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The compatible Grassmannian

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A B S T R A C T

Let A be a positive injective operator in a Hilbert space $(\mathcal{H}, \langle\cdot\,,\cdot\rangle)$, and denote by $[\cdot\,,\cdot]$ the inner product defined by A: $[f,g]=\langle Af,g\rangle$. A closed subspace $\mathcal{S}\subset\mathcal{H}$ is called A-compatible if there exists a closed complement for $\mathcal{S}$, which is orthogonal to $\mathcal{S}$ with respect to the inner product $[\cdot\,,\cdot]$. Equivalently, if there exists a necessarily unique bounded idempotent operator $Q_{\mathcal{S}}$ such that $R(Q_{\mathcal{S}})=\mathcal{S}$, which is symmetric for this inner product. The compatible Grassmannian Gr_A is the set of all A-compatible subspaces of $\mathcal{H}$. By parametrizing it via the one to one correspondence $\mathcal{S}\leftrightarrow Q_{\mathcal{S}}$, this set is shown to be a differentiable submanifold of the Banach space of all bounded operators in $\mathcal{H}$ which are symmetric with respect to the form $[\cdot\,,\cdot]$. A Banach–Lie group acts naturally on the compatible Grassmannian, the group of all invertible operators in $\mathcal{H}$ which preserve the form $[\cdot\,,\cdot]$. Each connected component in Gr_A of a compatible subspace $\mathcal{S}$ of finite dimension, turns out to be a symplectic leaf in a Banach Lie–Poisson space. For $1\leqslant p\leqslant\infty$, in the presence of a fixed $[\cdot\,,\cdot]$-orthogonal (direct sum) decomposition of $\mathcal{H}$, $\mathcal{H}=\mathcal{S}_0+\mathcal{N}_0$, we study the restricted compatible Grassmannian (an analogue of the restricted, or Sato Grassmannian). This restricted compatible Grassmannian is shown to be a submanifold of the Banach space of p-Schatten operators which are symmetric for the form $[\cdot\,,\cdot]$. It carries the locally transitive action of the Banach–Lie group of invertible operators which preserve $[\cdot\,,\cdot]$, and are of the form $G=1+K$, with K in the p-Schatten class. The connected components of this restricted Grassmannian are characterized by means of the Fredholm index of pairs of projections. Finsler metrics which are isometric for the group actions are introduced for both compatible Grassmannians, and minimality results for curves are proved.

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1. Introduction

Let $0 < A \leqslant 1$ be a positive operator with trivial kernel in a Hilbert space $(\mathcal{H}, \langle \cdot\,, \cdot \rangle)$. Then A defines an inner product $[\cdot\,, \cdot] = [\cdot\,, \cdot]_A$ on $\mathcal{H}$ by means of

$$[f, g] = \langle Af, g \rangle, \quad f, g \in \mathcal{H}.$$

Apparently, the form $[\cdot\,, \cdot]$ is continuous with respect to the topology of $\mathcal{H}$. If A is not invertible, $\mathcal{H}$ is not complete with this inner product. We shall denote by $\mathcal{L}$ the completion of $\mathcal{H}$ with this inner product. A closed linear subspace $\mathcal{S} \subset \mathcal{H}$ is called *compatible with* A (*or* A-*compatible*) if it admits a supplement which is orthogonal with respect to the inner product defined by A. In this paper, we study the *compatible Grassmannian*, namely

$$Gr_A = \{\mathcal{S} \subset \mathcal{H}\colon\ \mathcal{S} \text{ is compatible with } A\}.$$

Since A has trivial kernel, if $\mathcal{S}$ is compatible with A, then the supplement is unique, and it is given by $A(\mathcal{S})^{\perp}$. This allows us to identify each compatible subspace $\mathcal{S}$ with the projection $Q_{\mathcal{S}}$ with range $\mathcal{S}$ and kernel $A(\mathcal{S})^{\perp}$. Thus the compatible Grassmannian may be regarded as the following set

$$Gr_A = \{Q \in \mathcal{B}(\mathcal{H})\colon\ Q^2 = Q,\ Q^*A = AQ\},$$

where we will consider the uniform topology inherited from $\mathcal{B}(\mathcal{H})$. The proof of the afore-mentioned facts and examples of compatible and non-compatible subspaces can be found in [10,11], where a systematic study of compatible subspaces was carried out. The notion of compatible subspaces goes back to A. Sard [31], who introduced an equivalent definition under a different terminology to give an operator theoretic approach to problems in approximation theory (see [9]). On the other hand, S. Hassi and K. Nordström [20] studied general properties of self-adjoint projections with respect to a Hermitian form. For additional information on applications of compatible subspace to statistics, sampling, signal processing and frames the reader should see the references in [9,12].

Note that there are two notions of orthogonality, the one given by the usual inner product $\langle \cdot\,, \cdot \rangle$ of $\mathcal{H}$, and the one given by $[\cdot\,, \cdot]$, which we will call A-orthogonality. This leads us to consider, as useful tools in this work, the results on operators on spaces with two norms by M.G. Krein [22], J. Dieudonné [15], P.D. Lax [24], I.C. Gohberg and M.K. Zambickiĭ [18]. An operator will be called A-*symmetric* (resp. A-*anti-symmetric*, A-*unitary*) if it is bounded and symmetric (resp. anti-symmetric, unitary) with respect to the A inner product $[\cdot\,, \cdot]$. It is apparent that B is A-symmetric (resp. A-anti-symmetric) if and only if $B^*A = AB$ (resp. $B^*A = -AB$). It also follows that A-unitary operators form a subgroup $\mathbb{G}_A$ of the linear invertible group $Gl(\mathcal{H})$, which can be characterized as

$$\mathbb{G}_A = \{G \in Gl(\mathcal{H})\colon\ G^*AG = A\}.$$

Moreover, it is a Banach–Lie group endowed with the uniform topology of $\mathcal{B}(\mathcal{H})$ and its Lie algebra can be identified with the A-anti-symmetric operators (see Appendix A). However, $\mathbb{G}_A$ is not necessarily a complemented submanifold of $\mathcal{B}(\mathcal{H})$ if A is not invertible. We prove a result stating that the Banach–Lie algebra of $\mathbb{G}_A$ is complemented in $\mathcal{B}(\mathcal{H})$ if and only if the kernel of a certain nilpotent derivation is complemented (Theorem 2.5).

If $\mathcal{S}$ is compatible with A and $G \in \mathbb{G}_A$, then $G(\mathcal{S})$ is compatible with A and $Q_{G(\mathcal{S})} = GQ_{\mathcal{S}}G^{-1}$. Thus $\mathbb{G}_A$ acts on Gr_A by similarity.

One interesting example of a pair $\mathcal{H} \subset \mathcal{L}$ occurs when $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, and $\mathcal{H} = H_0^1(\Omega)$, $\mathcal{L} = L^2(\Omega, dx)$ are the standard Sobolev and Lebesgue spaces, respectively. In this case the operator $A \in \mathcal{B}(\mathcal{H})$ is the solution operator of the non-homogeneous Helmholtz equation

$$\begin{cases} u - \Delta u = f, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Namely, $Af = u_f$, is the solution of the above equation (see [1]). If $\mathcal{S}$ is a subspace of $H_0^1(\Omega)$, which is compatible with A, then $Q = Q_{\mathcal{S}}$ has the following property: $Q^* u_f = u_{Qf}$. In other words, Q^* projects solutions onto solutions with data in $\mathcal{S}$.

Our main results concern the differentiable structure of Gr_A. We show that Gr_A is a complemented submanifold of the real Banach space of bounded A-symmetric operators, and that the action of $\mathbb{G}_A$ is locally transitive and has smooth local cross sections (Corollary 4.5). The compatible Grassmannian has a remarkable property with respect to the space $\mathcal{Q}(\mathcal{H})$ of all idempotents in $\mathcal{B}(\mathcal{H})$, studied by G. Corach, H. Porta and L. Recht in [14]. In this paper a linear connection was introduced in $\mathcal{Q}(\mathcal{H})$, and its geodesics were computed: these are curves of the form $\delta(t) = e^{tX} Q e^{-tX}$ for specific operators X. We show here that if two elements Q_1 and Q_2 of Gr_A lie at norm distance less than $r = r_{Q_1}$, then the unique geodesic of $\mathcal{Q}(\mathcal{H})$ joining them lies inside Gr_A (see Proposition 4.8; Corollary 4.7).

In the case where $Q \in Gr_A$ has finite rank, and $\mathcal{H}$, $\mathcal{L}$ are (respectively) $H^1(\mathbb{R}^3)$ and $L^2(\mathbb{R}^3)$, the orbit of Q can be identified with the Grassmann manifold arising in quantum chemistry (see [8]). By this identification one can prove that the Grassmann manifold in quantum chemistry, and a corresponding Stiefel manifold, are complete Hilbert–Riemannian manifolds, or more generally, that there is a family of complete Finsler metrics on these manifolds. The completeness result crucially underpins the use of techniques from critical point theory in many nonlinear PDEs from quantum chemistry (see [3,16]). We show that the Grassmann manifold in quantum chemistry is a strong symplectic leaf in a Banach Lie–Poisson space (Corollary 5.2). These infinite dimensional Poisson structures were introduced by A. Odzijewicz and T. Ratiu in [26].

We also define a *restricted compatible Grassmannian* $Gr_{res,p}$, analogous to the restricted Grassmannian (also called Sato Grassmannian, [29,33]). Given $1 \leqslant p \leqslant \infty$ and a fixed A-orthogonal (direct sum) decomposition

$$\mathcal{H} = \mathcal{S}_0 + \mathcal{N}_0,$$

with $Q_0 = Q_{\mathcal{S}_0}$ (so that $\mathcal{N}_0 = N(Q_0)$, the kernel of Q_0), an element Q of Gr_A belongs to $Gr_{res,p}$ if

$$Q - Q_0 \in \mathcal{B}_p(\mathcal{H}),$$

where $\mathcal{B}_p(\mathcal{H})$ denotes the class of p-Schatten operators in $\mathcal{H}$. We show that the Banach–Lie group

$$\mathbb{G}_{p,A} = \{ G \in \mathbb{G}_A \colon \ G - 1 \in \mathcal{B}_p(\mathcal{H}) \}$$

acts on $Gr_{res,p}$, that the action is locally transitive, and that the orbits are the connected components of $Gr_{res,p}$. Moreover, these orbits are characterized by the Fredholm index of pairs of projections defined by J. Avron, R. Seiler and B. Simon in [4]. Given a pair (P_1, P_2) of orthogonal projections, the index of the pair is the index of the operator

$$P_2 P_1|_{R(P_1)} \to R(P_2),$$

if this operator is Fredholm (in which case the pair (P_1, P_2) is called a Fredholm pair), otherwise the index is infinite. If $Q_1, Q_2 \in Gr_{res,p}$, then their extensions $\bar{Q}_1, \bar{Q}_2$ are orthogonal projections on $\mathcal{L}$, and form a Fredholm pair. In Theorem 6.3 we show that they lie in the same connected component of $Gr_{res,p}$ (or equivalently, are conjugate by the action of $\mathbb{G}_{p,A}$) if and only if the index of the pair is zero (or equivalently, $index(\bar{Q}_1, \bar{Q}_0) = index(\bar{Q}_2, \bar{Q}_0)$). These components are differentiable manifolds in the local structure given by the p-norm. Also in the restricted compatible Grassmannian, it is shown that geodesics of $\mathcal{Q}(\mathcal{H})$ joining sufficiently close elements of $Gr_{res,p}$, lie inside $Gr_{res,p}$.

We introduce non-complete Finsler metrics for both Gr_A and $Gr_{res,p}$, in terms, respectively, of the operator norm and the p-norm in $\mathcal{L}$. Being isometric for the respective group actions, these metrics are a natural choice. We prove that the geodesics described above joining sufficiently close elements have minimal length (see Proposition 7.2; Corollary 7.3).

Let us give a brief outline of the contents of this paper. Preliminary results on operators on spaces with two norms are stated in Section 2, as well as the analysis of when the real space of bounded A-anti-symmetric operators is complemented in $\mathcal{B}(\mathcal{H})$. We conjecture that this happens if and only if A is invertible. In Section 3 we give examples of compatible subspaces. In Section 4 we examine the local structure of Gr_A, and the properties of the action of $\mathbb{G}_A$. The property of permanence of geodesics joining close elements is proved, and an estimate of this geodesic radius is given. In Section 5 we introduce a Banach–Lie–Poisson structure in the finite rank components of Gr_A. In Section 6 we introduce the restricted compatible Grassmannian $Gr_{res,p}$. We prove the local regular structure, the characterization of the components in terms of the index of pairs, and the permanence of geodesics. In Section 7 we introduce invariant Finsler metrics in Gr_A and $Gr_{res,p}$, and we prove the minimality results.

Notation 1.1. The Hilbert space $\mathcal{H}$ is endowed with two (inner product) norms, $\|\cdot\|$ will be the usual norm and $\|\cdot\|_A$ the one given by the A inner product. Clearly, $\|f\|_A \leqslant \|f\|$ for any $f \in \mathcal{H}$ (since $A \leqslant 1$). In addition, $\mathcal{B}_s^A(\mathcal{H})$ and $\mathcal{B}_{as}^A(\mathcal{H})$ will denote the Banach spaces of bounded (respectively) A-symmetric and A-anti-symmetric operators, regarded as subspaces of $\mathcal{B}(\mathcal{H})$. We shall refer to them hereafter as A-symmetric and A-anti-symmetric, always assuming that they bounded. The operator norm on $\mathcal{B}(\mathcal{H})$ will be denoted by $\|\cdot\|$, meanwhile $\|\cdot\|_{\mathcal{B}(\mathcal{L})}$ will be the operator norm of $\mathcal{B}(\mathcal{L})$.

2. Preliminaries

Note that any $G \in \mathbb{G}_A$ can be extended to a unitary operator $\bar{G}$ acting on $\mathcal{L}$. In [1] it was shown that $G \in \mathbb{G}_A$ if and only if $G = U|_{\mathcal{H}}$, where U is a unitary operator on $\mathcal{L}$ such that U leaves the dense subspace $\mathcal{H}$ fixed, i.e. $U(\mathcal{H}) = \mathcal{H}$.

