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Recognizing vertex intersection graphs of paths on bounded degree trees

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ABSTRACT

An (h, s, t)-representation of a graph G consists of a collection of subtrees of a tree T, where each subtree corresponds to a vertex of G such that (i) the maximum degree of T is at most h, (ii) every subtree has maximum degree at most s, (iii) there is an edge between two vertices in the graph G if and only if the corresponding subtrees have at least t vertices in common in T. The class of graphs that has an (h, s, t)-representation is denoted by $[h, s, t]$.

An undirected graph G is called a *VPT* graph if it is the vertex intersection graph of a family of paths in a tree. Thus, $[h, 2, 1]$ graphs are the *VPT* graphs that can be represented in a tree with maximum degree at most h. In this paper we characterize $[h, 2, 1]$ graphs using chromatic number. We show that the problem of deciding whether a given *VPT* graph belongs to $[h, 2, 1]$ is NP-complete, while the problem of deciding whether the graph belongs to $[h, 2, 1] - [h - 1, 2, 1]$ is NP-hard. Both problems remain hard even when restricted to *VPT* $\cap$ *Split*. Additionally, we present a non-trivial subclass of *VPT* $\cap$ *Split* in which these problems are polynomial time solvable.

1. Introduction

The intersection graph of a set family is a graph whose vertices are the members of the family, and the adjacency between them is defined by a non-empty intersection of the corresponding sets. Classic examples are interval graphs and chordal graphs.

An interval graph is the intersection graph of a family of intervals of the real line, or, equivalently, the vertex intersection graph of a family of subpaths of a path. A chordal graph is a graph without chordless cycles of length at least four. Gavril [6] proved that a graph is chordal if and only if it is the vertex intersection graph of a family of subtrees of a tree. Both classes have been widely studied [2].

In order to allow larger families of graphs to be represented by subtrees, several graph classes are defined imposing conditions on trees, subtrees and intersection sizes [14,15]. Let h, s and t be positive integers; an (h, s, t)-representation of a graph G consists in a host tree T and a collection $(T_v)_{v\in V(G)}$ of subtrees of T, such that (i) the maximum degree of T is at most h, (ii) every subtree T_v has maximum degree at most s, (iii) two vertices v and v' are adjacent in G if and only if the corresponding subtrees T_v and $T_{v'}$ have at least t vertices in common in T. The class of graphs that has an (h, s, t)-representation is denoted by $[h, s, t]$. When there is no restriction on the maximum degree of T or on the maximum degree of the subtrees, we use $h = \infty$ and $s = \infty$ respectively. Therefore, $[\infty, \infty, 1]$ is the class of chordal graphs and $[2, 2, 1]$ is

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the class of interval graphs. The classes $[\infty, 2, 1]$ and $[\infty, 2, 2]$ are called *VPT* and *EPT* respectively in [8]; and *UV* and *UE*, respectively in [17].

In recent years, the study of the classes $[h, s, t]$ has merited several publications in the literature. In [5], the minimum t such that a given graph belongs to $[3, 3, t]$ is studied. In [18], it is shown that $[3, 3, 1]$ is exactly the class of chordal graphs. In [12], $[4, 4, 2]$ graphs are characterized and a polynomial time algorithm for their recognition is given. In [11,4] respectively, the classes $[4, 2, 2]$ and $[4, 3, 2]$ are studied. Recognition, coloring and some other classic problems on the class $[\infty, 2, t]$ are treated in [9]. The relation between different classes is analyzed in [10]. In [7,19], it is shown that the problem of recognizing *VPT* graphs is polynomial time solvable. On the other hand, the recognition of *EPT* graphs is an NP-complete problem [8].

In this work, we focus on the classes $[h, 2, 1]$ for any fixed $h \geq 3$; they are all subclasses of *VPT*. We characterize $[h, 2, 1]$ graphs using chromatic number. We show that the problem of deciding whether a given *VPT* graph belongs to $[h, 2, 1]$ is NP-complete, while the problem of deciding whether the given graph belongs to $[h, 2, 1] - [h - 1, 2, 1]$ is NP-hard. Both problems remain hard even when restricted to $VPT \cap Split$. Additionally, we present a non-trivial subclass of $VPT \cap Split$ in which these problems are polynomial time solvable. The case $h = 2$ is not considered because $[2, 2, 1] = Interval$. Our results apply for any $h \geq 3$, they can be seen as a generalization of the case $h = 3$ which leads with the class $[3, 2, 1] = [3, 2, 2] = EPT \cap Chordal$ considered in [8,20].

