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# On asteroidal sets in chordal graphs

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ABSTRACT

We analyze the relation between three parameters of a chordal graph $G$: the number of non-separating cliques $nsc(G)$, the asteroidal number $an(G)$ and the leafage $l(G)$. We show that $an(G)$ is equal to the maximum value of $nsc(H)$ over all connected induced subgraphs $H$ of $G$. As a corollary, we prove that if $G$ has no separating simplicial cliques then $an(G) = l(G)$.

A graph $G$ is minimal $k$-asteroidal if $an(G) = k$ and $an(H) < k$ for every proper connected induced subgraph $H$ of $G$. The family of minimal $k$-asteroidal chordal graphs is unknown for every $k > 3$; for $k = 3$ it is the family described by Lekerkerker and Boland to characterize interval graphs. We prove that, for every minimal $k$-asteroidal chordal graph, all the above parameters are equal to $k$. In addition, we characterize the split graphs that are minimal $k$-asteroidal and obtain all the minimal 4-asteroidal split graphs. Finally, we applied our results on asteroidal sets to describe the clutters with $k$ edges that are minor-minimal in the sense that every minor has less than $k$ edges.



## 1. Introduction

Asteroidal triples were introduced by Lekkerkerker and Boland to characterize interval graphs [10]. An *asteroidal triple* of a graph $G$ is a set of three vertices such that each pair is connected by some path avoiding the closed neighborhood of the third vertex. Interval graphs are exactly the chordal graphs without asteroidal triples. Walter generalized the concept of asteroidal triple in order to characterize other subclasses of chordal graphs [14]. A subset of non-adjacent vertices of $G$ is an *asteroidal set* if the removal of the closed neighborhood of any one of its elements does not disconnect the remaining ones. The *asteroidal number* of $G$ is the cardinality of a largest asteroidal set of $G$. Many algorithmic and structural properties of graphs with bounded asteroidal number have been obtained [3,9]. The problem of determining the asteroidal number of a graph is NP-complete even when restricted to triangle-free 3-connected 3-regular planar graphs [8].

The class of chordal graphs has unbounded asteroidal number: for any integer $k \geq 2$, the graph obtained from $K_{1,k}$ by subdividing each edge with a new vertex is a chordal graph with $2k + 1$ vertices and asteroidal number $k$. However, there exists a polynomial time algorithm for computing the asteroidal number of a chordal graph [8].

Buneman, Gavril and Walter independently showed that a graph $G$ is chordal if and only if it is the intersection graph of a family of subtrees of a host tree [4,6,14]. The number of leaves of a host tree of $G$ with minimum number of leaves is the *leafage* of $G$. Recently, Habib and Stacho gave the first polynomial time algorithm for computing the leafage of a chordal graph [7]. In [11], the leafage is shown to be an upper bound of the asteroidal number.

A vertex subset of a connected graph is *separating* when its removal produces a disconnected graph. Decomposition by separating complete sets was introduced as an important tool for a *divide and conquer* approach of many hard graph problems. An $O(nm)$ time algorithm for the recognition of separating complete sets was given by Tarjan in [13] from a previous Whitesides' work [15]. Decomposition by separating cliques, or clique separators, was used by Monma and Wei to characterize intersection graphs of paths in a tree [12]. In this work we focus on non-separating cliques.

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In Section 2, we study the relation between the three parameters: the number of non-separating cliques, the asteroidal number and the leafage. We show that the asteroidal number of a chordal graph $G$ is equal to the maximum number of non-separating cliques over all connected induced subgraphs of $G$. This approach seems to be more natural, in pure graph theoretical terms, than the one by Lin, McKee and West where the asteroidal number is related to the width of the restricted simplicial neighborhood poset of the induced subgraphs [11]. A summary of those results is included in Section 2.1 in order to allow the readers to compare them with those in the present work.

In addition, as a corollary, we provide a sufficient condition for the equality between the asteroidal number and the leafage of a connected chordal graph, and we offer several examples of graphs satisfying the given condition. An interested open problem is to characterize those chordal graphs with equal asteroidal number and leafage.

A *minimal k-asteroidal graph* is a graph such that its asteroidal number is $k$ and the asteroidal number of every proper connected induced subgraph of it is less than $k$. The family of minimal 3-asteroidal chordal graphs was described in [10] to characterize interval graphs by forbidden induced subgraphs (see Fig. 2). For every $k > 3$, the family of minimal $k$-asteroidal chordal graph is unknown.

In Section 3, we prove that in the class of minimal $k$-asteroidal chordal graphs, all the above parameters are equal to $k$. We characterize the split graphs that are minimal $k$-asteroidal. Split graphs are the $2K_2$-free chordal graphs [2]. In addition, by linking split graphs with set families that satisfy a particular property, we obtain all the minimal 4-asteroidal split graphs (see Fig. 4).

A *clutter* or *Sperner system* is a family of sets none of which is a proper subset of other members of the family [1,5]. In Section 4, we applied the results of the previous sections to describe the clutters with $k$ members such that every one of its minors has less than $k$ members.

Finally, in Section 5, we pose some related open problems.

## 2. Non-separating cliques, asteroidal number and leafage

Let $G$ be a simple finite undirected graph. We use $V_G$ and $E_G$ to denote the vertex set and the edge set of $G$, respectively.

The *open neighborhood* $N_G(v)$ of a vertex $v$ is the set $\{w \in V_G | vw \in E_G\}$ and the *closed neighborhood* $N_G[v]$ is the set $N_G(v) \cup \{v\}$. For simplicity, when no confusion can arise, we omit the subindex $G$ and write $N(v)$ and $N[v]$.

A *complete set* is a subset of pairwise adjacent vertices. A *clique* is a maximal complete set. Many authors use the terms clique and maximal clique to refer to what we have called complete set and clique, respectively. We say that a vertex $v$ is *simplicial* when $N(v)$ is a complete set. The set of simplicial vertices of $G$ is denoted by $S(G)$. A *simplicial clique* is a clique containing a simplicial vertex. Notice that a clique $C$ is simplicial if and only if there exists a vertex $v$ such that $C = N[v]$.

For $S \subseteq V_G$, $G[S]$ is the subgraph of $G$ induced by $S$, and $G - S$ is a shorthand for $G[V_G - S]$. A *non-separating clique* is a clique $C$ such that $G - C$ has no more connected components than $G$.

