

## Research Article

# On a Conjecture regarding Fisher Information

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Fisher's information measure  $I$  plays a very important role in diverse areas of theoretical physics. The associated measures  $I_x$  and  $I_p$ , as functionals of quantum probability distributions defined in, respectively, coordinate and momentum spaces, are the protagonists of our present considerations. The product  $I_x I_p$  has been conjectured to exhibit a nontrivial lower bound in Hall (2000). More explicitly, this conjecture says that for any pure state of a particle in one dimension  $I_x I_p \geq 4$ . We show here that such is not the case. This is illustrated, in particular, for pure states that are solutions to the free-particle Schrödinger equation. In fact, we construct a family of counterexamples to the conjecture, corresponding to time-dependent solutions of the free-particle Schrödinger equation. We also conjecture that any normalizable time-dependent solution of this equation verifies  $I_x I_p \rightarrow 0$  for  $t \rightarrow \infty$ .

## 1. Introduction

A very important information measure, with manifold physical applications, was conceived by R. A. Fisher in the 1920s—for detailed discussions see [1–4]. Recent developments show that Fisher's information has a fundamental role in quantum mechanics [5–19]. In particular, it allows for the formulation of new quantum uncertainty principles [20–24]. It is usually abbreviated as  $I$  and can be thought of as a measure of the expected error in a measurement [1].

A particular instance of great relevance is that of translational families [1]. These are distribution functions whose form remains invariant under displacements of a shift parameter  $\theta$ . Thus, they are shift invariant distributions (in a Mach sense, there is no absolute origin for  $\theta$ ). The measure exhibits Galilean invariance [1]. Given a probability density  $f(\mathbf{x}, \theta)$ , with  $\mathbf{x} \in \mathbb{R}^D$  and  $\theta = (\theta_i)_{1 \leq i \leq n}$  a family of parameters, the concomitant Fisher matrix is [25]

$$I_{jk} := \int \frac{1}{f(\mathbf{x}, \theta)} \left( \frac{\partial f}{\partial \theta_j} \right) \left( \frac{\partial f}{\partial \theta_k} \right) d\mathbf{x}, \quad (1)$$

where  $d\mathbf{x} = \prod_{k=1}^D dx_k$  is the volume element in  $\mathbb{R}^D$ . In particular, for  $\theta \in \mathbb{R}^D$ , one defines translational families  $f(\mathbf{x} - \theta)$ , with elements  $I_{jk} = \int (1/f)(\partial_j f)(\partial_k f) d\mathbf{x}$ , where  $\partial_i$  represents the partial derivative with respect to the coordinate  $x_i$ . The trace of this matrix, given by  $I = \int (1/f)[\sum_{k=1}^D (\partial_k f)^2] d\mathbf{x}$ , is a good uncertainty indicator for probability distributions associated with quantum wave functions [26]. If  $\psi(\mathbf{x})$  is a normalized wave function in coordinate space ( $D$ -dimensions) and  $\tilde{\psi}(\mathbf{p}) = (2\pi)^{-D/2} \int e^{-i\mathbf{x}\cdot\mathbf{p}} \psi(\mathbf{x}) d\mathbf{x}$  is its momentum-counterpart, the corresponding probability densities are, respectively,  $\rho(\mathbf{x}) = |\psi(\mathbf{x})|^2$  and  $\tilde{\rho}(\mathbf{p}) = |\tilde{\psi}(\mathbf{p})|^2$ , with associated Fisher measures

$$I_x = \int \frac{1}{\rho} [\nabla_x \rho]^2 d\mathbf{x}, \quad (2)$$

$$I_p = \int \frac{1}{\tilde{\rho}} [\nabla_p \tilde{\rho}]^2 d\mathbf{p}, \quad (3)$$

which allow one to study uncertainty relations via the product  $I_x I_p$  [26].

For instance, one can demonstrate that if  $\psi(\mathbf{x})$  (or  $\tilde{\psi}(\mathbf{p})$ ) is real, then  $I_x I_p \geq 4D^2$  [5], with equality for coherent states of the harmonic oscillator (HO) [26]. For general, mixed states it is clear that the product  $I_x I_p$  does not possess a nontrivial lower bound (e.g., one can use thermal HO states, represented by Gaussian distributions in both coordinates and momenta, in the high temperature limit). In the case of pure states, though, the existence of such a lower bound for  $I_x I_p$  was an open question. Hall conjectured that the relation  $I_x I_p \geq 4$  might hold in general for pure states in one dimension [26, page 3]. We will next present a couple of counterexamples that show this conjecture to be incorrect. Our examples give rise to a new conjecture: for a bounded wave function one has  $I_x I_p \rightarrow 0$  when  $t \rightarrow \infty$ .

