Characterizing paths graphs on bounded degree trees by minimal forbidden induced subgraphs

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Article history:
Received 25 June 2013
Received in revised form 25 August 2014
Accepted 27 August 2014
Available online 20 September 2014

Keywords:
Intersection graphs
Representations on trees
VPT graphs
Critical graphs
Forbidden subgraphs

Abstract
An undirected graph \( G \) is called a VPT graph if it is the vertex intersection graph of a family of paths in a tree. The class of graphs which admit a VPT representation in a host tree with maximum degree at most \( h \) is denoted by \([h, 2, 1]\). The classes \([h, 2, 1]\) are closed under taking induced subgraphs, therefore each one can be characterized by a family of minimal forbidden induced subgraphs. In this paper we associate the minimal forbidden induced subgraphs for \([h, 2, 1]\) which are VPT with (color) \( h \)-critical graphs. We describe how to obtain minimal forbidden induced subgraphs from critical graphs, even more, we show that the family of graphs obtained using our procedure is exactly the family of VPT minimal forbidden induced subgraphs for \([h, 2, 1]\). The members of this family together with the minimal forbidden induced subgraphs for VPT (Lévêque et al., 2009; Tondato, 2009), are the minimal forbidden induced subgraphs for \([h, 2, 1]\), with \( h \geq 3 \). By taking \( h = 3 \) we obtain a characterization by minimal forbidden induced subgraphs of the class VPT \( \cap \) EPT = EPT \( \cap \) Chordal = \([3, 2, 2]\) = \([3, 2, 1]\) (see Golumbic and Jamison, 1985).

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1. Introduction

The intersection graph of a family is a graph whose vertices are the members of the family, and the adjacency between vertices is defined by a non-empty intersection of the corresponding sets. Classic examples are interval graphs and chordal graphs.

An interval graph is the intersection graph of a family of intervals of the real line, or, equivalently, the vertex intersection graph of a family of subpaths of a path. A chordal graph is a graph without chordless cycles of length at least four. Gavril [5] proved that a graph is chordal if and only if it is the vertex intersection graph of a family of subtrees of a tree. Both classes have been widely studied [3].

In order to allow larger families of graphs to be represented by subtrees, several graph classes are defined imposing conditions on trees, subtrees and intersection sizes [9,10]. Let \( h, s \) and \( t \) be positive integers; an \((h, s, t)\)-representation of a graph \( G \) consists in a host tree \( T \) and a collection \( (T_v)_{v \in V(G)} \) of subtrees of \( T \), such that (i) the maximum degree of \( T \) is at most \( h \), (ii) every subtree \( T_v \) has maximum degree at most \( s \), and (iii) two vertices \( v \) and \( v' \) are adjacent in \( G \) if and only if the corresponding subtrees \( T_v \) and \( T_{v'} \) have at least \( t \) vertices in common in \( T \). The class of graphs that have an \((h, s, t)\)-representation is denoted by \([h, s, t]\). When there is no restriction on the maximum degree of \( T \) or on the maximum degree of the subtrees, we use \( h = \infty \) and \( s = \infty \) respectively. Therefore, \([\infty, \infty, 1]\) is the class of chordal graphs and \([2, 2, 1]\) is the class of interval graphs. The classes \([\infty, 2, 1]\) and \([\infty, 2, 2]\) are called VPT and EPT respectively in [7]; and UV and UE, respectively in [13].
In [6,14], it is shown that the problem of recognizing VPT graphs is polynomial time solvable. Recently, in [1], generalizing a result given in [7], we have proved that the problem of deciding whether a given VPT graph belongs to \([h,2,1]\) is NP-complete even when restricted to the class VPT ∩ Split without dominated stable vertices. The classes \([h,2,1]\), \(h \geq 2\), are closed under taking induced subgraphs; therefore each one can be characterized by a family of minimal forbidden induced subgraphs. Such a family is known only for \(h = 2\) [11] and there are some partial results for \(h = 3\) [4]. In this paper we associate the VPT minimal forbidden induced subgraphs for \([h,2,1]\) with (color) \(h\)-critical graphs. We describe how to obtain minimal forbidden induced subgraphs from critical graphs, even more, we show that the family of graphs obtained using our procedure is exactly the family of VPT minimal forbidden induced subgraphs for \([h,2,1]\). The members of this family together with the minimal forbidden induced subgraphs for VPT (see Fig. 2) which were determined in [12,15], are the minimal forbidden induced subgraphs for \([h,2,1]\), with \(h \geq 3\). Notice that by taking \(h = 3\) we obtain a characterization by minimal forbidden induced subgraphs of the class VPT ∩ EPT = EPT ∩ Chordal = \([3,2,2]\) = \([3,2,1]\) [7].