The number of simplicial vertices, the number of simplicial cliques and the number of non-separating cliques of $G$ are denoted by $sv(G)$, $sc(G)$ and $nsc(G)$, respectively.

A graph is *chordal* if every cycle with more than three vertices has a chord, i.e., an edge joining two non-consecutive vertices of the cycle. We prove in the following lemma a fundamental property of chordal graphs.

**Lemma 2.1.** *Every non-separating clique of a chordal graph contains a simplicial vertex.*

**Proof.** Notice that the result is trivial when the graph has at most 2 cliques. Let $C$ be a non-separating clique of a connected chordal graph $G$ with at least 3 cliques. Assume that $C$ is not simplicial. We proceed by induction on the number of vertices of $G$. Since $G$ is a chordal graph which is not complete, $G$ has a pair of non-adjacent simplicial vertices [1], say $v$ and $w$. Let $G'$ be the connected chordal graph $G - v$. It is clear that $C$ is a clique of $G'$. We claim that $C$ is non-separating in $G'$. Indeed, if it is not, then there exist a pair of vertices $s$ and $t$ which are in different connected components of $G' - C$. This implies, since $s$ and $t$ are not separated by the removal of $C$ in $G$, that there exists an induced path between $s$ and $t$ containing the vertex $v$, which contradicts the fact that $v$ is simplicial.

Thus, by the inductive hypothesis, $C$ is a simplicial clique of $G'$. Let $h$ be a vertex such that $N_{G'}[h] = C$. Since $C$ is not simplicial in $G$, it follows that $N_G[h] = C \cup \{v\}$. But, since $v$ is simplicial in $G$, this implies that $N_G(v) \subseteq C$. Therefore, $v$ and $w$ are in different connected components of $G - C$, a contradiction. □

We obtain from the previous lemma that for every chordal graph $G$,

$$nsc(G) \leq sc(G) \leq sv(G). \quad (1)$$

A *stable set* is a subset of pairwise non-adjacent vertices. An *asteroidal set* is a stable set $A \subseteq V_G$ satisfying that for every $a \in A$, $A - \{a\}$ is contained in the same connected component of $G - N[a]$. The *asteroidal number* of $G$, denoted by $an(G)$, is the cardinality of a maximum asteroidal set of $G$ [9].

In [11, Lemma 7] shows that if $G$ is chordal then $S(G)$ contains a maximum asteroidal set of $G$. Since the vertices of an asteroidal set are pairwise non-adjacent, it is clear that

$$an(G) \leq sc(G). \quad (2)$$

Using the following lemma and relations (1) and (2), we prove that for any connected chordal graph $G$,

$$nsc(G) \leq an(G) \leq sc(G) \leq sv(G). \quad (3)$$

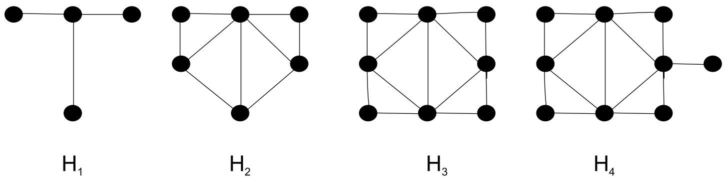


**Fig. 1.** Each graph in the above sequence is chordal and a connected induced subgraph of the next one.

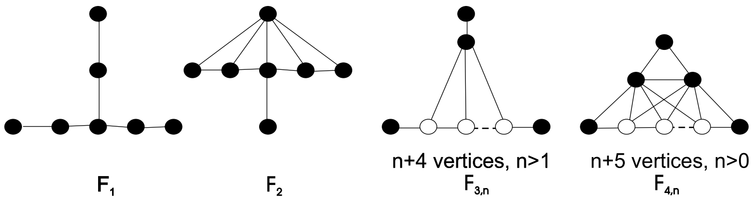


**Fig. 2.** Minimal 3-asteroidal chordal graphs.

**Table 1**
The number of simplicial cliques, the number of non-separating cliques and the width of the restricted simplicial neighborhood poset of each graph in Fig. 1.

| | $H_1$ | $H_2$ | $H_3$ | $H_4$ |
|---|---|---|---|---|
| $sc$ | 3 | 2 | 4 | 5 |
| $nsc$ | 0 | 2 | 4 | 3 |
| $w(P')$ | 1 | 2 | 4 | 4 |

**Lemma 2.2.** *Let $C_1, C_2, \ldots, C_k$ be different non-separating cliques of a connected chordal graph G. There exist simplicial vertices $a_1, a_2, \ldots, a_k$ of G such that $C_i = N[a_i]$ for every i with $1 \leq i \leq k$, and $\{a_1, a_2, \ldots, a_k\}$ is an asteroidal set of G.*

**Proof.** Consider any $i$, $1 \leq i \leq k$. Since every non-separating clique of a chordal graph contains a simplicial vertex, there exists a simplicial vertex $a_i$ of $G$ such that $C_i = N[a_i]$. Since $C_i$ is non-separating, $G - N[a_i]$ is connected. The proof follows from the fact that $a_j \notin N[a_i]$ for every $j \neq i$. □

The graph $H_4$ in Fig. 1 illustrates that the inequalities in (3) can be strict, in particular $nsc(H_4) = 3$, $an(H_4) = 4$ and $sc(H_4) = 5$. On the other hand, the simple case of the six-cycle $C_6$ shows that relation (3) is not true for general graphs. Notice that $nsc(C_6) = 6$, $an(C_6) = 3$ and $sc(C_6) = 0$.

A direct consequence of the asteroidal set definition is that if $H$ is a connected induced subgraph of a graph $G$, then

$$an(H) \leq an(G). \tag{4}$$

Therefore, the asteroidal number of a graph is an upper bound for the asteroidal number of its induced subgraphs. A relation of this type cannot be established for the parameters $sc$ and $nsc$, not even in the case of chordal graphs. The graphs $H_1, H_2, H_3$ and $H_4$ in Fig. 1 are chordal and each one is a connected induced subgraph of the next one; however, the values of the parameters $sc$ and $nsc$ for these graphs are neither decreasing nor increasing, as we see in Table 1.