## 2. Counterexamples

Our first example is taken from the considerations (in a different context) of [5]. A free-particle's one-dimensional wave packet  $\psi(x, t)$  (unit mass) evolves according to Schrödinger's equation

$$\mathbf{i} \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2}. \quad (4)$$

Setting the initial conditions

$$\begin{aligned} \psi(x, 0) &= A_0 \exp\left[-\frac{x^2}{2\Delta^2}\right], \\ \tilde{\psi}(p, 0) &= \tilde{A}_0 \exp\left[-\frac{\Delta^2 p^2}{2}\right], \end{aligned} \quad (5)$$

with  $A_0 = \Delta^{-1/2} \pi^{-1/4}$ ,  $\tilde{A}_0 = \Delta^{1/2} \pi^{-1/4}$ , and  $\Delta > 0$ , that correspond to a Gaussian packet, one finds the solution

$$\psi(x, t) = A(t) \exp\left[-\frac{x^2}{2\Delta^2(1 + \mathbf{i}t/\Delta^2)}\right], \quad (6)$$

where  $A(t) = A_0(1 + \mathbf{i}t/\Delta^2)^{-1/2}$ . The associated probability densities are

$$\begin{aligned} \rho(x, t) &= \frac{\Delta}{\sqrt{\pi(\Delta^4 + t^2)}} \exp\left[-\frac{\Delta^2 x^2}{\Delta^4 + t^2}\right], \\ \bar{\rho}(p, t) &= \frac{\Delta}{\sqrt{\pi}} \exp\left[-\Delta^2 p^2\right]. \end{aligned} \quad (7)$$

The product  $I_x I_p = 4\Delta^4(\Delta^4 + t^2)^{-1}$  obeys the relation  $I_x I_p < 4$  for  $t > 0$ . Also, one has  $I_x I_p \rightarrow 0$  for  $t \rightarrow \infty$ .

We pass now to another free-particle solution, given by the first partial derivative of  $\psi(x, t)$  with respect to  $x$ :  $\psi^{(1)}(x, t) \propto \partial_x \psi(x, t)$  (see (4)), correctly normalized. It is easy to see that  $\psi^{(1)}(x, t)$  is a solution by deriving both members of (4); that is,

$$\mathbf{i} \frac{\partial}{\partial t} \frac{\partial \psi}{\partial x} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \frac{\partial \psi}{\partial x}. \quad (8)$$

The new solution is

$$\psi^{(1)}(x, t) = A^{(1)}(t) \exp\left[-\frac{x^2}{2\Delta^2(1 + \mathbf{i}t/\Delta^2)}\right], \quad (9)$$

with  $A^{(1)}(t) = -2^{1/2} \pi^{1/4} \Delta^{3/2} (\Delta^2 + \mathbf{i}t)^{-3/2}$ . The two corresponding densities are

$$\begin{aligned} \rho^{(1)}(x, t) &= \frac{2\Delta^3}{\sqrt{\pi(\Delta^4 + t^2)^3}} x^2 \exp\left[-\frac{\Delta^2 x^2}{\Delta^4 + t^2}\right], \\ \bar{\rho}^{(1)}(p, t) &= \frac{2\Delta^3}{\sqrt{\pi}} p^2 \exp\left[-\Delta^2 p^2\right]. \end{aligned} \quad (10)$$

The product is  $I_x^{(1)} I_p^{(1)} = 36\Delta^4(\Delta^4 + t^2)^{-1}$ , verifying  $I_x^{(1)} I_p^{(1)} < 4$  for  $t > 2\sqrt{2}\Delta^2$ , and  $I_x^{(1)} I_p^{(1)} \rightarrow 0$  when  $t \rightarrow \infty$ .