We shall denote by $\sigma_{\mathcal{H}}(T)$ the spectrum of T as an operator in $\mathcal{H}$, and by $\sigma_{\mathcal{L}}(\bar{T})$ the spectrum of its extension (when it exists) to $\mathcal{L}$. We will say that λ belongs to the point spectrum of T if $0 < \dim(\ker(T - \lambda 1))$, and λ is said to have finite multiplicity if this dimension is finite. Let us recall the following facts, adapted from their original broader context to our case:

Theorem 2.1. *(See M.G. Krein [22], P.D. Lax [24].) Let B be an A-symmetric operator. The following assertions hold:*

i) *$\bar{B}$ exists and it is bounded as an operator on $\mathcal{L}$.*
ii) *$\sigma_{\mathcal{L}}(\bar{B}) \subseteq \sigma_{\mathcal{H}}(B)$.*
iii) *If λ belongs to the point spectrum of B as an operator on $\mathcal{H}$, then λ belongs to the point spectrum of $\bar{B}$ as an operator on $\mathcal{L}$. Moreover, if λ has finite multiplicity, then the λ-eigenspace over $\mathcal{H}$ and $\mathcal{L}$ is the same.*
iv) *If B is a compact operator on $\mathcal{H}$, then $\bar{B}$ is a compact operator on $\mathcal{L}$ and $\sigma_{\mathcal{L}}(\bar{B}) = \sigma_{\mathcal{H}}(B)$.*

Remark 2.2. It is not difficult to see that if B is A-symmetric, then $\|\bar{B}\|_{\mathcal{B}(\mathcal{L})} \leqslant \|B\|$. Also note that A itself is A-symmetric, and that its extension $\bar{A}$ remains positive definite.

Operators which are A-symmetric are often called *symmetrizable*.

A generalization of the above results can be found in [18]. A bounded operator acting on $\mathcal{H}$ is called *proper* if it has a bounded adjoint with respect to the inner product on $\mathcal{L}$. This means that $B \in \mathcal{B}(\mathcal{H})$ is proper if and only if for every $f \in \mathcal{H}$, there is a vector $g \in \mathcal{H}$ satisfying $[Bh, f] = [h, g]$ for all $h \in \mathcal{H}$. This allows to define $B^+ f = g$. Actually, B^+ is the restriction to $\mathcal{H}$ of the $\mathcal{L}$-adjoint of B. In particular, B is A-symmetric if and only if $B = B^+$.

Theorem 2.3. *(See I.C. Gohberg, M.K. Zambickiĭ [18].) Let B be a proper operator. The following assertions hold:*

i) *$\bar{B}$ exists and it is bounded as an operator on $\mathcal{L}$.*
ii) *$\sigma_{\mathcal{L}}(\bar{B}) \subseteq \sigma_{\mathcal{H}}(B) \cup \overline{\sigma_{\mathcal{H}}(B^+)}$, where the second bar indicates complex conjugation.*
iii) *If B is a compact operator on $\mathcal{H}$, then $\bar{B}$ is a compact operator on $\mathcal{L}$. Moreover, $\sigma_{\mathcal{L}}(\bar{B}) = \sigma_{\mathcal{H}}(B)$ and the eigenspaces in $\mathcal{L}$ and $\mathcal{H}$ corresponding to the nonzero eigenvalues coincide.*

The following result will also be useful.

Theorem 2.4. *(See J. Dieudonné [15].) Let B be an A-symmetric operator. Then there is a unique symmetric operator X in $\mathcal{H}$ such that $A^{1/2}B = XA^{1/2}$.*

We finish this section on preliminaries by giving a characterization of the case when $\mathcal{B}_s^A(\mathcal{H})$ is a complemented (real) subspace of $\mathcal{B}(\mathcal{H})$. Note that $\mathcal{B}_{as}^A(\mathcal{H}) = i\mathcal{B}_s^A(\mathcal{H})$, and therefore the first subspace is complemented if and only if the second is, and $\mathbb{S}$ is a supplement for $\mathcal{B}_s^A(\mathcal{H})$ if and only if $i\mathbb{S}$ is a supplement for $\mathcal{B}_{as}^A(\mathcal{H})$.

Consider the Hilbert space $\mathcal{H} \times \mathcal{H}$ and the operator A_0 on $\mathcal{H} \times \mathcal{H}$ given by

$$A_0 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}.$$

Note that $A_0^2 = 0$. Recall that A is injective. Then a straightforward computation shows that the operator $B \in \mathcal{B}(\mathcal{H} \times \mathcal{H})$ commutes with A_0 if and only if

$$B = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix},$$

with $XA = AZ$.

Theorem 2.5. *The real subspace $\mathcal{B}_s^A(\mathcal{H})$ is complemented in $\mathcal{B}(\mathcal{H})$ if and only if the commutant of A_0 is complemented (as a complex subspace) in $\mathcal{B}(\mathcal{H} \times \mathcal{H})$.*

Proof. By the form of the commutant of A_0, namely

$$\begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} X & 0 \\ 0 & Z \end{pmatrix},$$

with $XA = AZ$, it is apparent that it is complemented if and only if the complex subspace

$$\mathcal{D} = \{(X, Z) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \colon \ XA = AZ\}$$

is complemented in $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$. Note that $\mathcal{D}$ decomposes as a direct sum by means of

$$(X, Z) = \frac{1}{2}(X + Z^*, X^* + Z) + \frac{1}{2}(X - Z^*, Z - X^*),$$

where $X^* + Z \in \mathcal{B}_s^A(\mathcal{H})$ and $Z - X^* \in \mathcal{B}_{as}^A(\mathcal{H})$. Indeed,

$$\bigl(X^* + Z\bigr)^* A = XA + Z^*A = AZ + (AZ)^* = AZ + (XA)^* = A\bigl(Z + X^*\bigr),$$

and similarly for $Z - X^*$. Thus $\mathcal{D} = \mathcal{D}_s \oplus \mathcal{D}_{as}$, where

$$\mathcal{D}_s = \bigl\{\bigl(V^*, V\bigr)\colon\ V \in \mathcal{B}_s^A(\mathcal{H})\bigr\}$$

and

$$\mathcal{D}_{as} = \bigl\{\bigl(W^*, -W\bigr)\colon\ W \in \mathcal{B}_{as}^A(\mathcal{H})\bigr\}.$$

It is apparent that $\mathcal{D}_s \cap \mathcal{D}_{as} = \{0\}$.

Suppose first that $\mathcal{B}_s^A(\mathcal{H}) \oplus \mathbb{S} = \mathcal{B}(\mathcal{H})$ for some closed real subspace $\mathbb{S} \subset \mathcal{B}(\mathcal{H})$, so that also $\mathcal{B}_{as}^A(\mathcal{H}) \oplus i\mathbb{S} = \mathcal{B}(\mathcal{H})$. Then

$$\mathbb{T} = \bigl\{\bigl(R^*, R\bigr) + \bigl(-T^*, T\bigr)\colon\ R \in \mathbb{S},\ T \in i\mathbb{S}\bigr\}$$

is a supplement for $\mathcal{D}$ (note that the sum $(R^*, R) + (-T^*, T)$ is direct), and moreover, it is a complex subspace of $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$. Indeed, if $(R^*, R) + (-T^*, T) \in \mathbb{T}$, then

$$i\bigl(\bigl(R^*, R\bigr) + \bigl(-T^*, T\bigr)\bigr) = \bigl((iT)^*, iT\bigr) + \bigl((iR)^*, -iR\bigr) \in \mathbb{T},$$

because $iT \in \mathbb{S}$ and $iR \in i\mathbb{S}$.

Conversely, suppose that $\mathcal{D}$ is complemented in $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$,

$$\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) = \mathbb{T} \oplus \mathcal{D} = \mathbb{T} \oplus \mathcal{D}_{as} \oplus \mathcal{D}_s.$$

Then the real subspace $\mathcal{D}_s = \{(V^*, V)\colon\ V \in \mathcal{B}_s^A(\mathcal{H})\}$ is complemented in $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$. Let $E : \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathcal{D}_s \subset \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ be an idempotent projecting onto $\mathcal{D}_s$. Put

$$\mathbb{D} = \bigl\{\bigl(T^*, T\bigr)\colon\ T \in \mathcal{B}(\mathcal{H})\bigr\}.$$

Then $\mathcal{D}_s \subset \mathbb{D}$ and

$$E|_{\mathbb{D}} : \mathbb{D} \to \mathcal{D}_s \subset \mathbb{D}$$

is an idempotent operator, and the set of pairs (V^*, V), $V \in \mathcal{B}_s^A(\mathcal{H})$ are complemented in the set of pairs (T^*, T), $T \in \mathcal{B}(\mathcal{H})$. It follows that $\mathcal{B}_s^A(\mathcal{H})$ is complemented in $\mathcal{B}(\mathcal{H})$. □

Apparently, if A is invertible, the kernel of the nilpotent derivation on $\mathcal{B}(\mathcal{H} \times \mathcal{H})$ given by $\delta_{A_0}(X) = XA_0 - A_0X$ is complemented. We conjecture that this is also a necessary condition. Derivations with closed (and complemented) range have been characterized. However, to our knowledge, there are no results characterizing derivations with complemented kernel.

3. Examples of compatible subspaces

First note that there exist non-compatible subspaces if A is non-invertible [10]. It is not difficult to see that if $\mathcal{S}$ is finite dimensional, then it is A-compatible. The same holds for finite co-dimensional subspaces. If P is an orthogonal projection which commutes with A, then P is A-symmetric (for instance, if P is a spectral projection of A). Therefore $R(P)$ is compatible. We shall call these special type of compatible subspaces (ranges of orthogonal projection commuting with A), *commuting subspaces*.

In [1] several examples of A-unitary operators where considered. One way to obtain compatible subspaces is by taking $\mathcal{S} = G(\mathcal{S}_0)$, where $\mathcal{S}_0$ is commuting and $G \in \mathbb{G}_A$.

Example 3.1. Let $\mathcal{H} = H_0^1(0,1)$, the subspace of the Sobolev space obtained as the closure with respect to the inner product

$$\langle f, g\rangle = \int_0^1 f(t)\bar{g}(t)\,dt + \int_0^1 f'(t)\bar{g}'(t)\,dt \tag{1}$$

of smooth functions on $(0,1)$ with compact support. The second inner product $[\cdot\,,\cdot]$ is the usual $L^2(0,1)$ inner product, so that $\mathcal{L} = L^2(0,1)$. In this case the operator A which implements $[\cdot\,,\cdot]$ is the solution operator of the Sturm–Liouville problem

$$\begin{cases} u - u'' = f \\ u(0) = u(1) = 0, \end{cases}$$

that is, $Af = u$, the unique solution u of the equation above for a given $f \in \mathcal{H}$. In this case A is compact, and it is diagonalized with the eigenvectors $s_k(t) = \frac{\sqrt{2}}{\sqrt{k^2\pi^2+1}} \sin(k\pi t)$ with eigenvalues $\lambda_k = (k^2\pi^2+1)^{-1}$. Thus A is an infinite diagonal matrix with different positive numbers in the diagonal. Therefore any operator commuting with A is also diagonal. In particular, any commuting subspace $\mathcal{S}_0$ is generated by a collection of eigenvectors s_k. Pick any infinite proper collection of s_k, containing s_1 and s_2. Consider the operator $G = M_{e^{i\pi t}}|_{H_0^1(0,1)}$, i.e. $(Gf)(t) = e^{i\pi t} f(t)$. Clearly, $M_{e^{i\pi t}}$ is a unitary operator on $L^2(0,1)$, and also it is apparent that

$$M_{e^{i\pi t}}\big(H_0^1(0,1)\big) \subset H_0^1(0,1)$$

and

$$M_{e^{i\pi t}}^{-1}\big(H_0^1(0,1)\big) = M_{e^{-i\pi t}}\big(H_0^1(0,1)\big) \subset H_0^1(0,1),$$

i.e., by the above remark, $G \in \mathbb{G}_A$. An elementary computation shows that $\langle G(s_1), s_j\rangle \neq 0$ for all j odd, and $\langle G(s_2), s_k\rangle \neq 0$ for all k even. It follows that $G(\mathcal{S}_0)$ is a compatible subspace, which is non-commuting.

Example 3.2. Consider $\mathcal{H} = H^1(\mathbb{R}) \subset L^2(\mathbb{R})$, or rather, its Fourier transform

$$\mathcal{H} = \big\{f \in L^2(\mathbb{R}):\ \big(1+x^2\big)^{1/2} f(x) \in L^2(\mathbb{R})\big\},$$

with the complete inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} \big(1+x^2\big) f(x)\bar{g}(x)\,dx.$$

The second inner product is the usual L^2 inner product, and the operator A implementing it in $\mathcal{H}$ is $Af(x) = \frac{1}{1+x^2} f(x)$. Consider the reflection $Rf(x) = f(-x)$. A simple computation shows that R is symmetric on $\mathcal{H}$, and that it commutes with A. Then the closed subspace $\mathcal{S}_0 = \{f \in \mathcal{H}:\ Rf = f\}$ of a.e. even functions, is a commuting subspace. For $a \in \mathbb{R}$, consider the translation operator $T_a f(x) = f(x-a)$. Note that T_a is a unitary operator on L^2, which leaves $\mathcal{H}$ fixed. Therefore $T_a \in \mathbb{G}_A$. Note also that

$$T_aRT_{-a}f(x) = f(-x+2a),$$

so that $\mathcal{S}_a = T_a(\mathcal{S}_0) = \{f \in \mathcal{H}\colon\ f(x) = f(-x+2a) \text{ a.e.}\}$ is a compatible subspace. Another simple computation shows that T_aRT_{-a} is not symmetric if $a \neq 0$. The subspace $\mathcal{S}_a$ is non-commuting if $a \neq 0$. Indeed,

$$Q_{\mathcal{S}_a} = T_a Q_{\mathcal{S}_0} T_{-a} = \frac{1}{2}(1 + T_aRT_{-a})$$

is not symmetric.

Both examples are constructed as translations of a commuting subspace. A natural question would be, if every compatible subspace $\mathcal{S}$ is of the form $\mathcal{S} = G(\mathcal{S}_0)$, where $\mathcal{S}_0$ is a commuting subspace and $G \in \mathbb{G}_A$. The following simple example shows that in general this is not the case.

Example 3.3. Let $\mathcal{H} = L^2(0,1)$ with the usual inner product, and

$$\mathcal{L} = \{f \in L^2(0,1)\colon\ f(t)t^{1/2} \in L^2(0,1)\}$$

given by $A = M_t$, $(Af)(t) = tf(t)$, i.e.

$$[f,g] = \int_0^1 tf(t)\overline{g(t)}\,dt.$$

The operator A generates a maximal abelian von Neumann algebra in $\mathcal{B}(\mathcal{H})$ (namely $L^\infty(0,1)$ acting as multiplication operators). It follows that any non-trivial projection commuting with A is of the form $P = M_{\chi_\Delta}$, where χ_Δ is the characteristic function of a measurable set Δ of positive Lebesgue measure. It follows that any commuting subspace is infinite and co-infinite dimensional. Pick any finite dimensional subspace $\mathcal{S}$. Then $\mathcal{S}$ is compatible, but it is not of the form $G(\mathcal{S}_0)$ for any commuting subspace $\mathcal{S}_0$.