Moreover, by relations (3) and (4), if $H$ is a connected induced subgraph of a chordal graph $G$ then

$$nsc(H) \leq an(H) \leq an(G) \leq sc(G) \leq sv(G). \tag{5}$$

Hence, the asteroidal number of a chordal graph $G$ is also an upper bound for the number of non-separating cliques of every connected induced subgraph of $G$. The next theorem shows that this bound is sharp in the sense that there always exists a connected induced subgraph $H$ such that $nsc(H) = an(G)$.

**Theorem 2.3.** *Let $\{a_1, a_2, \ldots, a_k\} \subseteq S(G)$ be an asteroidal set of a chordal graph G. There exists a connected induced subgraph H of G such that $N_H[a_i]$, $1 \leq i \leq k$, are k different non-separating cliques of H.*

**Proof.** When $k \leq 2$ the proof is straightforward, then assume $k \geq 3$. Let $A = \{a_1, a_2, \ldots, a_k\}$ be a simplicial asteroidal set of a chordal graph $G$. For every pair of vertices $a_l, a_j$ of $A$ and each $a_s$ in $A - \{a_l, a_j\}$, consider one induced path $P_{\{l,j\},s}$ in $G$, between $a_l$ and $a_j$ avoiding the closed neighborhood of $a_s$. Let $H$ be the subgraph of $G$ induced by the union of the vertices of all the considered paths. Notice that $H$ is connected.

Consider any $i$, $1 \leq i \leq k$. Since $a_i$ is simplicial, $N_H[a_i]$ is a clique of $H$. Moreover, we claim that $N_H[a_i]$ is non-separating. Indeed, suppose, for a contradiction, that $H - N_H[a_i]$ is disconnected. Notice that, by the choice of $H$, the vertices in $A - \{a_i\}$ are all contained in the same connected component $C$ of $H - N_H[a_i]$. Then, if $H - N_H[a_i]$ is not connected, there exists a vertex $x$ of $H - N_H[a_i]$ which does not belong to the connected component $C$. Therefore, $x$ must be interior to a path $P_{\{l,j\},s}$

with $l, j, s \neq i$. Since $N_H[a_i]$ disconnects $a_l$ from $x$ and $a_j$ from $x$, there must exist in $P_{\{l,j\},s}$ a vertex of $N_H[a_i]$ between $a_l$ and $x$ and another between $x$ and $a_j$. This contradicts the fact that $P_{\{l,j\},s}$ is an induced path. It follows that $H - N_H[a_i]$ is connected and, consequently, $N_H[a_i]$ is a non-separating clique of $H$. □

Using the previous theorem and the relation (5), we prove the following result.

**Theorem 2.4.** *The asteroidal number $an(G)$ of a connected chordal graph $G$ is equal to the maximum value of $nsc(H)$ over all connected induced subgraphs $H$ of $G$.*

Gavril showed that a graph $G$ is chordal if and only if there exist a *host tree* $T$ and a family $\mathcal{T} = (T_v)_{v \in V_G}$ of subtrees of $T$ such that $G$ is the intersection graph of the family $\mathcal{T}$ [6]. The *leafage* of a chordal graph $G$, denoted by $l(G)$, is the number of leaves of a host tree of $G$ with a minimum number of leaves.

In [11] (see Section 2.1), the leafage of a chordal graph is studied and it is proved that for any chordal graph $G$,

$$an(G) \leq l(G) \leq sc(G). \tag{6}$$

The following corollary provides a sufficient condition for the equality between the leafage and the asteroidal number of a connected chordal graph. This condition is not necessary; for example, graph $H_4$ in Fig. 1 satisfies that $an(H_4) = l(H_4) = 4$, however $nsc(H_4) = 3$ and $sc(H_4) = 5$.

**Corollary 2.5.** *Let $G$ be a connected chordal graph such that every simplicial clique of $G$ is non-separating. Then $nsc(G) = an(G) = l(G) = sc(G)$.*

**Proof.** By Theorem 2.4 and relation (6), $nsc(G) \leq an(G) \leq l(G) \leq sc(G)$. If every simplicial clique of $G$ is non-separating then $nsc(G) = sc(G)$ and the proof follows. □

The reader can verify easily that the graphs of the following families satisfy the condition from Corollary 2.5. In Section 3, we will prove that the minimal $k$-asteroidal chordal graphs also do.

- Graphs obtained from a connected chordal graph $G$ by attaching to it a pendant vertex to every vertex $v$ of $G$ such that $N_G[v]$ is a separating clique.
- Path powers, that is, graphs $P_n^k$ where $V(P_n^k) = \{1, 2, \ldots, n\}$ and $E(P_n^k) = \{ij : 1 \leq i, j \leq n \wedge |i - j| \leq k\}$.
- Connected (bull, claw)-free chordal graphs, where the *bull* is the graph obtained by adding a vertex adjacent to the two central vertices of a path with four vertices and a the *claw* is the graph $H_1$ in Fig. 1.

### *2.1. Restricted simplicial neighborhood poset*

In [11], the asteroidal number of a chordal graph is related to the width of a poset. We recall that concept and provide some separating examples so that the reader can appreciate the difference with those used in the present work.

The *derived graph* of $G$, written $G'$, is the graph $G - S(G)$; and the *modified neighborhood* of a simplicial vertex $v$, written $R'(v)$, is the vertex set $N(v) - S(G)$. Let $S'(G)$ be the set of simplicial vertices $v$ of $G$ such that $G' - R'(v)$ is connected. The *restricted simplicial neighborhood poset* of $G$ is the set $\{R'(v), v \in S'(G)\}$, denoted by $P'(G)$, ordered by inclusion.

An *antichain* in a poset is a set of pairwise incomparable elements. The *width* of a poset is the size of an anti-chain with maximum size. The width of the poset $P'(G)$ is denoted by $w(P'(G))$.

**Theorem 2.6** (*[11]*)**.** *The asteroidal number $an(G)$ of a connected chordal graph $G$ is equal to the maximum of $w(P'(H))$ over all connected induced subgraphs $H$ of $G$.*

Notice that if $C_1, \ldots, C_k$ are non-separating cliques of a chordal graph $G$ and $s_1, \ldots, s_k$ are simplicial vertices such that $C_i = N[s_i]$ for $1 \leq i \leq k$, then $R'(s_1), \ldots, R'(s_k)$ is an anti-chain of $P'(G)$. Thus, for any chordal graph $G$,

$$nsc(G) \leq w(P'(G)).$$

Table 1 shows that, in general, this inequality is strict. In that table, the number of non-separating cliques and the width of the posets corresponding to the graphs of Fig. 1 are provided; we observe that $nsc(H_1) < w(P'(H_1))$ and that $nsc(H_4) < w(P'(G))$.

