of trees compatible with $K(G)$ such that every tree of the sequence different from T_1 is built from its predecessor by adding one edge and removing another. Let i be the first number between 1 and $k-1$ such that T_i is a clique tree of G and T_{i+1} is not. Let C_1C_2 be the edge that is added and C_3C_4 be the edge that is removed to get T_{i+1} from T_i. Then, $C_3, C_4 \in T_i[C_1, C_2]$. The last sentence and Theorem 4.1 (part c) imply that $N_{K(G)}[C_1] \cap N_{K(G)}[C_2] \subseteq N_{K(G)}[C_3] \cap N_{K(G)}[C_4]$. We also infer from Theorem 4.4 that $|N_{K(G)}[C_1] \cap N_{K(G)}[C_2]| = |N_{K(G)}[C_3] \cap N_{K(G)}[C_4]|$ $(*)$ and thus the two intersections are equal. Since T_i is a clique tree, it follows from Proposition 3.1 that $C_1 \cap C_2 \subseteq C_3 \cap C_4$. On the other side, the fact that T_{i+1} is not a clique tree and Theorem 3.2 imply that $|C_1 \cap C_2| < |C_3 \cap C_4|$. Set $S = C_3 \cap C_4$. Then, by Theorem 3.4, the set S is a minimal vertex separator, $C_1 \cap C_2 \subsetneq S$ and the condition that $C \cap C_1 \neq \emptyset$ and $C \cap C_2 \neq \emptyset$ for every $C \in \mathcal{C}(G)$ with $C \cap S \neq \emptyset$ is deduced from $(*)$.

Conversely, suppose that there exist $S \in \mathtt{S}(G)$ and $C_1, C_2 \in \mathcal{C}(G)$ such that $C_1 \cap C_2 \subsetneq S$, and $C \cap C_1 \neq \emptyset$ and $C \cap C_2 \neq \emptyset$ for every $C \in \mathcal{C}(G)$ with $C \cap S \neq \emptyset$. Let C_3, C_4 be a separating pair of G such that $C_3 \cap C_4 = S$ and T be a clique tree of G such that $C_3C_4 \in E(T)$. Consider the following cases:

(1) $C_3, C_4 \in T[C_1, C_2]$: the hypothesis and the fact that T is compatible with $K(G)$ imply that $N_{K(G)}[C_1] \cap N_{K(G)}[C_2] = N_{K(G)}[C_3] \cap N_{K(G)}[C_4]$. Then, $T + C_1C_2 - C_3C_4$ is compatible with $K(G)$. However, $|C_1 \cap C_2| < |C_3 \cap C_4|$, so $T + C_1C_2 - C_3C_4$ is not a clique tree of G.
(2) Otherwise, C_1 and C_2 are in the same connected component of $T - C_3C_4$. This fact implies that $C_3 \in T[C_1, C_4] \cap T[C_2, C_4]$ or that $C_4 \in T[C_1, C_3] \cap T[C_2, C_3]$.

Suppose that $C_3 \in T[C_1, C_4] \cap T[C_2, C_4]$. Then, $C_1 \cap C_4 \subseteq C_3 \cap C_4$ and $C_2 \cap C_4 \subseteq C_3 \cap C_4$.

We now prove that $C_1 \cap C_4 \neq C_3 \cap C_4$ or that $C_2 \cap C_4 \neq C_3 \cap C_4$. Suppose to the contrary that $C_1 \cap C_4 = C_3 \cap C_4$ and $C_2 \cap C_4 = C_3 \cap C_4$. Let $T' = T + C_1C_4 - C_3C_4$. Then T' is a clique tree of G. Since $C_1 \in T'[C_2, C_4]$, it follows from Proposition 3.1 that $C_2 \cap C_4 \subseteq C_1 \cap C_2$. By our supposition, this inclusion implies that $C_3 \cap C_4 \subseteq C_1 \cap C_2$, which contradicts $C_1 \cap C_2 \subsetneq S$. Consequently, $C_1 \cap C_4 \subsetneq C_3 \cap C_4$ or $C_2 \cap C_4 \subsetneq C_3 \cap C_4$. Furthermore, for every $C \in \mathcal{C}(G)$, if $C \cap S \neq \emptyset$ then $C \cap C_1 \neq \emptyset$, $C \cap C_2 \neq \emptyset$, $C \cap C_3 \neq \emptyset$ and $C \cap C_4 \neq \emptyset$. If $C_1 \cap C_4 \subsetneq C_3 \cap C_4$, then the fact that $C_3, C_4 \in T[C_1, C_4]$ implies that case (1) can be applied with C_1, C_4 instead of C_1, C_2. The case that $C_2 \cap C_4 \subsetneq C_3 \cap C_4$ is analogous.

The proof is similar in the case that $C_4 \in T[C_1, C_3] \cap T[C_2, C_3]$. □

A careful reading of the proof of Theorem 4.6 reveals that it is based on the fact that, for a chordal graph G that is not basic chordal, there exists a clique tree T of G and edges $e_1 \in E(T)$ and $e_2 \in E(K(G))$ such that $T - e_1 + e_2$ is not a clique tree of G but is compatible with $K(G)$. Conversely, the existence of such tree T and the edges e_1 and e_2 clearly imply that G is not basic chordal.

The edge e_1 gives us S (intersection of the cliques of G that form the edge) in the statement of Theorem 4.6 and e_2 gives us C_1 and C_2 (the endpoints of the edge).

It is also interesting to note that determining whether S, C_1 and C_2 under the conditions of Theorem 4.6 exist can be done in polynomial time because $|\mathtt{S}(G)|$ and $|\mathcal{C}(G)|$ are of order $O(|V(G)|)$ for a chordal graph G. Hence, the problem of basic chordal graph recognition can be solved in polynomial time.

5. Subtree inducing sets and the concept of basis

Given that clique trees and compatible trees are characterized by the fact that some particular sets induce subtrees in them, it is natural to wonder if there are even more sets inducing subtrees. This is the question that we study in this section and the conclusions to be obtained will prove to have many implications for the main problem of this paper.

Let G be a graph. If G is chordal, then $\mathcal{SC}(G)$ will denote the family of all subsets F of $\mathcal{C}(G)$ such that $T[F]$ is a subtree of T for every clique tree T of G. For instance, each member of the dual clique family $D\mathcal{C}(G)$ is in $\mathcal{SC}(G)$, which is a consequence of the definition of clique tree given in the beginning of Section 3. The definition of $\mathcal{SC}(G)$ implies that it can be represented as a family of subtrees of each clique tree of G. Since every family of subtrees of a tree is Helly [6, see p. 92 of that reference], we deduce that $\mathcal{SC}(G)$ is Helly. The letters $\mathcal{S}$ and $\mathcal{C}$ in the expression $\mathcal{SC}(G)$ come from the words subtree and chordal.

