# A GEOMETRIC APPROACH TO DISCRETE CONNECTIONS ON PRINCIPAL BUNDLES

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Abstract. This work revisits, from a geometric perspective, the notion of discrete connection on a principal bundle, introduced by M. Leok, J. Marsden and A. Weinstein. It provides precise definitions of discrete connection, discrete connection form and discrete horizontal lift and studies some of their basic properties and relationships. An existence result for discrete connections on principal bundles equipped with appropriate Riemannian metrics is proved.

## 1. Introduction

The study of symmetries is a central part of many areas of Mathematics and Physics. In the Differential Geometric setting, principal bundles provide a powerful instrument to model many symmetric systems. Connections on principal bundles are very convenient tools, especially in the topologically nontrivial case. Among other things, using connections, the equations of motion of mechanical systems can be written globally and, also, connections capture the complexity of the bundle via, for instance, the associated curvature.

Roughly speaking, mechanical systems are continuous time dynamical systems on the tangent bundle $TQ$ of a manifold $Q$, whose dynamics is defined using a variational principle. In the same spirit, discrete mechanical systems are usually introduced as discrete time dynamical systems on $Q \times Q$ whose dynamics is defined using a variational principle [11]. The main motivation is that, for continuous time, one has velocities (tangent vectors), whereas when the time is discrete, one has pairs of points (close to one another).

Let $G$ be a Lie group acting on $Q$ via the (left) action $l^Q : G \times Q \to Q$ in such a way that the quotient mapping $\pi : Q \to Q/G$ is a principal $G$-bundle. The *vertical bundle* $\mathcal{V}$ is defined, at each $q \in Q$, by $\mathcal{V}(q) := T_q l_G^Q(q) = \{\xi_Q(q) : \xi \in Lie(G)\} \subset T_q Q$, where $l_G^Q(q)$ is the $G$-orbit through $q$ and $\xi_Q(q)$ is the infinitesimal generator

of $l^Q$ at $q$. A *connection* $\mathcal{A}$ on the principal $G$-bundle $\pi$ is a $G$-equivariant distribution $Hor_{\mathcal{A}}$ on $Q$ complementing $\mathcal{V}$. That is, at each $q \in Q$, there is a subspace $Hor_{\mathcal{A}}(q) \subset T_qQ$ such that, for each $v_q \in T_qQ$, there is a unique decomposition

$$(1.1) \qquad v_q = \underbrace{\xi_Q(q)}_{\in \mathcal{V}(q)} + \underbrace{v_q - \xi_Q(q)}_{\in Hor_{\mathcal{A}}(q)}.$$

for some $\xi \in \mathfrak{g} := Lie(G)$.

Trying to extend techniques that were used to analyze the reduction of symmetric mechanical systems ([2] and [1]) to the case of discrete mechanical systems, M. Leok, J. Marsden and A. Weinstein introduced in [9] and [8] a notion of *discrete connection* on a principal bundle that mimics the notion of connection. Their definition states that a discrete connection $\mathcal{A}_d$ is a $G$-equivariant subset $Hor_{\mathcal{A}_d} \subset Q \times Q$ that is complementary to the *discrete vertical bundle* $\mathcal{V}_d := \{(q, l_g^Q(q)) \in Q \times Q : g \in G\}$ where, for $g \in G$, $l_g^Q : Q \to Q$ is defined by $l_g^Q(q) := l^Q(g, q)$. Complementary means that every $(q_0, q_1) \in Q \times Q$ can be decomposed uniquely in a vertical part and a horizontal part in such a way that

$$(1.2) \qquad (q_0, q_1) = \underbrace{(q_0, l_g^Q(q_0))}_{\in \mathcal{V}_d} \cdot \underbrace{(q_0, l_{g^{-1}}^Q(q_1))}_{\in Hor_{\mathcal{A}_d}}$$

for some $g \in G$. The composition of vertical and arbitrary pairs (based at the same point $q_0$) is defined by $(q_0, l_g^Q(q_0)) \cdot (q_0, q_1) := (q_0, l_g^Q(q_1))$. The basic intuition is that tangent vectors in $TQ$ become finite curves and, eventually, pairs of points, elements of $Q \times Q$. Vertical vectors based at $q \in Q$ are those tangent vectors pointing in the direction of the group action which, for finite time, leads to pairs of the form $(q, l_g^Q(q))$ for $g \in G$. With this motivation, (1.2) is the discrete analogue of (1.1).

Discrete connections have been successfully used to study the reduction of discrete mechanical systems (see [10], [4], and [3]).

In their work, Leok, Marsden and Weinstein do not provide a thorough definition of discrete connection, although they discuss the equivalence between this notion and other approaches. For instance, they relate a discrete connection to what they call a discrete connection form and also to a discrete horizontal lift. They also give an interpretation in terms of splittings of a certain discrete Atiyah sequence, although the groupoid setting for this very intriguing approach is not detailed. The purpose of the current paper is to give a precise definition of discrete connection on a principal bundle and analyze some of its more basic properties. Additional geometric properties like parallel transport, holonomy and curvature will be discussed elsewhere.

In Section 2 we define discrete connections and study some elements associated to them: domain and slices. We can associate other objects to a discrete connections $\mathcal{A}_d$ that, in turn, can be used to characterize completely $\mathcal{A}_d$. Two such objects are the discrete connection form and the discrete horizontal lift, that are analyzed in Sections 3 and 4 respectively. Last, in Section 5, we prove an existence result for discrete connections on principal bundles equipped with an adequate Riemannian metric.

*Notation:* when $l^Q$ is the left $G$-action on $Q$, $l^{Q \times Q}$ is the induced diagonal $G$-action on $Q \times Q$. The quotient mapping from a space to its space of orbits

is denoted by $\pi$ and $p_j$ is the projection from a Cartesian product onto its $j$-th factor. Given the maps $f_i : X_i \to Y_i$ (for $i = 1, 2$), their Cartesian product is $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ with $(f_1 \times f_2)(x_1, x_2) := (f_1(x_1), f_2(x_2))$.

## 2. Geometric definition

In order to use discrete connections in geometry, it is important that they are well defined and that the objects that we associate to them (discrete connection form, horizontal lift, etc.) be smooth.

**Definition 2.1.** Let $Hor \subset Q \times Q$ be an $l^{Q \times Q}$-invariant submanifold containing the diagonal $\Delta_Q \subset Q \times Q$. We say that $Hor$ defines the *discrete connection* $\mathcal{A}_d$ on the principal bundle $\pi : Q \to Q/G$ if $(id_Q \times \pi)|_{Hor} : Hor \to Q \times (Q/G)$ is an injective local diffeomorphism. We denote $Hor$ by $Hor_{\mathcal{A}_d}$.

**Remark 2.2.** It is possible to consider the slightly more general notion of *affine discrete connection* that replaces the condition $\Delta_Q \subset Hor$ with the requirement that $Hor$ contains the graph of a smooth map $\gamma : Q \to Q$. Discrete connections correspond to the case $\gamma := id_Q$. This more general notion has been used in [4] in order to construct discrete connections associated to nonvanishing conserved discrete momenta of discrete mechanical systems.