The paper is organized as follows: in Section 2, we provide basic definitions and basic results. In Section 3, we characterize $[h, 2, 1]$ graphs for $h \geq 3$. In Section 4, we present the time complexity analysis for the recognition problem. Finally, in Section 5 we pose some open questions.

2. Preliminaries

Throughout this paper, graphs are connected, finite and simple. The *vertex set* and the *edge set* of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The *open neighborhood* of a vertex v, represented by $N_G(v)$, is the set of vertices adjacent to v. The *closed neighborhood* $N_G[v]$ is $N_G(v) \cup \{v\}$. The *degree* of v, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. For simplicity, when no confusion can arise, we omit the subindex G and write $N(v)$, $N[v]$ or $d(v)$.

A *complete set* is a subset of mutually adjacent vertices. A *clique* is a maximal complete set. The *family of cliques* of G is denoted by $\mathcal{C}(G)$. A *stable set* is a subset of vertices no two of which are adjacent.

An $(\infty, 2, 1)$-*representation* of G, also called a *VPT representation*, is a pair $\langle \mathcal{P}, T \rangle$, where T is a *host tree* and $\mathcal{P}$ is a family $(P_v)_{v \in V(G)}$ of subpaths of T satisfying that two vertices v and v' of G are adjacent if and only if P_v and $P_{v'}$ have at least one vertex in common.

Since a family of paths in a tree satisfies the Helly property [1], if C is a clique of G then there exists a vertex q of T such that $C = \{v \in V(G) : q \in V(P_v)\}$.

On the other hand, if q is any vertex of the host tree T, the set $\{v \in V(G) : q \in V(P_v)\}$, denoted by C_q, is a complete set of G, but not necessarily a clique. In order to avoid this drawback we introduce the notion of full representation.

If q is a vertex of T, the connected components of $T - q$ are called the *branches of* T *at* q. A path is *contained* in a branch if all its vertices are vertices of the branch. Notice that if $N_T(q) = \{q_1, q_2, \ldots, q_h\}$ then T has exactly h branches at q. The branch containing q_i is denoted by T_i; we say that q_i is the *root* of T_i. Two branches T_i and T_j are *linked* by a path $P_v \in \mathcal{P}$ if both vertices q_i and q_j belong to $V(P_v)$.

Definition 1. A VPT representation $\langle \mathcal{P}, T \rangle$ is full at a vertex q of T if, for every two branches T_i and T_j of T at q, there exist paths $P_v, P_w, P_u \in \mathcal{P}$ such that: (i) the branches T_i and T_j are linked by P_v; (ii) P_w is contained in T_i and intersects P_v in at least one vertex; and (iii) P_u is contained in T_j and intersects P_v in at least one vertex. A representation $\langle \mathcal{P}, T \rangle$ is full if it is full at every $q \in V(T)$ with $d_T(q) \geq 4$.

The following theorem and its Corollary 3 show that a *VPT* representation which is not full can be modified to obtain a full VPT representation without increasing the maximum degree of the host tree; and, even more, decreasing the degree of the vertices of T at which the representation is not full.

Theorem 2. *Let $\langle \mathcal{P}, T \rangle$ be a VPT representation of G. Assume there exists a vertex $q \in V(T)$ with $d_T(q) = h \geq 4$ and two branches of T at q which are linked by no path of $\mathcal{P}$. Then there exists a VPT representation $\langle \mathcal{P}', T' \rangle$ of G with $V(T') = V(T) \cup \{q'\}$, $q' \notin V(T)$, and*

$$d_{T'}(x) = \begin{cases} 3, & \text{if } x = q' \\ h - 1, & \text{if } x = q \\ d_T(x), & \text{if } x \in V(T') \setminus \{q, q'\}. \end{cases}$$

Proof. Let q_1 and q_2 be the neighbors of q that are roots of two non linked branches. We obtain the $\langle \mathcal{P}', T' \rangle$ representation of G as follows (see Fig. 1): the set of vertices of T' is $V(T) \cup \{q'\}$, where q' is a new vertex not in $V(T)$. The set of edges is $(E(T) \setminus \{qq_1, qq_2\}) \cup \{q'q_1, q'q_2, q'q\}$. Observe that the degree of each vertex $x \in V(T')$ is the required in the statement of the present theorem. Now we define the paths P'_v for $v \in V(G)$: if q_1 and q or q_2 and q belong to $V(P_v)$ then $V(P'_v) = V(P_v) \cup \{q'\}$.