Similarly, for G dually chordal, $\mathcal{SDC}(G)$ will denote the family of all subsets F of $V(G)$ such that $T[F]$ is a subtree of T for every tree T compatible with G. The family $\mathcal{SDC}(G)$ is Helly as well (the arguments to prove this are similar to those applied to $\mathcal{SC}(G)$) and, among the most known sets in $\mathcal{SDC}(G)$, we have cliques, closed neighborhoods and minimal vertex separators of G [1,2,12]. The letters $\mathcal{S}$, $\mathcal{D}$ and $\mathcal{C}$ in the expression $\mathcal{SDC}(G)$ come from the words subtree, dually and chordal.

A comparison between these definitions and Proposition 4.3 yield that $\mathcal{SDC}(K(G)) \subseteq \mathcal{SC}(G)$ for every chordal graph G.

It is not always easy to list all the members of $\mathcal{SC}(G)$ or $\mathcal{SDC}(G)$ because there is no polynomial bound for the cardinality of these families. But it would be desirable to find a procedure that would generate them all in the case that only some members are known.

Given a family $\mathcal{F}$ of sets, the union $\bigcup_{F\in\mathcal{F}} F$ is said to be *connected* if the intersection graph $L(\mathcal{F})$ is connected. We denote it by $\bigcup^{c}_{F\in\mathcal{F}} F$. It is not difficult to verify that families such as $\mathcal{SC}(G)$ and $\mathcal{SDC}(G)$ are closed under intersections and connected unions.

Suppose that $\mathcal{F}$ is closed under connected unions. Call a subfamily $\mathcal{F}'$ of $\mathcal{F}$ *generating* if every member of $\mathcal{F}$ with more than one element can be expressed as the connected union of some members of $\mathcal{F}'$. We call a subfamily $\mathcal{B}$ a *basis* for $\mathcal{F}$ if it is generating and no proper subfamily of $\mathcal{B}$ generates $\mathcal{F}$.

One condition to determine whether one member of a family is in some basis is the following:

Proposition 5.1. *Let $\mathcal{B}$ be a basis for $\mathcal{F}$ and D be a member of $\mathcal{F}$ with more than one element. The following are equivalent:*

(a) *D is in $\mathcal{B}$.*
(b) *for every subfamily $\mathcal{F}'$ of $\mathcal{F}$, the equality $D = \bigcup^{c}_{F\in\mathcal{F}'} F$ implies that $D \in \mathcal{F}'$.*

Proof. (a) $\Rightarrow$ (b): let $\mathcal{F}'$ be a subfamily of $\mathcal{F}$ such that $D = \bigcup^{c}_{F\in\mathcal{F}'} F$, and $F_1, \ldots, F_n$ be the non-unit members of $\mathcal{F}'$, so $D = \bigcup^{c}_{1\le i\le n} F_i$ as well. For each value of i between 1 and n, let $D_{i1}, \ldots, D_{im}$ be members of $\mathcal{B}$ such that $F_i = \bigcup^{c}_{1\le j\le m} D_{ij}$. Consequently, $D = \bigcup^{c}_{\substack{1\le i\le n\\ 1\le j\le m}} D_{ij}$. If no D_{ij} is equal to D, then the last equality would imply that $\mathcal{B} - \{D\}$ also generates $\mathcal{F}$, thus contradicting that $\mathcal{B}$ is a basis. Therefore, there exist numbers i and j such that $D_{ij} = D$.

Since $D_{ij} \subseteq F_i \subseteq D$, we have that $F_i = D$.

(b) $\Rightarrow$ (a): as $\mathcal{B}$ is a basis, let $D_1, \ldots, D_n$ be members of $\mathcal{B}$ such that $D = \bigcup^{c}_{1\le i\le n} D_i$. By the hypothesis, $D_i = D$ for some value of i. Hence, $D \in \mathcal{B}$. □

Since condition (b) is independent of the basis that we are considering, the following conclusion is immediate:

Corollary 5.2. *$\mathcal{F}$ has a unique basis.*

Before we continue, it is advisable to note that the last concepts that were defined are related to the original problem:

Theorem 5.3. *Let G be a chordal graph. The following statements are equivalent:*

(a) *every tree compatible with $K(G)$ is a clique tree of G.*
(b) *$\mathcal{SC}(G) = \mathcal{SDC}(K(G))$.*
(c) *$\mathcal{SC}(G)$ and $\mathcal{SDC}(K(G))$ have the same basis.*

Proof. (a) $\Rightarrow$ (b): Proposition 4.3 and the hypothesis imply that the clique trees of T are exactly the compatible trees of $K(G)$. Therefore, the definitions of $\mathcal{SC}(G)$ and $\mathcal{SDC}(K(G))$ imply that they are equal.

(b) $\Rightarrow$ (a) The equality between $\mathcal{SC}(G)$ and $\mathcal{SDC}(K(G))$ implies that $D\mathcal{C}(G) \subseteq \mathcal{SDC}(K(G))$. Therefore, every tree compatible with $K(G)$ is a clique tree of G.

(b) $\Leftrightarrow$ (c): trivial. □

The name basic chordal for the graphs that we are studying was inspired by part (c) of Theorem 5.3.

It is possible to use all the information from Section 3 to find, given a chordal graph G, the basis of $\mathcal{SC}(G)$.

Proposition 5.4. *Let G be a chordal graph and $A \in \mathcal{SC}(G)$. If C_1, C_2 is a separating pair contained in A, then $\mathcal{C}_{C_1\cap C_2} \subseteq A$.*

Proof. Let C be any element of $\mathcal{C}_{C_1\cap C_2}$. If $C = C_1$ or $C = C_2$, then it is clear that $C \in A$. Thus assume that $C \ne C_1$ and $C \ne C_2$. Let T be a clique tree of G such that $C_1C_2 \in E(T)$. We can also suppose without loss of generality that $C_2 \in T[C, C_1]$, so $C \cap C_1 \subseteq C_1 \cap C_2$. As $C \in \mathcal{C}_{C_1\cap C_2}$, it follows that $C_1 \cap C_2 \subseteq C \cap C_1$. Hence $C_1 \cap C_2 = C \cap C_1$. Let $T' = T - C_1C_2 + CC_1$. We infer from Theorem 3.2 that T' is a clique tree of G. Since $A \in \mathcal{SC}(G)$, we have that $T'[A]$ is a subtree of T'. Furthermore, $C \in T'[C_1, C_2]$ and $C_1, C_2 \in A$. Thus $C \in A$.