In [11], it is also proved that if the derived graph $G'$ is a complete graph then $an(G) = l(G) = w(P'(G))$. In order to compare this result with Corollary 2.5, notice that the graph $H_1$ of Fig. 1 satisfies that its derived graph $H_1'$ is complete but it does not satisfy $nsc(H_1) = sc(H_1)$; whereas the graph $H_2$ of that figure satisfies that $nsc(H_2) = sc(H_2)$ but $H_2'$ is not a complete graph.

## 3. Minimal $k$-asteroidal chordal graphs

Let $k \geq 3$ be an integer. A *minimal $k$-asteroidal chordal graph* is a chordal graph $G$ such that $an(G) = k$ and $an(H) < k$ for every proper induced connected subgraph $H$ of $G$. The minimal 3-asteroidal chordal graphs are depicted in Fig. 2; they

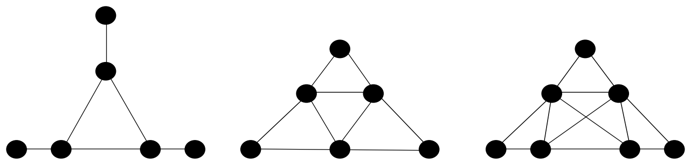


**Fig. 3.** The minimal 3-asteroidal split graphs.

are the well-known forbidden induced subgraphs for interval graphs within the class of chordal graphs [10]. The family of minimal $k$-asteroidal chordal graphs is unknown for every $k > 3$.

Notice that, for every graph in Fig. 2, the number of non-separating cliques is equal to the number of simplicial vertices. In the following Theorem 3.1, we prove that this fact holds for all minimal $k$-asteroidal chordal graphs.

**Theorem 3.1.** *Let G be a minimal k-asteroidal chordal graph. Then*

$$nsc(G) = an(G) = l(G) = sc(G) = sv(G) = k.$$

**Proof.** Since $G$ is minimal, $G$ is connected. By Theorem 2.4, there exists an induced connected subgraph $H$ of $G$ such that $nsc(H) = an(G) = k$. Therefore, by inequality (5), $an(H) = k$. Since $G$ is minimal and $H$ is an induced connected subgraph of $G$, we have $H = G$. It follows that $nsc(G) = an(G) = k$.

Let $C_1, \ldots, C_k$ be the non-separating cliques of $G$. Since $G$ is chordal, every non-separating clique contains a simplicial vertex. Hence, there exist simplicial vertices $s_1, \ldots, s_k$ such that $C_i = N[s_i]$ for $1 \le i \le k$. It follows that $k \le sv(G)$.

Suppose $k < sv(G)$. Let $s$ be a simplicial vertex of $G$, $s \neq s_i$ for $1 \le i \le k$. Since the removal of a simplicial vertex does not disconnect a graph, $G - s$ is a connected induced subgraph of $G$. Since $G$ is minimal, $s \notin N[s_i]$ for $1 \le i \le k$; thus every $C_i$ is a clique of $G - s$. We claim that each $C_i$, $1 \le i \le k$, is a non-separating clique of $G - s$. Indeed, $(G - s) - C_i = (G - C_i) - s$; since $C_i$ is a non-separating clique of $G$, and $s$ a simplicial vertex, $(G - C_i) - s$ is connected. It follows that $nsc(G - s) \ge k$. But, by relation (3), $nsc(G - s) \le an(G - s)$, thus $an(G - s) \ge k$. This fact contradicts the minimality of $G$. We have proved that $sv(G) = k$ and then $nsc(G) = an(G) = l(G) = sc(G) = sv(G) = k$. □

As a step in the direction of determining the minimal $k$-asteroidal chordal graphs, we study minimal $k$-asteroidal split graphs. A *split graph* is a chordal graph with a chordal complement, or, equivalently, a graph whose vertex set admits a partition into a stable set $S$ and a complete set $K$. Without loss of generality, we assume that $K = \cup_{s \in S} N(s)$. Observe that each vertex in $S$ is simplicial. On the other hand, if $v$ is simplicial and $v \in K$, then there exists $s \in S$ such that $N[s] = N[v] = K \cup \{s\}$. Therefore,

$$sc(G) = |S|. \tag{7}$$

Minimal 3-asteroidal split graphs are the three graphs in Fig. 3. As in the case of chordal graphs, the family of minimal $k$-asteroidal split graphs is unknown for every $k > 3$. The following theorem characterizes minimal $k$-asteroidal split graphs.

**Theorem 3.2.** *Let G be a split graph with partition $(S, K)$ and $k \ge 3$. The graph G is minimal k-asteroidal if and only if*

1. *$|S| = k$;*
2. *for all distinct vertices $s, s' \in S$, there exists $v \in K$ such that $v$ is adjacent to $s$ and non-adjacent to $s'$; and*
3. *for every $v \in K$, there exist $s, s' \in S$ such that $v$ is the only vertex that is adjacent to $s$ and non-adjacent to $s'$.*

**Proof.** Notice that every asteroidal set of a non-trivial split graph is contained in the stable set. Let $G$ be a minimal $k$-asteroidal split graph. By Theorem 3.1 and relation (7), $|S| = k$. Hence, $S$ is an asteroidal set. Let $s, s'$ be vertices of $S$. Since $N[s']$ does not disconnect $s$ from the remaining vertices of $S$, there exists $v \in K$ adjacent to $s$ and non-adjacent to $s'$.

Let $v$ be a vertex of $K$. Since $G$ is minimal, $S$ is not a $k$-asteroidal set of $G - v$. Therefore, there exist $s$, $s'$ and $s''$ vertices of $S$ such that $s$ and $s''$ are not in the same connected component of $(G - v) - N[s']$. Since $s$ and $s''$ are in the same connected component of $G - N[s']$, then $v$ is the only vertex non-adjacent to $s'$ that is adjacent to $s$ or to $s''$.