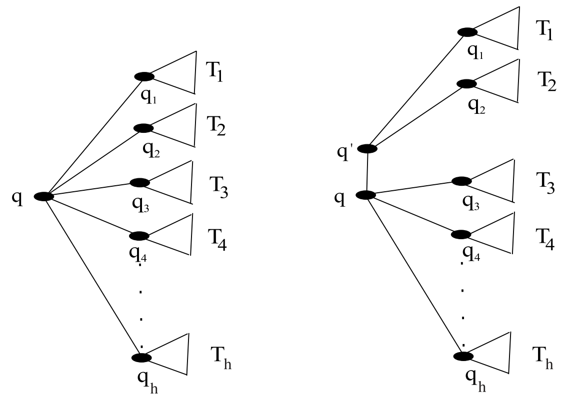

Fig. 1. The degree of q in the tree T on the left is h. The degree of q in the tree T' on the right is $h-1$.

In any other case, $V(P'_v) = V(P_v)$. Since no path of $\mathcal{P}$ contains both vertices q_1 and q_2, each $V(P'_v)$ effectively induces a path in T'. Moreover, since all the paths where vertex q' was added had vertex q in common, it is clear that, for any pair of vertices $v, w \in V(G)$, $V(P_v) \cap V(P_w) \neq \emptyset$ if and only if $V(P'_v) \cap V(P'_w) \neq \emptyset$. It follows that $\langle \mathcal{P}', T' \rangle$ is a *VPT* representation of G and the proof is completed. □

Corollary 3. *Any* $[h, 2, 1]$ *graph admits a full* $(h, 2, 1)$*-representation.*

Proof. Let $\langle \mathcal{P}, T \rangle$ be an $(h, 2, 1)$-representation of G. We can assume, without loss of generality, that if x is an end vertex of a path $P_v \in \mathcal{P}$ then there exists a path $P_u \in \mathcal{P}$ intersecting P_v only in x, in other case the vertex x can be removed from P_v. This implies that any path of $\mathcal{P}$ linking two branches intersects paths contained in those branches. Now, the proof proceeds inductively applying Theorem 2 in a vertex q of T of degree at least four at which $\langle \mathcal{P}, T \rangle$ is not full. □

A graph G is *split* if $V(G)$ can be partitioned into a stable set S and a clique K [2]. The pair (S, K) is a *split partition* of G. The vertices in S are called *stable vertices*, and K is called the *central clique* of G. A vertex s is a *dominated stable vertex* if $s \in S$ and there exists $s' \in S$ such that $N(s) \subseteq N(s')$. Notice that if G is split then $\mathcal{C}(G) = \{K, N[s] \text{ for } s \in S\}$.

Lemma 4. *Let* $G \in VPT \cap Split$ *with split partition* (S, K) *and let* $\langle \mathcal{P}, T \rangle$ *be a full VPT representation of* G*. If* $q \in V(T)$ *and* $C_q \neq K$*, then* $d_T(q) \leq 3$.

Proof. Assume, for a contradiction, that there exists $q \in V(T)$ such that $C_q \neq K$ and $d_T(q) \geq 4$. Since, $\langle \mathcal{P}, T \rangle$ is full and $d_T(q) \geq 4$, C_q is a clique of G, thus there exists $s \in S$ such that $C_q = N[s]$. Since in C_q there exist at least three vertices which do not belong to K, then s has at least two neighbors which are not in K. This contradicts the fact that $G \in Split$. □

A graph is *k-colorable* if its vertices can be colored with at most k colors in such a way that no two adjacent vertices share the same color. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k such that G is k-colorable.

Theorem 5 (*[16]*)**.** *For any fixed* $k \geq 3$*, the problem of deciding whether a given graph G is k-colorable is NP-complete.*

A graph is *perfect* if and only if it contains no odd cycle of length at least 5, or its complement, as induced subgraphs [3].

Theorem 6 (*[13]*)**.** *For any fixed* $k \geq 3$*, the problem of deciding whether a given perfect graph G is k-colorable is polynomial time solvable.*

3. Characterization of $[h, 2, 1]$, for $h \geq 3$

In this section, we present a characterization of *VPT* graphs which are representable in a host tree with maximum degree at most h. The characterization is given in terms of the chromatic number of the branch graphs.