The paper is organized as follows: in Section 2, we provide basic definitions and basic results. In Section 3, we give necessary conditions for VPT minimal non-\([h,2,1]\) graphs. In Section 4, we show a procedure to construct minimal non-\([h,2,1]\) graphs. In Section 5, we describe the family of all minimal non-\([h,2,1]\) graphs.

2. Preliminaries

Throughout this paper, graphs are connected, finite and simple. The vertex set and the edge set of a graph \(G\) are denoted by \(V(G)\) and \(E(G)\) respectively. The open neighborhood of a vertex \(v\), represented by \(N_c(v)\), is the set of vertices adjacent to \(v\). The closed neighborhood \(N_C(v)\) is \(N_c(v) \cup \{v\}\). The degree of \(v\), denoted by \(d_c(v)\), is the cardinality of \(N_c(v)\). For simplicity, when no confusion can arise, we omit the subindex \(C\) and write \(N(v), N[v] \) or \(d(v)\). Two vertices \(x, y \in V(G)\) are called true twins if \(N[x] = N[y]\).

A complete set is a subset of mutually adjacent vertices. A clique is a maximal complete set. The family of cliques of \(G\) is denoted by \(C(G)\). A stable set, also called an independent set, is a subset of pairwise non-adjacent vertices.

A graph \(G\) is \(k\)-colorable if its vertices can be colored with at most \(k\) colors in such a way that no two adjacent vertices share the same color. The chromatic number of \(G\), denoted by \(\chi(G)\), is the smallest \(k\) such that \(G\) is \(k\)-colorable. A vertex \(v \in V(G)\) or an edge \(e \in E(G)\) is a critical element of \(G\) if \(\chi(G - v) < \chi(G)\) or \(\chi(G - e) < \chi(G)\) respectively. A graph \(G\) with chromatic number \(h\) is \(h\)-vertex critical (resp. \(h\)-critical) if each of its vertices (resp. edges) is a critical element.

A VPT representation of \(G\) is a pair \((\mathcal{P}, T)\) where \(\mathcal{P}\) is a family \((P_v)_{v \in V(G)}\) of subpaths of a host tree \(T\) satisfying that two vertices \(v\) and \(v'\) of \(G\) are adjacent if and only if \(P_v\) and \(P_{v'}\) have at least one vertex in common; in such case we say that \(P_v\) intersects \(P_{v'}\). When the maximum degree of the host tree is at most \(h\) the VPT representation of \(G\) is called an \((h,2,1)\)-representation of \(G\). The class of graphs which admit an \((h,2,1)\)-representation is denoted by \([h,2,1]\).

Since a family of vertex paths in a tree satisfies the Helly property [2], if \(C\) is a clique of \(G\) then there exists a vertex \(q\) of \(T\) such that \(C = \{v \in V(G) \mid q \in V(P_v)\}\). On the other hand, if \(q\) is any vertex of the host tree \(T\), the set \(\{v \in V(G) \mid q \in V(P_v)\}\), denoted by \(C_q\), is a complete set of \(G\), but not necessarily a clique. In order to avoid this drawback, we introduce the notion of full representation at \(q\).