Conversely, let $G$ be a split graph satisfying conditions (1), (2) and (3). By (1) and (2), $S$ is a $k$-asteroidal set. Let us see that $G$ is minimal. Suppose that for some vertex $v$, $G - v$ has a $k$-asteroidal set. Observe that $v$ cannot be a vertex of $S$, then $v \in K$ and the asteroidal set of $G - v$ is $S$. This is contradicted by the fact that, by (3), there exist $s$ and $s'$ such that, in $G - v$, $N[s']$ disconnects $s$ from any other vertex of $S$. □

The *cross complement* of a split graph $G$ with partition $(S, K)$ is the graph $\widetilde{G}$ that is obtained replacing in $G$ the set of edges between $S$ and $K$ by its complement. Then $\widetilde{G}$ has a split partition $(S, K)$ and for every $s \in S$, $N_{\widetilde{G}}(s) = K - N_G(s)$. Observe that $\widetilde{(\widetilde{G})} = G$.

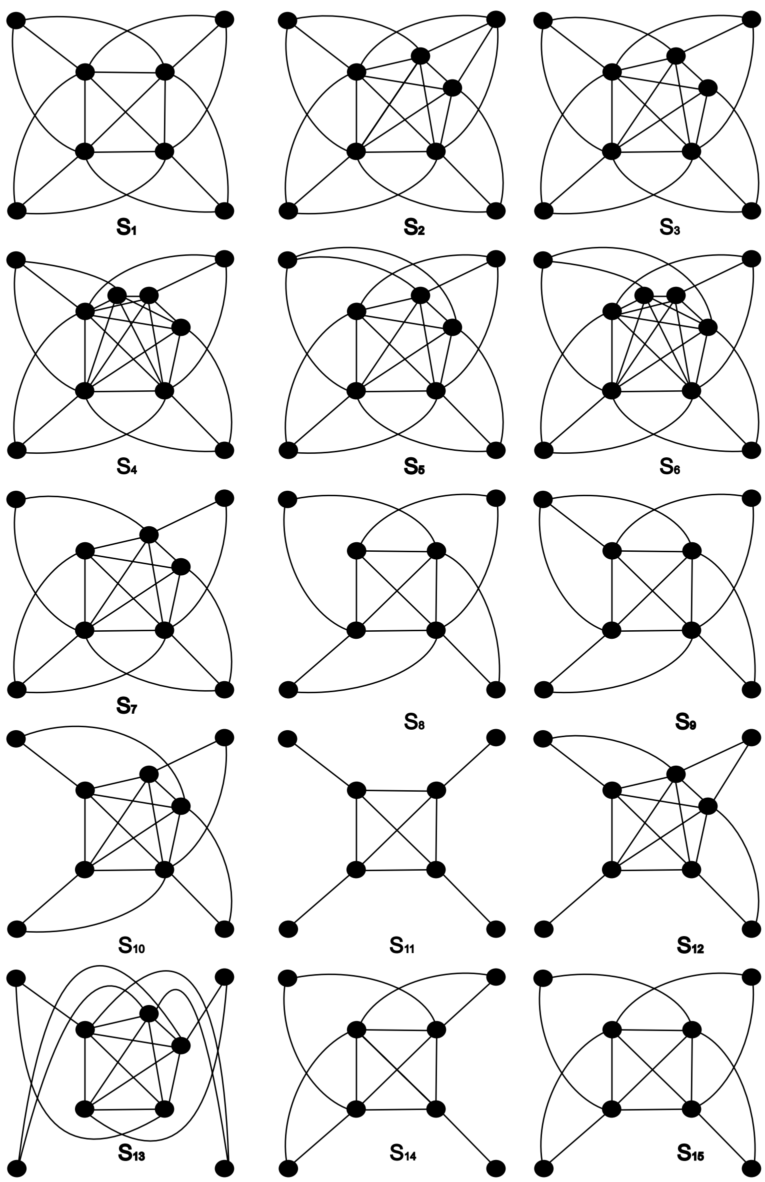


**Fig. 4.** Minimal 4-asteroidal split graphs.

**Corollary 3.3.** *A split graph is minimal k-asteroidal if and only if its cross complement is minimal k-asteroidal.*

The following lemma connects split graphs with set families; we will use it to obtain the minimal 4-asteroidal split graphs.

Let $\mathcal{F}$ be a family of proper subsets of $\{1, 2, \ldots, k\}$. Classify its members in columns $c_1, c_2, \ldots, c_k$ as follows: a member $F$ is in column $c_j$ if and only if $j \notin F$. Observe that each member $F$ is in $k - |F|$ columns.

Say that family $\mathcal{F}$ satisfy *property* $\mathcal{P}$ whenever

(a) for all distinct $i, j \in \{1, 2, \ldots, k\}$, in column $c_j$ there is at least one member $F$ containing $i$; and
(b) for every member $F$, there exist $i, j \in \{1, 2, \ldots, k\}$ such that $F$ is the only member in column $c_j$ containing $i$.

For simplicity, to sort the members of a set family into columns we use | to limit the columns and/to separate two different sets in the same column; in addition we give a set by just enumerating its elements. For example, consider the family of proper subsets of $\{1, 2, 3, 4\}$ with members $\{1\}$, $\{2\}$, $\{2, 3\}$ and $\{2, 3, 4\}$. The column classification is

2,3,4 / 2 / 2,3 | 1 | 1 / 2 | 1 / 2 / 2,3.

This means that column $c_1$ contains the sets $\{2, 3, 4\}$, $\{2\}$ and $\{2, 3\}$; that $c_2$ contains the set $\{1\}$; that $c_3$ contains the sets $\{1\}$ and $\{2\}$; and that $c_4$ contains the sets $\{1\}$, $\{2\}$ and $\{2, 3\}$.

**Lemma 3.4.** *There exists a one-to-one correspondence between the minimal k-asteroidal split graphs and the families of proper subsets of* $\{1, 2, \ldots, k\}$ *satisfying property* $\mathcal{P}$.

**Proof.** Let $G$ be a minimal $k$-asteroidal split graph with partition $(S, K)$. By Theorem 3.2, the stable set $S$ has size $k$, thus we can consider $S = \{1, 2, \ldots, k\}$. Let $\mathcal{F}$ be the set family $(F_v)_{v\in K}$ where $F_v = N(v) \cap S$. By Theorem 3.2, each $F_v$ is a proper subset of $S$ and $\mathcal{F}$ satisfies property $\mathcal{P}$.