Remark 3.4. In [8] it was shown that for any pair of finite dimensional subspaces $\mathcal{S}_1, \mathcal{S}_2$ of the same dimension, there exists an element $G \in \mathbb{G}_A$ such that $G(\mathcal{S}_1) = \mathcal{S}_2$. Moreover, it was remarked that the element G constructed is of the form $1+$ finite rank. In [1] it was shown that such elements G are exponentials: there exists a finite rank element in $Z \in \mathcal{B}_{as}^A(\mathcal{H})$ such that $G = e^Z$. Then the curve $e^{tZ}Q_{\mathcal{S}_1}e^{-tZ}$ joins $Q_{\mathcal{S}_1}$ and $Q_{\mathcal{S}_2}$. Thus the subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ lie in the same connected component of Gr_A.

There is an alternative characterization for compatible subspaces in terms of extensions to the Hilbert space $\mathcal{L}$:

Proposition 3.5. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Denote by $\overline{\mathcal{S}}$ its closure in $\mathcal{L}$. Then $\mathcal{S}$ is A-compatible if and only if $\overline{\mathcal{S}} \cap \mathcal{H} = \mathcal{S}$ and the orthogonal projection $P_{\overline{\mathcal{S}}} \in \mathcal{B}(\mathcal{L})$ satisfies $P_{\overline{\mathcal{S}}}(\mathcal{H}) \subseteq \mathcal{H}$.*

Proof. Suppose that $\mathcal{S} \subset \mathcal{H}$ is compatible, then there exists $Q_{\mathcal{S}}$, the unique idempotent with range $\mathcal{S}$ which is A-symmetric. Note that $Q_{\mathcal{S}}$ extends to an orthogonal projection $\bar{Q}_{\mathcal{S}}$ on $\mathcal{L}$, with $\mathcal{S} \subset R(\bar{Q}_{\mathcal{S}}) \subset \overline{\mathcal{S}}$, i.e. $P_{\overline{\mathcal{S}}} = \bar{Q}_{\mathcal{S}}$. Thus

$$P_{\overline{\mathcal{S}}}(\mathcal{H}) = Q_{\mathcal{S}}(\mathcal{H}) = \mathcal{S} \subset \mathcal{H}.$$

Clearly, $\mathcal{S} \subset \overline{\mathcal{S}} \cap \mathcal{H}$. If $f \in \overline{\mathcal{S}} \cap \mathcal{H}$, then $f = P_{\overline{\mathcal{S}}}(f) = Q_{\mathcal{S}}(f) \in \mathcal{S}$.

Conversely, suppose that $\mathcal{S} = \overline{\mathcal{S}} \cap \mathcal{H}$ and that $P_{\overline{\mathcal{S}}}(\mathcal{H}) \subset \mathcal{H}$. Then $P_{\overline{\mathcal{S}}}|_{\mathcal{H}}$ is an A-symmetric idempotent in $\mathcal{H}$. Its range is $P_{\overline{\mathcal{S}}}(\mathcal{H}) = \overline{\mathcal{S}} \cap \mathcal{H} = \mathcal{S}$. □

The following example shows that the hypothesis $\overline{\mathcal{S}} \cap \mathcal{H} = \mathcal{S}$ in the above proposition does not follow from the assumption $P_{\overline{\mathcal{S}}}(\mathcal{H}) \subseteq \mathcal{H}$.

Example 3.6. Let $H^1(0,1)$ be the space of functions $f \in L^2(0,1)$ that have weak derivative $f' \in L^2(0,1)$. It is a Hilbert space with the norm given by Eq. (1) and $H_0^1(0,1)$ is a proper closed subspace of $H^1(0,1)$ (see e.g. [32]). Set $\mathcal{S} = H_0^1(0,1)$, $\mathcal{H} = H^1(0,1)$ and $\mathcal{L} = L^2(0,1)$. Since $H_0^1(0,1)$ is a dense subspace of $L^2(0,1)$, then $\overline{\mathcal{S}} \cap \mathcal{H} = \mathcal{L} \cap \mathcal{H} = \mathcal{H} \neq \mathcal{S}$. However, the orthogonal projection onto $\overline{\mathcal{S}} = \mathcal{L}$, which is the identity map, trivially leaves $\mathcal{H}$ invariant.

Let B be an A-symmetric operator. Pick $f \in \overline{\ker(B)} \cap \mathcal{H}$. Then there is a sequence $(f_n)_n$ such that $Bf_n = 0$ and $f_n \to f$ in the norm of $\mathcal{L}$. Since B is $\mathcal{L}$-continuous by Theorem 2.1, one obtains that $Bf = \lim Bf_n = 0$, and consequently, $f \in \ker(B)$. Hence $\overline{\ker(B)} \cap \mathcal{H} = \ker(B)$. Despite of being satisfied one of the assumptions in Proposition 3.5, the following example shows that the kernel of an A-symmetric operator is not a compatible subspace in general.

Example 3.7. Consider again the Sobolev space $\mathcal{H} = H^1(0,1)$ as in Example 3.6. Take a bounded smooth function $\theta : (0,1) \to \mathbb{R}$, with bounded derivative, such that θ is equal to zero in a proper subinterval $(0,a]$ of $(0,1)$ and positive in the complement $(a,1)$. Recall that $H^1(0,1)$ is a dense subspace of $\mathcal{L} = L^2(0,1)$, and each function in $H^1(0,1)$ admits an absolutely continuous representative. Consider the multiplication operator

$$\bar{M}_\theta : L^2(0,1) \to L^2(0,1), \qquad \bar{M}_\theta f = \theta f.$$

Since θ is a real-valued function, this operator is self-adjoint. Moreover, $\bar{M}_\theta$ leaves $H^1(0,1)$ invariant because the function θ is smooth. Therefore $\bar{M}_\theta$ is the extension of an A-symmetric operator M_θ acting on $H^1(0,1)$. Clearly, the kernel of M_θ is given by

$$\ker(M_\theta) = \bigl\{ f \in H^1(0,1)\colon\ f \equiv 0 \text{ on } [a,1) \bigr\}.$$

On the other hand, $\ker(\bar{M}_\theta)$ is orthogonal to $R(\bar{M}_\theta)$ due to the fact that $\bar{M}_\theta$ is self-adjoint. The constant function $f_0 \equiv 1$ belongs to $H^1(0,1)$, and can be written as the sum of two characteristic functions

$$f_0 = \chi_{(0,a]} + \chi_{(a,1)}.$$

Clearly $\chi_{(0,a]} \in \overline{\ker(M_\theta)} = \ker(\bar{M}_\theta)$ and this sum is orthogonal with respect to the inner product of $L^2(0,1)$. Therefore $P_{\overline{\ker(M_\theta)}}(f_0) = \chi_{(0,a]} \notin H^1(0,1)$, and consequently, $\ker(M_\theta)$ is not a compatible subspace.

Remark 3.8. Another useful criteria to check that a subspace $\mathcal{S}$ is compatible with A is provided by the notion of Dixmier angle between subspaces. Recall that the Dixmier angle between two subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ is the angle in $[0,\pi/2]$ whose cosine is given by

$$c_0(\mathcal{S}_1,\mathcal{S}_2) = \sup\bigl\{ \bigl|\langle f,g\rangle\bigr|\colon\ f \in \mathcal{S}_1,\ \|f\| < 1,\ g \in \mathcal{S}_2,\ \|g\| < 1 \bigr\}.$$

Then a subspace $\mathcal{S}$ is compatible with A if and only if $c_0(\mathcal{S}^\perp, \overline{A(\mathcal{S})}) < 1$, where the bar here stands for the closure with respect to the topology of $\mathcal{H}$ (see [12]).

4. Differentiable structure of Gr_A

Let us transcribe the following result contained in the appendix of the paper [30] by I. Raeburn, which is a consequence of the implicit function theorem in Banach spaces.

Lemma 4.1. *Let G be a Banach–Lie group acting smoothly on a Banach space X. For a fixed $x_0 \in X$, denote by $\pi_{x_0} : G \to X$ the smooth map $\pi_{x_0}(g) = g \cdot x_0$. Suppose that*

1. *π_{x_0} is an open mapping, regarded as a map from G onto the orbit $\{g \cdot x_0\colon\ g \in G\}$ of x_0 (with the relative topology of X).*
2. *The differential $d(\pi_{x_0})_1 : (TG)_1 \to X$ splits: its kernel and range are closed complemented subspaces.*

Then the orbit $\{g \cdot x_0\colon\ g \in G\}$ is a smooth submanifold of X, and the map

$$\pi_{x_0} : G \to \{g \cdot x_0\colon\ g \in G\}$$

is a smooth submersion.

We shall apply this lemma to our situation. Fix $Q_0 \in Gr_A$, and consider the map

$$\pi_{Q_0} : \mathbb{G}_A \to \mathcal{B}_s^A(\mathcal{H}), \qquad \pi_{Q_0}(G) = GQ_0G^{-1}.$$

Clearly, it is a C^∞ map, its differential at the identity 1 is

$$d(\pi_{Q_0})_1 = \delta_{Q_0} : \mathcal{B}_{as}^A(\mathcal{H}) \to \mathcal{B}_s^A(\mathcal{H}), \qquad \delta_{Q_0}(X) = XQ_0 - Q_0X,$$

where the Banach–Lie algebra of $\mathbb{G}_A$ is identified with $\mathcal{B}_{as}^A(\mathcal{H})$, the space of A-anti-symmetric operators.

Proposition 4.2. *The range and the kernel of δ_{Q_0} are complemented. The range of δ_{Q_0} consists of all A-symmetric operators that are co-diagonal with respect to Q_0, i.e.*

$$R(\delta_{Q_0}) = \big\{Y \in \mathcal{B}_s^A(\mathcal{H})\colon\ Q_0YQ_0 = (1 - Q_0)Y(1 - Q_0) = 0\big\}.$$

The kernel of δ_{Q_0} consists of all A-anti-symmetric operators that are diagonal with respect to Q_0, i.e.

$$\ker(\delta_{Q_0}) = \big\{X \in \mathcal{B}_{as}^A(\mathcal{H})\colon\ Q_0X = XQ_0\big\}.$$

Proof. Let us prove first the assertion on $R(\delta_{Q_0})$. Clearly, if $Y = XQ_0 - Q_0X$ for some operator $X \in \mathcal{B}_{as}^A(\mathcal{H})$, then Y is A-symmetric, and $Q_0YQ_0 = 0 = (1-Q_0)Y(1-Q_0)$. Conversely, let Y be an A-symmetric operator which is co-diagonal with respect to Q_0. Then

$$\begin{aligned} Y &= Q_0YQ_0 + Q_0Y(1 - Q_0) + (1 - Q_0)YQ_0 + (1 - Q_0)Y(1 - Q_0) \\ &= Q_0Y(1 - Q_0) + (1 - Q_0)YQ_0 = Q_0Y + YQ_0. \end{aligned}$$

Let $X = YQ_0 - Q_0Y$. Clearly X is A-anti-symmetric (being the commutator of two A-symmetric operators). Moreover, note that

$$\delta_{Q_0}(X) = (YQ_0 - Q_0Y)Q_0 - Q_0(YQ_0 - Q_0Y) = YQ_0 + Q_0Y = Y.$$

The assertion on the kernel of δ_{Q_0} is trivial. Let us prove that these spaces are complemented. To this end, consider the linear map

$$E = E_{Q_0} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \qquad E(X) = Q_0 X Q_0 + (1 - Q_0) X (1 - Q_0).$$

Clearly, E is idempotent. Note that $E(\mathcal{B}_s^A(\mathcal{H})) \subset \mathcal{B}_s^A(\mathcal{H})$ and $E(\mathcal{B}_{as}^A(\mathcal{H})) \subset \mathcal{B}_{as}^A(\mathcal{H})$. Indeed, if $X \in \mathcal{B}_s^A(\mathcal{H})$, then

$$\begin{aligned} E(X)^* A &= Q_0^* X^* Q_0^* + \left(1 - Q_0^*\right) X^* \left(1 - Q_0^*\right) A = Q_0^* X^* A Q_0 + \left(1 - Q_0^*\right) X^* A (1 - Q_0) \\ &= Q_0^* A X Q_0 + \left(1 - Q_0^*\right) A X (1 - Q_0) = A Q_0 X Q_0 + A (1 - Q_0) X (1 - Q_0) = A E(X), \end{aligned}$$

and similarly if $X \in \mathcal{B}_{as}^A(\mathcal{H})$. Let E_s and E_{as} be the idempotents obtained by restricting E to (respectively) $\mathcal{B}_s^A(\mathcal{H})$ and $\mathcal{B}_{as}^A(\mathcal{H})$. The range of E_{as} consists of A-anti-symmetric operators which commute with Q_0, i.e. $R(E_{as}) = \ker(\delta_{Q_0})$, and thus it is complemented in $\mathcal{B}_{as}^A(\mathcal{H})$. The kernel of E_s consists of A-symmetric operators which are co-diagonal with respect to Q_0, thus $\ker(E_s) = R(\delta_{Q_0})$, and therefore this space is complemented in $\mathcal{B}_s^A(\mathcal{H})$. $\square$

The next result implies in particular that the map π_{Q_0} is open, when it is regarded as a map from $\mathbb{G}_A$ onto the orbit $\{G Q_0 G^{-1} \colon G \in \mathbb{G}_A\}$. This result is based on general facts of the space of idempotents on $\mathcal{B}(\mathcal{H})$. The main ideas are contained in [28,13] and [14]. We state them in the following remark.

Remark 4.3. The set $\mathcal{Q}(\mathcal{H})$ of idempotents is a complemented analytic submanifold of $\mathcal{B}(\mathcal{H})$. It carries the action of the general linear group $Gl(\mathcal{H})$: $G \cdot Q = G Q G^{-1}$. The map induced by this action and a fixed $Q_0 \in \mathcal{Q}(\mathcal{H})$,

$$\pi_{Q_0} : Gl(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})_{Q_0} = \text{connected component of } Q_0 \text{ in } \mathcal{Q}(\mathcal{H}),$$

$\pi_{Q_0}(G) = G Q_0 G^{-1}$, is an analytic submersion. Its differential at 1 is the derivation $d(\pi_{Q_0})_1(X) = X Q_0 - Q_0 X$. In particular, the tangent space of $\mathcal{Q}(\mathcal{H})$ at Q_0 is the Banach space $\{X Q_0 - Q_0 X \colon X \in \mathcal{B}(\mathcal{H})\}$.