The inclusion $\mathcal{C}_{C_1\cap C_2} \subseteq A$ follows. □

Theorem 5.5. *Let G be a chordal graph. Then, $\{\mathcal{C}_S : S \in \mathtt{S(G)}\}$ is the basis of $\mathcal{SC}(G)$.*

Proof. Let A be any member of $\mathcal{SC}(G)$ with $|A| > 1$ and T be a clique tree of G. Let $e_1, \ldots, e_k$ be all the edges of $T[A]$ and S_i be the minimal vertex separator of G equal to the intersection of the endpoints of e_i for $1 \le i \le k$. By Proposition 5.4, we have that $\mathcal{C}_{S_i} \subseteq A$ for all i between 1 and k. Hence $\bigcup_{i=1}^{k} \mathcal{C}_{S_i} \subseteq A$.

Moreover, $\mathcal{C}_{S_i}$ contains the endpoints of e_i for $1 \le i \le k$. We infer that $A \subseteq \bigcup_{i=1}^{k} \mathcal{C}_{S_i}$. Therefore, $A = \bigcup_{i=1}^{k} \mathcal{C}_{S_i}$. This union is connected because $T[A]$ is a tree.

We conclude that $\{\mathcal{C}_S : S \in \mathtt{S(G)}\}$ is a generator of $\mathcal{SC}(G)$.

Let now S be a fixed minimal vertex separator and C_1C_2 be any edge of T such that $C_1 \cap C_2 = S$. Suppose that $\{M_1, \ldots, M_n\}$ is a subfamily of $\mathcal{SC}(G)$ such that $\mathcal{C}_S = \bigcup_{1\le i\le n}^{c} M_i$ and let T_1 and T_2 be the subtrees generated by removing the edge C_1C_2 from $T[\mathcal{C}_S]$, with $C_1 \in V(T_1)$ and $C_2 \in V(T_2)$. As $\bigcup_{i=1}^{n} M_i$ is connected, there must be a number j between 1 and n such that $M_j \cap V(T_1) \neq \emptyset$ and $M_j \cap V(T_2) \neq \emptyset$. Since $T[M_j]$ is a subtree of $T[\mathcal{C}_S]$, we have that $C_1, C_2 \in M_j$. Now apply Proposition 5.4 to get that $\mathcal{C}_{C_1\cap C_2} \subseteq M_j$, that is, $\mathcal{C}_S \subseteq M_j$. Hence $\mathcal{C}_S = M_j$. Thus, by Proposition 5.1, we deduce that $\mathcal{C}_S$ is a member of the basis of $\mathcal{SC}(G)$.

Therefore, $\{\mathcal{C}_S : S \in \mathcal{S}(G)\}$ is the basis of $\mathcal{SC}(G)$. □

By noting that the numbers of members of $\mathcal{C}(G)$ and of $\mathtt{S}(\mathtt{G})$ are of order $O(|V(G)|)$ for a chordal graph G, we conclude that computing the basis for $\mathcal{SC}(G)$ can be done efficiently in polynomial time.

If we define the *dimension* of G to be the number of members of the basis of $\mathcal{SC}(G)$, then Theorem 5.5 implies that the dimension equals $|\mathtt{S}(\mathtt{G})|$, which is at most the number of cliques in G minus one. On the other hand, if G is dually chordal, define the *dual dimension* of G as the number of members of the basis of $\mathcal{SDC}(G)$.

It is clear from Theorem 5.3 that if a graph G is basic chordal then the dimension of G is equal to the dual dimension of $K(G)$. But the converse is not necessarily true.

Consider the chordal graph G in Fig. 3. Its separating pairs are C_1, C_3 and C_2, C_3 and C_3, C_4, so $\mathtt{S}(\mathtt{G}) = \{\{2,3\}, \{2,5\}, \{5,6\}\}$. By Theorem 5.5, the basis of $\mathcal{SC}(G)$ is $\{\{C_1, C_3\}, \{C_2, C_3\}, \{C_3, C_4\}\}$. Due to the simplicity of $K(G)$, it is not difficult to verify that the basis of $\mathcal{SDC}(K(G))$ is $\{\{C_1, C_2, C_3\}, \{C_1, C_3\}, \{C_1, C_3, C_4\}\}$. In this case, both the dimension of G and the dual dimension of $K(G)$ are equal to 3. However, the bases are different, so there is at least one tree compatible with $K(G)$ that is not a clique tree of G. We have already seen such a tree in Fig. 3.

It is a logical next step to attempt, given a dually chordal graph G, to find the basis of $\mathcal{SDC}(G)$. We do it by finding chordal graphs whose clique trees are related to the compatible trees of G. Knowing how to calculate the bases for those chordal graphs will be fundamental for us.

We say that a chordal graph H is *in correspondence* with G if H is basic chordal and $K(H) = G$. The reader must be aware that we do not differentiate here between the graph equality and graph isomorphism.

Lemma 5.6 ([8]). *Let $\mathcal{F}$ be a Helly and separating family. Then, $\mathcal{C}(L(\mathcal{F})) = D\mathcal{F}$.*

Theorem 5.7. *Let G be a dually chordal graph and H be a chordal graph. Then, H is in correspondence with G if and only if H is the intersection graph of a separating subfamily $\mathcal{F}$ of $\mathcal{SDC}(G)$ such that the two-section graph $S(\mathcal{F})$ is equal to G.*

Proof. Suppose that H is in correspondence with G. Set $\mathcal{F} = \{\mathcal{C}_v\}_{v\in V(H)}$. We know that H is equal to the intersection graph of $\mathcal{F}$, which is a subfamily of $\mathcal{SC}(H)$. It is not difficult to verify that $S(\mathcal{F}) = K(H) = G$, and that $\mathcal{F}$ is a separating family. Furthermore, by Theorem 5.3, we have that $\mathcal{SC}(H) = \mathcal{SDC}(K(H)) = \mathcal{SDC}(G)$, so $\mathcal{F} \subseteq \mathcal{SDC}(G)$.

Conversely, assume that H is the intersection graph of a separating family $\mathcal{F}$ such that $S(\mathcal{F}) = G$ and all its members belong to $\mathcal{SDC}(G)$. As $\mathcal{F} \subseteq \mathcal{SDC}(G)$, we can see $\mathcal{F}$ as a family of subtrees of a fixed compatible tree T of G. Furthermore, every intersection graph of subtrees of a tree is chordal [14]. Therefore, H is chordal.

Now we show that $K(H)$ is isomorphic to G. Recall that we know from the definition of $\mathcal{SDC}(G)$ that it is a Helly family. Since $\mathcal{F}$ is a subfamily of $\mathcal{SDC}(G)$, we infer that $\mathcal{F}$ is also Helly. Moreover, we are assuming that $\mathcal{F}$ is separating. Thus, we can apply Lemma 5.6 to get that $\mathcal{C}(H) = \mathcal{C}(L(\mathcal{F})) = D\mathcal{F}$. It is simple to prove that two different vertices u and v are adjacent in G if and only if D_u and D_v are adjacent in $K(H)$, that is, the function $f : V(G) \to V(K(H))$ such that $f(v) = D_v$ for all $v \in V(G)$ is a graph isomorphism between G and $K(H)$.