In general, one can show that the whole family of solutions of (4) given by successive derivatives of  $\psi(x, t)$ , that is, the set  $\{\psi^{(n)}(x, t) \mid \psi^{(n)}(x, t) = N_n \partial_x^n \psi(x, t), n = 0, 1, 2, \dots\}$ , verifies that  $I_x I_p \rightarrow 0$  when  $t \rightarrow \infty$ , with  $N_n$  the pertinent normalization constants. Thus, the family  $\{\psi^{(n)}(x, t)\}_{n \in \mathbb{N}_0}$  yields infinite counterexamples to Hall's conjecture. To see this, one needs first to rewrite the Fisher measure in wave function's terms, so that (2) becomes equivalent to

$$I_x = 4 \int (\nabla_x |\psi|)^2 dx, \quad (11)$$

or, in one dimension,  $I_x = \int (\partial_x |\psi|)^2 dx$ . Further,  $|\psi| = \psi^* \psi$ . Thus,  $I_x$  can be expressed in terms of  $\psi$  and  $\psi^{(1)}$ . In one dimension one has

$$I_x = 4 \int (\psi^{(1)*} \psi + \psi^* \psi^{(1)})^2 dx. \quad (12)$$

In general, for  $\psi^{(k)}$ , the Fisher's measure associated with the distribution  $|\psi^{(k)}|^2$  becomes

$$I_x^{(k)} = 4 \int (\psi^{(k+1)*} \psi^{(k)} + \psi^{(k)*} \psi^{(k+1)})^2 dx. \quad (13)$$

We show now that the integrand tends to zero for  $t \rightarrow \infty$ . Thus,  $I_x^{(k)} \rightarrow 0$  in such a limit. Actually, we will show that  $\psi^{(k)}(x, t) \rightarrow 0$  for  $t \rightarrow \infty$ . The  $k$ th derivative of  $\psi(x, t) \equiv \psi^{(0)}(x, t)$  is proportional to the  $k$ th derivative of a Gaussian distribution, given by

$$\begin{aligned} \psi^{(k)}(x, t) &= N_k(t) \frac{\partial^k}{\partial x^k} \psi^{(0)}(x, t) \\ &= N_k(t) \frac{\partial^k}{\partial x^k} \left( A(t) e^{-c(t)^2 x^2} \right) \\ &= N_k(t) A(t) (-1)^k c(t)^2 H_k(c(t)x) e^{-c(t)^2 x^2} \\ &= N_k(t) (-1)^k c(t)^2 H_k(c(t)x) \psi^{(0)}(x, t), \end{aligned} \quad (14)$$

where  $c(t)^2 = (2(\Delta^2 + \mathbf{i}t))^{-1}$  and  $H_k(y)$  is the Hermite polynomial of degree  $k$  in the variable  $y$ . The time-dependent

parameters  $c(t)$ ,  $\psi^{(0)}(x, t)$ , and  $A(t)$  vanish for  $t \rightarrow \infty$ . What is the behavior of  $N_k$ ? Let us see what happens with  $\tilde{\psi}^{(k)}(p, t)$ , the  $k$ th solution in momentum space, corresponding to the Fourier transform of  $\psi^{(k)}(x, t)$ . We have

$$\begin{aligned}\tilde{\psi}^{(k)}(p, t) &= \frac{1}{\sqrt{2\pi}} \int e^{-ixp} \psi^{(k)}(x, t) dx \\ &= N_k(t) A(t) \int e^{-ixp} \frac{\partial^k}{\partial x^k} e^{-c(t)^2 x^2} dx \\ &= N_k(t) A(t) (\mathbf{i}p)^k \frac{1}{\sqrt{2c(t)}} \exp\left[-\frac{p^2}{4c(t)^2}\right] \\ &= N_k(t) \sqrt{c(t)} (\mathbf{i}p)^k \tilde{\psi}^{(0)}(p, t).\end{aligned}\quad (15)$$

Demanding normalization leads to

$$\begin{aligned}1 &= \int \tilde{\psi}^{(k)}(p, t) \tilde{\psi}^{(k)*}(p, t) dp \\ &= |N_k(t)|^2 |c(t)| \int p^{2k} |\tilde{\psi}^{(0)}|^2 dp \\ &= |N_k(t)|^2 |c(t)| \frac{\Delta}{\sqrt{\pi}} \int p^{2k} e^{-\Delta^2 p^2} dp \\ &= |N_k(t)|^2 |c(t)| \frac{\Delta}{\sqrt{\pi}} \Gamma\left(k + \frac{1}{2}\right) \Delta^{-2k-1}.\end{aligned}\quad (16)$$

Thus,

$$|N_k(t)|^2 = \frac{\sqrt{\pi} \Delta^{2k}}{\Gamma(k + 1/2)} 2\sqrt{\Delta^4 + t^2}.\quad (17)$$