There is a natural linear connection, induced by the action of the general linear group, and the direct decomposition of $\mathcal{B}(\mathcal{H})$ at every $Q_0 \in \mathcal{Q}(\mathcal{H})$, as Q_0-diagonal operators plus Q_0-co-diagonal operators. Namely, the set of Q_0-diagonal operators is precisely the kernel of $d(\pi_{Q_0})_1$, thus the (smooth, equivariant) distribution

$$\mathcal{Q} \ni Q_0 \mapsto \{Q_0\text{-co-diagonal operators}\}$$

defines the structure of a homogeneous reductive space on $\mathcal{Q}(\mathcal{H})_{Q_0} = Gl(\mathcal{H})/\mathrm{Stab}_{Q_0}$, compare [21], Section X.2. Associated to every $Gl(\mathcal{H})$-invariant G-structure is then a canonical invariant connection, whose associated geometric quantities (torsion and curvature tensors, geodesics) can be computed in terms of this distribution. For instance, geodesics of the connection, starting at the point Q_0, are of the form

$$\delta(t) = e^{tX} Q_0 e^{-tX},$$

where X is a Q_0-co-diagonal element in $\mathcal{B}(\mathcal{H})$.

The affine correspondence $Q \leftrightarrow \epsilon_Q = 2Q - 1$ allows to view idempotents alternatively as symmetries, i.e. operators ϵ such that $\epsilon^2 = 1$. Some calculations are easier with symmetries than with idempotents. For instance, X is co-diagonal with respect to Q if and only if it anti-commutes with ϵ_Q. In particular, the above geodesic, regarded as a curve of symmetries, is

$$\epsilon_\delta(t) = e^{tX} \epsilon_Q e^{-tX} = e^{2tX} \epsilon_Q = \epsilon_Q e^{-2tX}.$$

Finally, another feature of the diagonal, co-diagonal decomposition is that it enables one to produce local cross sections of the action of the general linear group, by means of the exponential map. Namely, the differential at 1 of the C^∞ map

$$\{X \in \mathcal{B}(\mathcal{H}):\ QXQ = (1-Q)X(1-Q) = 0\} \to \mathcal{Q}(\mathcal{H}), \quad X \mapsto e^X Q e^{-X}$$

is the linear isomorphism

$$\{X \in \mathcal{B}(\mathcal{H}):\ QXQ = (1-Q)X(1-Q) = 0\} \to \{XQ - QX:\ X \in \mathcal{B}(\mathcal{H})\}, \quad X \mapsto XQ - QX.$$

Indeed, it is a simple computation to prove that this linear map is its own inverse. Thus, by the inverse mapping theorem in Banach spaces, the former map is a local diffeomorphism. There exists an open set $\mathcal{V}_Q \subset \mathcal{Q}(\mathcal{H})$, $Q \in \mathcal{V}_Q$, such that for any $R \in \mathcal{V}_Q$ there is a unique Q-co-diagonal X_R such that $e^{X_R} Q e^{-X_R} = R$. The map $R \to X_R$ is analytic, and therefore an analytic cross section is defined:

$$\mathcal{V}_Q \ni R \mapsto e^{X_R} \in Gl(\mathcal{H}).$$

Let us call it the exponential cross section. Note that the open set $\mathcal{V}_Q$ can be chosen to be a ball in $\mathcal{B}(\mathcal{H})$, intersected with $\mathcal{Q}(\mathcal{H})$: $\mathcal{V}_Q = \{R \in \mathcal{Q}(\mathcal{H}):\ \|R - Q\| < r_Q\}$. Moreover, we can further shrink r_Q so that if $R \in \mathcal{V}_Q$, $\|e^{2X_R} - 1\| < 1$. We fix this value of r_Q for each Q (we will estimate this radius below).

Proposition 4.4. *For each $Q_0 \in Gr_A$, the exponential cross section just described, when restricted to $\mathcal{V}_{Q_0} \cap Gr_A$, takes values in $\mathbb{G}_A$. Thus the map $\pi_{Q_0} : \mathbb{G}_A \to \{GQ_0G^{-1}:\ G \in \mathbb{G}_A\}$ has continuous local cross sections, and in particular it is an open map.*

Proof. Let $Q \in \mathcal{V}_{Q_0} \cap Gr_A$. Then $Q = e^{X_Q} Q_0 e^{-X_Q}$, and thus we get

$$\epsilon_Q = e^{2X_Q} \epsilon_{Q_0} = \epsilon_{Q_0} e^{-2X_Q}.$$

Then $e^{2X_Q} = \epsilon_Q \epsilon_{Q_0}$ and $\epsilon_{Q_0} \epsilon_Q = e^{-2X_Q}$, so we have that

$$\left(e^{2X_Q}\right)^* A = \left(2Q_0^* - 1\right)\left(2Q^* - 1\right)A = A(2Q_0 - 1)(2Q - 1) = Ae^{-2X_Q}.$$

Hence $e^{2X_Q} \in \mathbb{G}_A$. Note that by the assumption on r_{Q_0}, it follows that $\|e^{2X_Q} - 1\| < 1$. In [1], it was shown that if $G \in \mathbb{G}_A$ and $\|G - 1\| < 1$, then the usual series of the logarithm of G (which is absolutely convergent) converges to an element which belongs to $\mathcal{B}_{as}^A(\mathcal{H})$. It follows that $X_Q \in \mathcal{B}_{as}^A(\mathcal{H})$. Hence $e^{X_Q} \in \mathbb{G}_A$. □

Note that what in fact was proved above, is that the smooth map

$$\mathcal{V}_{Q_0} \cap Gr_A \ni Q \mapsto X_Q$$

takes values in $\mathcal{B}_{as}^A(\mathcal{H})$, the Banach–Lie algebra of $\mathbb{G}_A$. We may use Lemma 4.1 to prove the following:

Corollary 4.5. *The orbit $\mathbb{G}_A \cdot Q_0 = \{GQ_0G^{-1}:\ G \in \mathbb{G}_A\}$ is a complemented C^∞ submanifold of $\mathcal{B}_s^A(\mathcal{H})$. The map $\pi_{Q_0} : \mathbb{G}_A \to \mathbb{G}_A \cdot Q_0$ is a C^∞ submersion. Moreover, the orbit $\mathbb{G}_A \cdot Q_0$ is a union of connected components of Gr_A, and therefore, Gr_A is a complemented submanifold of $\mathcal{B}_s^A(\mathcal{H})$.*

Proof. The assertion that the orbit $\mathbb{G}_A \cdot Q_0$ is a union of connected components of Gr_A needs proof. By Proposition 4.4, which provides the existence of continuous local cross sections for the action of $\mathbb{G}_A$, it follows that this action is locally transitive, a fact which implies the assertion. Also it implies that an element of $Q \in Gr_A$, which verifies $\|Q - Q_0\| < r_{Q_0}$, can be connected to Q_0 by the curve $e^{tX_{Q_0}} Q_0 e^{-tX_{Q_0}}$ in Gr_A. □

Remark 4.6. In the case where $\mathrm{rank}(Q_0) < \infty$, it can be shown that $\mathbb{G}_A \cdot Q_0$ is also a complemented submanifold of $\mathcal{B}(\mathcal{H})$ (see [8]).

Note that the curve in the above proof is a geodesic of $\mathcal{Q}(\mathcal{H})$. Therefore we have also the following consequence:

Corollary 4.7. *Let $Q, Q_0 \in Gr_A$ such that $\|Q - Q_0\| < r_{Q_0}$. Then there is a unique geodesic δ (up to reparametrization) of $\mathcal{Q}(\mathcal{H})$ which joins them, and $\delta(t) \in Gr_A$ for all $t \in \mathbb{R}$.*

We may compute an estimate for r_{Q_0}. As remarked in the Section 2, if $B, C \in \mathcal{B}_s^A(\mathcal{H})$, then they have symmetric extensions $\bar{B}, \bar{C}$ in $\mathcal{L}$, and $\|\bar{B} - \bar{C}\|_{\mathcal{B}(\mathcal{L})} \leqslant \|B - C\|$. It follows that if $Q, Q_0 \in Gr_A$ satisfy $\|Q - Q_0\| < 1$, then $\bar{Q}, \bar{Q_0}$ are self-adjoint projections in $\mathcal{L}$, lying at distance less than 1. In [27] it was shown that two self-adjoint projections at distance less than 1 are joined by a unique geodesic of the manifold of self-adjoint projections. Thus there exists a unique anti-symmetric operator $Z_{\bar{Q}}$ acting on $\mathcal{L}$, which is co-diagonal with respect to $\bar{Q}_0$, such that

$$e^{2Z_{\bar{Q}}} \epsilon_{\bar{Q}_0} = \epsilon_{\bar{Q}_0} e^{-2Z_{\bar{Q}}} = \epsilon_{\bar{Q}}.$$

Note that $e^{2Z_{\bar{Q}}} = \epsilon_{\bar{Q}} \epsilon_{\bar{Q}_0}$. By construction, $\epsilon_{\bar{Q}}$ and $\epsilon_{\bar{Q}_0}$ leave $\mathcal{H} \subset \mathcal{L}$ fixed. Therefore $e^{2Z_{\bar{Q}}}(\mathcal{H}) \subset \mathcal{H}$. Similarly, its inverse $e^{-2Z_{\bar{Q}}} = \epsilon_{\bar{Q}_0} \epsilon_{\bar{Q}}$ leaves $\mathcal{H}$ invariant. It follows that $e^{2Z_{\bar{Q}}}$ is a unitary operator in $\mathcal{L}$ such that $e^{2Z_{\bar{Q}}}(\mathcal{H}) = \mathcal{H}$. Therefore

$$e^{2Z_{\bar{Q}}}|_{\mathcal{H}} \in \mathbb{G}_A.$$

Let us further shrink the distance between Q_0 and Q so that the exponent $Z_{\bar{Q}}$ induces an operator in the Banach–Lie algebra of $\mathbb{G}_A$. To this effect, it will suffice that

$$\left\| e^{2Z_{\bar{Q}}}\big|_{\mathcal{H}} - 1 \right\| < 1 \quad \text{and} \quad \left\| e^{-2Z_{\bar{Q}}}\big|_{\mathcal{H}} - 1 \right\| < 1.$$

Indeed, on one hand, since $e^{2Z_{\bar{Q}}}|_{\mathcal{H}} - 1$ is extendable to $\mathcal{L}$, by Theorem 2.3,

$$\sigma_{\mathcal{L}}\big(e^{2Z_{\bar{Q}}} - 1\big) \subset \sigma_{\mathcal{H}}\big(e^{2Z_{\bar{Q}}}\big|_{\mathcal{H}} - 1\big) \cup \overline{\sigma_{\mathcal{H}}\big(\big(e^{2Z_{\bar{Q}}}\big|_{\mathcal{H}}\big)^+ - 1\big)}.$$

Note that

$$\big(e^{2Z_{\bar{Q}}}\big|_{\mathcal{H}}\big)^+ = e^{-2Z_{\bar{Q}}}\big|_{\mathcal{H}} = \epsilon_{\bar{Q}_0} \epsilon_{\bar{Q}}|_{\mathcal{H}} = \epsilon_{Q_0} \epsilon_Q.$$

If $\rho_{\mathcal{H}}$ and $\rho_{\mathcal{L}}$ denote the spectral radii, then

$$\begin{aligned} \left\| e^{2Z_{\bar{Q}}} - 1 \right\|_{\mathcal{B}(\mathcal{L})} &= \rho_{\mathcal{L}}\big(e^{2Z_{\bar{Q}} - 1}\big) \\ &\leqslant \max\big\{ \rho_{\mathcal{H}}\big(e^{2Z_{\bar{Q}}}\big|_{\mathcal{H}} - 1\big), \rho_{\mathcal{H}}\big(e^{-2Z_{\bar{Q}}}\big|_{\mathcal{H}} - 1\big) \big\} \\ &\leqslant \max\big\{ \big\| e^{2Z_{\bar{Q}}}\big|_{\mathcal{H}} - 1 \big\|, \big\| e^{-2Z_{\bar{Q}}}\big|_{\mathcal{H}} - 1 \big\| \big\} < 1. \end{aligned}$$

This implies in particular that $2Z_{\bar{Q}}$ is the usual determination of the logarithm (note that $\|Z_{\bar{Q}}\| < \pi/6$). On the other hand, by [1, Lemma 3.3], $\|e^{2Z_{\bar{Q}}}|_{\mathcal{H}} - 1\| < 1$ implies that the usual logarithm series for $e^{2Z_{\bar{Q}}}|_{\mathcal{H}}$ converges in $\mathcal{B}(\mathcal{H})$ to an element in the Banach–Lie algebra of $\mathbb{G}_A$. Therefore it follows that $X_Q := Z_{\bar{Q}}|_{\mathcal{H}} \in \mathcal{B}_{as}^A(\mathcal{H})$, as wanted. Note that $e^{2Z_{\bar{Q}}}|_{\mathcal{H}} = \epsilon_Q \epsilon_{Q_0}$, and then

$$\left\| e^{2Z_{\bar{Q}}}\ \big|_{\mathcal{H}} - 1 \right\| = \|\epsilon_Q \epsilon_{Q_0} - 1\| = \left\| (\epsilon_Q - \epsilon_{Q_0}) \epsilon_{Q_0} \right\| \leqslant \|\epsilon_Q - \epsilon_{Q_0}\| \|\epsilon_{Q_0}\| = 2\|\epsilon_{Q_0}\| \|Q - Q_0\|.$$

Similarly,

$$\left\|e^{-2Z_{\bar{Q}}}\,\big|_{\mathcal{H}} - 1\right\| = \|\epsilon_{Q_0}\epsilon_Q - 1\| = \left\|\epsilon_{Q_0}(\epsilon_Q - \epsilon_{Q_0})\right\| \leqslant 2\|\epsilon_{Q_0}\|\|Q - Q_0\|.$$

We may summarize the above discussion in the following

Proposition 4.8. *With the current notations,* $r_{Q_0} = \frac{1}{2\|\epsilon_{Q_0}\|} = \frac{1}{2}\frac{1}{\|Q_0\|+(\|Q_0\|^2-1)^{1/2}}$.