For every $F \in \mathcal{F}$, consider the member $\mathcal{C}_F$ of $D\mathcal{C}(H)$. Then, $\mathcal{C}_F = \{C \in \mathcal{C}(H) : F \in C\} = \{D_v \in D\mathcal{F} : v \in F\}$. Since $F \in \mathcal{SDC}(G)$, it follows from the isomorphism between G and $K(H)$ that $\{D_v \in D\mathcal{F} : v \in F\} \in \mathcal{SDC}(K(H))$. Consequently, $D\mathcal{C}(H) \subset \mathcal{SDC}(K(H))$. Therefore, every tree compatible with $K(H)$ is a clique tree of H, which completes the proof. □

As an example of Theorem 5.7, consider the graph $K(G)$ of Fig. 3. We know $K(G)$ as the clique graph of a chordal graph that is not basic chordal. Let us now find a chordal graph that is in correspondence with $K(G)$. It is easy to verify that the sets $\{C_1\}, \{C_2\}, \{C_3\}, \{C_4\}, \{C_1, C_2, C_3\}$ and $\{C_1, C_3, C_4\}$ determine a separating subfamily $\mathcal{F}$ of $\mathcal{SDC}(K(G))$ such that the two section graph of $\mathcal{F}$ equals $K(G)$. Therefore, by Theorem 5.7, the graph $L(\mathcal{F})$ appearing in Fig. 4 is in correspondence with $K(G)$.

Theorem 5.7 provides the ideal framework so that all what is known from Section 3 can be used to obtain properties of compatible trees similar to those of clique trees. Some other properties are also obtained.

Theorem 5.8. *Let G be a dually chordal graph, $\mathcal{F}$ be a separating subfamily of $\mathcal{SDC}(G)$ such that $S(\mathcal{F}) = G$ and $D\mathcal{F} = \{D_v\}_{v\in V(G)}$. Then:*

(a) *given $u, v \in V(G)$, there exists a tree T compatible with G such that $uv \in E(T)$ if and only if D_u and D_v form a separating pair of $L(\mathcal{F})$.*
(b) *if T is a tree compatible with G, and u and v are two vertices of G, then there exists a tree T' that is compatible with G and has $uv \in E(T')$ if and only if there are two vertices x and y adjacent in $T[u, v]$ such that $D_x \cap D_y = D_u \cap D_v$.*
(c) *assign to each $uv \in E(G)$ the number $|D_u \cap D_v|$ to obtain the weighted graph G^w. Then, a tree T is compatible with G if and only if it is a maximum weight spanning tree of G^w weighing $\sum_{F\in\mathcal{F}} |F| - |\mathcal{F}|$.*

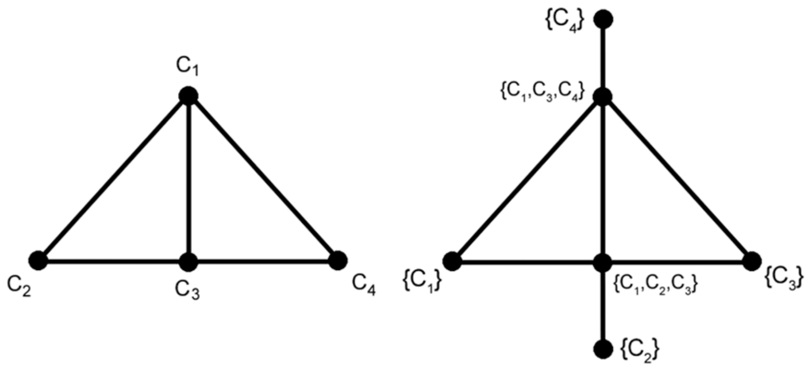

Fig. 4. The dually chordal graph $K(G)$ of Fig. 3 (left) and a chordal graph whose clique trees are exactly the compatible trees of $K(G)$ (right).

(d) *if T is compatible with G, $A \in \mathcal{SDC}(G)$, $uv \in E(T)$ and $\{u, v\} \subseteq A$, then* $\bigcap_{F\in D_u\cap D_v} F \subseteq A$.
(e) *if T is compatible with G, then* $\left\{\bigcap_{F\in D_u\cap D_v} F : uv \in E(T)\right\}$ *is the basis of* $\mathcal{SDC}(G)$.

Proof. (a) Apply Theorem 5.7 and then use Theorem 3.4 on $L(\mathcal{F})$.
(b) Apply Theorem 5.7 and then use Theorem 3.5 on $L(\mathcal{F})$.
(c) Apply Theorem 5.7 and then use Theorem 3.2 on $L(\mathcal{F})$, noting that

$$\sum_{C\in\mathcal{C}(L(\mathcal{F}))} |C| - |V(L(\mathcal{F}))| = \sum_{v\in V(G)} |D_v| - |\mathcal{F}|$$
$$= \sum_{v\in V(G)} \sum_{F\in\mathcal{F}} |\{v\}\cap F| - |\mathcal{F}| = \sum_{F\in\mathcal{F}} \sum_{v\in V(G)} |\{v\}\cap F| - |\mathcal{F}| = \sum_{F\in\mathcal{F}} |F| - |\mathcal{F}|.$$

(d) Apply part (a) to the edge uv, and Theorems 5.7 and 5.3 and Proposition 5.4 on $L(\mathcal{F})$, noting that

$$\mathcal{C}_{D_u\cap D_v} = \{C \in C(L(\mathcal{F})) : D_u \cap D_v \subseteq C\} = \{D_w : D_u \cap D_v \subseteq D_w\}$$
$$= \{D_w : \forall F \in \mathcal{F}, F \in D_u \cap D_v \to w \in F\} = \left\{D_w : w \in \bigcap_{F\in D_u\cap D_v} F\right\}.$$

(e) Part (d) implies that $\left\{\bigcap_{F\in D_u\cap D_v} F : uv \in E(T)\right\}$ generates $\mathcal{SDC}(G)$. The minimality is a consequence of part (a) and Theorems 5.7, 5.3 and 5.5 applied to $L(\mathcal{F})$. □

The most typical example of a family with the characteristics mentioned in Theorem 5.8 is the one consisting of the cliques of G and the unit sets of vertices. Applying some of the results of Theorem 5.8 to it leads to the following conclusions:

Theorem 5.9. *Let G be a dually chordal graph. Assign to each edge $uv \in E(G)$ the number $|\mathcal{C}_u \cap \mathcal{C}_v|$ to obtain the weighted graph G^w. Then, a tree T is compatible with G if and only if it is a maximum weight spanning tree of G^w weighing* $\sum_{C\in\mathcal{C}(G)} |C| - |\mathcal{C}(G)|$.