Definition 7 (*[8]*)**.** Let $C \in \mathcal{C}(G)$. The branch graph of G for the clique C, denoted by $B(G/C)$, is defined as follows: its vertices are the vertices of $V(G) \setminus C$ which are adjacent to some vertex of C. Two vertices v and w are adjacent in $B(G/C)$ if and only if

(1) $vw \notin E(G)$;
(2) there exists a vertex $x \in C$ such that $xv \in E(G)$ and $xw \in E(G)$;
(3) there exists a vertex $y \in C$ such that $yv \in E(G)$ and $yw \notin E(G)$; and
(4) there exists a vertex $z \in C$ such that $zv \notin E(G)$ and $zw \in E(G)$.

As will be seen in what follows, branch graphs of VPT graphs can be used to describe intrinsic properties of representations.

Lemma 8. *Let C be a clique of a VPT graph G, $\langle (P_v)_{v\in V(G)}, T\rangle$ be a VPT representation of G and q be a vertex of T such that $C = C_q$. If v is a vertex of $B(G/C)$ then P_v is contained in some branch of T at q. If two vertices v and w are adjacent in $B(G/C)$ then P_v and P_w are not contained in a same branch of T at q.*

Proof. By the definition of branch graph, if $v \in V(B(G/C))$ then $v \notin C$. It follows that $q \notin V(P_v)$, thus P_v is contained in some branch of T at q.

Let w be adjacent to v in $B(G/C)$. Suppose to the contrary that P_v and P_w are contained in the same branch of T at q. Let x and y be the vertices of P_v and P_w respectively at minimum distance from q. Since there exists a vertex of C adjacent to v and w, there exists a path in T containing q, x and y. Therefore, without loss of generality, we can assume that x is between q and y or that $x = y$. In both cases, $N(w)\cap C \subseteq N(v)\cap C$, which contradicts the fact that v and w are adjacent in $B(G/C)$. □

Lemma 9. *Let $\langle \mathcal{P}, T\rangle$ be a VPT representation of G. Let $C \in \mathcal{C}(G)$ and $q \in V(T)$ such that $C = C_q$. If $d_T(q) = h$, then $B(G/C)$ is h-colorable.*

Proof. Let $T_1, T_2, \ldots, T_h$ be the branches of T at q. By Lemma 8, if we color each vertex v of $B(G/C)$ with the index i of the branch T_i containing the path P_v, then we obtain a proper coloring of $B(G/C)$. Since there are h branches, $B(G/C)$ is h-colorable. □

Theorem 10. *Let $G \in VPT$ and $h \geq 3$. The graph G belongs to $[h, 2, 1]$ if and only if $B(G/C)$ is h-colorable for every $C \in \mathcal{C}(G)$. The direct implication is true also for $h = 2$.*

Proof. Let $\langle \mathcal{P}, T\rangle$ be an $(h, 2, 1)$-representation of G with $h \geq 2$. If $C \in \mathcal{C}(G)$ then there exists $q \in V(T)$ such that $C = C_q$. Since $d_T(q) \leq h$, by Lemma 9, $B(G/C)$ is h-colorable.

The reciprocal implication for $h = 3$ was proven by Golumbic and Jamison in [8]; then we assume $h \geq 4$. In a similar way to [8], we will prove that if $B(G/C)$ is h-colorable for every clique C of G, then G admits an $(h, 2, 1)$-representation.

Let $\langle \mathcal{P}, T\rangle$ be a full VPT representation of G. It exists by Corollary 3.

We proceed by induction on the number k of vertices of T whose degree exceeds h. If $k = 0$ we are done. If $k > 0$, there exists a vertex q of T with degree $d > h$. Since the representation is full C_q is a clique of G.

Say $N_T(q) = \{q_1, q_2, \ldots, q_d\}$ and, for each i, let T_i be the branch of T at q containing q_i. We can assume that, for each T_i, there exists a vertex $v_i \in V(G)$ such that the corresponding path P_{v_i} is contained in the branch T_i and $q_i \in V(P_{v_i})$. For otherwise, we can contract the edge qq_i to obtain a new *VPT* representation of G without changing the intersection of the paths, and repeat this procedure as many times as needed until the assumption holds. Notice that during this operation some vertices of T disappear, and that the degree of q may increase, but the number of vertices whose degree exceeds h does not grow.

We call v_i the leader of T_i for $1 \leq i \leq d$. Observe that each leader v_i is a vertex of the branch graph $B(G/C_q)$. Let μ_i be the color of v_i in a proper h-coloring of $B(G/C_q)$. We color each branch T_i with the color μ_i of its leader v_i.