Remember that

$$\psi^{(0)}(x, t) = A(t) e^{-c(t)^2 x^2},\quad (18)$$

with  $A(t) = \pi^{-1/4} \sqrt{2\Delta} c(t)$ , and

$$\tilde{\psi}^{(0)}(p, t) = \frac{\sqrt{\Delta}}{\pi^{1/4}} \exp\left[-\frac{p^2}{4c(t)^2}\right].\quad (19)$$

One finds the following limits for the absolute values of the wave functions:

$$\begin{aligned}|\psi^{(k)}(x, t)|^2 &= |N_k(t) c(t)^2 H_k(c(t)x) \psi^{(0)}(x, t)|^2 \\ &\sim |c(t)|^{-1} |c(t)|^4 |c(t)|^{2k} |c(t)|^2 e^{-2\Re(c(t)^2)x^2} \\ &\sim |c(t)|^{2k+5} \exp\left[-\frac{\Delta^2 x^2}{\Delta^4 + t^2}\right] \xrightarrow{t \rightarrow \infty} 0,\end{aligned}\quad (20)$$

$$\begin{aligned}|\tilde{\psi}^{(k)}(p, t)|^2 &= |N_k(t) \sqrt{c(t)} (\mathbf{i}p)^k \tilde{\psi}^{(0)}(p, t)|^2 \\ &\sim |c(t)|^{-1} |c(t)| \left| \exp\left[-\frac{p^2}{4c(t)^2}\right] \right|^2 \\ &= e^{-\Delta^2 p^2}.\end{aligned}\quad (21)$$

Equation (20) indicates that all functions  $\psi^{(k)}(x, t)$  vanish for  $t \rightarrow \infty$ . Accordingly, from (13) we find  $I_x^{(k)} \rightarrow 0$  for  $t \rightarrow \infty$  for all  $k = 0, 1, 2, \dots$ . Further, (21) shows that  $\tilde{\psi}^{(k)}(p, t)$  does not vanish in this limit. In fact,  $|\tilde{\psi}^{(k)}(p, t)|$  does not depend on  $t$ .

So as to understand what happens with  $I_p^{(k)}$  let us see an expression analogous to (13) in momentum space:

$$I_p^{(k)} = 4 \int (\tilde{\psi}^{(k+1)*} \tilde{\psi}^{(k)} + \tilde{\psi}^{(k)*} \tilde{\psi}^{(k+1)})^2 dx.\quad (22)$$

Expanding the integrand using (15) we have

$$\begin{aligned}\tilde{\psi}^{(k+1)*} \tilde{\psi}^{(k)} + \tilde{\psi}^{(k)*} \tilde{\psi}^{(k+1)} \\ = 2\Im(N_k N_{k+1}^*) |c(t)| p^{2k+1} |\tilde{\psi}^{(0)}(p, t)|^2.\end{aligned}\quad (23)$$

Introducing this into (22), and remembering that both  $N_k$  and  $c(t)$  are independent of  $p$ , we find

$$\begin{aligned}I_p^{(k)} &= 16\Im(N_k N_{k+1}^*)^2 |c(t)|^2 \frac{\Delta^2}{\pi} \int p^{4k+2} e^{-2\Delta^2 p^2} \\ &= 16\Im(N_k N_{k+1}^*)^2 |c(t)|^2 \frac{\Delta^2}{\pi} \frac{\Gamma(2k+3/2)}{(2\Delta^2)^{2k+3/2}}.\end{aligned}\quad (24)$$

Since  $|N_k(t)|^2 \sim |c(t)|^{-1}$  (see (16)), one has  $|N_k(t) N_{k+1}|^2 |c(t)|^2 \sim 1$  and thus  $I_p^{(k)}$  becomes bounded. Thus, there exists  $I_{p, \max}^{(k)} \in \mathbb{R}_{>0}$  such that  $I_p^{(k)} \leq I_{p, \max}^{(k)}$  for all  $k = 0, 1, 2, \dots$ . We conclude that  $I_x^{(k)} I_p^{(k)} \rightarrow 0$  for  $t \rightarrow \infty$  for the whole family of solutions  $\{\psi^{(k)}(x, t)\}_{k \in \mathbb{N}_0}$ .

### 3. Conclusions

We conclude by reiterating that we have found an infinite number of counterexamples to the conjecture  $I_x I_p \geq 4$ , for pure states, put forward in [26]. On the basis of these results, we conjecture that, for any normalizable wave function  $\psi(x, 0)$ , the corresponding time-dependent solution  $\psi(x, t)$  of the free-particle Schrödinger equation satisfies  $I_x I_p \rightarrow 0$  for  $t \rightarrow \infty$ .

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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