Proof. Set $\mathcal{F} = \mathcal{C}(G) \cup \{\{v\} : v \in V(G)\}$. Apply part (c) of Theorem 5.8 to this family and note that $|D_u \cap D_v| = |\mathcal{C}_u \cap \mathcal{C}_v|$ for every $uv \in E(G)$. Furthermore,

$$\sum_{F\in\mathcal{F}} |F| - |\mathcal{F}| = \sum_{C\in\mathcal{C}(G)} |C| + |V(G)| - (|\mathcal{C}(G)| + |V(G)|) = \sum_{C\in\mathcal{C}(G)} |C| - |\mathcal{C}(G)|. \quad \square$$

Theorem 5.9 has long been known [7] and was proved independently by many authors.

Theorem 5.10. *Let G be a dually chordal graph and T be compatible with G. Then,* $\left\{\bigcap_{C\in\mathcal{C}_u\cap\mathcal{C}_v} C : uv \in E(T)\right\}$ *is the basis of* $\mathcal{SDC}(G)$.

Proof. Apply part (e) of Theorem 5.8 to $\mathcal{F}$ defined like in the previous theorem. Note again that $D_u \cap D_v = \mathcal{C}_u \cap \mathcal{C}_v$ for every $uv \in E(G)$. □

As an example, consider the dually chordal graph G and its compatible tree T in Fig. 2. Now we use the edges of T to obtain the basis of $\mathcal{SDC}(G)$. Consider the edge 12 in T. The only clique containing 1 and 2 is {1, 2, 3}, so this set is in the basis. Now consider the edge 23. The cliques that contain both 2 and 3 are {1, 2, 3} and {2, 3, 5, 6}. Their intersection equals {2, 3}. Thus, this set is also in the basis. Similarly, after considering the remaining edges of the compatible tree, we get that the other members of the basis are {2, 5}, {5, 6}, {2, 4, 5} and {5, 6, 7}.

Finding the basis was easy for the graph in the example because it is small, but it could be hard to compute the basis for larger graphs because there is no polynomial bound for the number of cliques of an arbitrary dually chordal graph. A connection with neighborhoods saves us from that difficulty:

Proposition 5.11. *Let u and v be adjacent vertices of a graph G. Then,* $\bigcap_{C\in \mathcal{C}_u\cap \mathcal{C}_v} C = \bigcap_{w\in N[u]\cap N[v]} N[w]$.

Proof. We prove the double inclusion.

Let $x \in \bigcap_{C\in \mathcal{C}_u\cap \mathcal{C}_v} C$ and $w \in N[u]\cap N[v]$. Then, $\{u, v, w\}$ is complete and there exists a clique C such that $\{u, v, w\} \subseteq C$. Thus $C \in \mathcal{C}_u \cap \mathcal{C}_v$, so $x \in C$. Therefore, $x \in N[w]$. We can conclude that $\bigcap_{C\in \mathcal{C}_u\cap \mathcal{C}_v} C \subseteq \bigcap_{w\in N[u]\cap N[v]} N[w]$.

Conversely, let $x \in \bigcap_{w\in N[u]\cap N[v]} N[w]$ and $C \in \mathcal{C}_u \cap \mathcal{C}_v$. Then, $C \subseteq N[u] \cap N[v]$ and, by the description of x, this vertex is in the closed neighborhood of each element of C. As a consequence, $x \in C$. We can infer from this reasoning that $\bigcap_{w\in N[u]\cap N[v]} N[w] \subseteq \bigcap_{C\in \mathcal{C}_u\cap \mathcal{C}_v} C$. □

Another alternative for computing the basis of $\mathcal{SDC}(G)$ is to replace $\mathcal{C}(G)$ by a subfamily $\mathcal{F}$ whose members are selected as follows: for each edge uv of G, take $C \in \mathcal{C}(G)$ such that $\{u, v\} \subseteq C$. It is straightforward that $|\mathcal{F}| \leq |E(G)|$ and that $S(\mathcal{F}) = \mathcal{C}(G)$.

Similarly to [12], we say that a set A of vertices of a graph G is *positive boolean* if it can be obtained by repeated intersections and unions of closed neighborhoods of vertices. We define A to be *connected positive boolean* if connected unions are used instead of common unions.

A combination of Theorem 5.10 and Proposition 5.11 reveals that, for a dually chordal graph G, the members of the basis of $\mathcal{SDC}(G)$ are connected positive boolean, and so are the connected unions of them. Moreover, since the closed neighborhood of each vertex induces a subtree of every compatible tree, we can conclude:

Theorem 5.12. *Let G be a dually chordal graph and A be a subset of* $V(G)$*. Then,* $A \in \mathcal{SDC}(G)$ *if and only if it is connected positive boolean.*

Theorem 4.2 implies that a graph is dually chordal if and only if it has a spanning tree T such that each of its cliques induces a subtree of T. In view of Theorem 5.12, we can derive a more general characterization.

Theorem 5.13. *Let G be a graph and* $\mathcal{F}$ *be a family of connected positive boolean subsets of* $V(G)$ *such that* $S(\mathcal{F}) = G$*. Given a tree T with* $V(T) = V(G)$*, the following statements are equivalent:*

(a) *T is compatible with G.*
(b) *every member of* $\mathcal{F}$ *induces a subtree of T.*

Proof. (a) ⇒ (b): by Theorem 5.12, every member of $\mathcal{F}$ is in $\mathcal{SDC}(G)$, so it induces a subtree of T.

(b) ⇒ (a): let T be a spanning tree of G such that every member of $\mathcal{F}$ induces a subtree of T. Let u and v be two adjacent vertices and w be another vertex such that $w \in T[u, v]$. Since $S(\mathcal{F}) = G$, we can take $F \in \mathcal{F}$ such that $\{u, v\} \subseteq F$. It also holds that $T[F]$ is a subtree of T, and thus $w \in F$. Therefore, w is adjacent to u and to v in $S(\mathcal{F})$, that is, $uw \in E(G)$ and $vw \in E(G)$. Consequently, by Theorem 4.1, we have that T is compatible with G. □

Corollary 5.14. *Let G be a graph and* $\mathcal{F}$ *be a family of connected positive boolean subsets of* $V(G)$ *such that* $S(\mathcal{F}) = G$*. Then, G is dually chordal if and only if there exists a spanning tree T of G such that every member of* $\mathcal{F}$ *induces a subtree of T.*

6. One more characterization and additional properties of basic chordal graphs

In this final part of the paper, we use all what we have learned from the previous section to find a new necessary and sufficient condition for a graph to be basic chordal (Theorem 6.4). Its statement is preceded by some lemmas.