We can assume that if two branches T_i and T_j are linked then they have different colors. Indeed, suppose to the contrary that T_i and T_j are linked by a path $P_v \in \mathcal{P}$ and have the same color. Then, their leaders v_i and v_j have the same color in $B(G/C_q)$, which implies that

$$v_i \text{ and } v_j \text{ are non adjacent in } B(G/C_q). \tag{1}$$

By the definition of branch graph and (1), since v_i and v_j are non adjacent in G, $v \in C_q$ and v is adjacent to v_i and to v_j in G; we can assume, without loss of generality, that $N(v_i) \cap C_q \subseteq N(v_j) \cap C_q$. It means that the branch T_i is linked only to the branch T_j; thus we can change the color of T_i to either of the $h - 1$ remaining colors. By repeating this procedure as many times as necessary, we obtain an h coloring of the branches such that any two linked branches have different color.

Now, we obtain a new *VPT* representation $\langle P', T'\rangle$ of G as follows. The tree T' is obtained from T by means of the following procedure (in Fig. 2 we offer an example):

for every i, $1 \leq i \leq d$, remove the edge qq_i;
for every i, $1 \leq i \leq d$, add a vertex q_i' adjacent to q_i; and
for every j, $1 \leq j \leq h$, add the edges necessaries to obtain an induced path with end vertex q connecting the vertices q_i' such that the corresponding branch T_i has color μ_j.

The rest of the tree T remains unchanged. Notice that $d_{T'}(q) = h$, $d_{T'}(q_i') \leq 3$ for every i, and the remaining vertices have the same degree as vertices of T' than as vertices of T.

The only paths of $\mathcal{P}$ which are modified to obtain the paths of $\mathcal{P}'$ are those containing q. If a path $P_v \in \mathcal{P}$ has q as an endpoint and intersect a branch T_i, then we obtain P_v' by replacing in P_v the edge qq_i by the unique subpath of T' linking q and q_i. If P_v has q as an internal vertex, and intersect two branches T_i and T_j, then, we obtain P_v' by replacing in P_v the edges qq_i and qq_j by the only subpath of T' linking q_i, q and q_j. Notice that such subpath exists because linked branches have different colors, thus q_i and q_j are in different branches of T' at q.

It is easy to see that the construction described above leaves the intersection graph of paths unchanged, while the number of vertices of the host tree whose degree is greater than h, decreases. Thus, by induction, the implication is proven. □

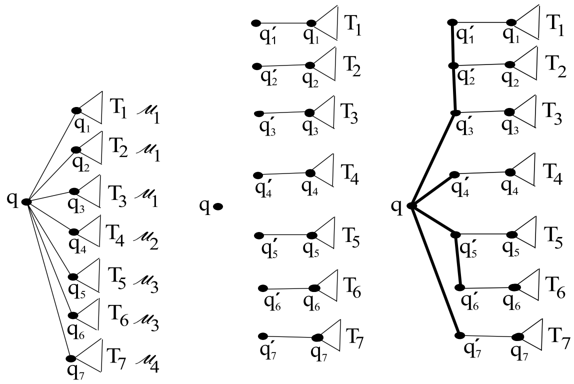

Fig. 2. $d_T(q) = 7$ and $B(G/C_q)$ is 4-colorable.

Observe that the reciprocal implication of Theorem 10 is false for $h = 2$; consider, by instance, the graph obtained from $K_{1,3}$ by subdividing each edge with a new vertex. This graph, that we call T_2^3, is a well known forbidden induced subgraph for the class of interval graphs. It is easy to see that $T_2^3 \in VPT$ and that $B(G/C)$ is 2-colorable for every clique C; however, $T_2^3 \notin [2, 2, 1]$.

Theorem 11. *Let $G \in VPT$ and $h \geq 4$. The graph G belongs to $[h, 2, 1] - [h-1, 2, 1]$ if and only if $\text{Max}_{C \in \mathcal{C}(G)}(\chi(B(G/C))) = h$. The reciprocal implication is also true for $h = 3$.*

Proof. By Theorem 10, $G \in [h, 2, 1]$ if and only if

$$\underset{C \in \mathcal{C}(G)}{\text{Max}} (\chi(B(G/C))) \leq h.$$

And, by the same Theorem 10, $G \notin [h-1, 2, 1]$ if and only if

$$\underset{C \in \mathcal{C}(G)}{\text{Max}} (\chi(B(G/C))) > h - 1. \quad \square$$

4. Complexity

In this section, we prove that, for any $h \geq 3$, the problem of deciding whether a given graph belongs to $[h, 2, 1]$ is NP-complete. We also show that recognizing $[h, 2, 1] - [h-1, 2, 1]$ graphs is NP-hard for any $h \geq 4$. Our results prove that both problems remain difficult even when the input graphs are restricted to be *VPT*, split and without dominated stable vertices.