Let $(F_i)_{i\in I}$ be a family of proper subsets of $\{1, 2, \ldots, k\}$ satisfying property $\mathcal{P}$. Consider $G$ the split graph with partition $(S, I)$ such that $N(i) \cap S = F_i$ for every $i \in I$. It is easy to see that the vertices of $G$ satisfy the conditions in Theorem 3.2, then $G$ is a minimal $k$-asteroidal split graph.

Since the two constructions are inverse to each other, the proof follows. □

**Theorem 3.5.** *The minimal* 4*-asteroidal split graphs are the graphs* $S_i$ *for* $1 \le i \le 15$ *depicted in* Fig. 4.

**Proof.** We will determine the families $\mathcal{F}$ of proper subsets of $\{1, 2, 3, 4\}$ satisfying property $\mathcal{P}$ of Lemma 3.4 and the corresponding split graphs.

Notice that if $\mathcal{F}$ satisfies (a) and (b), then no member can be added to $\mathcal{F}$ without losing property (b).

Let $j, i, r, t$ be all distinct. In order to satisfy (a), column $c_j$ must contain one of the following subfamilies of $\mathcal{F}$ (we omit the two families which are symmetrical to that in the second case, and the two being symmetrical to that in the third case):

- the set $\{i, r, t\}$; in this case we say that $c_j$ is an $\mathcal{A}$ column;
- the sets $\{i, r\}$ and $\{r, t\}$; in this case we say that $c_j$ is a $\mathcal{B}_r$ column;
- the sets $\{i, r\}$ and $\{t\}$; in this case we say that $c_j$ is a $\mathcal{C}_t$ column;
- the sets $\{i\}$, $\{r\}$ and $\{t\}$; in this case we say that $c_j$ is a $\mathcal{D}$ column.

We consider different cases depending on the types of columns. Without loss of generality, we will arbitrarily choose the column number.

1. There are four $\mathcal{A}$ columns.

   2,3,4 | 1,3,4 | 1,2,4 | 1,2,3.

   It satisfies (a) and (b). *We obtain the graph* $S_1$ *in* Fig. 4.
2. There are exactly three $\mathcal{A}$ columns and the remaining one is
   (a) a $\mathcal{B}_r$ column.

   2,3,4 / 2,3 | 1,3,4 | 1,2,4 / 1,2 | 1,2 / 2,3.

   It satisfies (a) and (b). *We obtain the graph* $S_2$ *in* Fig. 4.

   (b) a $\mathcal{C}_t$ column.

   2,3,4 / 3 | 1,3,4 / 3 | 1,2,4 / 1,2 | 1,2 / 3.

   It satisfies (a) and (b). *We obtain the graph* $S_3$ *in* Fig. 4.

   (c) a $\mathcal{D}$ column.

   2,3,4 / 2 / 3 | 1,3,4 / 1 / 3 | 1,2,4 / 1 / 2 | 1 / 2 / 3.

   It satisfies (a) and (b). *We obtain the graph* $S_4$ *in* Fig. 4.
3. There are exactly two $\mathcal{A}$ columns and between the remaining two there is
   (a) at least one $\mathcal{B}_r$ column. We have to consider two subcases depending on $r$.

   (i) $c_r$ is one of the two $\mathcal{A}$ columns.

   2,3,4 / 2,4 | 1,3,4 | 1,2 / 2,4 | 1,2.

   Observe that, in order to satisfy (a), in the fourth column, there must be a set containing 3. This set cannot have three elements because there are exactly two $\mathcal{A}$ columns; then it can be: $\{1, 3\}$; $\{2, 3\}$; or $\{3\}$. We claim that it is $\{1, 3\}$. Indeed, since the set $\{2, 3, 4\}$ is only in the first column, in order to satisfy (b), 3 cannot be in another set in the first column; this means that neither $\{2, 3\}$ nor $\{3\}$ can be members of the family. We have

   2,3,4 / 2,4 | 1,3,4 /1,3 | 1,2 / 2,4 | 1,2 / 1,3.

   It satisfies (a) and (b). *We obtain the graph* $S_5$ *in* Fig. 4.

   (ii) $c_r$ is none of the $\mathcal{A}$ columns.

   2,3,4 / 2,4 | 1,3,4 / 1,4 | 1,4 / 2,4 | .

   As in the previous case, in order to satisfy (b), since $\{2, 3, 4\}$ and $\{2, 4\}$ are in the first column; and $\{1, 3, 4\}$ and $\{1, 4\}$ are in the second, none of the sets $\{1, 3\}$, $\{2, 3\}$ and $\{3\}$ can be members of the family. Thus, it is impossible to satisfy (a) and (b).

   (b) No $\mathcal{B}_r$ column but at least one $\mathcal{D}$ column.

   2,3,4 / 2 / 4 | 1,3,4 /1 /4 | 1/ 2/ 4 | 1 /2.

   As in the previous case, because of the sets in the first and second columns, none of the sets $\{1, 3\}$, $\{2, 3\}$ and $\{3\}$ can be members of the family. Thus, it is impossible to satisfy (a) and (b).

(c) Neither $\mathcal{B}_r$ nor $\mathcal{D}$ columns. Then, $\mathcal{C}_t$ columns. We consider two cases depending on $t$,

(i) $c_t$ is one of the two $\mathcal{A}$ columns.

2,3,4 / 2,4 | 1,3,4 / 1 | 1/ 2, 4 | 1.

Again, since {2, 3, 4} is only in the first column and it is together with the set {2, 4}; in the fourth column, 3 must be in the subset {1, 3}; that is

2,3,4 / 2,4 | 1,3,4 / 1 / 1,3 | 1/ 2, 4 | 1 / 1,3.

Finally, since there are no $\mathcal{B}_r$ columns; in the fourth column, 2 must be in a unitary set;

2,3,4 / 2,4 / 2 | 1,3,4 / 1 / 3,1 | 1/ 2, 4 / 2 | 1 / 3,1 / 2.

It satisfies (a) and (b). *We obtain the graph $S_6$ in* Fig. 4.

(ii) $c_t$ is none of the two $\mathcal{A}$ columns.

2,3,4 / 4 | 1,3,4 / 4 | 4 / 1,2 | 1,2.

Since there is no $\mathcal{B}_r$ column, in the fourth column, 3 must be in a unitary set;

2,3,4 / 4 / 3 | 1,3,4 / 4 / 3 | 4 / 1,2 | 1,2 / 3.

It satisfies (a) and (b). *We obtain the graph $S_7$ in* Fig. 4.