The following Lemma, whose proof is straightforward, provides a convenient way to characterize discrete connections.

**Lemma 2.3.** *The requirement that $(id_Q \times \pi)|_{Hor} : Hor \to Q \times (Q/G)$ be an injective local diffeomorphism in Definition 2.1 is equivalent to the following two assertions being true simultaneously.*

  *(1) $(id_Q \times \pi)|_{Hor} : Hor \to Q \times (Q/G)$ is a local diffeomorphism and*
  *(2) $l_g^{Q \times Q_2}(Hor) \cap Hor = \emptyset$ for all $g \neq e$ in $G$. Here $l_g^{Q \times Q_2}(q_0, q_1) := (q_0, l_g^Q(q_1))$.*

Recall that a smooth map $f : X \to Y$ is *transversal* to the submanifold $Z \subset Y$ if $\text{Im}(df(x)) + T_{f(x)}Z = T_{f(x)}Y$, for all $x \in f^{-1}(Z)$; this situation is denoted by $f \pitchfork Z$. When $f \pitchfork Z$ and $f^{-1}(Z) \neq \emptyset$, $f^{-1}(Z) \subset X$ is a regular submanifold (see Theorem on page 28 of [5]). When $Z_1, Z_2 \subset Y$ are submanifolds, they *intersect transversely* when $i_{Z_1} \pitchfork Z_2$, where $i_{Z_1} : Z_1 \to Y$ is the inclusion map; this is denoted by $Z_1 \pitchfork Z_2$. In particular, if not empty, $Z_1 \cap Z_2 \subset Y$ is a submanifold when $Z_1 \pitchfork Z_2$. We refer to [5] for more on transversality. For any $q \in Q$ define the smooth map $i_q : Q \to Q \times Q$ by $i_q(q') = (q, q')$. The next result proves the basic properties of a discrete connection.

**Proposition 2.4.** *Let $Hor_{\mathcal{A}_d}$ be a discrete connection on the principal $G$-bundle $\pi : Q \to Q/G$. Then, the following statements are true.*

  *(1) Let $\mathfrak{U} := l_G^{Q \times Q_2}(Hor_{\mathcal{A}_d}) = \{(q_0, l_g^Q(q_1)) \in Q \times Q : (q_0, q_1) \in Hor_{\mathcal{A}_d}, g \in G\} \subset Q \times Q$, $\mathfrak{U}' := (id_Q \times \pi)(Hor_{\mathcal{A}_d}) \subset Q \times (Q/G)$, and $\mathfrak{U}'' := (\pi \times \pi)(Hor_{\mathcal{A}_d}) \subset (Q/G) \times (Q/G)$. Then $\mathfrak{U}, \mathfrak{U}'$ and $\mathfrak{U}''$ are open subspaces of the corresponding spaces.*
  *(2) $\mathfrak{U}$ is $G \times G$-invariant for the product action on $Q \times Q$. Also, $\mathfrak{U} = (\pi \times \pi)^{-1}(\mathfrak{U}'')$.*
  *(3) For any $q \in Q$, $i_q \pitchfork Hor_{\mathcal{A}_d}$. Furthermore, $Hor^2(q) := i_q^{-1}(Hor_{\mathcal{A}_d}) \subset Q$ is a submanifold of dimension $\dim(Q) - \dim(G)$.*

(4) *For any $q \in Q$, $Hor^2(q) \pitchfork \mathcal{V}_d(q)$, where $\mathcal{V}_d(q) := l_G^Q(q)$. More precisely, $Hor^2(q) \cap \mathcal{V}_d(q) = \{q\}$ and $T_q Hor^2(q) \oplus T_q \mathcal{V}_d(q) = T_q Q$.*

*Proof.* Being $\mathcal{A}_d$ a discrete connection, $\mathfrak{U}'$ is open and, as $\mathfrak{U} := l_G^{Q \times Q_2}(Hor_{\mathcal{A}_d}) = (id_Q \times \pi)^{-1}(\mathfrak{U}')$ with $id_Q \times \pi$ continuous, $\mathfrak{U}$ is also open. Being $\pi$ a fiber bundle, it is an open map and so is $\pi \times \pi$; consequently, $(\pi \times \pi)(\mathfrak{U}) = (\pi \times \pi)(Hor_{\mathcal{A}_d}) = \mathfrak{U}''$ is open, proving part 1. The $G \times G$-invariance of $\mathfrak{U}$ follows from the $G$-invariance of $Hor_{\mathcal{A}_d}$ by direct computation. This $G \times G$-invariance together with $(\pi \times \pi)(\mathfrak{U}) = \mathfrak{U}''$ lead to the proof of point 2.

Let $q' \in i_q^{-1}(Hor_{\mathcal{A}_d})$. For $(v, v') \in T_{(q,q')}(Q \times Q)$, as $d((id_Q \times \pi)|_{Hor_{\mathcal{A}_d}})(q, q')$ is an isomorphism, there is a unique $(\widetilde{v}, \widetilde{v'}) \in T_{(q,q')} Hor_{\mathcal{A}_d}$ such that $d((id_Q \times \pi)|_{Hor_{\mathcal{A}_d}})(q, q')(\widetilde{v}, \widetilde{v'}) = d(id_Q \times \pi)(q, q')(v, v')$. It is easy to check that $\widetilde{v} = v$, so that $(v, v') = (v, \widetilde{v'}) + (0, v' - \widetilde{v'})$, where the first term is in $T_{(q,q')} Hor_{\mathcal{A}_d}$ and the second in $\text{Im}(di_q(q')))$. This proves the transversality condition in point 3. The rest of this point is a consequence of the transversality condition (see Theorem on page 28 of [5]) and the fact that $q \in i_q^{-1}(Hor_{\mathcal{A}_d})$.

By condition 2 in Lemma 2.3, $Hor^2(q) \cap \mathcal{V}_d(q) = \{l_g^Q(q) : (q, l_g^Q(q)) \in Hor_{\mathcal{A}_d}\} = \{q\}$. For $v' \in T_q Q$, as $d((id_Q \times \pi)|_{Hor_{\mathcal{A}_d}})(q, q)$ is an isomorphism, there is a unique $(\widetilde{v}, \widetilde{v'}) \in T_{(q,q)} Hor_{\mathcal{A}_d}$ such that $d((id_Q \times \pi)|_{Hor_{\mathcal{A}_d}})(q, q)(\widetilde{v}, \widetilde{v'}) = d((id_Q \times \pi))(q, q)(0, v')$. Then, $\widetilde{v} = 0$ and $\widetilde{v'} - v' \in \ker(d\pi(q)) = T_q(l_G^Q(q))$. Hence $di_q(q)(v') = (0, v') = (0, \widetilde{v'}) + (0, v' - \widetilde{v'})$, with $(0, \widetilde{v'}) \in T_{(q,q)} Hor_{\mathcal{A}_d}$. Consequently, $v' = \widetilde{v'} + (v' - \widetilde{v'})$ with $v' - \widetilde{v'} \in T_q \mathcal{V}_d(q) = T_q(l_G^Q(q))$ and $\widetilde{v'} \in T_q Hor^2(q) = (di_q(q))^{-1}(T_{(q,q)} Hor_{\mathcal{A}_d})$, proving the transversality part of point 4. The direct sum property follows from the dimensions of the subspaces. $\qquad\square$

Given a discrete connection $\mathcal{A}_d$, the open subset $\mathfrak{U} \subset Q \times Q$ defined in part 1 of Proposition 2.4 will be called the *domain* of $\mathcal{A}_d$. The submanifolds $Hor^2(q) \subset Q$ introduced in part 3 of the same result will be the *horizontal slices*.