First we state the following fundamental property of *VPT* ∩ *Split* graphs which is used in the proof of Theorems 15 and 16.

Lemma 12. *Let $G \in VPT \cap Split$ with split partition (S, K), and let $s \in S$. Then, $\chi(B(G/N[s])) \leq 3$.*

Proof. Let $\langle P, T\rangle$ be a full VPT representation of G. There exists $q \in V(T)$ such that $C_q = N[s]$. Since $N[s] \neq K$, by Lemma 4, $d_T(q) \leq 3$. Thus, by Lemma 9, $\chi(B(G/N[s])) \leq d_T(q) \leq 3$. $\square$

For the NP-completeness proof, we use a reduction from the well known chromatic number problem cited in Theorem 5.

Given a graph G we will construct in polynomial time a graph $\widehat{G} \in VPT \cap Split$ without dominated stable vertices, in such a way that $\chi(G) = h$ if and only if $\widehat{G} \in [h, 2, 1] - [h-1, 2, 1]$.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, we define the graph $\widehat{G}$ by means of its *VPT* representation $\langle \mathcal{P}, T\rangle$ as follows: the tree T is a star with a central vertex q and leaves q_i for $1 \leq i \leq n$. The path family $\mathcal{P}$ contains:

a one vertex path P_i with $V(P_i) = \{q_i\}$, for each $1 \leq i \leq n$;
a three vertex path P_{ij} with $V(P_{ij}) = \{q_i, q, q_j\}$, for each $1 \leq i < j \leq n$ such that $v_i v_j \in E(G)$; and
a two vertex path P_{iq} with $V(P_{iq}) = \{q, q_i\}$, for each $1 \leq i \leq n$ such that $d_G(v_i) = 1$.

We call each vertex of $\widehat{G}$ as the corresponding path of $\mathcal{P}$.

In Fig. 3, we offer an example of a graph G, the *VPT* representation of $\widehat{G}$, and the graph $\widehat{G}$.

Notice that $\widehat{G}$ is a split graph with the vertex set partitioned into a stable set of size $n = |V(G)|$ containing the vertices corresponding to the one vertex paths P_i; and a central clique of size $|E(G)| + |\{v \in V(G); d_G(v) = 1\}|$ containing the vertices corresponding to the remaining paths. Since all these paths contain the vertex q of T, the central clique is C_q. The other cliques of $\widehat{G}$ are the cliques C_{q_i} for $1 \leq i \leq n$, each one containing the vertices corresponding to the paths containing the vertex q_i of T respectively. The graph $\widehat{G}$ has no more cliques. In addition, every stable vertex P_i of $\widehat{G}$ is non-dominated.

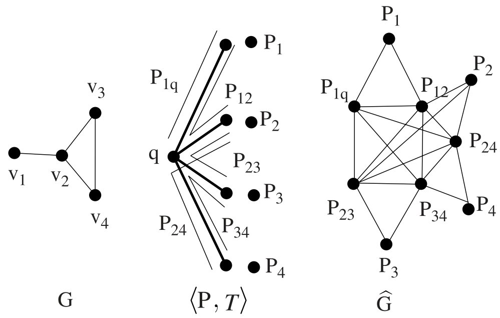

Fig. 3. A graph G, the *VPT* representation of $\widehat{G}$ and the graph $\widehat{G}$.

The main properties of $\widehat{G}$ are stated in the following two theorems.

Theorem 13. *If $\widehat{G}$ is the graph obtained from G as above, then $B(\widehat{G}/C_q) = G$.*

Proof. Notice that $B(\widehat{G}/C_q)$ has exactly n vertices: P_i for $1 \leq i \leq n$.

We will see that P_i and P_j are adjacent in $B(\widehat{G}/C_q)$ if and only if v_i and v_j are adjacent in G. If $P_iP_j \in E(B(\widehat{G}/C_q))$ then there exists a vertex of C_q adjacent to both P_i and P_j. Thus, the path $P_{ij} \in \mathcal{P}$, which implies that $v_iv_j \in E(G)$.