Let \((\mathcal{P}, T)\) be a VPT representation of \(G\) and let \(q\) be a vertex of \(T\) with degree \(h\). The connected components of \(T - q\) are called the branches of \(T\) at \(q\). A path is contained in a branch if all its vertices are vertices of the branch. Notice that if \(N_T(q) = \{q_1, q_2, \ldots, q_h\}\), then \(T\) has exactly \(h\) branches at \(q\). The branch containing \(q_i\) is denoted by \(T_i\). Two branches \(T_i\) and \(T_j\) are linked by a path \(P_{ij} \in \mathcal{P}\) if both vertices \(q_i, q_j\) belong to \(V(P_{ij})\).

**Definition 1.** A VPT representation \((\mathcal{P}, T)\) is full at a vertex \(q\) of \(T\) if, for every two branches \(T_i\) and \(T_j\) at \(q\), there exist paths \(P_{i1}, P_{j1}, P_{ij} \in \mathcal{P}\) such that: (i) the branches \(T_i, T_j\) are linked by \(P_{ij}\); (ii) \(P_{ij}\) is contained in \(T_i\) and intersects \(P_{i1}\) in at least one vertex; and (iii) \(P_{ij}\) is contained in \(T_j\) and intersects \(P_{j1}\) in at least one vertex.

A clear consequence of the previous definition is that if \((\mathcal{P}, T)\) is full at a vertex \(q\) of \(T\), with \(d_T(q) = h \geq 3\), then \(C_q\) is a clique of \(G\).

The following theorem from [1] shows that any VPT representation which is not full at some vertex \(q\) of \(T\) with \(d_T(q) \geq 4\) can be modified to obtain a new VPT representation without increasing the maximum degree of the host tree; while decreasing the degree of the vertex \(q\).

**Theorem 2 ([1]).** Let \((\mathcal{P}, T)\) be a VPT representation of \(G\). Assume there exists a vertex \(q \in V(T)\) with \(d_T(q) = h \geq 4\) and two branches of \(T\) at \(q\) which are linked by no path of \(\mathcal{P}\). Then there exists a VPT representation \((\mathcal{P}', T')\) of \(G\) with \(V(T') = V(T) \cup \{q', q'\}\), \(q' \not\in V(T)\), and

\[
d_{T'}(x) = \begin{cases} 
3 & \text{if } x = q'; \\
h - 1 & \text{if } x = q; \\
d_T(x) & \text{if } x \in V(T') \setminus \{q, q'\}.
\end{cases}
\]
Definition 3 ([7]). Let \( C \in \mathcal{C}(G) \). The branch graph of \( G \) for the clique \( C \), denoted by \( B(G/C) \), is defined as follows: its vertices are the vertices of \( V(G) \setminus C \) which are adjacent to at least one vertex of \( C \). Two vertices \( u \) and \( v \) are adjacent in \( B(G/C) \) if and only if

1. \( uv \not\in E(G) \);
2. there exists a vertex \( x \in C \) such that \( xu \in E(G) \) and \( xv \in E(G) \);
3. there exists a vertex \( y \in C \) such that \( yu \in E(G) \) and \( yv \not\in E(G) \);
4. there exists a vertex \( z \in C \) such that \( zu \not\in E(G) \) and \( zv \in E(G) \).

It is clear that if \( C \in \mathcal{C}(G) \) and \( v \in V(G) \setminus C \), then \( C \in \mathcal{C}(G - v) \). The following claim, whose proof is trivial, describes the branch graph of \( G - v \) for the clique \( C \) in terms of \( B(G/C) \).

Claim 4. Let \( C \in \mathcal{C}(G) \) and \( v \in V(G) \setminus C \): (i) if \( v \not\in V(B(G/C)) \) then \( B((G - v)/C) = B(G/C) \); (ii) if \( v \in V(B(G/C)) \) then \( B((G - v)/C) = B(G/C) - v \).