**Proposition 2.5.** *Let $\mathcal{A}_d$ be a discrete connection with domain $\mathfrak{U}$ on the principal $G$-bundle $\pi : Q \to Q/G$. For any $(q_0, q_1) \in \mathfrak{U}$, there is a unique $g \in G$ such that (1.2) holds.*

*Proof.* The existence of $g$ follows from the definitions of $\mathfrak{U}$ and the composition $\cdot$. The uniqueness of $g$ is a consequence of $(id_Q \times \pi)|_{Hor_{\mathcal{A}_d}}$ being a injective. $\qquad\square$

## 3. Discrete connection form

A convenient way of describing and using a connection on a principal $G$-bundle $\pi : Q \to Q/G$ is through the associated *connection $1$-form*, that is a Lie algebra valued 1-form $\mathcal{A} : TQ \to \mathfrak{g}$ such that $\mathcal{A}(v_q) = \xi$, where $\xi$ is as in (1.1) (see [7] for more details). In the same spirit, when $\mathcal{A}_d$ is a discrete connection on the same bundle, the element $g \in G$ in (1.2), captures the vertical part of a pair $(q_0, q_1)$ in the sense that "what is left" is horizontal. The next definition makes this notion more precise.

**Definition 3.1.** Given a discrete connection $\mathcal{A}_d$ with domain $\mathfrak{U}$ on the principal $G$-bundle $\pi : Q \to Q/G$, we define its *associated discrete connection form*

$$(3.1) \qquad \mathcal{A}_d : \mathfrak{U} \subset Q \times Q \to G \quad \text{by} \quad \mathcal{A}_d(q_0, q_1) := g,$$

where $g$ is the element of $G$ that appears in the decomposition (1.2).

When $\pi : Q \to Q/G$ is a principal $G$-bundle, the fibered product of $\pi$ with itself —that is, the pairs $(q_0, q_1)$ such that $\pi(q_0) = \pi(q_1)$— is denoted by $Q \times_{\pi \times \pi} Q$. Let $\kappa : Q \times_{\pi \times \pi} Q \to G$ be defined by $\kappa(q_0, q_1) := g$ if and only if $l_g^Q(q_0) = q_1$. It is easy to check that $\kappa$ is a smooth function.

**Lemma 3.2.** *Given a discrete connection $\mathcal{A}_d$ with domain $\mathfrak{U}$ on the principal $G$-bundle $\pi : Q \to Q/G$, its associated discrete connection form $\mathcal{A}_d : \mathfrak{U} \to G$ is smooth.*

*Proof.* The function $\mathcal{A}_d$ is a composition of smooth functions. Indeed, for all $(q_0, q_1) \in \mathfrak{U}$, we have $\mathcal{A}_d(q_0, q_1) = \kappa\big(p_2(((id_Q \times \pi)|_{Hor_{\mathcal{A}_d}})^{-1}((id_Q \times \pi)(q_0, q_1))), q_1\big)$.
$\square$

**Example 3.3.** Let $R$ be a smooth connected manifold and $G$ a Lie group. Define $Q := R \times G$ and consider the left $G$-action on $Q$ defined by $l_g^Q(r, g') := (r, gg')$. This action turns $Q$ into the principal $G$-bundle $p_1 : Q \to R$. Let $\mathcal{U}'' \subset R \times R$ be an open subset containing the diagonal $\Delta_R$ and $C : \mathcal{U}'' \to G$ be a smooth function such that $C(r_0, r_0) = e$ for all $r_0 \in R$. Consider

(3.2)
$$Hor := \{((r_0, g_0), (r_1, g_1)) \in Q \times Q : (r_0, r_1) \in \mathcal{U}'' \text{ and}$$
$$g_0 = g_1 C(r_0, r_1)\} \subset Q \times Q.$$

$Hor \subset Q \times Q$ is a regular submanifold because $Hor$ is the graph of the smooth map $((r_0, g_0), r_1) \mapsto ((r_0, g_0), (r_1, g_0(C(r_0, r_1))^{-1}))$. It is immediate that $Hor$ is $l^{Q \times Q}$-invariant and contains $\Delta_Q$. Furthermore, as $(id_Q \times \pi)|_{Hor} : Hor \to Q \times (Q/G)$ specializes to

$$((r_0, g_0), (r_1, g_0(C(r_0, r_1))^{-1})) \mapsto ((r_0, g_0), r_1)$$

that is a diffeomorphism with inverse $((r_0, g_0), r_1) \mapsto ((r_0, g_0), (r_1, g_0(C(r_0, r_1))^{-1}))$, we conclude that $Hor$ defines a discrete connection $\mathcal{A}_d^C$ on the (trivial) principal $G$-bundle $p_1 : Q \to R$. It is easy to check that the domain of $\mathcal{A}_d^C$ is $\mathfrak{U} = (p_1 \times p_1)^{-1}(\mathcal{U}'') = \{((r_0, g_0), (r_1, g_1)) \in Q \times Q : (r_0, r_1) \in \mathcal{U}''\}$, $\mathfrak{U}' = \{((r_0, g_0), r_1) \in Q \times (Q/G) : (r_0, r_1) \in \mathcal{U}''\}$ and $\mathfrak{U}'' = \mathcal{U}'' \subset R \times R$. In the special case when $\mathcal{U}'' = R \times R$ and $C(r_0, r_1) = e$ for all $r_0, r_1 \in R$, the connection $\mathcal{A}_d^e$ is called the *trivial discrete connection.*

Given $(q_0, q_1) = ((r_0, g_0), (r_1, g_1)) \in \mathfrak{U}$, it is easy to see that $(q_0, l_{g^{-1}}^Q(q_1)) \in Hor_{\mathcal{A}_d^C}$ if and only if $g = g_1 C(r_0, r_1) g_0^{-1}$. Therefore, the associated discrete connection form is

(3.3)
$$\mathcal{A}_d^C((r_0, g_0), (r_1, g_1)) = g_1 C(r_0, r_1) g_0^{-1}.$$

**Theorem 3.4.** *Let $\mathcal{A}_d$ be a discrete connection on the principal $G$-bundle $\pi : Q \to Q/G$ with domain $\mathfrak{U}$. Then, for all $(q_0, q_1) \in \mathfrak{U}$ and $g_0, g_1 \in G$,*