Proof. The norm of ϵ_{Q_0} was computed in [17]:

$$\|\epsilon_{Q_0}\| = \|Q_0\| + \left(\|Q_0\|^2 - 1\right)^{1/2}. \qquad \square$$

5. Finite rank orbits as symplectic leaves

The foundations of Banach Poisson differential geometry were investigated by A. Odzijewicz and T. Ratiu in [26]. One of the key features of this new theory is that Banach Poisson manifolds provide an appropriate setting for a unified approach to the Hamiltonian and the quantum mechanical description of physical systems. In addition to the seminal article [26], we also refer the reader to the monograph [25] for a detailed exposition on Banach Poisson manifolds; meanwhile several interesting examples and applications can be found in [6,7,19].

An important class of infinite dimensional linear Poisson manifolds is given by Banach Lie–Poisson spaces: a Banach space $\mathfrak{b}$ is called a *Banach Lie–Poisson space* if $\mathfrak{b}^*$ is a Banach–Lie algebra endowed with a Lie bracket $[\,\cdot,\cdot\,]$ such that $\mathrm{ad}_x^*(\mathfrak{b}) \subseteq \mathfrak{b}$ for all $x \in \mathfrak{b}^*$. Recall that $\mathrm{ad}_x^* : \mathfrak{b}^{**} \to \mathfrak{b}^{**}$ is the coadjoint representation, i.e. the dual map to the adjoint representation defined by $\mathrm{ad}_x : \mathfrak{b}^* \to \mathfrak{b}^*$, $\mathrm{ad}_x(y) = [x, y]$. Actually, this is an equivalent characterization of Banach Lie–Poisson spaces (see [26, Theorem 4.2]), whereas the original definition involves the notion of Banach Poisson manifolds that we have omitted here. Notable examples of Banach Lie–Poisson spaces are the dual of any reflexive Banach Lie algebra, the space of trace class operators on a Hilbert space, and more generally, the predual of a von Neumann algebra.

Let $\mathcal{B}_p(\mathcal{H})$ denote the class of p-Schatten operators on $\mathcal{H}$ $(1 \leqslant p \leqslant \infty)$, and let Tr be the usual trace on the Hilbert space $\mathcal{H}$. Recall that the p-norm of an operator in $X \in \mathcal{B}_p(\mathcal{H})$ is given by $\|X\|_p = \mathrm{Tr}(|X|^p)^{1/p}$. If $p = \infty$, $\mathcal{B}_\infty(\mathcal{H})$ denotes the compact operators on $\mathcal{H}$. Denote by $\mathbb{G}_{p,A}$ the group consisting of operators in $\mathbb{G}_A$ which are $\mathcal{B}_p(\mathcal{H})$-perturbations of the identity:

$$\mathbb{G}_{p,A} = \left\{G \in \mathbb{G}_A\colon\ G - 1 \in \mathcal{B}_p(\mathcal{H})\right\}.$$

The group $\mathbb{G}_{p,A}$ is a connected Banach–Lie group with the topology defined by the metric $(G_1, G_2) \mapsto \|G_1 - G_2\|_p$ (see Appendix A). Indeed, in [1] it was proved that $\mathbb{G}_{p,A}$ is an exponential group. This means that $\mathbb{G}_{p,A} = exp(\mathfrak{g}_p)$, where $\mathfrak{g}_p$ is its Lie algebra, i.e.

$$\mathfrak{g}_p = \mathcal{B}_{as}^A(\mathcal{H}) \cap \mathcal{B}_p(\mathcal{H}).$$

We will prove that the Lie algebra $\mathfrak{g}_p$ of the Banach Lie group $\mathbb{G}_{A,p}$ is a Banach Lie–Poisson space. To this end, we will need the following proposition, which shows that the usual duality relationships are preserved in the class of A-anti-symmetric operators.

Proposition 5.1. *The map*

$$\mathfrak{g}_p \to (\mathfrak{g}_q)^*, \qquad Z \mapsto \mathrm{Tr}(Z\,\cdot\,) \tag{2}$$

is an isometric isomorphism of real Banach spaces if $1 < p < \infty$ *and* $p^{-1} + q^{-1} = 1$. *Moreover, the same map implements the real isometric isomorphisms* $(\mathfrak{g}_\infty)^* \simeq \mathfrak{g}_1$ *and* $(\mathfrak{g}_1)^* \simeq \mathcal{B}^A_{as}(\mathcal{H})$.

Proof. We will only prove the case where $1 < p < \infty$; the other two cases are analogous. We first check that the map is well defined. Clearly, any functional of the form $X \mapsto \operatorname{Tr}(ZX)$, $Z \in \mathfrak{g}_p$, $X \in \mathfrak{g}_q$, is continuous. To show that this functional is real note that ZX is a proper operator. According to Theorem 2.3, and noting that ZX is a compact operator on $\mathcal{H}$, it follows that $\sigma_{\mathcal{H}}(ZX) = \sigma_{\mathcal{L}}(\bar{Z}\bar{X})$, taking into account the multiplicity of the nonzero eigenvalues. Moreover, ZX is a trace class operator on $\mathcal{H}$ by Hölder's inequality and its extension $\bar{Z}\bar{X}$ is also trace class on $\mathcal{L}$ by Lalesco's inequality [23]. Then by Lidskii's theorem we find that

$$\operatorname{Tr}(ZX) = \sum_{i=1}^{\infty} \lambda_i(ZX) = \operatorname{Tr}_{\mathcal{L}}(\bar{Z}\bar{X}),$$

where $\lambda_i(ZX)$ are the eigenvalues counted with multiplicity and $\operatorname{Tr}_{\mathcal{L}}$ is the usual trace in $\mathcal{B}(\mathcal{L})$. Since the operators Z, X can be extended to anti-symmetric compact operators on $\mathcal{L}$ and the trace of the product of two anti-symmetric operators is real, we get that $\operatorname{Tr}(ZX)$ is real.

Let us prove that the map in Eq. (2) is surjective. Let φ be a functional in $(\mathfrak{g}_q)^*$. Extend this functional to the complex vector space $S := \mathbb{C}\mathfrak{g}_q$ by defining $\varphi_1(wX) = w\varphi(X)$, $w \in \mathbb{C}$. It is well defined: if $zX = wY$, then $-\bar{z}X = (zX)^+ = (wY)^+ = -\bar{w}Y$. Thus, $\operatorname{Re}(z)X = \operatorname{Re}(w)Y$ and $\operatorname{Im}(z)X = \operatorname{Im}(w)Y$. Suppose that $\operatorname{Re}(z) \neq 0$, then

$$\varphi_1(zX) = z\varphi(X) = z\varphi\left(\frac{\operatorname{Re}(w)}{\operatorname{Re}(z)}Y\right) = z\frac{\operatorname{Re}(w)}{\operatorname{Re}(z)}\varphi(Y) = w\varphi(Y) = \varphi_1(wY).$$

Also note that

$$\|\varphi\| = \sup_{\|X\|_q=1,\, X\in\mathfrak{g}_q} \left|\varphi(X)\right| \leqslant \sup_{\|X\|_q=1,\, X\in S} \left|\varphi_1(X)\right| = \|\varphi_1\|.$$

Pick $X \in \mathfrak{g}_q$ and $w \in \mathbb{C}$ such that $\|wX\|_q = 1$. Then,

$$\left|\varphi_1(wX)\right| = |w|\left|\varphi(X)\right| = \left|\varphi\left(\frac{X}{\|X\|_q}\right)\right| \leqslant \|\varphi\|.$$

Hence we have $\|\varphi_1\| = \|\varphi\|$. Next φ_1 can be extended to a functional $\tilde{\varphi}_1$ on $\mathcal{B}_q(\mathcal{H})$ with the same norm, by the Hahn–Banach theorem. Since $(\mathcal{B}_q(\mathcal{H}))^* \simeq \mathcal{B}_p(\mathcal{H})$, it follows that $\tilde{\varphi}_1 = \operatorname{Tr}(Z\,\cdot\,)$, for some $Z \in \mathcal{B}_p(\mathcal{H})$.

Let us prove that Z is an A-anti-symmetric operator. To this end, we consider the following rank one proper operators: $(f \otimes g)(h) = [h, g]f$ for any $f, g, h \in \mathcal{H}$. It is easily seen that $(f \otimes g)^+ = g \otimes f$. Set $X = f \otimes g$. Then these operators may be decomposed as the sum of real and imaginary parts with respect to the adjoint on $\mathcal{L}$, that is,

$$\operatorname{Re}(X) = \frac{1}{2}\big((f \otimes g) + (g \otimes f)\big), \qquad \operatorname{Im}(X) = \frac{1}{2i}\big((f \otimes g) - (g \otimes f)\big)$$

and $X = \operatorname{Re}(X) + i\operatorname{Im}(X)$, with $\operatorname{Re}(X)$ and $\operatorname{Im}(X)$ A-symmetric. Since φ is real-valued, we find that

$$\begin{aligned}\tilde{\varphi}_1\big(X^+\big) &= \tilde{\varphi}_1\big(\operatorname{Re}(X) - i\operatorname{Im}(X)\big) = -i\tilde{\varphi}_1\big(i\operatorname{Re}(X)\big) - \tilde{\varphi}_1\big(i\operatorname{Im}(X)\big)\\ &= -i\varphi\big(i\operatorname{Re}(X)\big) - \varphi\big(i\operatorname{Im}(X)\big) = \overline{i\varphi\big(i\operatorname{Re}(X)\big) - \varphi\big(i\operatorname{Im}(X)\big)} = -\overline{\tilde{\varphi}_1(X)}.\end{aligned}$$

We thus get

$$\operatorname{Tr}\big(Z(g\otimes f)\big)=\tilde{\varphi}_1(g\otimes f)=-\overline{\tilde{\varphi}_1(f\otimes g)}=-\overline{\operatorname{Tr}\big(Z(f\otimes g)\big)}.$$

Now it suffices to note that for all $f,g\in\mathcal{H}$,

$$[Zg,f]=\operatorname{Tr}\big(Z(g\otimes f)\big)=-\overline{\operatorname{Tr}\big(Z(f\otimes g)\big)}=-[g,Zf],$$

which proves that Z is A-anti-symmetric. To finish the proof of this lemma, we point out that the isomorphism is isometric because $\|\varphi\|=\|\varphi_1\|=\|\tilde{\varphi}_1\|=\|Z\|_p$. □

Orbits of finite rank projections in the compatible Grassmannian are related with variational spaces in many-particle Hartree–Fock theory. To briefly show this relationship, we introduce the following infinite dimensional Stiefel type manifold: for $n\in\mathbb{N}$,

$$\mathcal{C}_n=\big\{(h_1,h_2,\dots,h_n)\in\mathcal{H}^n\colon\ [h_i,h_j]=\delta_{ij}\big\}.$$

We emphasize that n-tuples of vectors in $\mathcal{C}_n$ have orthonormal coordinates with respect to the inner product defined by A on the Hilbert space $\mathcal{L}$. Next we consider the following equivalence relation:

$$(h_1,h_2,\dots,h_n)\sim(g_1,g_2,\dots,g_n)\quad\text{if}\ \sum_{i=1}^n U_{ij}h_i=g_j,\ j=1,\dots,n,\ \text{for some}\ (U_{ij})\in\mathcal{U}\big(\mathbb{C}^n\big),$$

where $\mathcal{U}(\mathbb{C}^n)$ is the unitary group of all $n\times n$ matrices. In the case where $\mathcal{H}=H^1(\mathbb{R}^3)$ is the first order Sobolev space of $\mathbb{R}^3$, $\mathcal{L}=L^2(\mathbb{R}^3)$ and $(Af)(x)=\frac{1}{1+|x|^2}f(x)$, the above defined $\mathcal{C}_n$ is usually known as the *Stiefel manifold in quantum chemistry* and the quotient space $\mathcal{C}_n/\sim$ is called the *Grassmann manifold in quantum chemistry*.

The geometric structure of these Banach manifolds was studied in [8], and does not depend on the specific afore-mentioned function spaces. The main reason for studying geometric properties of these manifolds is to provide a rigorous setting to apply critical point theory in Hartree–Fock type equations (see [3]). In particular, it was shown that $\mathcal{C}_n/\sim$ is homeomorphic to the orbit $\mathbb{G}_A\cdot Q_0$ of an A-symmetric projection Q_0 of rank n. This allows to endow $\mathcal{C}_n/\sim$ with a manifold structure by making this homeomorphism into a diffeomorphism.

According to Remark 3.4, the orbit of an A-symmetric projection Q_0 of rank n is connected, and it is given by

$$\mathbb{G}_A\cdot Q_0=\big\{Q\in\mathcal{B}(\mathcal{H})\colon\ Q^2=Q,\ Q^*A=AQ,\ \operatorname{rank}(Q)=n\big\}.$$

The following result can be seen as a generalization of the fact that the unitary orbit of a finite rank orthogonal projection is a strong symplectic leaf in the Banach Lie–Poisson space of trace class operators.

Corollary 5.2. *For $1\leqslant p\leqslant\infty$, the space $\mathfrak{g}_p$ is a real Banach Lie–Poisson space, and for any $Q_0\in Gr_A$ such that rank$(Q_0)<\infty$, the orbit $\mathbb{G}_A\cdot Q_0$ is a strong symplectic leaf in $\mathfrak{g}_p$ endowed with the form*

$$\omega_{Gr_A}\big(GQ_0G^{-1}\big)\big(\big[GXG^{-1},GQ_0G^{-1}\big],\big[GYG^{-1},GQ_0G^{-1}\big]\big)=\operatorname{Tr}\big(Q_0[X,Y]\big),$$

where $X,Y\in\mathfrak{g}_p$ and $G\in\mathbb{G}_A$.