4. There is exactly one $\mathcal{A}$ column and between the remaining three there is

(a) at least one $\mathcal{B}_r$ column. We consider two subcases depending on $r$,

(i) $c_r$ is the $\mathcal{A}$ column.

2,3,4 | 1,3 / 1,4 | 1,4 | 1,3.

If {2} is a member of the family, then

2,3,4 / 2 | 1,3 / 1,4 | 1,4 / 2 | 1,3 /2.

It satisfies (a) and (b). *We obtain the graph $S_8$ in* Fig. 4.

If {2} is not a member but {1, 2} is, then

2,3,4 | 1,3 / 1,4 | 1,4 / 1,2 | 1,3 / 1,2.

It satisfies (a) and (b). *We obtain the graph $S_9$ in* Fig. 4.

Finally, if neither {2} nor {1, 2} are members, then we need the sets {2, 3} and {2, 4} to get 2 in the third and fourth columns but, in this case, property (b) is not satisfied because of {2, 3, 4}, {2, 3} and {2, 4} in the first column.

(ii) $c_r$ is not the $\mathcal{A}$ column.

2,3,4 / 3,4 | 1,3 / 3,4 | | 1,3.

Again, in order that {2, 3, 4} satisfies (b), any other member containing 2 must contain 1. And in order that {3, 4} satisfies (b), any other member containing 4 must contain 2. It follows that {1, 2, 4} must be a member of the family; this contradicts that there is exactly one $\mathcal{A}$ column.

(b) No $\mathcal{B}_r$ column but some $\mathcal{C}_t$ column. We consider two subcases depending on $t$,

(i) $c_t$ is the $\mathcal{A}$ column.

2,3,4 /3,4 | 1/ 3,4 | 1 | 1.

Again, in order that {2, 3, 4} satisfies (b), any other member containing 2 must contain 1. And in order that {3, 4} satisfies (b), any other member containing 4 must contain 2 or any other member containing 3 must contain 2. In any case, as in the previous case, there is a second $\mathcal{A}$ column.

(ii) $c_t$ is not the $\mathcal{A}$ column.

2,3,4 / 4 | 1,3/ 4 | 4 | 1,3.

Since there is no $\mathcal{B}_r$ column, the sets {1, 4}, {3, 4}, {1, 2} and {2, 3} are not members of the family; then in order to satisfy (a), {1} and {2} must be

2,3,4 / 4 / 2 | 1,3 / 4 / 1 | 4 / 2 / 1 | 1,3 / 2 /1.

It satisfies (a) and (b). *We obtain the graph $S_{10}$ in* Fig. 4.

(c) Neither $\mathcal{B}_r$ nor $\mathcal{C}_t$ columns. Then each one of the remaining three columns are $\mathcal{D}$ columns. In this case, property (b) is not satisfied.

5. No $\mathcal{A}$ columns. We consider two cases,

(a) There is some unitary set. By Corollary 3.3, the cross complement of the corresponding graph is a minimal 4-asteroidal split graph and the column classification for the family corresponding to it has some $\mathcal{A}$ column and no unitary sets. Notice that it is the situation of the cases where we obtain the graphs $S_1, S_2, S_5$ and $S_9$.

Thus, *we obtain the graphs $S_{11}, S_{12}, S_{13}$ and $S_{14}$ in* Fig. 4 which are the cross complements of the graphs $S_1, S_2, S_5$ and $S_9$, respectively.

(b) There are no unitary sets; thus there are neither $\mathcal{C}_t$ nor $\mathcal{D}$ columns. It follows that there are only $\mathcal{B}_r$ columns. We consider the following subcases,

(i) there are two $\mathcal{B}_r$ columns with the same subindex $r$,

2,3 / 3,4 | 1,3 / 3,4 | | 2,3 / 1,3.

In order to satisfy property (a), one of the sets {1, 2} or {1, 4} must be a member of the family; without loss of generality, we assume {1, 2} is.

2,3 / 3,4 | 1,3 / 3,4 | 1,2 | 2,3 / 1,3 / 1,2.

Again, in order to satisfy property (a), one of the sets {1, 4} or {2, 4} must be a member of the family.

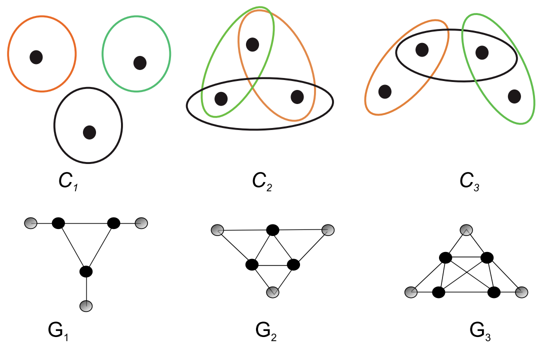


**Fig. 5.** From each clutter $\mathcal{C}_i$ we obtain a split graph $G_i$, and vice versa.

If $\{1, 4\}$ is a member, property (b) is not satisfied because of $\{1, 3\}$:

| | | | |
|---|---|---|---|
| 2,3 / 3,4 | 1,3 / 3,4 / 1,4 | 1,2 / 1,4 | 2,3 / 1,3 / 1,2. |

If $\{2, 4\}$ is a member, property (b) is not satisfied because of $\{2, 3\}$:

| | | | |
|---|---|---|---|
| 2,3 / 3,4 / 2,4 | 1,3 / 3,4 | 1,2 / 2,4 | 2,3 / 1,3 / 1,2. |

(ii) No two $\mathcal{B}_r$ columns have the same subindex $r$. Without loss of generality, assume $c_1$ is a $\mathcal{B}_2$ column,

| | | | |
|---|---|---|---|
| 2,3 / 2,4 | | 2,4 | 2,3. |

Observe that if $c_3$ is a $\mathcal{B}_1$ column, then we have the sets $\{1, 2\}$, $\{1, 4\}$ and $\{2, 4\}$. Thus $c_3$ is also a $\mathcal{B}_2$ column, this contradicts no two $\mathcal{B}_r$ columns with the same subindex $r$. It follows that $c_3$ must be a $\mathcal{B}_4$ column;

| | | | |
|---|---|---|---|
| 2,3 / 2,4 | 1,4 | 2,4 / 1,4 | 2,3. |

Then, $c_2$ must be either a $\mathcal{B}_1$ or a $\mathcal{B}_3$ column.
In the former, we get

| | | | |
|---|---|---|---|
| 2,3 / 2,4 | 1,4 / 1,3 | 2,4 / 1,4 | 2,3 /1,3. |

It satisfies (a) and (b). *We obtain the graph $S_{15}$ in* Fig. 4.
In the second case, we get

| | | | |
|---|---|---|---|
| 2,3 / 2,4 / 3,4 | 1,4 / 1,3 / 3,4 | 2,4 / 1,4 | 2,3 / 1,3. |

Observe that, in this case, $c_2$ and $c_3$ are $\mathcal{B}_4$ columns, this contradicts no two $\mathcal{B}_r$ columns with the same subindex $r$. □

## 4. Minor-minimal *k*-edge clutters

In this section we present an application of our previous results in the study of clutters or Sperner families.