(3.4)
$$\mathcal{A}_d(l_{g_0}^Q(q_0), l_{g_1}^Q(q_1)) = g_1 \mathcal{A}_d(q_0, q_1) g_0^{-1}.$$

*In addition, $Hor_{\mathcal{A}_d} = \{(q_0, q_1) \in \mathfrak{U} : \mathcal{A}_d(q_0, q_1) = e\}$. Conversely, given a smooth function $\mathcal{A} : \mathcal{U} \to G$, where $\mathcal{U} \subset Q \times Q$ is an open subset that contains the diagonal $\Delta_Q \subset Q \times Q$ and is invariant under the product $G \times G$-action on $Q \times Q$, such that (3.4) holds (with $\mathcal{A}_d$ replaced by $\mathcal{A}$) and $\mathcal{A}(q_0, q_0) = e$ for all $q_0 \in Q$, then $Hor := \{(q_0, q_1) \in \mathcal{U} : \mathcal{A}(q_0, q_1) = e\}$ defines a discrete connection whose associated discrete connection form is $\mathcal{A}$.*

*Proof.* For $(q_0, q_1) \in \mathfrak{U}$, let $h := \mathcal{A}_d(q_0, q_1)$. By the $G \times G$-invariance of $\mathfrak{U}$, we have that $(l^Q_{g_0}(q_0), l^Q_{g_1}(q_1)) \in \mathfrak{U}$ for any $g_0, g_1 \in G$; let $\widetilde{h} := \mathcal{A}_d(l^Q_{g_0}(q_0), l^Q_{g_1}(q_1))$. By definition, $(q_0, l^Q_{h^{-1}}(q_1)) \in Hor_{\mathcal{A}_d}$ and by the $G$-invariance of $Hor_{\mathcal{A}_d}$, we see that $(l^Q_{g_0}(q_0), l^Q_{(g_1 h g_0^{-1})^{-1}}(l^Q_{g_1}(q_1))) \in Hor_{\mathcal{A}_d}$. On the other hand, also by definition, $(l^Q_{g_0}(q_0), l^Q_{\widetilde{h}^{-1}}(l^Q_{g_1}(q_1))) \in Hor_{\mathcal{A}_d}$. Using that $(id_Q \times \pi)|_{Hor_{\mathcal{A}_d}}$ is one to one, we conclude that $g_1 h g_0^{-1} = \widetilde{h}$, proving that (3.4) holds.

By Proposition 2.5, the element $g$ appearing in (1.2) is unique, hence $g = e$ characterizes the horizontality of $(q_0, q_1) \in \mathfrak{U}$. This proves that $Hor_{\mathcal{A}_d} = \{(q_0, q_1) \in \mathfrak{U} : \mathcal{A}_d(q_0, q_1) = e\}$.

Conversely, given $\mathcal{U}$, $\mathcal{A}$ and $Hor$ as in the statement, we show that $Hor$ defines a discrete connection. Since $\mathcal{A}(q_0, q_0) = e$ for all $q_0 \in Q$, we have that $\Delta_Q \subset Hor$. It is easy to check explicitly that $d\mathcal{A} : T_{(q_0,q_1)}\mathcal{U} \to T_e G = \mathfrak{g}$ is onto for all $(q_0, q_1) \in Hor$. Hence, $e$ is a regular value of $\mathcal{A}$ and $Hor := \mathcal{A}^{-1}(\{e\}) \subset Q \times Q$ is a submanifold. The $l^{Q \times Q}$-invariance of $Hor$ follows readily using (3.4).

Assume now that $(q_0, l^Q_g(q_1)) \in Hor$ with $(q_0, q_1) \in Hor$. Then $e = \mathcal{A}(q_0, l^q_g(q_1)) = g\mathcal{A}(q_0, q_1) = ge = g$, showing that condition 2 in Lemma 2.3 holds. That $(id_Q \times \pi)|_{Hor}$ is a local diffeomorphism is checked locally. Let $U \subset Q/G$ be an open subset trivializing $\pi$ and $\sigma \in \Gamma(U, Q|_U)$, a local section over $U$ of the principal $G$-bundle $\pi : Q \to Q/G$. Define $\Phi_\sigma : Q \times U \to Q \times (Q|_U)$ by $\Phi_\sigma(q_0, r_1) := (q_0, l^Q_{\mathcal{A}(q_0, \sigma(r_1))^{-1}}(\sigma(r_1)))$. It can be checked that $\Phi_\sigma$ is a smooth map whose image is contained in $Hor$. Furthermore, $\Phi_\sigma$ is the inverse of $(id_Q \times \pi)|_{Hor \cap ((Q|_U) \times (Q|_U))}$, showing that condition 1 in Lemma 2.3 holds. By Lemma 2.3, $(id_Q \times \pi)|_{Hor}$ is an injective local diffeomorphism and we conclude that $Hor$ defines a discrete connection $\mathcal{A}_d$ on $\pi : Q \to Q/G$. Direct evaluation shows that the domain of $\mathcal{A}_d$ is $\mathfrak{U} = \mathcal{U}$ and that the discrete connection form associated to $\mathcal{A}_d$ is $\mathcal{A}$. $\square$

Motivated by the previous analysis we introduce the following concept.

**Definition 3.5.** Let $\pi : Q \to Q/G$ be a principal $G$-bundle and $\mathcal{U} \subset Q \times Q$ be an open subset that contains the diagonal $\Delta_Q \subset Q \times Q$ and is invariant under the product $G \times G$-action on $Q \times Q$. A smooth function $\mathcal{A} : \mathcal{U} \to G$ is called a *discrete connection form* if $\mathcal{A}(q_0, q_0) = e$ for all $q_0 \in Q$ and it satisfies

$$\mathcal{A}(l^Q_{g_0}(q_0), l^Q_{g_1}(q_1)) = g_1 \mathcal{A}(q_0, q_1) g_0^{-1} \quad \text{for all} \quad (q_0, q_1) \in \mathfrak{U}, \quad g_0, g_1 \in G.$$

Using the new notion and taking Lemma 3.2 into account, we rewrite Theorem 3.4 as follows.

**Theorem 3.6.** *Let $\mathcal{A}_d$ be a discrete connection on the principal $G$-bundle $\pi : Q \to Q/G$ with domain $\mathfrak{U}$. Then, its associated discrete connection form $\mathcal{A}_d$ is a discrete connection form. Conversely, given a discrete connection form $\mathcal{A} : \mathcal{U} \to G$ on the same principal bundle $\pi$, with $\mathcal{U}$ as in Definition 3.5, the subset $Hor := \{(q_0, q_1) \in \mathcal{U} : \mathcal{A}(q_0, q_1) = e\} \subset Q \times Q$ defines a discrete connection on $\pi$ whose associated discrete connection form is $\mathcal{A}$.*

**Remark 3.7.** Theorem 3.6 establishes a correspondence between discrete connections and discrete connection forms. The assignments $\mathcal{A} \mapsto Hor_{\mathcal{A}} := \mathcal{A}^{-1}(\{e\})$ and $Hor_{\mathcal{A}_d} \mapsto \mathcal{A}_d$ given by (3.1) are the corresponding opposite operations. This correspondence justifies using the name $\mathcal{A}_d$ for both the discrete connection and the discrete connection form.