Proof. According to Remark 3.4, any pair of A-symmetric projections of the same rank are conjugated by an invertible operator G such that $G-1$ has finite rank. Then for $1 \leqslant p \leqslant \infty$ the orbit may be described as

$$\mathbb{G}_A \cdot Q_0 = \{GQ_0G^{-1}\colon\ G \in \mathbb{G}_{p,A}\}.$$

Now we are going to verify the hypothesis of [26, Theorem 7.4], which in particular will imply that $\mathfrak{g}_p$ is a real Banach Lie–Poisson space for $1 \leqslant p \leqslant \infty$ and the formula above for the symplectic form. By Proposition 5.1, the predual of $\mathfrak{g}_q$ is given by $\mathfrak{g}_p$, where $1 \leqslant q < \infty$ and $q^{-1} + p^{-1} = 1$. Apparently, the coadjoint orbit satisfies $\{GXG^{-1}\colon\ X \in i\mathfrak{g}_p\} \subseteq i\mathfrak{g}_p$ for any $G \in \mathbb{G}_{q,A}$. Since Q_0 has finite rank, it follows that $Q_0 \in i\mathfrak{g}_p$. We will show later, in Corollary 6.5 (which is proved independently from the facts discussed in this section) that the projection map induced by the action is a submersion. This means that $\{G \in \mathbb{G}_{q,A}\colon\ GQ_0 = Q_0G\}$ is a Banach–Lie subgroup of $\mathbb{G}_{q,A}$. Again due to the fact that Q_0 has finite rank, it follows that $\mathbb{G}_q \cdot Q_0 = \mathbb{G}_p \cdot Q_0$, and by Corollary 6.5, the inclusion map $\mathbb{G}_{p,A} \cdot Q_0 \hookrightarrow i\mathfrak{g}_p = Q_0 + i\mathfrak{g}_p$ is an injective immersion. By [26, Theorem 7.5] this latter fact implies that the symplectic form is strong. □

6. The restricted compatible Grassmannian

Given a fixed p, $1 \leqslant p \leqslant \infty$, and a compatible subspace $\mathcal{S}_0 \subset \mathcal{H}$, with infinite dimension and co-dimension, with A-symmetric idempotent $Q_0 = Q_{\mathcal{S}_0}$, we shall define the p-restricted compatible Grassmannian, induced by the direct sum decomposition

$$\mathcal{H} = \mathcal{S}_0 + \mathcal{N}_0,$$

where $\mathcal{N}_0 = \ker(Q_0)$. This decomposition is non-orthogonal with respect to the inner product on $\mathcal{H}$. We shall adopt the following definition, which in the case of the usual restricted Grassmannian, i.e. $A = 1$, $\mathcal{H} = \mathcal{L}$, is a property equivalent to the definition given, for instance, in [29]. A compatible subspace $\mathcal{S}$ belongs to the *p-restricted compatible Grassmannian* $Gr_{res,p} = Gr^A_{res,p,Q_0}(\mathcal{H})$ if

$$Q_{\mathcal{S}} - Q_0 \in \mathcal{B}_p(\mathcal{H}).$$

We may think of $Gr_{res,p}$ as the set of all A-symmetric projections satisfying the above equation. We endow $Gr_{res,p}$ with the topology defined by $(Q_1, Q_2) \mapsto \|Q_1 - Q_2\|_p$. Note that $Gr_{res,p}$ is smoothly acted on by the group $\mathbb{G}_{p,A}$. Indeed, if $Q \in Gr_{res,p}$ and $G \in \mathbb{G}_{p,A}$, it is clear that $GQG^{-1} \in Gr_A$. Moreover, since $G = 1+K$ and $G^{-1} = 1 + K'$ with $K, K' \in \mathcal{B}_p(\mathcal{H})$,

$$GQG^{-1} - Q_0 = (1+K)Q(1+K') - Q_0 = KQ + QK' + KQK' + (Q - Q_0) \in \mathcal{B}_p(\mathcal{H}).$$

As with the usual restricted Grassmannian, the restricted compatible Grassmannian $Gr_{res,p}$ is not connected. We shall see that the connected components are parametrized by the integers.

Remark 6.1. Recall that $\rho_{\mathcal{H}}$ denotes the spectral radius in $\mathcal{H}$. Note that the exponential map between the open sets (in the p-norm)

$$\{X \in \mathfrak{g}_p\colon\ \rho_{\mathcal{H}}(X) < \pi\} \to \{G \in \mathbb{G}_{p,A}\colon\ -1 \notin \sigma_{\mathcal{H}}(G)\}$$

is an analytic diffeomorphism. To prove this assertion, first note that if $X \in \mathfrak{g}_p$ and $\rho_{\mathcal{H}}(X) < \pi$, then $-1 \notin \sigma_{\mathcal{H}}(e^X)$. Suppose that $e^X = e^Y$ with $X, Y \in \mathfrak{g}_p$ and X, Y as above. According to Theorem 2.1, their extensions $\bar{X}, \bar{Y}$ to $\mathcal{L}$ verify that $\|\bar{X}\|_{\mathcal{B}(\mathcal{L})} = \rho_{\mathcal{L}}(\bar{X}) \leqslant \rho_{\mathcal{H}}(X) < \pi$, and $\|\bar{Y}\|_{\mathcal{B}(\mathcal{L})} < \pi$. On the other hand, $e^{\bar{X}}, e^{\bar{Y}}$ are unitary operators in $\mathcal{L}$ whose restrictions to $\mathcal{H}$, namely e^X, e^Y, coincide. It follows that $e^{\bar{X}} = e^{\bar{Y}}$.

The bound on the $\mathcal{B}(\mathcal{L})$ norms implies that $\bar{X} = \bar{Y}$, and therefore $X = Y$. The spectrum of $G \in \mathbb{G}_{p,A}$ lies in the unit circle and consists (with the possible exception of $1 \in \sigma_{\mathcal{H}}(G)$) of eigenvalues with finite multiplicity. Thus if $-1 \notin \sigma_{\mathcal{H}}(G)$, one can define the inverse of the exponential, by means of the usual determination of the logarithm (which is singular in the real negative axis, that does not intersect $\sigma_{\mathcal{H}}(G)$). This map is clearly analytic in $1 + \mathcal{B}_p(\mathcal{H})$. Apparently, this logarithm takes values in $\{X \in \mathfrak{g}_p\colon\ \rho_{\mathcal{H}}(X) < \pi\}$.

Given $Q_1 \in Gr_{res,p}$, we denote by $(Gr_{res,p})_{Q_1}$ the connected component of Q_1 in the restricted compatible Grassmannian.

Proposition 6.2. *The following assertions hold*:

i) *The action of* $\mathbb{G}_{p,A}$ *on* $(Gr_{res,p})_{Q_1}$ *is transitive.*
ii) *The map*

$$\mathbb{G}_{p,A} \to (Gr_{res,p})_{Q_1} \subseteq Q_1 + \mathcal{B}_p(\mathcal{H}), \qquad G \mapsto GQ_1G^{-1}$$

has continuous local cross sections, when both spaces are regarded with the p-norm.

Proof. i) Since $\mathbb{G}_{p,A}$ is connected, $\mathbb{G}_{p,A} \cdot Q_1$ is contained in the component of Q_1 in $Gr_{res,p}$. We first prove that $\mathbb{G}_{p,A} \cdot Q_1$ is open and closed in $Gr_{res,p}$. To this end, note that Q_1 is an interior point of $Gr_{res,p}$. Let $Q \in Gr_{res,p}$ such that $\|Q - Q_1\|_p < r_{Q_1}$, where r_{Q_1} was defined in Section 4. Since $\|Q - Q_1\| \leqslant \|Q - Q_1\|_p < r_{Q_1}$, there exists an A-anti-symmetric operator X_Q which is co-diagonal with respect to Q_1 and such that $Q = e^{X_Q} Q_1 e^{-X_Q}$.

We claim that $X_Q \in \mathcal{B}_p(\mathcal{H})$: note that $Q, Q_1 \in Gr_{res,p}$ implies that $Q - Q_1 \in \mathcal{B}_p(\mathcal{H})$. Then

$$\epsilon_Q - \epsilon_{Q_1} = 2(Q - Q_1) \in \mathcal{B}_p(\mathcal{H}).$$

On the other hand, by construction,

$$\epsilon_Q - \epsilon_{Q_1} = e^{2X_Q}\epsilon_{Q_1} - \epsilon_{Q_1} = \left(e^{2X_Q} - 1\right)\epsilon_{Q_1}.$$

It follows that $e^{2X_Q} - 1 \in \mathcal{B}_p(\mathcal{H})$, that is, $e^{2X_Q} \in \mathbb{G}_{p,A}$. Since $\rho_{\mathcal{H}}(e^{2X_Q} - 1) \leqslant \|e^{2X_Q} - 1\| < 1$, then $-1 \notin \sigma_{\mathcal{H}}(e^{2X_Q})$. By Remark 6.1 we get that $X_Q \in \mathcal{B}_p(\mathcal{H})$, and our claim is proved. Hence we obtain that $Q = e^{X_Q} Q_1 e^{-X_Q} \in \mathbb{G}_{p,A} \cdot Q_1$.

Let $G_0 Q_1 G_0^{-1}$ be any other element in this orbit, with $G_0 = 1 + K_0$, $G_0^{-1} = 1 + K_0'$, and $K_0, K_0' \in \mathcal{B}_p(\mathcal{H})$. We are going to show that it is also an interior point of the restricted Grassmannian. If $Q \in Gr_{res,p}$ satisfies

$$\left\|Q - G_0 Q_1 G_0^{-1}\right\|_p < \frac{r_{Q_1}}{\|G_0\|\|G_0^{-1}\|},$$

then

$$\left\|G_0^{-1} Q G_0 - Q_1\right\|_p = \left\|G_0^{-1}\left(Q - G_0 Q_1 G_0^{-1}\right) G_0\right\|_p \leqslant \left\|G_0^{-1}\right\| \left\|Q - G_0 Q_1 G_0^{-1}\right\|_p \|G_0\| < r_{Q_1}.$$

It follows that $G_0^{-1} Q G_0 = G^{-1} Q_1 G$ for some $G \in \mathbb{G}_{p,A}$, and we thus get $Q \in \mathbb{G}_{p,A} \cdot Q_1$. Thus the orbit $\mathbb{G}_{p,A} \cdot Q_1$ is open in the connected component of Q_1 in $Gr_{res,p}$. Note that this assertion is valid for any element Q of this component. Then the complement of $\mathbb{G}_{p,A} \cdot Q_1$ inside this component, which is a union of disjoint and open orbits, is itself open. Thus $\mathbb{G}_{p,A} \cdot Q_1$ is also closed in $(Gr_{res,p})_{Q_1}$, and therefore $\mathbb{G}_{p,A} \cdot Q_1 = (Gr_{res,p})_{Q_1}$.

ii) We will only define a cross section in a neighborhood of Q_1. A standard translation procedure following the idea of i) can be used to construct a local cross section at any other point of the connected component $(Gr_{res,p})_{Q_1}$.

Let $Q \in Gr_{res,p}$ such that $\|Q - Q_1\|_p < r_{Q_1}$. The operator $X_Q \in \mathfrak{g}_p$ that we have just defined in i) is actually given by the usual logarithm series

$$X_Q = \frac{1}{2}\log\bigl((2Q_1 - 1)(2Q - 1)\bigr) = \frac{1}{2}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}\bigl((2Q_1 - 1)(2Q - 1) - 1\bigr)^n,$$

which is convergent in the p-norm because $r_{Q_1} < 1$. Since the series is the uniform limit of the partial sums, which are continuous functions in the p-norm, we can conclude that the map

$$\bigl\{Q \in Gr_{res,p}\colon\ \|Q - Q_1\|_p < r_{Q_1}\bigr\} \ni Q \mapsto X_Q \in \mathfrak{g}_p$$

is continuous in the p-norm. □

Let us characterize the connected components of $Gr_{res,p}$ in terms of the Fredholm index, as with the usual restricted Grassmannian. To this effect, several results in the paper [4] by J. Avron, R. Seiler and B. Simon will be useful. First and foremost, the notion of index for a pair of orthogonal projections.

Note the following fact. If $Q_1, Q_2 \in Gr_{res,p}$, their self-adjoints extensions $\bar{Q}_1$ and $\bar{Q}_2$ verify that $\bar{Q}_1 - \bar{Q}_2 \in \mathcal{B}_p(\mathcal{L})$. Indeed, by Theorem 2.1, $\bar{Q}_1 - \bar{Q}_2$ are self-adjoint compact operators, whose eigenvalues coincide with the eigenvalues of $Q_1 - Q_2$. Moreover, the absolute values of these eigenvalues, by a classical result [23], are bounded by the singular values of $Q_1 - Q_2$. It follows that $\bar{Q}_1 - \bar{Q}_2 \in \mathcal{B}_p(\mathcal{L})$. Therefore, $\bar{Q}_1, \bar{Q}_2$ belong to the classical restricted Grassmannian in the Hilbert space $\mathcal{L}$, given by the orthogonal polarization

$$\mathcal{L} = R(\bar{Q}_0) \oplus N(\bar{Q}_0).$$

In particular, this implies that

$$\bar{Q}_2\bar{Q}_1|_{R(\bar{Q}_1)} : R(\bar{Q}_1) \to R(\bar{Q}_2)$$

are Fredholm operators. According to [4], the index of this operator is the index $index(\bar{Q}_1, \bar{Q}_2)$ of the pair $\bar{Q}_1, \bar{Q}_2$.

Theorem 6.3. *Let $Q_1, Q_2 \in Gr_{res,p}$. Then Q_1 and Q_2 belong to the same connected component if and only if $index(\bar{Q}_1, \bar{Q}_0) = index(\bar{Q}_2, \bar{Q}_0)$.*

Proof. Let $Q(t)$, $t \in [t_1, t_2]$, be a continuous path in $Gr_{res,p}$ such that $Q(t_1) = Q_1$ and $Q(t_2) = Q_2$. The inequality

$$\bigl\|\bar{Q}(t) - \bar{Q}(s)\bigr\|_{\mathcal{B}(\mathcal{L})} \leqslant \bigl\|Q(t) - Q(s)\bigr\|,$$

implies that the path $\bar{Q}(t)$ is continuous in the restricted Grassmannian of $\mathcal{L}$. This implies that $index(\bar{Q}_1, \bar{Q}_0) = index(\bar{Q}_2, \bar{Q}_0)$.