A *hypergraph* $\mathcal{C}$ is an ordered pair $(V(\mathcal{C}), E(\mathcal{C}))$, where $V(\mathcal{C})$ is a finite set and $E(\mathcal{C})$ is a family of subsets of $V(\mathcal{C})$. The elements of $V(\mathcal{C})$ and $E(\mathcal{C})$ are called *vertices* and *edges* of $\mathcal{C}$, respectively. An hypergraph is a *clutter* if none of its edges is a proper subset of another edge. Clutters are also called *Sperner families* in the literature [1,5].

Let $v$ be a vertex of a clutter $\mathcal{C}$. The *contraction* $\mathcal{C}/v$ and the *deletion* $\mathcal{C} \setminus v$ are clutters defined as follows: both have $V(\mathcal{C}) - \{v\}$ as their vertex sets; $E(\mathcal{C}/v)$ is the set of inclusion-minimal elements of $\{F - \{v\}, F \in E(\mathcal{C})\}$; and $E(\mathcal{C} \setminus v)$ is the set $\{F \in E(\mathcal{C}) : v \notin F\}$. A clutter obtained from $\mathcal{C}$ by a sequence of contractions and deletions is called a *minor* of $\mathcal{C}$.

Let $\mathcal{C}$ be a clutter with $|E(\mathcal{C})| = k$. Notice that every minor of $\mathcal{C}$ has at most $k$ edges. When no minor of $\mathcal{C}$ has $k$ edges, we say that $\mathcal{C}$ is a *minor-minimal $k$-edge clutter*. The only three minor-minimal 3-edge clutters are represented on the top of Fig. 5.

The following theorems show that all the minor-minimal 4-edge clutters are represented in Fig. 4.

**Theorem 4.1.** *There exists a one-to-one correspondence between the minor-minimal $k$-edge clutters and the minimal $k$-asteroidal split graphs.*

**Proof.** Let $\mathcal{C}$ be a minor-minimal $k$-edge clutter with edges $F_1, F_2, \ldots, F_k$. We obtain a split graph $G$ with partition $(S, K)$ by considering $S = \{1, 2, \ldots, k\}$, $K = V(\mathcal{C})$ and $N_G(i) = F_i$ for every $i \in S$. Since every vertex of $\mathcal{C}$ must belong to some edge of $\mathcal{C}$, is clear that $K = \cup_{i \in S} N_G(i)$. We offer three different examples in Fig. 5.

Since $F_i \not\subseteq F_j$ for $i \neq j$, $S$ is a $k$-asteroidal set of $G$. We claim that $G$ is minimal $k$-asteroidal. Indeed, suppose for a contradiction that there exists a vertex $v$ of $G$ such that $G - v$ has a $k$-asteroidal set. It is clear that $v$ must be a vertex of $K$ and that $S$ must be the $k$-asteroidal set of $G - v$. Therefore, $N(i) - \{v\} \not\subseteq N(j) - \{v\}$ for every $i \neq j$, this is $F_i - \{v\} \not\subseteq F_j - \{v\}$ for every $i \neq j$. Thus, the clutter $\mathcal{C}/v$ is a minor of $\mathcal{C}$ with $k$ edges, which contradicts the minimality of $\mathcal{C}$.

Let $G$ be a minimal $k$-asteroidal split graph with partition $(S, K)$. By Theorem 3.1, $S = \{s_1, \ldots, s_k\}$ is the only $k$-asteroidal set of $G$. We obtain a clutter $\mathcal{C}$ with $k$ edges by setting $V(\mathcal{C}) = K$ and $E(\mathcal{C}) = \{N_G(s_i), 1 \leq i \leq k\}$. Notice that this clutter is the set family used in the proof of Lemma 3.4.

Let $v \in V(\mathcal{C})$, we will show that $\mathcal{C} \setminus v$ and $\mathcal{C}/v$ have less than $k$ edges.

Since $v$ is not a universal vertex of $G$ and since $E(\mathcal{C} \setminus v)$ is the set $\{F \in E(\mathcal{C}) : v \notin F\}$, it is clear that $\mathcal{C} \setminus v$ has less than $k$ edges.

On the other hand, since $G$ is minimal $k$-asteroidal, $S$ is not a $k$-asteroidal set of $G - v$ which means that there exist $s_i, s_j, s_k \in S$ such that $s_j$ and $s_k$ are not in the same connected component of $(G - v) - N[s_i]$. This implies, since $s_j$ and $s_k$ are in the same connected component of $G - N[s_i]$, that either $N(s_j) - \{v\} \subseteq N(s_i)$ or $N(s_k) - \{v\} \subseteq N(s_i)$. Since $E(\mathcal{C}/v)$ is the set of inclusion-minimal elements of $\{F - v, F \in E(\mathcal{C})\}$, we obtain that $E(\mathcal{C}/v)$ has less than $k$ edges. □

**Theorem 4.2.** *The open neighborhoods of the vertices of the maximum stable set of each graph in Fig. 4 represent the minor-minimal 4-edge clutters.*

## 5. Related open problems

We left open the problem of determining the family of minimal 4-asteroidal chordal graphs. We are also working in getting the minimal $(k + 1)$-asteroidal split graphs from the minimal $k$-asteroidal split graphs by an inductive step.

In addition, we pose the problem of characterizing chordal graphs such that every simplicial clique is non-separating.

Finally, we propose the combinatorial problem of computing the number of minimal $k$-asteroidal split graphs for $k \geq 5$. By the previous results, we know that this number is 1 for $k = 1$ and $k = 2$, 3 for $k = 3$ and 15 for $k = 4$.

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