Reciprocally, assume $v_iv_j \in E(G)$, then P_{ij} is a vertex of C_q and it is adjacent to P_i and to P_j in $\widehat{G}$. We claim that there exists a vertex of C_q adjacent to P_i and non adjacent to P_j. Indeed, if $d_G(v_i) = 1$ then the required vertex of C_q is P_{iq}. If $d_G(v_i) > 1$, v_i must have a neighbor v_l with $l \neq j$, then the required vertex of C_q is P_{il}. In an analogous way, we can prove that there exists a vertex of C_q adjacent to P_j and non adjacent to P_i. Since P_i and P_j are non adjacent in $\widehat{G}$, by the branch graph definition, we obtain that P_i and P_j are adjacent in $B(\widehat{G}/C_q)$. We conclude that $B(\widehat{G}/C_q) = G$. □

Theorem 14. *Let $\widehat{G}$ be the graph obtained from G as above. For any $h \geq 4$, $\widehat{G}$ belongs to $[h, 2, 1] - [h-1, 2, 1]$ if and only if $\chi(G) = h$.*

Proof. Since $h \geq 4$, by Lemma 12 and Theorem 13, $\text{Max}_{C \in \mathcal{C}(\widehat{G})} \chi(B(\widehat{G}/C)) = \chi(B(\widehat{G}/C_q)) = \chi(G)$. Hence, by Theorem 11, $\widehat{G}$ belongs to $[h, 2, 1] - [h-1, 2, 1]$ if and only if $\chi(G) = h$. □

Theorem 15. *For any $h \geq 4$, the problem of deciding whether a given graph belongs to $[h, 2, 1] - [h-1, 2, 1]$ is an NP-hard, even when restricted to the class of VPT ∩ Split graphs without dominated stable vertices.*

In addition, since an $(h, 2, 1)$-representation is a polynomial certificate of belonging to $[h, 2, 1]$; using Theorem 10 and the construction above, we have proved the following result.

Theorem 16. *For any $h \geq 3$, the problem of deciding whether a given graph belongs to $[h, 2, 1]$ is NP-complete, even when restricted to the class of VPT ∩ Split graphs without dominated stable vertices.*

4.1. A polynomial time solvable subclass

We have proved that deciding whether a given *VPT ∩ Split* graph without dominated stable vertices admits a representation as intersection of paths of a tree with maximum degree h is an NP-complete problem. In what follows we describe a non-trivial subclass of *VPT ∩ Split* where this problem is polynomial time solvable.

For $n \geq 4$, an *n-sun*, denoted by S_n, is a split graph with stable set $\{s_1, s_2, \ldots, s_n\}$, central clique $\{v_1, v_2, \ldots, v_n\}$, $N(s_i) = \{v_i, v_{i+1}\}$ for $1 \leq i \leq n-1$, and $N(s_n) = \{v_n, v_1\}$. See Fig. 4.

We say that G belongs to *SVS* (special *VPT* subclass) whenever

- $G \in VPT \cap Split$,
- for all $v \in K$, $|N(v) \cap S| \leq 2$, where (S, K) is a split partition of G,
- if S_k, with $k \in \{4, 2n+1 \text{ for } n \geq 2\}$, is induced in G then there exists $v \in K$ such that v is adjacent to two non-consecutive vertices of the stable set of S_k.

The class *SVS* is not trivial, in the sense that it includes graphs in $[h, 2, 1]$ for all $h \geq 4$.

For example, for $n \geq 4$, let A_n (see [10]) be the split graph with stable set $S = \{s_1, \ldots, s_n\}$, central clique $K = \{v_{ij}, 1 \leq i < j \leq n\}$ and $N(v_{ij}) = \{s_i, s_j\}$, for all $1 \leq i < j \leq n$. It is easy to see that A_n belongs to *SVS*, and $B(A_n/K)$ is the complete graph with set of vertices $\{s_1, \ldots, s_n\}$. Hence, by Theorem 11, $A_n \in [n, 2, 1] - [n-1, 2, 1]$. (As an example see Fig. 5.)

The following two lemmas are used in the proof of the main Theorem 19 which proves that in the class *SVS*, for any $h \geq 4$, the graphs belonging to $[h, 2, 1]$ can be recognized efficiently.