**Remark 3.8.** The situation described in Example 3.3 corresponds to the general discrete connection on the (trivial) principal $G$-bundle $p_1 : R \times G \to R$. Indeed, if $\mathcal{A}_d$ is such a connection with domain $\mathfrak{U}$, using (3.4), we have that $\mathcal{A}_d((r_0, g_0), (r_1, g_1)) = g_1 \mathcal{A}_d((r_0, e), (r_1, e)) g_0^{-1}$. Define $C(r_0, r_1) := \mathcal{A}_d((r_0, e), (r_1, e))$ for all $(r_0, r_1) \in \mathfrak{U}'' = (\pi \times \pi)(\mathfrak{U})$, so that

$$Hor_{\mathcal{A}_d} = \mathcal{A}_d^{-1}(\{e\}) = \{((r_0, g_0), (r_1, g_1)) \in \mathfrak{U} : g_1 C(r_0, r_1) = g_0\}.$$

Comparison of this last expression with (3.2) shows that the discrete connection form satisfies $\mathcal{A}_d = \mathcal{A}_d^C$.

As all principal $G$-bundles are locally of the form $p_1 : R \times G \to R$, Example 3.3 provides a local description of arbitrary discrete connections on principal $G$-bundles.

## 4. Discrete horizontal lift

As in the case of connections on principal bundles, discrete connections establish local diffeomorphisms between slices of the horizontal submanifold and the base space of the bundle. The inverse operation is the *discrete horizontal lift*.

**Definition 4.1.** Let $\mathcal{A}_d$ be a discrete connection with domain $\mathfrak{U}$ on the principal $G$-bundle $\pi : Q \to Q/G$ ; let $\mathfrak{U}' \subset Q \times (Q/G)$ be the open set defined in part 1 of Proposition 2.4. The *associated discrete horizontal lift* $h_d : \mathfrak{U}' \to Q \times Q$ is the inverse map of the diffeomorphism $(id_Q \times \pi)|_{Hor_{\mathcal{A}_d}} : Hor_{\mathcal{A}_d} \to \mathfrak{U}'$. Explicitly

$$(4.1) \qquad h_d(q_0, r_1) = (q_0, q_1) \quad \Leftrightarrow \quad (q_0, q_1) \in Hor_{\mathcal{A}_d} \text{ and } \pi(q_1) = r_1.$$

It is convenient to write $h_d^{q_0}(r_1) := h_d(q_0, r_1)$ and define $\overline{h_d^{q_0}} := p_2 \circ h_d^{q_0}$.

**Example 4.2.** The horizontal lift associated to the connections $\mathcal{A}_d^C$ introduced in Example 3.3 is given by $h_d : (R \times G) \times R \to (R \times G) \times (R \times G)$ with

$$h_d^{(r_0, g_0)}(r_1) = ((r_0, g_0), (r_1, g_0 C(r_0, r_1)^{-1})).$$

**Remark 4.3.** When $(q_0, q_1) \in \mathfrak{U}$, if $g := \mathcal{A}_d(q_0, q_1)$, we know that $(q_0, l_{g^{-1}}^Q(q_1)) \in Hor_{\mathcal{A}_d}$. Then, $h_d^{q_0}(\pi(q_1)) = h_d^{q_0}(\pi(l_{g^{-1}}^Q(q_1))) = (q_0, l_{g^{-1}}^Q(q_1))$ and $\overline{h_d^{q_0}}(\pi(q_1)) = l_{g^{-1}}^Q(q_1)$, so that $l_g^Q(\overline{h_d^{q_0}}(\pi(q_1))) = q_1$. Hence,

$$(4.2) \qquad \mathcal{A}_d(q_0, q_1) = \kappa(\overline{h_d^{q_0}}(\pi(q_1)), q_1), \quad \text{for all} \quad (q_0, q_1) \in \mathfrak{U}.$$

The following result establishes the basic properties of the discrete horizontal lift associated to a discrete connection. It also proves that a discrete connection can be reconstructed given its associated horizontal lift.

**Theorem 4.4.** *Let $\mathcal{A}_d$ be a discrete connection on the principal $G$-bundle $\pi : Q \to Q/G$ with domain $\mathfrak{U}$. Then the following assertions are true.*

(1) *$\mathfrak{U}' \subset Q \times (Q/G)$ is $G$-invariant for the $G$-action defined by $l_g^{Q \times (Q/G)}(q_0, r_1) := (l_g^Q(q_0), r_1)$ for all $g \in G$.*

(2) *$h_d : \mathfrak{U}' \to Q \times Q$ is smooth and $G$-equivariant for the $G$-actions $l^{Q \times (Q/G)}$ and $l^{Q \times Q}$.*

(3) *$h_d$ is a section over $\mathfrak{U}'$ of $(id_Q \times \pi) : Q \times Q \to Q \times (Q/G)$; that is, $(id_Q \times \pi) \circ h_d = id_{\mathfrak{U}'}$.*

(4) *For every $q_0 \in Q$, $(q_0, \pi(q_0)) \in \mathfrak{U}'$ and $h_d^{q_0}(\pi(q_0)) = (q_0, q_0)$.*

*Conversely, assume that $\mathcal{U}' \subset Q \times (Q/G)$ is an open set that satisfies condition 1 (with $\mathfrak{U}'$ replaced by $\mathcal{U}'$) and $h : \mathcal{U}' \to Q \times Q$ is a map such that conditions 2, 3 and 4 are satisfied (with $\mathfrak{U}'$ and $h_d$ replaced by $\mathcal{U}'$ and $h$). Then, there exists a unique discrete connection $\mathcal{A}_d$ with domain $\mathfrak{U} = (id_Q \times \pi)^{-1}(\mathcal{U}')$ on $\pi : Q \to Q/G$ such that $\mathcal{U}' = \mathfrak{U}'$ and $h = h_d$.*

*Proof.* In general, $id_Q \times \pi$ is $G$-equivariant for the actions $l^{Q \times Q}$ and $l^{Q \times (Q/G)}$. When $\mathcal{A}_d$ is a discrete connection, $Hor_{\mathcal{A}_d}$ is $G$-invariant, so that $(id_Q \times \pi)|_{Hor_{\mathcal{A}_d}}$ is a $G$-equivariant diffeomorphism and, consequently, its image $\mathfrak{U}'$ is $G$-invariant and its inverse $h_d$ is smooth and $G$-equivariant. This proves points 1 and 2. Point 3 follows immediately from the definition of $h_d$, while point 4 is a consequence of $\Delta_Q \subset Hor_{\mathcal{A}_d}$.