Lemma 6.1. *Let G be a chordal graph and* C_1, C_2 *be a separating pair. If C is a clique such that* $C \cap C_1 \neq \emptyset$ *and* $C \cap C_2 \neq \emptyset$*, then* $C \cap C_1 \cap C_2 \neq \emptyset$*.*

Proof. Let T be a clique tree of G such that $C_1C_2 \in E(T)$. Such tree T exists because of Theorem 3.4. Then, $C_1 \in T[C, C_2]$ or $C_2 \in T[C, C_1]$. In the first case, $C\cap C_2 \subseteq C_1\cap C_2$; in the second, $C\cap C_1 \subseteq C_1\cap C_2$. Both inclusions imply that $C\cap C_1\cap C_2 \neq \emptyset$. □

Lemma 6.2. *Let G be a dually chordal graph, T be a tree compatible with G,* $uv \in E(T)$ *and* $\mathcal{F}$ *be any separating subfamily of* $\mathcal{SDC}(G)$ *such that* $S(\mathcal{F}) = G$*. Set* $B_{\mathcal{F}} = \bigcap_{F\in D_u\cap D_v} F$*. Then,* $B_{\mathcal{F}}$ *does not depend on the choice of* $\mathcal{F}$*.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be separating subfamilies of $\mathcal{SDC}(G)$ with two-section graphs equal to G. The set $\{u, v\}$ is contained in both $B_{\mathcal{F}}$ and $B_{\mathcal{F}'}$, so part (d) of Theorem 5.8 can be applied on both $\mathcal{F}$ and $\mathcal{F}'$ to conclude that $B_{\mathcal{F}} \subseteq B_{\mathcal{F}'}$ and $B_{\mathcal{F}'} \subseteq B_{\mathcal{F}}$, respectively. Therefore, $B_{\mathcal{F}} = B_{\mathcal{F}'}$. □

Given a graph G and $S \in \mathtt{S}(G)$, define B_S as the set of cliques of G that intersect every clique intersecting S, that is, $B_S = \bigcap_{C\cap S\neq\emptyset} N_{K(G)}[C]$. This set should not be mistaken for $B_{\mathcal{F}}$ from the previous lemma. The definition clearly implies that $\mathcal{C}_S \subseteq B_S$ for every $S \in \mathtt{S}(G)$. Now we prove:

Lemma 6.3. *Let G be a chordal graph. Then,* $\{B_S : S \in \mathtt{S}(G)\}$ *is the basis of* $\mathcal{SDC}(K(G))$*.*

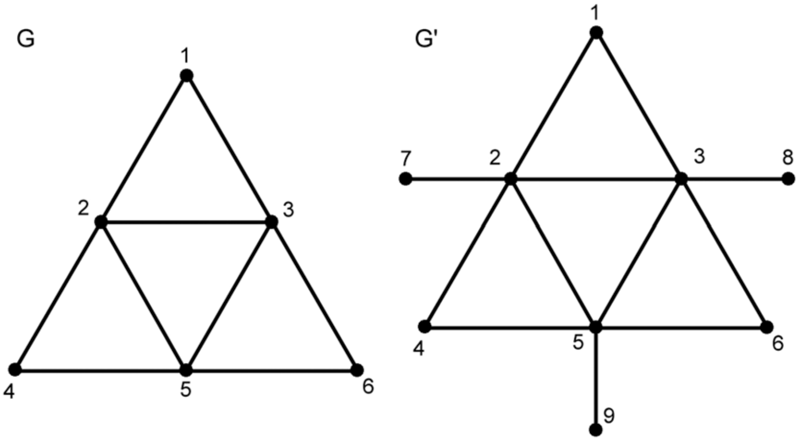

Fig. 5. Every tree compatible with $K(G')$ is a clique tree of G'. The same cannot be said about G.

Proof. Let S be any minimal vertex separator of G and C_1, C_2 be a separating pair such that $S = C_1 \cap C_2$. Then, by Lemma 6.1, $\bigcap_{C\cap S\neq\emptyset} N_{K(G)}[C] = \bigcap_{C\in N_{K(G)}[C_1]\cap N_{K(G)}[C_2]} N_{K(G)}[C]$. Now apply Proposition 5.11 to conclude that $B_S = \bigcap_{\substack{D\in\mathcal{C}(K(G))\\ C_1,C_2\in D}} D$.

Finally, take a clique tree T for G and complete the proof by applying Proposition 4.3 and Theorems 3.3–3.5 and 5.10. □

If G is a basic chordal graph, then we deduce from Theorems 5.3 and 5.5 and Lemma 6.3 that $\{B_S : S \in \mathtt{S}(\mathtt{G})\} = \{\mathcal{C}_S : S \in \mathtt{S}(\mathtt{G})\}$. Thus, for each $S \in \mathtt{S}(\mathtt{G})$, there exists $S' \in \mathtt{S}(\mathtt{G})$ such that $B_S = \mathcal{C}_{S'}$. In principle, S' does not need to equal S. However, the equality always holds, thus leading to the following characterization:

Theorem 6.4. *Let G be a chordal graph. Then, G is basic chordal if and only if $B_S = \mathcal{C}_S$ for every $S \in \mathtt{S}(\mathtt{G})$.*

Proof. Suppose that $B_S = \mathcal{C}_S$ for every $S \in \mathtt{S}(\mathtt{G})$. Therefore, by Theorem 5.5 and Lemma 6.3, the bases for $\mathcal{SC}(G)$ and $\mathcal{SDC}(K(G))$ are equal and it follows from Theorem 5.3 that G is basic chordal.

Conversely, suppose that G is a basic chordal graph. Let $S \in \mathtt{S}(\mathtt{G})$, T be a clique tree of G and $C_1C_2 \in E(T)$ be such that $C_1 \cap C_2 = S$. Set $\mathcal{F}_1 = \mathcal{C}(K(G)) \cup \{\{C\} : C \in K(G)\}$ and $\mathcal{F}_2 = \{\mathcal{C}_v : v \in V(G)\}$. Use the expression found for B_S in Lemma 6.3 and apply Lemma 6.2 to $C_1, C_2, \mathcal{F}_1, \mathcal{F}_2$ to get that

$$B_S = \bigcap_{\substack{D\in\mathcal{C}(K(G))\\ C_1,C_2\in D}} D = B_{\mathcal{F}_1} = B_{\mathcal{F}_2} = \bigcap_{C_1,C_2\in\mathcal{C}_v} \mathcal{C}_v = \bigcap_{v\in C_1\cap C_2} \mathcal{C}_v = \bigcap_{v\in S} \mathcal{C}_v = \mathcal{C}_S. \quad \square$$

Note that, since we know that the number of minimal vertex separators does not surpass the number of vertices in chordal graphs, Theorem 6.4 can be used to design a polynomial-time algorithm for recognizing basic chordal graphs.