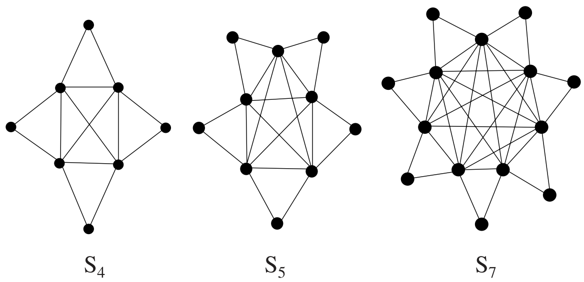

Fig. 4. The sun graphs S_4, S_5 and S_7.

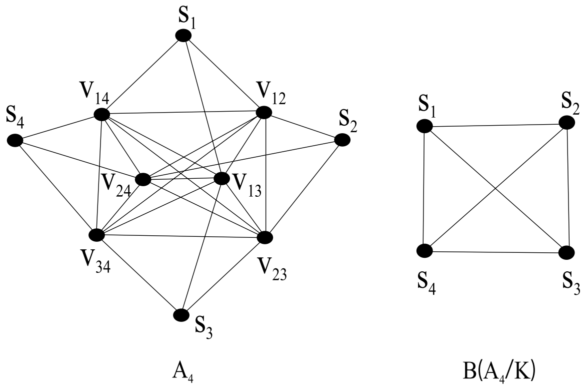

Fig. 5. The graph A_4 belongs to *SVS* and $A_4 \in [4, 2, 1] - [3, 2, 1]$.

Lemma 17. *Let $G \in VPT \cap Split$ with split partition (S, K) such that for all $v \in K$, $|N(v) \cap S| \leq 2$, and let $n \geq 4$. If $B(G/K)$ has an induced C_n then G has an induced S_n.*

Proof. Let $\langle \mathcal{P}, T \rangle$ be a *VPT* representation of G and $q \in V(T)$ such that $K = C_q$. Let C_n be an induced cycle of $B(G/K)$ with vertices $s_1, s_2, \ldots, s_n$. It is clear that every $s_i \in S$. Since s_i is adjacent to s_{i+1} in $B(G/K)$, there exists $v_i \in K$ such that v_i is adjacent to s_i and to s_{i+1} in G. Since, for all $v \in K$, $|N(v) \cap S| \leq 2$, if $i \neq i'$ then $v_i \neq v_{i'}$, thus $s_1, s_2, \ldots, s_n, v_1, v_2, \ldots, v_n$ induce an n-sun in G and the proof is completed. □

Lemma 18. *If $G \in SVS$ with split partition (S, K), then $B(G/K)$ is perfect.*

Proof. Suppose to the contrary that $B(G/K)$ is not perfect, then $B(G/K)$ has an odd cycle or the complement of an odd cycle as induced subgraphs. Since the complement of C_5 is C_5; and the complement of any odd cycle of size 7 or more has an induced C_4, it follows that $B(G/K)$ has an induced C_k, for some $k \in \{4, 2n + 1 \text{ for } n \geq 2\}$. Therefore, by Lemma 17, G has an induced S_k. Since $G \in SVS$, there exists $v \in K$ such that v is adjacent to two non-consecutive vertices s and s' of the stable set of S_k. Notice that the existence of v implies that the vertices s and s' are adjacent in $B(G/K)$. This contradicts the fact that C_k is an induced cycle of $B(G/K)$. □

Theorem 19. *For any fixed $h \geq 4$, the problem of deciding whether a given graph $G \in SVS$ belongs to $[h, 2, 1] - [h - 1, 2, 1]$ is polynomial time solvable.*

Proof. Given $G \in SVS$, in order to determinate if $G \in [h, 2, 1] - [h - 1, 2, 1]$, by Theorem 11 and Lemma 12, it is enough to calculate the chromatic number of $B(G/K)$, where K is the central clique of G. Notice that the branch graph $B(G/K)$ can be obtained in polynomial time. On the other hand, by Lemma 18, $B(G/K)$ is perfect. Thus, by Theorem 6, its chromatic number can be calculated in polynomial time. □

5. Future work

In this paper, we give, for any $h \geq 3$, a characterization of $[h, 2, 1]$ graphs, and we prove that recognizing the graphs belonging to this class is NP-complete. In addition, we show a family, called *SVS*, in which this problem is polynomial time solvable. We are working in describing a larger subclass of *VPT* graphs where this problem remains polynomial. On the other hand, we are analyzing the possibility of extending the techniques used in the present paper to characterize the classes $[h, 2, 2]$.

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