Conversely, suppose that $index(\bar{Q}_1, \bar{Q}_0) = index(\bar{Q}_2, \bar{Q}_0)$. By Theorem 3.4 in [4], this implies that

$$index(\bar{Q}_1, \bar{Q}_2) = index(\bar{Q}_1, \bar{Q}_0) + index(\bar{Q}_0, \bar{Q}_2) = index(\bar{Q}_1, \bar{Q}_0) - index(\bar{Q}_2, \bar{Q}_0) = 0.$$

Since $Q_1 - Q_2$ is compact, the eigenspaces corresponding to the nonzero eigenvalues of $Q_1 - Q_2$ are finite dimensional. By Theorem 2.1, we know that

$$\ker(\bar{Q}_1 - \bar{Q}_2 - 1) = \ker(Q_1 - Q_2 - 1) \quad \text{and} \quad \ker(\bar{Q}_1 - \bar{Q}_2 + 1) = \ker(Q_1 - Q_2 + 1).$$

As remarked in [4], the hypothesis $index(\bar{Q}_1, \bar{Q}_2) = 0$ implies that

$$\dim \ker(\bar{Q}_1 - \bar{Q}_2 - 1) = \dim \ker(\bar{Q}_1 - \bar{Q}_2 + 1).$$

Let W_0 be a unitary transformation (for the inner product $[\cdot\,,\cdot]$ of $\mathcal{L}$) from $\ker(\bar{Q}_1 - \bar{Q}_2 - 1)$ onto $\ker(\bar{Q}_1 - \bar{Q}_2 + 1)$. Let $\mathcal{N}$ be the subspace

$$\mathcal{N} = \ker(\bar{Q}_1 - \bar{Q}_2 - 1) \oplus \ker(\bar{Q}_1 - \bar{Q}_2 + 1) \subset \mathcal{H},$$

and $U_0 : \mathcal{N} \to \mathcal{N}$ be the unitary operator

$$U_0(f \oplus g) = W_0^{-1} g \oplus W_0 f.$$

Since $\mathcal{N}$ is finite dimensional, it has a supplement $\mathcal{H}_0$ which is orthogonal for the inner product $[\cdot\,,\cdot]$ of $\mathcal{L}$ (see Proposition 3.5):

$$\mathcal{N} \oplus \mathcal{H}_0 = \mathcal{H} \quad \text{and} \quad \mathcal{N} \oplus \overline{\mathcal{H}}_0 = \mathcal{L}.$$

Straightforward computations show that

$$Q_1\big(\ker(Q_1 - Q_2 - 1)\big) \subset \mathcal{N} \text{ and } Q_1\big(\ker(Q_1 - Q_2 + 1)\big) \subset \ker(Q_1 - Q_2 - 1),$$

and therefore $Q_1(\mathcal{N}) \subset \mathcal{N}$. Similarly, $Q_2(\mathcal{N}) \subset \mathcal{N}$. Therefore Q_1 and Q_2 leave $\mathcal{H}_0$ invariant. To complete the proof, we have to construct an invertible operator G_0 on $\mathcal{H}_0$, intertwining $Q_1|_{\mathcal{H}_0}$ and $Q_2|_{\mathcal{H}_0}$, such that G_0 is isometric for the inner product $[\cdot\,,\cdot]$ of $\mathcal{L}$ and belongs to $1 + \mathcal{B}_p(\mathcal{H}_0)$. Clearly, this would imply that $U_0 \oplus G_0$ belongs to $\mathbb{G}_{p,A}$ and intertwines Q_1 and Q_2.

Let $B = 1 - Q_1 - Q_2$.

Clearly B is reduced by the decomposition $\mathcal{N} \oplus \mathcal{H}_0 = \mathcal{H}$. The extension $\bar{B}$ of B to $\mathcal{L}$ is invertible (see [4]). Indeed, since $B^2 = 1 - (Q_1 - Q_2)^2$, the spectrum of $\bar{B}^2$ consists of 1 and eigenvalues of finite multiplicity, corresponding to the squares of the nonzero eigenvalues of $Q_1 - Q_2$. Since the eigenvalues 1 and -1 have been erased in $\mathcal{H}_0$, it follows that B is invertible in $\mathcal{H}_0$. Set

$$S = Q_2 Q_1 + (1 - Q_2)(1 - Q_1).$$

Note that S also is reduced by $\mathcal{H}_0 \oplus \mathcal{N} = \mathcal{H}$. Denote by $S_0 = S|_{\mathcal{H}_0}$. It is easily seen that

$$S = (1 - 2Q_2)B = B(1 - 2Q_1),$$

which implies that S_0 is invertible in $\mathcal{H}_0$. Moreover,

$$S = 1 + 2Q_2Q_1 - Q_1 - Q_2 = 1 + Q_2(Q_1 - Q_2) + (Q_2 - Q_1)Q_1 \in 1 + \mathcal{B}_p(\mathcal{H}),$$

and then $S_0 \in 1_{\mathcal{H}_0} + \mathcal{B}_p(\mathcal{H}_0)$. Let us obtain from S_0 an $\mathcal{L}$-isometry in $1_{\mathcal{H}_0} + \mathcal{B}_p(\mathcal{H}_0)$, intertwining Q_1 and Q_2. Let $S_0' = Q_1Q_2 + (1 - Q_1)(1 - Q_2)$ acting on $\mathcal{H}_0$. Clearly it is invertible, and intertwines Q_2 with Q_1. Moreover, $\bar{S}_0' = \bar{S}_0^{+}$ (as in Theorem 2.3). Put

$$R = S_0' S_0 = Q_1 Q_2 Q_1 + (1 - Q_1)(1 - Q_2)(1 - Q_1),$$

which is invertible and commutes with Q_1. Moreover, it is symmetrizable on $\mathcal{H}_0$, that is, $R^* P_{\mathcal{H}_0} A P_{\mathcal{H}_0} = P_{\mathcal{H}_0} A P_{\mathcal{H}_0} R$. Also note that the extension $\bar{R} = |\bar{S}_0|^2$ is positive and invertible on $\overline{\mathcal{H}}_0$. The spectrum of R consists of finite multiplicity positive eigenvalues and the scalar 1. The element R is of the form $1 + \mathcal{B}_p(\mathcal{H}_0)$, therefore it is an invertible element in the Banach algebra $\mathbb{C} + \mathcal{B}_p(\mathcal{H}_0)$ (endowed with the norm $|z1 + K| = (|z|^p + \|K\|_p^p)^{1/p}$). Let Γ be a path in $\mathbb{C}$, which contains the spectrum $\sigma_{\mathcal{H}_0}(R)$ in its interior, leaves 0 outside, and is symmetric with respect to the x-axis. Consider the invertible element $T \in \mathcal{B}(\mathcal{H}_0)$ given by the Riesz integral

$$T = \frac{1}{2\pi i} \int_{\Gamma} e^{\frac{1}{2}\log(z)} (z\,1 - R)^{-1}\, dz,$$

where $\log(z)$ is the usual determination of the complex logarithm. Then

$$T^* P_{\mathcal{H}_0} A P_{\mathcal{H}_0} = -\frac{1}{2\pi i} \int_{\bar{\Gamma}} e^{\frac{1}{2}\log(\bar{z})} \left(\bar{z}\,1 - R^*\right)^{-1} P_{\mathcal{H}_0} A P_{\mathcal{H}_0}\, d\bar{z}.$$

Since R is symmetrizable, for any integer power $k \geqslant 0$,

$$\left(R^*\right)^k P_{\mathcal{H}_0} A P_{\mathcal{H}_0} = P_{\mathcal{H}_0} A P_{\mathcal{H}_0} R^k.$$

Therefore

$$\left(\bar{z}\,1 - R^*\right)^{-1} P_{\mathcal{H}_0} A P_{\mathcal{H}_0} = P_{\mathcal{H}_0} A P_{\mathcal{H}_0} (\bar{z}\,1 - R)^{-1},$$

and thus

$$T^* P_{\mathcal{H}_0} A P_{\mathcal{H}_0} = -P_{\mathcal{H}_0} A P_{\mathcal{H}_0} \frac{1}{2\pi i} \int_{\bar{\Gamma}} e^{\frac{1}{2}\log(\bar{z})} (\bar{z}\,1 - R)^{-1}\, d\bar{z}.$$

Since Γ is symmetric with respect to the x-axis, if $z(t)$, $t \in I$ is a counterclockwise parametrization of Γ, then $\bar{z}(t)$, $t \in I$ is a clockwise reparametrization of the same path. Thus,

$$T^* P_{\mathcal{H}_0} A P_{\mathcal{H}_0} = P_{\mathcal{H}_0} A P_{\mathcal{H}_0} T,$$

i.e. T is a symmetrizable operator in $\mathcal{H}_0$, and it is of the form $w1 + \mathcal{B}_p(\mathcal{H}_0)$. Also it is apparent that T commutes with Q_1, because R does. The extension of T to $\mathcal{L}$ is the square root of $\bar{R}$, and therefore $w = 1$, i.e. $R \in 1 + \mathcal{B}_p(\mathcal{H}_0)$. Then $G_0 = S_0 T^{-1}$ is an invertible element acting in $\mathcal{H}_0$, which induces an invertible element in $\overline{\mathcal{H}}_0$, given by

$$\bar{G} = \bar{S}_0 \bar{T}^{-1} = \bar{S}_0 |\bar{S}_0|^{-1},$$

which is a unitary operator in $\overline{\mathcal{H}}_0$. Clearly, G_0 intertwines $Q_1|_{\mathcal{H}_0}$ and $Q_2|_{\mathcal{H}_0}$, and the proof is complete. □

There is an analogue of Proposition 4.2 in this context, which will allow us to prove the local regularity of the component $(Gr_{res,p})_{Q_1}$. Recall that δ_{Q_1} is defined by $\delta_{Q_1}(X) = XQ_1 - Q_1 X$. Note that if $X \in \mathfrak{g}_p$, then $\delta_{Q_1}(X) \in i\mathfrak{g}_p$. Indeed, it was already seen that it belongs to $\mathcal{B}_s^A(\mathcal{H})$, and it is apparent that $\delta_{Q_1}(X) \in \mathcal{B}_p(\mathcal{H})$, if $X \in \mathcal{B}_p(\mathcal{H})$.

Lemma 6.4. *The map*

$$\delta_{Q_1}|_{\mathfrak{g}_p}: \mathfrak{g}_p \to i\mathfrak{g}_p, \qquad \delta_{Q_1}(X) = XQ_1 - Q_1X$$

has complemented range and kernel.

Proof. Note that by a similar argument as above, $\delta_{Q_1}(i\mathfrak{g}_p) \subset \mathfrak{g}_p$. To avoid confusion, let us denote by δ_1 the map from $\mathfrak{g}_p$ to $i\mathfrak{g}_p$, and by δ_2 the map from $i\mathfrak{g}_p$ to $\mathfrak{g}_p$, both induced by restricting δ_{Q_1}. Note that

$$\delta_1\delta_2\delta_1 = \delta_1 \quad \text{and} \quad \delta_2\delta_1\delta_2 = \delta_2.$$

Indeed, if $X \in \mathcal{B}(\mathcal{H})$, an elementary computation shows that

$$\delta_{Q_1}^3(X) = XQ_1 - Q_1X = \delta_{Q_1}(X),$$

and the claim follows. These formulas imply that δ_2 is a pseudo-inverse for δ_1. Therefore $\delta_2\delta_1$ is an idempotent operator acting in $\mathfrak{g}_p$, whose kernel clearly contains the kernel of δ_1. On the other hand, if X lies in the kernel of $\delta_2\delta_1$, then apparently $0 = \delta_1\delta_2\delta_1(X) = \delta_1(X)$, i.e. $\ker(\delta_2\delta_1) = \ker(\delta_1)$, and therefore $\ker(\delta_1)$ is complemented in $\mathfrak{g}_p$.

Similarly, $\delta_1\delta_2$ is an idempotent operator in $i\mathfrak{g}_p$, whose range is contained in the range of δ_2. If $Y = \delta_1(X)$ for some $X \in \mathfrak{g}_p$, then

$$\delta_1\delta_2(Y) = \delta_1\delta_2\delta_1(X) = \delta_1(X) = Y,$$

i.e. $R(\delta_1\delta_2) = R(\delta_1)$, and therefore it is complemented in $i\mathfrak{g}_p$. □

Therefore Lemma 4.1 applies:

Corollary 6.5. *The component $(Gr_{res,p})_{Q_1}$ of the compatible restricted Grassmannian is a complemented submanifold of $Q_0 + i\mathfrak{g}_p$ and the map*

$$\mathbb{G}_{A,p} \to (Gr_{res,p})_{Q_1}, \qquad G \mapsto GQ_1G^{-1}$$

is a C^∞ submersion.

There is yet another consequence of the fact that the logarithm $Q \mapsto X_Q$ takes values in $\mathfrak{g}_p$ if $\|Q - Q_1\|_p < r_{Q_1}$:

Corollary 6.6. *Let Q be an element in the component of Q_1 in $Gr_{res,p}$. If $\|Q - Q_1\|_p < r_{Q_1}$, then the unique geodesic $\delta(t) = e^{tX_Q}Q_1e^{-tX_Q}$ of the full compatible Grassmannian Gr_A which joins Q and Q_1, lies in fact inside the restricted Grassmannian $Gr_{res,p}$.*

7. Finsler metrics in Gr_A

We shall endow the tangent spaces of Gr_A with a continuous Finsler metric. Recall first that the tangent space of Gr_A at Q_0 is given by

$$(TGr_A)_{Q_0} = \{XQ_0 - Q_0X\colon\ X \in \mathcal{B}_{as}^A(\mathcal{H})\}.$$

Note that the elements of the group $\mathbb{G}_A$ are isometries for the $\mathcal{L}$ inner product. Moreover, the action is isometric if one chooses for $(TGr_A)_Q$ the norm of $\mathcal{B}(\mathcal{L})$. Namely, for $v \in (TGr_A)_Q$, put

$$|v|_Q = \|\bar{v}\|_{\mathcal{B}(\mathcal{L})}.$$

Then if $v = XQ_0 - Q_0X \in (TGr_A)_{Q_0}$ and $G \in \mathbb{G}_A$

$$|G \cdot v|_{G \cdot Q} = \left\|G(XQ - QX)G^{-1}\right\|_{\mathcal{B}(\mathcal{L})} = \|XQ - QX\|_{\mathcal{B}(\mathcal{L})} = |v|_Q.$$

We denote the length functional induced by this metric with

$$L_{Gr_A}(\gamma) = \int_0^1 \|\dot{\gamma}\|_{\mathcal{B}(\mathcal{L})}\, dt,$$

where $\gamma : [0,1] \to Gr_A$ is a piecewise smooth curve. The rectifiable distance associated to this metric is defined as usual, i.e.

$$d(Q_0, Q_1) = \inf\left\{L_{Gr_A}(\gamma)\colon\ \gamma \text{ is a piecewise smooth curve in } Gr_A \text{ joining } Q_0 \text{ and } Q_1\right\}.$$

Remark 7.1. It is easily seen that this rectifiable distance defines a topology weaker than the operator norm of $\mathcal{B}(\mathcal{H})$. On the other hand, the metric space (Gr_A, d) is not complete. In order to show this latter fact, pick a vector $f \in \mathcal{L} \backslash \mathcal{H}$ such that $\|f\|_A = 1$. Then there exists a sequence of vectors $(f_n)_n$ in $\mathcal{H}$ such that $\|f\|_A = 1$ and $\|f_n - f\|_A \to 0$. Next consider the rank one operators defined by $(f_n \otimes f_n)(h) = [h, f_n]f_n$, $h \in \mathcal{H}$. Clearly, $f_n \otimes f_n \in Gr_A$.