Now assume that $\mathcal{U}'$ and $h$ are as in the statement. Let $\mathfrak{U} := (id_Q \times \pi)^{-1}(\mathcal{U}')$, that is open due to the openness of $\mathcal{U}'$ and the smoothness of $id_Q \times \pi$. Furthermore, it follows from the $G$-invariance of $\mathcal{U}'$ that $\mathfrak{U}$ is $G \times G$-invariant and, by the first part of condition 4, it contains the diagonal $\Delta_Q \subset Q \times Q$. Motivated by (4.2), define $\mathcal{A}_d : \mathfrak{U} \to G$ by $\mathcal{A}_d(q_0, q_1) := \kappa(p_2(h(q_0, \pi(q_1))), q_1)$. Being a composition of smooth functions, $\mathcal{A}_d$ is smooth. Straightforward computations show that $\mathcal{A}_d$ satisfies condition (3.4) and $\mathcal{A}_d(q_0, q_0) = e$ for all $q_0 \in Q$. All together, by Theorem 3.4, $\mathcal{A}_d$ defines a discrete connection with domain $\mathfrak{U}$ on $\pi : Q \to Q/G$. It is immediate that $\mathfrak{U}' = (id_Q \times \pi)(\mathfrak{U}) = \mathcal{U}'$ and, using (4.2) as well as the definition of $\mathcal{A}_d$, we conclude that $h = h_d$, proving the last part of the Theorem. The uniqueness assertion follows from the fact that the discrete connection form is determined by the horizontal lift using (4.2). $\qquad\square$

Motivated by the previous analysis, it is convenient to introduce the following notion.

**Definition 4.5.** Let $G$ be a Lie group acting on $Q$ by $l^Q$ in such a way that $\pi : Q \to Q/G$ is a principal $G$-bundle and let $\mathcal{U}' \subset Q \times (Q/G)$ be an open subset. A smooth function $h : \mathcal{U}' \to Q \times Q$ is a *discrete horizontal lift* on $\pi$ if the following conditions hold.

(1) $\mathcal{U}' \subset Q \times (Q/G)$ is $G$-invariant for the $G$-action $l^{Q \times (Q/G)}$.
(2) $h : \mathcal{U}' \to Q \times Q$ is smooth and $G$-equivariant for the $G$-actions $l^{Q \times (Q/G)}$ and $l^{Q \times Q}$.
(3) $h$ is a section of $(id_Q \times \pi) : Q \times Q \to Q \times (Q/G)$ over $\mathcal{U}'$; that is, $(id_Q \times \pi) \circ h = id_{\mathcal{U}'}$.
(4) For every $q_0 \in Q$, $(q_0, \pi(q_0)) \in \mathcal{U}'$ and $h(q_0, \pi(q_0)) = (q_0, q_0)$.

We can rewrite Theorem 4.4 using the new concept as follows.

**Theorem 4.6.** *Let $\mathcal{A}_d$ be a discrete connection on the principal $G$-bundle $\pi : Q \to Q/G$ with domain $\mathfrak{U}$. Then $h_d : \mathfrak{U}' \to Q \times Q$ as defined by (4.1) is a discrete horizontal lift on $\pi$. Conversely, if $h : \mathcal{U}' \to Q \times Q$ is a discrete horizontal lift on $\pi : Q \to Q/G$, there exists a unique discrete connection $\mathcal{A}_d$ with domain $\mathfrak{U} = (id_Q \times \pi)^{-1}(\mathcal{U}')$ on $\pi$ such that $\mathcal{U}' = \mathfrak{U}'$ and $h = h_d$.*

**Remark 4.7.** We see from Theorem 4.6 that mapping $\mathcal{A}_d \mapsto h_d$ and $h \mapsto \mathcal{A}$ with $\mathcal{A}(q_0, q_1) := \kappa(p_2(h(q_0, \pi(q_1))), q_1)$ establishes a bijection between discrete horizontal lifts and discrete connections forms —or discrete connections— on a principal bundle.

**Remark 4.8.** Given a discrete connection $\mathcal{A}_d$ on the principal bundle $\pi : Q \to Q/G$, its discrete connection form $\mathcal{A}_d$ and discrete horizontal lift $h_d$ may not be defined everywhere. Indeed, if $h_d : Q \times (Q/G) \to Q \times Q$ (that is, $\mathfrak{U}' = Q \times (Q/G)$), then for any $q \in Q$, the map $r \mapsto \overline{h_d^q}(r)$ is a global section of the principal bundle $\pi$, so that the bundle is trivial. Hence, for nontrivial principal $G$-bundles $h_d$ and, consequently, $\mathcal{A}_d$ can only be defined in some open set of the total space.

**Remark 4.9.** The equations of motion of a $G$-symmetric mechanical system on $Q$ can be written using a connection $\mathcal{A}$ on the principal $G$-bundle $\pi : Q \to Q/G$. In [1], Cendra, Marsden and Ratiu use the same type of connection to construct an isomorphic model for the reduced space $TQ/G$. Given a connection $\mathcal{A}$, they define $\alpha_{\mathcal{A}} : TQ/G \to T(Q/G) \oplus \widetilde{\mathfrak{g}}$, an isomorphism of vector bundles over $Q/G$, by

$$\alpha_{\mathcal{A}}([q, \dot{q}]_G) := \big(d\pi(q, \dot{q}), [(q, \mathcal{A}(q, \dot{q}))]_G\big),$$

where $\widetilde{\mathfrak{g}}$ is the adjoint vector bundle of $\mathfrak{g}$. This identification allows them to establish a reduced variational principle and the associated reduced equations of motion.

Similarly, a discrete connection $\mathcal{A}_d$ on the principal bundle $\pi : Q \to Q/G$ can be used to construct an isomorphic model for the discrete reduced space $(Q \times Q)/G$. Given $\mathcal{A}_d$, we define $\alpha_{\mathcal{A}_d} : (Q \times Q)/G \to (Q/G \times Q/G) \times_{Q/G} \tilde{G}$, an isomorphism of bundles over $Q/G$, by

$$\alpha_{\mathcal{A}_d}([(q_0, q_1)]_G) := ((\pi(q_0), \pi(q_1)), [(q_0, \mathcal{A}_d(q_0, q_1))]_G),$$

where $\tilde{G} := (Q \times G)/G$ is the adjoint bundle of $Q$ by $G$ with respect to the $G$-action on $Q \times G$ given by $l_g^{Q \times G}(q, h) := (l_g^Q(q), ghg^{-1})$. This identification of spaces is used in [4], [3], and [10] to study the reduction of discrete mechanical systems with symmetries. It is important to observe that, according to Remark 4.8, this identification is only local for nontrivial $G$-bundles.

## 5. Existence of discrete connections

All principal $G$-bundles carry connections that can be constructed, for instance, using Riemannian metrics on the bundle. In this section we prove an existence result for discrete connections given appropriate Riemannian metrics.