Although Theorem 6.4 was discovered thanks to basis theory, going back to Theorem 4.6 reveals that this result could have also been used to give a simple proof of Theorem 6.4. Here we show how:

Theorem 6.5. *Let G be a chordal graph. The following statements are equivalent:*

1. *there exist $S \in \mathtt{S}(\mathtt{G})$ and $C_1, C_2 \in \mathcal{C}(G)$ such that $C_1 \cap C_2 \subsetneq S$ and, for all $C \in \mathcal{C}(G)$ with $C \cap S \neq \emptyset$, the intersections $C \cap C_1$ and $C \cap C_2$ are not empty.*
2. *there exists $S \in \mathtt{S}(\mathtt{G})$ such that $B_S \neq \mathcal{C}_S$.*

Proof. Suppose that 1. is true and let C_1, C_2, S have the characteristics mentioned there. We now prove that $B_S \neq \mathcal{C}_S$.

Since $C_1 \cap C_2 \subsetneq S$, it is impossible that $S \subseteq C_1$ and $S \subseteq C_2$ are both true, otherwise $S \subseteq C_1 \cap C_2$. Suppose without loss of generality that S is not contained in C_1. Then, $C_1 \notin \mathcal{C}_S$. However, by the hypothesis, $C_1 \in B_S$. Therefore, $B_S \neq \mathcal{C}_S$.

Conversely, suppose that 2. is true and take $S \in \mathtt{S}(\mathtt{G})$ such that $B_S \neq \mathcal{C}_S$. Since $\mathcal{C}_S \subseteq B_S$, there exists $D \in \mathcal{C}(G)$ such that $D \in B_S$ and $D \notin \mathcal{C}_S$.

Let T be a clique tree of G and C_3, C_4 be such that $C_3C_4 \in E(T)$ and $C_3 \cap C_4 = S$. Then, $C_4 \in T[C_3, D]$ or $C_3 \in T[C_4, D]$. Suppose without loss of generality that $C_4 \in T[C_3, D]$. We infer from Proposition 3.1 that $C_3 \cap D \subseteq C_3 \cap C_4 = S$. The equality $C_3 \cap D = S$ cannot hold because it implies that $S \subseteq D$, which is in contradiction with $D \notin \mathcal{C}_S$. Therefore, $C_3 \cap D \subsetneq S$.

If C is any clique of G such that $C \cap S \neq \emptyset$, then $C \cap C_3 \neq \emptyset$ because $S \subseteq C_3$, and $C \cap D \neq \emptyset$ because $D \in B_S$. Therefore, we can set $C_1 = C_3$ and $C_2 = D$ to verify that 1. is true. □

As an example of Theorem 6.4, consider the two graphs G and G' in Fig. 5.

$K(G)$ equals the complete graph on four vertices, so every spanning tree of $K(G)$ is a compatible tree. However, it is simple to use Theorem 3.2 to verify that G has only one clique tree. Therefore, G is not basic chordal. In order to conclude the same through Theorem 6.4, note that $\mathtt{S(G)} = \{\{2, 3\}, \{2, 5\}, \{3, 5\}\}$ and $\mathcal{C}(G) = \{\{1, 2, 3\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 5, 6\}\}$. Then, the comparison between the sets B_S and $\mathcal{C}_S$, for every $S \in \mathtt{S(G)}$, is given by the following table:

S	B_S	$\mathcal{C}_S$	$B_S = \mathcal{C}_S$
{2, 3}	$\mathcal{C}(G)$	{{1, 2, 3}, {2, 3, 5}}	×
{2, 5}	$\mathcal{C}(G)$	{{2, 3, 5}, {2, 4, 5}}	×
{3, 5}	$\mathcal{C}(G)$	{{2, 3, 5}, {3, 5, 6}}	×

Nevertheless, if we just add three new vertices of degree one to obtain the graph G' then the result is different. This time, $\mathcal{C}(G') = \mathcal{C}(G) \cup \{\{2, 7\}, \{3, 8\}, \{5, 9\}\}$ and $\mathtt{S(G')} = \mathtt{S(G)} \cup \{\{2\}, \{3\}, \{5\}\}$. The corresponding table is:

S	B_S	$\mathcal{C}_S$	$B_S = \mathcal{C}_S$
{2, 3}	{{1, 2, 3}, {2, 3, 5}}	{{1, 2, 3}, {2, 3, 5}}	✓
{2, 5}	{{2, 3, 5}, {2, 4, 5}}	{{2, 3, 5}, {2, 4, 5}}	✓
{3, 5}	{{2, 3, 5}, {3, 5, 6}}	{{2, 3, 5}, {3, 5, 6}}	✓
{2}	{{1, 2, 3}, {2, 7}, {2, 3, 5}, {2, 4, 5}}	{{1, 2, 3}, {2, 7}, {2, 3, 5}, {2, 4, 5}}	✓
{3}	{{1, 2, 3}, {3, 8}, {2, 3, 5}, {3, 5, 6}}	{{1, 2, 3}, {3, 8}, {2, 3, 5}, {3, 5, 6}}	✓
{5}	{{2, 4, 5}, {5, 9}, {2, 3, 5}, {3, 5, 6}}	{{2, 4, 5}, {5, 9}, {2, 3, 5}, {3, 5, 6}}	✓

Therefore, G' is basic chordal. Since G is an induced subgraph of G', we conclude that the class of basic chordal graphs is not hereditary. In fact, a construction like the one used to obtain G' allows to show that every chordal graph is an induced subgraph of some basic chordal graph.

Proposition 6.6. *Let G be a chordal graph and V' be the set of vertices of G that are not simplicial. Let G' be the graph constructed from G by adding, for each $v \in V'$, a vertex v^* and the edge vv^*. Then, G' is basic chordal.*

Proof. If $V' = \emptyset$, then G is a complete graph, $G = G'$ and it is easy to check that G is basic chordal.

Assume for the rest of the proof that $V' \neq \emptyset$. Thus $\mathtt{S(G')} = \mathtt{S(G)} \cup \{\{v\} : v \in V'\}$. We now prove that, for every $S \in \mathtt{S(G')}$, we have $B_S = \mathcal{C}_S$, where the two sets are computed with respect to G'. Let $S \in \mathtt{S(G')}$ and $C \in B_S$ and $v \in S$. As v is the vertex of a minimal vertex separator, it follows that v is not simplicial, and hence $v \in V'$. Then, $\{v, v^*\}$ is a clique of G' intersecting S. Since $C \in B_S$, we infer that $C \cap \{v, v^*\} \neq \emptyset$. Thus, $v \in C$. We can conclude from this reasoning that $S \subseteq C$, that is, $C \in \mathcal{C}_S$. Therefore, $B_S \subseteq \mathcal{C}_S$. By the definition of B_S, the inclusion $\mathcal{C}_S \subseteq B_S$ is always true. Thus $B_S = \mathcal{C}_S$. By Theorem 6.4, the graph G' is basic chordal. □

Many of the results and proofs about chordal graphs in this section tacitly assume that the graphs are not complete, otherwise the family of minimal vertex separators is empty. However, it is easy to verify that the theorems and propositions remain true for the case of complete graphs.