Given $n, m \geqslant 1$, we can find an operator $G_{n,m} \in \mathbb{G}_A$ such that $G_{n,m}(f_n \otimes f_n)G_{n,m}^{-1} = f_m \otimes f_m$. Indeed, $G_{n,m}$ can be chosen satisfying $\operatorname{rank}(G_{n,m} - 1) \leqslant 2$ (see [8, Lemma 3.4]). According to [1, Proposition 4.5] the operators $G_{n,m}$ have logarithms $X_{n,m}$ of finite rank in $\mathcal{B}_{as}^A(\mathcal{H})$. Consider the curves $\gamma_{n,m}(t) = e^{tX_{n,m}}(f_n \otimes f_n)e^{-tX_{n,m}}$, then

$$\begin{aligned} d(f_n \otimes f_n, f_m \otimes f_m) &\leqslant L_{Gr_A}(\gamma_{n,m}) \\ &\leqslant \left\|X_{n,m}(f_n \otimes f_n) - (f_n \otimes f_n)X_{n,m}\right\|_{\mathcal{B}(\mathcal{L})} \\ &\leqslant 2\|X_{n,m}\|_{\mathcal{B}(\mathcal{L})} \longrightarrow_{n,m\to\infty} 0, \end{aligned}$$

where this convergence to zero is due to the well-known fact that orthogonal projections in $\mathcal{B}(\mathcal{L})$ have local continuous cross sections and $\|f_n \otimes f_n - f \otimes f\|_{\mathcal{B}(\mathcal{L})} \to 0$. Hence we have proved that $(f_n \otimes f_n)_n$ is a Cauchy sequence in (Gr_A, d).

Now we will show that $(f_n \otimes f_n)_n$ does not converge in (Gr_A, d). Suppose that there is $Q_0 \in Gr_A$ such that $d(f_n \otimes f_n, Q_0) \to 0$. Since straight lines are shortest paths in any vector space,

$$\|\bar{Q}_0 - f_n \otimes f_n\|_{\mathcal{B}(\mathcal{L})} \leqslant \int_0^1 \|\bar{\dot{\gamma}}\|_{\mathcal{B}(\mathcal{L})}\, dt$$

for any piecewise curve $\gamma : [0,1] \to Gr_A$ joining $\bar{Q}_0$ and $f_n \otimes f_n$. Therefore

$$\|\bar{Q}_0 - f_n \otimes f_n\|_{\mathcal{B}(\mathcal{L})} \leqslant d(Q_0, f_n \otimes f_n) \to 0.$$

Thus we get that $\bar{Q}_0 = f \otimes f$. By our choice of the vector f, it follows that $\bar{Q}_0$ does not leave $\mathcal{H}$ invariant. This contradicts our assumption that $Q_0 \in Gr_A$.

With this choice for the metric one can prove that the geodesics in Corollary 4.7 are curves of minimal length in Gr_A.

Proposition 7.2. *Let $Q_0, Q_1 \in Gr_A$ such that $\|Q_0 - Q_1\| < r_{Q_0}$. Then the unique geodesic of $\mathcal{Q}(\mathcal{H})$ which joins them, and lies inside Gr_A, has minimal length among all possible piecewise smooth curves in Gr_A joining Q_0 and Q_1.*

Proof. Let $\gamma(t)$, $t \in I$, be a piecewise smooth curve in Gr_A. Then $\bar{\gamma}(t)$, which is a curve in the set of orthogonal projections in $\mathcal{B}(\mathcal{L})$, is also piecewise smooth. Indeed, the extension map

$$\mathcal{B}_s^A(\mathcal{H}) \to \mathcal{B}(\mathcal{L}), \qquad X \mapsto \bar{X}$$

is contractive (see Remark 2.2). Note that by Corollary 4.7, the condition $\|Q_0 - Q_1\| < r_{Q_0}$ implies the existence of the geodesic $\delta(t) = e^{tZ_{Q_0}} Q_0 e^{-tZ_{Q_0}}$ joining Q_0 and Q_1. The extension $\bar{\delta}$ of this curve is given by

$$\bar{\delta}(t) = e^{t\bar{Z}_{Q_0}} \bar{Q}_0 e^{-t\bar{Z}_{Q_0}} = e^{tZ_{\bar{Q}_0}} \bar{Q}_0 e^{-tZ_{\bar{Q}_0}},$$

which is the unique geodesic joining $\bar{Q}_0$ and $\bar{Q}_1$ in the manifold of symmetric projections in $\mathcal{B}(\mathcal{L})$, and $\|\bar{Z}_{Q_0}\|_{\mathcal{B}(\mathcal{L})} \leqslant \pi/6 < \pi/2$. Then by the result in [27],

$$length(\bar{\delta}) \leqslant length(\bar{\gamma}),$$

where the lengths are measured with the usual operator norm of $\mathcal{B}(\mathcal{L})$. On the other hand, by the definition of the metric in Gr_A, it follows that

$$L_{Gr_A}(\gamma) = \int_I \|\bar{\dot{\gamma}}(t)\|_{\mathcal{B}(\mathcal{L})}\, dt = \int_I \|\dot{\bar{\gamma}}(t)\|_{\mathcal{B}(\mathcal{L})}\, dt = length(\bar{\gamma}),$$

and the same holds true for δ. This completes the proof. □

A natural Finsler metric can also be defined in the restricted compatible Grassmannian $Gr_{res,p}$. As seen in the last section, the connected components are homogeneous spaces of the Banach–Lie group $\mathbb{G}_{A,p}$. Let us introduce a metric which is invariant for the action of this group. Namely, a tangent vector v to the component of Q_1 of $Gr_{res,p}$ at Q, is the velocity vector $v = \dot{\alpha}(0)$ of a curve which takes values in the A-symmetric part of $Q_1 + \mathcal{B}_p(\mathcal{H})$. Therefore tangent vectors are A-symmetric operators in $\mathcal{B}_p(\mathcal{H})$. Then put

$$|v|_Q = \|\bar{v}\|_p,$$

the Schatten p-norm of the extension $\bar{v} \in \mathcal{B}_p(\mathcal{L})$. Associated with this metric there is a non-complete rectifiable distance (same proof as Remark 7.1). Elements G in $\mathbb{G}_{A,p}$ extend to unitary operators $\bar{G}$ in $\mathcal{L}$, which due to the results of M.G. Krein and P.D. Lax, are of the form $1 + \mathcal{B}_p(\mathcal{L})$. Let $\gamma(t)$ be a smooth curve in $Gr_{res,p}$ with $\gamma(0) = Q_1$. Then $\bar{\gamma}$ is a curve in the restricted Grassmannian of $\mathcal{L}$ given by the polarization $\mathcal{L} = R(\bar{Q}_0) \oplus N(\bar{Q}_0)$. If $\|Q - Q_1\|_p < r_{Q_1}$, by Corollary 6.6, there exists a unique geodesic $\delta(t) = e^{tX_Q} Q_1 e^{-tX_Q}$ such that $\delta(0) = 1$. The assumption $r_{Q_1} < 1$, implies in particular, that $\|\bar{Q} - \bar{Q}_1\|_{\mathcal{B}(\mathcal{L})} \leqslant \|Q - Q_1\|_p < 1$. Therefore by [2, Theorem 5.5], the geodesic $\bar{\delta}$ has minimal length in the restricted Grassmannian of $\mathcal{L}$. Thus we have the following:

Corollary 7.3. *Let $Q, Q_1 \in Gr_{res,p}$ such that $\|Q - Q_1\|_p < r_{Q_1}$. Then the unique geodesic δ of the compatible Grassmannian which joins them (and which remains inside $Gr_{res,p}$), has minimal length for the Finsler metric defined above.*

Appendix A. Smooth structure of $\mathbb{G}_A$

In this section, we show that $\mathbb{G}_A$ is Banach Lie group endowed with the uniform topology. This fact was proven in [1] in the case that $\mathcal{H}$ is a Sobolev space. However, it is not difficult to see that the same proof works in our general setting. We include this result here for the sake of completeness.

Recall that $\mathbb{G}_A$ is the group of A-unitaries, which can also be described as

$$\mathbb{G}_A = \{G \in Gl(\mathcal{H}):\ G^*AG = A\}.$$

The natural candidate to be the Lie algebra of $\mathbb{G}_A$ is the real subspace $\mathcal{B}_{as}^A(\mathcal{H})$ of A-anti-symmetric operators.

Lemma 7.4. *$X \in \mathcal{B}_{as}^A(\mathcal{H})$ if and only if $e^{tX} \in \mathbb{G}_A$ for all $t \in \mathbb{R}$.*

Proof. If $X \in \mathcal{B}_{as}^A(\mathcal{H})$, $(X^*)^k A = (-1)^k A X^k$. Then $(e^{tX})^* A = e^{tX^*} A = A e^{-tX} = A(e^{tX})^{-1}$, i.e. $e^{tX} \in \mathbb{G}_A$. Conversely, if $e^{tX} \in \mathbb{G}_A$ for all t, we may differentiate the identity $e^{tX^*}A = Ae^{-tX}$ at $t = 0$, to obtain $X^*A = -AX$. □

Lemma 7.5. *Let $G \in \mathbb{G}_A$. Let $\mathcal{L}$ be a half-line in the complex plane from 0 to infinity. If $\sigma_{\mathcal{H}}(G) \cap \mathcal{L} = \emptyset$, then there exists $X \in \mathcal{B}_{as}^A(\mathcal{H})$ such that $e^X = G$.*

Proof. Changing G to $e^{i\theta}G$, we can assume that $\mathcal{L}$ is the negative real axis. Since $0 \notin \sigma_{\mathcal{H}}(G)$ and $0 \notin \sigma_{\mathcal{H}}(G^{-1})$, then it is possible to find a simple closed curve γ, which does not intersect $\mathcal{L}$ and contains $\sigma_{\mathcal{H}}(G)$ and $\sigma_{\mathcal{H}}(G^{-1})$ in its interior. In addition, we can choose γ satisfying $\bar{\gamma} = \gamma$. From the assumption $\sigma_{\mathcal{H}}(G) \cap \mathcal{L} = \emptyset$, it follows that $\sigma_{\mathcal{H}}(G^{-1}) \cap \mathcal{L} = \emptyset$ and that there is a well-defined branch of the logarithm, and $X = \log(G)$ can be defined using the Riesz functional calculus. If γ is counterclockwise oriented, then

$$\begin{aligned} X^*A &= -\frac{1}{2\pi i}\int_\gamma \overline{\log(z)}\,(\bar{z} - G^*)^{-1} A\,dz = -\frac{1}{2\pi i}\int_\gamma \log(\bar{z}) A(\bar{z} - G^{-1})^{-1}\,dz \\ &= \frac{1}{2\pi i}\int_{\bar{\gamma}} \log(z) A(z - G^{-1})^{-1}\,dz = A\log(G^{-1}) = -AX. \end{aligned}$$

Hence $X \in \mathcal{B}_{as}^A(\mathcal{H})$, and the proof is complete. □

Proposition 7.6. *The group $\mathbb{G}_A$ is a Banach Lie group endowed with the uniform topology of $\mathcal{B}(\mathcal{H})$. Its Lie algebra is given by the subspace of A-anti-symmetric operators.*

Proof. We first note that $\mathbb{G}_A$ is a closed subgroup of the linear invertible group $Gl(\mathcal{H})$. According to Lemma 7.4, we know that $\mathcal{B}_{as}^A(\mathcal{H}) = \{X \in \mathcal{B}(\mathcal{H}):\ e^{tX} \in \mathbb{G}_A,\ \forall t \in \mathbb{R}\}$. It follows that there is a manifold structure making $\mathbb{G}_A$ into a Banach Lie group with Lie algebra given by $\mathcal{B}_{as}^A(\mathcal{H})$ (see e.g. [5, Corollary 3.7]). In order to see that the manifold topology of $\mathbb{G}_A$ coincides with the topology inherited from $Gl(\mathcal{H})$, we have to find an open neighborhood $\mathcal{U}$ of zero in $\mathcal{B}(\mathcal{H})$ such that the exponential map $\exp : \mathcal{U} \to \exp(\mathcal{U})$ is a diffeomorphism and satisfies $\exp(\mathcal{U} \cap \mathcal{B}_{as}^A(\mathcal{H})) = \exp(\mathcal{U}) \cap \mathbb{G}_A$ (see [5, Proposition 4.4]). To this end, recall that the following restriction of the exponential map is a diffeomorphism:

$$\exp : \mathcal{U} = \left\{X \in \mathcal{B}(\mathcal{H}):\ \sigma_{\mathcal{H}}(X) \subseteq \mathbb{R} + i\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\} \to \{G \in Gl(\mathcal{H}):\ \mathrm{Re}(z) > 0,\ \forall z \in \sigma_{\mathcal{H}}(G)\}.$$

Moreover, the logarithm of any operator $G \in \exp(\mathcal{U}) \cap \mathbb{G}_A$ belongs to $\mathcal{B}_{as}^A(\mathcal{H})$ by Lemma 7.5. This completes the proof. □

Following the same ideas one can prove the corresponding result for the groups

$$\mathbb{G}_{p,A} = \{G \in \mathbb{G}_A\colon\ G - 1 \in \mathcal{B}_p(\mathcal{H})\},$$

where $1 \leqslant p \leqslant \infty$, and $\mathcal{B}_p(\mathcal{H})$ is the p-Schatten ideal.

Corollary 7.7. *The group $\mathbb{G}_{p,A}$ is a Banach Lie group endowed with the p-norm topology. Its Lie algebra is given by $\mathcal{B}_{as}^A(\mathcal{H}) \cap \mathcal{B}_p(\mathcal{H})$.*

Proof. Note that $\mathbb{G}_{p,A}$ is a closed subgroup of the Banach Lie group

$$Gl_p(\mathcal{H}) = \{G \in Gl(\mathcal{H})\colon\ G - 1 \in \mathcal{B}_p(\mathcal{H})\},$$

which has Lie algebra given by $\mathcal{B}_p(\mathcal{H})$. Moreover, its topology is defined by the p-norm (see [5, Proposition 9.28]). To prove our statement about $\mathbb{G}_{p,A}$, it suffices to follow the same argument of Proposition 7.6 with $Gl_p(\mathcal{H})$ in place of $Gl(\mathcal{H})$. □

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