Let $\pi : Q \to Q/G$ be a principal $G$-bundle and $\langle , \rangle_Q$ a $G$-invariant Riemannian metric on $Q$. The vertical bundle $\mathcal{V}$ has an orthogonal complement, the *horizontal bundle* $\mathcal{H} \subset TQ$; it is easy to check that $\mathcal{H}$ defines a connection $\mathcal{A}^{\langle , \rangle_Q}$ on the principal $G$-bundle $\pi$. Then there is a unique metric $\langle , \rangle_{Q/G}$ on $Q/G$ that turns $\pi$ into a Riemannian submersion, *i.e.*, for each $q \in Q$, $d\pi(q)|_{\mathcal{H}_q} : \mathcal{H}_q \to T_{\pi(q)}(Q/G)$ is an isometry.

By Theorems 8.7 in Chapter III and 3.6 in Chapter IV in [7], for every $r \in Q/G$, there is an open set $W_r \subset Q/G$ such that any two points in $W_r$ can be joined by a unique length minimizing geodesic (with respect to $\langle , \rangle_{Q/G}$) lying in $W_r$. Furthermore, for each $r' \in W_r$ there is a normal coordinate neighborhood centered at $r'$ containing $W_r$. Define

$$(5.1) \qquad \mathcal{U} := \cup_{r \in Q/G} (\pi \times \pi)^{-1}(W_r) \subset Q \times Q,$$

that is open because the sets $W_r$ are open and $\pi$ is continuous. Also, from the definition, $\mathcal{U}$ is $G \times G$-invariant (for the product $G$-action) and contains the diagonal $\Delta_Q$.

Let $(q_0, q_1) \in \mathcal{U}$. As $\pi(q_0), \pi(q_1) \in W_r$ for some $r \in Q/G$, they can be joined by a unique geodesic $\gamma : [0,1] \to W_r$ such that $\gamma(0) = \pi(q_0)$ and $\gamma(1) = \pi(q_1)$. Being $\mathcal{A}^{\langle,\rangle_Q}$ a connection on the principal $G$-bundle $\pi : Q \to Q/G$, by Proposition 3.1 in [7], there is an $\mathcal{A}^{\langle,\rangle_Q}$-horizontal lift[1] $\widetilde{\gamma} : [0,1] \to Q|_{W_r}$ of $\gamma(t)$ to $Q$ such that $\widetilde{\gamma}(0) = q_0$ and $\widetilde{\gamma}(1) \in Q_{\pi(q_1)}$. In addition, as $\gamma$ is a geodesic and $\pi$ is a Riemannian submersion, by Proposition 3.1 in [6], $\widetilde{\gamma}$ is a geodesic for $\langle,\rangle_Q$. The value $\widetilde{\gamma}(1)$ is independent of the open set $W_r$ chosen for the construction. Indeed, if we pick another open set $W_{r'}$ containing $\pi(q_0)$ and $\pi(q_1)$ we would have two length minimizing geodesics $\gamma_r$ and $\gamma_{r'}$ joining $\pi(q_0)$ to $\pi(q_1)$ and contained in $W_r$ and $W_{r'}$ respectively. Then, by Theorem 10.4 in [12], $\gamma_r([0,1]) = \gamma_{r'}([0,1])$, so that both geodesics are contained in $W_r \cap W_{r'}$ and, by the uniqueness of the geodesics in $W_r$, $\gamma_r(t) = \gamma_{r'}(t)$ for all $t \in [0,1]$.

**Remark 5.1.** The open subsets $W_r \subset Q/G$ introduced above are not necessarily unique. The same, in principle, applies to the open subset $\mathcal{U} \subset Q \times Q$ defined by (5.1).

Choosing a family $\{W_r : r \in Q/G\}$ as above and constructing $\mathcal{U}$ with (5.1), define $\mathcal{A}_d^{\langle,\rangle_Q} : \mathcal{U} \to G$ by

$$(5.2) \qquad\qquad \mathcal{A}_d^{\langle,\rangle_Q}(q_0, q_1) := \kappa(\widetilde{\gamma}(1), q_1).$$

**Theorem 5.2.** *Let $(Q, \langle,\rangle_Q)$ be a Riemannian manifold where the Lie group $G$ acts by isometries in such a way that $\pi : Q \to Q/G$ is a principal $G$-bundle. Then, there is a discrete connection $\mathcal{A}_d^{\langle,\rangle_Q}$ on $\pi$ whose domain is $\mathfrak{U} = \mathcal{U}$ and whose discrete connection form is given by (5.2).*

*Proof.* From the previous construction, $\mathcal{U} \subset Q \times Q$ is an open set, $G \times G$-invariant for the product $G$-action and contains the diagonal $\Delta_Q$. In addition, by the smooth dependence of the geodesics on both the initial and final point as well as the initial point and velocity, $\mathcal{A}_d^{\langle,\rangle_Q}$ is smooth. For any $q_0 \in Q$, the unique length minimizing geodesic joining $\pi(q_0)$ to itself in $Q/G$ is the constant path, so that its horizontal lift is, again, the constant path and we conclude that

$$\mathcal{A}_d^{\langle,\rangle_Q}(q_0, q_0) := \kappa(\widetilde{\gamma}(1), q_0) = \kappa(q_0, q_0) = e.$$

Let $(q_0, q_1) \in \mathcal{U}$ and $g_0, g_1 \in G$. for any $r \in Q/G$ such that $\pi(q_0), \pi(q_1) \in W_r$, there is a unique length minimizing geodesic $\gamma : [0,1] \to Q/G$ contained in $W_r$ and such that $\gamma(0) = \pi(q_0)$ and $\gamma(1) = \pi(q_1)$. As $\pi(q_0) = \pi(l_{g_0}^Q(q_0))$ we can consider the horizontal lifts $\widetilde{\gamma}_{q_0}$ and $\widetilde{\gamma}_{l_{g_0}^Q(q_0)}$ of $\gamma$ starting at $q_0$ and $l_{g_0}^Q(q_0)$ respectively. As $G$ acts on $Q$ by isometries, for any $g_0 \in G$, $l_{g_0}^Q$ is an isometry of $Q$. Then, $l_{g_0}^Q(\widetilde{\gamma}_{q_0}(t))$ is a horizontal geodesic in $Q$ starting at $l_{g_0}^Q(q_0)$. Furthermore, $\pi(l_{g_0}^Q(\widetilde{\gamma}_{q_0}(t))) = \pi(\widetilde{\gamma}_{q_0}(t)) = \gamma(t)$, so that $l_{g_0}^Q(\widetilde{\gamma}_{q_0}(t))$ is the horizontal lift of $\gamma$ starting at $l_{g_0}^Q(q_0)$, that is, $\widetilde{\gamma}_{l_{g_0}^Q(q_0)} = l_{g_0}^Q \circ \widetilde{\gamma}_{q_0}$. Then

$$\mathcal{A}_d^{\langle,\rangle_Q}(l_{g_0}^Q(q_0), l_{g_1}^Q(q_1)) = \kappa(\widetilde{\gamma}_{l_{g_0}^Q(q_0)}(1), l_{g_1}^Q(q_1)) = \kappa(l_{g_0}^Q(\widetilde{\gamma}_{q_0}(1)), l_{g_1}^Q(q_1))$$

$$= g_1 \kappa(\widetilde{\gamma}_{q_0}(1), q_1) g_0^{-1} = g_1 \mathcal{A}_d^{\langle,\rangle_p}(q_0, q_1) g_0^{-1},$$

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[1]As we will be considering only one connection on $\pi$, in what follows, we omit its reference when we consider horizontal lifts of paths.

so that $\mathcal{A}_d^{\langle,\rangle_Q}$ satisfies condition (3.4). The main result follows from Theorem 3.4.