We finish the paper with a characterization of the basic chordal graphs with diameter not larger than two.

Recall that the *distance* between two vertices u and v in a graph G, denoted $d(u, v)$, is the length of any shortest path of G joining u and v, and the *diameter* of G, denoted $diam(G)$, is defined to be the maximum distance between two vertices of G, that is, $diam(G) = \max\{d(u, v) : u, v \in V(G)\}$. Vertex v is *universal* if $N[v] = V(G)$.

Theorem 6.7. *Let G be a chordal graph such that $diam(G) \leq 2$. Then, G is basic chordal if and only if every vertex of G is either simplicial or universal.*

Proof. Suppose that every vertex of G is simplicial or universal. If G is complete, then it is straightforward that G is basic chordal. If G is not complete, then the intersection of any two distinct cliques of G is equal to the set of universal vertices of G. Thus, by Theorem 3.3, the set of universal vertices of G is the only minimal vertex separator of G. Let S denote this separator. It is easy to verify that $B_S = \mathcal{C}_S = \mathcal{C}(G)$. Therefore, by Theorem 6.4, we have that G is basic chordal.

Conversely, suppose that G is basic chordal. Now we show that $K(G)$ is complete. This is direct in case that G is complete.

Otherwise, let C_1 and C_2 be any two distinct cliques of G. We have to prove that $C_1 \cap C_2 \neq \emptyset$. Let T be a clique tree of G. If C_1 and C_2 are adjacent in T, then it is clear that they are not disjoint. If they are not adjacent in T, then let $C \in T(C_1, C_2)$. Let C_3 be a leaf of T such that $C_1 \in T[C, C_3]$ and let C_4 be another leaf such that $C_2 \in T[C, C_4]$. Hence, C_3 and C_4 are simplicial cliques of G. Let v and w be simplicial vertices in C_3 and C_4, respectively. Since $diam(G) \leq 2$, we have $N[v] \cap N[w] \neq \emptyset$, that is, $C_3 \cap C_4 \neq \emptyset$. By the construction, $\{C_1, C_2\} \subseteq T[C_3, C_4]$. Thus, it follows from Proposition 3.1 that $C_3 \cap C_4 \subseteq C_1 \cap C_2$. Therefore, $C_1 \cap C_2 \neq \emptyset$.

As a consequence of the previous argument, $K(G)$ is a complete graph. Hence, every spanning tree of $K(G)$ is a compatible tree for this graph. As G is basic chordal, we infer that every spanning tree of $K(G)$ is a clique tree of G.

We deduce from the last fact that the members of $\mathcal{SC}(G)$ are $\mathcal{C}(G)$ and its unit subsets.

Let v be a vertex of G. Thus, $\mathcal{C}_v \in \mathcal{SC}(G)$. If $\mathcal{C}_v = \mathcal{C}(G)$, then v is a universal vertex of G. If $\mathcal{C}_v$ is a unit set, then v is simplicial. This concludes the proof. □

Note that, this theorem can also be used to prove that the graph G in Fig. 5 is not basic chordal. Its diameter equals two but the vertices 2, 3 and 5 are neither simplicial nor universal.

7. Final remarks and future work

Initially, dually chordal graphs were studied independently and simultaneously by many authors under a diversity of names. When those studies were put in contact, the necessity of a unification became evident and it was advisable to have only one name to refer to the class. A better understanding of the class made the name dually chordal graph become increasingly convincing. We think that this work strengthens the sense of duality between chordal and dually chordal graphs through a deep study of their characterizing trees. We also see how our effort to characterize basic chordal graphs allows other properties to become evident, thus giving us more insights into the nature of chordal and dually chordal graphs.

Putting restrictions on the clique trees of a chordal graph gives rise to new subclasses of chordal graphs. Three well-studied subclasses appearing in that context are *UV* graphs, *DV* graphs and *RDV* graphs [13]. A chordal graph is *UV* if it possesses a clique tree such that each set $\mathcal{C}_v$ induces a path in the tree. Such a tree is given the name of *UV-clique tree*. The graph is *DV* if there is a clique tree whose edges can be oriented so that each set $\mathcal{C}_v$ induces a directed path. If such tree can be rooted at certain vertex, then the graph is *RDV*. For these last two graphs, we use the terms *DV-clique tree* and *RDV-clique tree*. The class of clique graphs of *UV* graphs is just that of dually chordal graphs. But the clique graphs of *DV* graphs and *RDV* graphs give rise to new subclasses of dually chordal graphs, called *dually DV* and *dually RDV graphs*. These classes also have characteristic spanning trees, namely, the *DV-compatible tree* and the *RDV-compatible tree*. Therefore, it makes sense to study the correspondence between *DV* (*RDV*)-clique trees and *DV* (*RDV*)-compatible trees, which in our opinion will give rise to new subclasses of basic chordal graphs. This subject will be part of our future work.

References

[1] A. Brandstädt, F. Dragan, V. Chepoi, V. Voloshin, Dually chordal graphs, SIAM Journal on Discrete Mathematics 11 (1998) 437–455.
[2] P. De Caria, M. Gutierrez, On minimal vertex separators of dually chordal graphs: properties and characterizations, Discrete Applied Mathematics 160 (2012) 2627–2635.
[3] G.A. Dirac, On rigid circuit graphs, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 25 (1961) 71–76.
[4] F. Dragan, HT-graphs: centers, connected r-domination and Steiner trees, Computer Science Journal of Moldova 1 (1993) 64–83.
[5] D.R. Fulkerson, O.A. Gross, Incidence matrices and interval graphs, Pacific Journal of Mathematics 15 (1965) 835–855.
[6] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, second ed., in: Annals of Discrete Mathematics, vol. 57, Elsevier, 2004.
[7] M. Gutierrez, Tree-Clique Graphs, Estudos e comunicações do Instituto de Matemática 63 (1996) 7–26. Universidade Federal do Rio de Janeiro.
[8] M. Gutierrez, J. Meidanis, Algebraic theory for the clique operator, Journal of the Brazilian Computer Society 7 (2001) 53–64.
[9] H. Habib, J. Stacho, Reduced clique graphs of chordal graphs, European Journal of Combinatorics 33 (2012) 712–735.
[10] J.B. Kruskal, On the shortest spanning tree of a graph and the traveling salesman problem, Proceedings of the American Mathematical Society 7 (1956) 48–50.
[11] T.A. McKee, How chordal graphs work, Bulletin of the ICA 9 (1993) 27–39.
[12] T.A. McKee, When fundamental cycles span cliques, Congressus Numerantium 191 (2009) 213–218.
[13] C.L. Monma, V.K. Wei, Intersection graphs of paths in a tree, Journal of Combinatorial Theory. Series B 41 (1986) 141–181.
[14] Y. Shibata, On the tree representation of chordal graphs, Journal of Graph Theory 12 (1988) 421–428.