$\square$

**Remark 5.3.** The submanifold $Hor_{\mathcal{A}_d^{\langle,\rangle_Q}}$ underlying the discrete connection $\mathcal{A}_d^{\langle,\rangle_Q}$ constructed in Theorem 5.2 is $Hor_{\mathcal{A}_d^{\langle,\rangle_Q}} = (\tau_Q \times \exp)(\mathcal{D} \cap \mathcal{H})$ where $\tau_Q : TQ \to Q$ is the canonical projection, $\mathcal{D} \subset TQ$ is an open subset —containing the image of the zero section and contained in the domain of the exponential mapping— and $\mathcal{H}$ is the horizontal distribution.

**Example 5.4.** Let $\mathbb{H}$ be the $\mathbb{R}$-algebra of *quaternions* together with its canonical inner product $\langle,\rangle_{\mathbb{H}}$ and basis $\{\hat{1}, \hat{\imath}, \hat{\jmath}, \hat{k}\}$. The submanifold of unit norm quaternions is the sphere $S^3$ with the round metric, while the unit norm imaginary quaternions form $S^2$ with the round metric. Define $\phi : S^3 \to S^2$ by $\phi(q) := \overline{q}\hat{\imath}q$, where $\overline{q}$ denotes the conjugated quaternion (change the sign of the imaginary part of $q$). It is a well known fact that $\phi$ is a principal $U(1)$-bundle for the $U(1)$-action on $S^3$ given by the isometries $l_{e^{i\theta}}^{S^3}(q) := (\cos(\theta)\hat{1} + \sin(\theta)\hat{\imath})q$. This bundle is the *Hopf bundle*. The map $\phi$ is a Riemannian submersion if we consider *twice* the round metric in $S^2$.

The construction of a discrete connection $\mathcal{A}_d^{\langle,\rangle_{S^3}}$ associated to the Hopf bundle described at the beginning of the section is as follows. When $q \in \mathbb{H} - \{0\}$ we denote by $\langle q \rangle \subset \mathbb{H}$ the 1-dimensional *real* subspace of $\mathbb{H}$ generated by $q$. For $q \in S^3$, the vertical bundle is $\mathcal{V}(q) = \langle \hat{\imath}q \rangle \subset T_q S^3 = \langle q \rangle^\perp \subset \mathbb{H}$, so that the horizontal bundle is $\mathcal{H}_q = \langle \hat{\imath}q \rangle^\perp \subset T_q S^3 \subset \mathbb{H}$. For $r \in S^2$, the open subsets $W_r := \{r' \in S^2 : \langle r, r' \rangle_{\mathbb{H}} > 0\} \subset S^2$ have the required geodesic uniqueness properties. Then, $\mathcal{U} := \cup_{r \in S^2}(\phi \times \phi)^{-1}(W_r) = \{(q_0, q_1) \in S^3 \times S^3 : \phi(q_0) \neq -\phi(q_1)\}$; the last identity is due to the fact that if $\phi(q_0) \neq -\phi(q_1)$ then $\phi(q_0), \phi(q_1) \in W_r$ for $r := \frac{\phi(q_0)+\phi(q_1)}{\|\phi(q_0)+\phi(q_1)\|}$. Using the explicit form of the geodesics in $S^2$ and $S^3$ as maximal circles, after some work, it can be seen that the domain of the discrete connection form $\mathcal{A}_d^{\langle,\rangle_{S^3}}$ is

$$\mathfrak{U} := \mathcal{U} = \{(q_0, q_1) \in S^3 \times S^3 : P_{\hat{1}}(q_1\overline{q_0})^2 + P_{\hat{\imath}}(q_1\overline{q_0})^2 \neq 0\},$$

and the connection form is

$$\mathcal{A}_d^{\langle,\rangle_{S^3}}(q_0, q_1) = \frac{P_{\hat{1}}(q_1\overline{q_0}) + P_{\hat{\imath}}(q_1\overline{q_0})i}{|P_{\hat{1}}(q_1\overline{q_0}) + P_{\hat{\imath}}(q_1\overline{q_0})i|} \in U(1),$$

where $P_{\hat{1}}, P_{\hat{\imath}} : \mathbb{H} \to \mathbb{R}$ are the projections on the corresponding coordinates.

**Remark 5.5.** In the same general setting as in Theorem 5.2, Example 4.1 of [9] constructs a function $\mathcal{A}_d : Q \times Q \to G$ as follows. Given two points $q_0, q_1 \in Q$, let $q_{01}$ be the geodesic in $Q$ satisfying $q_{01}(0) = q_0$ and $q_{01}(1) = 1$ (this may actually restrict the domain of $\mathcal{A}_d$ to an open neighborhood of the diagonal in $Q \times Q$). Then, let $x_{01} := \pi \circ q_{01}$ and $\widetilde{q}_{01}$ the horizontal lift (with respect to the horizontal distribution $\mathcal{H}$) of $x_{01}$ to $Q$ starting at $q_0$. Finally, define

$$(5.3) \qquad \mathcal{A}_d(q_0, q_1) := \kappa(\widetilde{q}_{01}(1), q_1).$$

Pairs in the "horizontal manifold" $\mathcal{A}_d^{-1}(\{e\})$ are the endpoints of horizontal geodesics in $Q$. It follows that this manifold is essentially the same as $Hor_{\mathcal{A}_d^{\langle,\rangle_Q}}$ corresponding to the discrete connection constructed by Theorem 5.2 (see Remark 5.3). Still, the function $\mathcal{A}_d$ defined in (5.3) may not be a discrete connection form: it may fail to

satisfy condition (3.4). For instance, in the context of the Hopf bundle considered in Example 5.4, it can be checked explicitly that, when $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$,

$$\mathcal{A}_d \left( \hat{1}, l^{S^3}_{e^{i\theta}} \left( \frac{\hat{1} + \hat{j}}{\sqrt{2}} \right) \right) = e^{i\beta_\theta} \quad \text{for} \quad \beta_\theta := \frac{\sin(\theta) \arccos(\frac{\cos(\theta)}{\sqrt{2}})}{\sqrt{2 - \cos(\theta)^2}},$$

whereas

$$e^{i\theta} \mathcal{A}_d \left( \hat{1}, \frac{\hat{1} + \hat{j}}{\sqrt{2}} \right) = e^{i\theta}.$$

Evaluation of the first derivatives of those expressions at $\theta = 0$ confirms that (3.4) does not hold in this case.

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