Lemma 5 ([1]). Let \( C \) be a clique of a VPT graph \( G \), \( (\mathcal{P}, T) \) be a VPT representation of \( G \) and \( q \) be a vertex of \( T \) such that \( C = C_q \). If \( v \) is a vertex of \( B(G/C) \), then \( P_v \) is contained in some branch of \( T \) at \( q \). If two vertices \( u \) and \( v \) are adjacent in \( B(G/C) \), then \( P_u \) and \( P_v \) are not contained in the same branch of \( T \) at \( q \).

In [1] we proved the following two results which show that there is a relationship between the VPT graphs that can be represented in a tree with maximum degree at most \( h \) and the chromatic number of their branch graphs.

Lemma 6 ([1]). Let \( C \) be a clique of a VPT graph \( G \), \( (\mathcal{P}, T) \) be a VPT representation of \( G \) and \( q \) be a vertex of \( T \) such that \( C = C_q \). If \( d_T(q) = h \), then \( B(G/C) \) is \( h \)-colorable.

Theorem 7 ([1]). Let \( G \in \text{VPT} \) and \( h \geq 4 \). The graph \( G \) belongs to \([h, 2, 1] - [h - 1, 2, 1] \) if and only if \( \max_{C \in \mathcal{C}(G)}(\chi(B(G/C))) = h \). The reciprocal implication is also true for \( h = 3 \).

Definition 8. A clique \( K \) of a graph \( G \) is called principal if

\[
\max_{C \in \mathcal{C}(G)}(\chi(B(G/C))) = \chi(B(K)).
\]

A graph \( G \) is split if \( V(G) \) can be partitioned into a stable set \( S \) and a clique \( K \). The pair \((S, K)\) is the split partition of \( G \) and this partition is unique up to isomorphisms. The vertices in \( S \) are called stable vertices, and \( K \) is called the central clique of \( G \). We say that a vertex \( s \) is a dominated stable vertex if \( s \in S \) and there exists \( s' \in S \) such that \( N(s) \subseteq N(s') \). Notice that if \( G \) is split then \( \mathcal{C}(G) = \{K\} \cup \{N[s] \mid s \in S\} \). We will write \text{Split} for the class of split graphs.

Lemma 9. If \((S, K)\) is the split partition of \( G \in \text{VPT} \cap \text{Split} \), then \( K \) is a principal clique of \( G \).

Proof. Let \( s \in S \), we know that \( N[s] \in \mathcal{C}(G) \). Observe that \( V(B(G/N[s])) = (K - N(s)) \cup S' \), with \( S' = \{x \in S \mid N(x) \cap N(s) \neq \emptyset\} \). We claim that the vertices of \( K - N(s) \) are isolated in \( B(G/N[s]) \). Indeed, let \( x \in K - N(s) \); if \( y \in K - N(s) \), then \( xy \not\in E(B(G/N[s])) \) because \( xy \not\in E(G) \) and, if \( y \in S' \) then \( xy \not\in E(B(G/N[s])) \) because \( N(y) \subseteq N(x) \). Then, the chromatic number of \( B(G/N[s]) \) is equal to the chromatic number of the subgraph of \( B(G/N[s]) \) induced by \( S' \), which clearly is a subgraph of \( B(K) \). Thus \( \chi(B(G/N[s])) \leq \chi(B(K)) \). Hence \( \max_{C \in \mathcal{C}(G)}(\chi(B(G/C))) = \chi(B(K)) \), that is, \( K \) is a principal clique of \( G \). \( \square \)

3. Necessary conditions for VPT minimal non-[h, 2, 1] graphs

In this section we give necessary conditions for being a VPT minimal non-[h, 2, 1] graph; recall that:

Definition 10. A minimal non-[h,2,1] graph is a minimal forbidden induced subgraph for the class \([h, 2, 1]\), this means any graph \( G \) such that \( G \not\in [h, 2, 1] \) and \( G - v \not\in [h, 2, 1] \) for every vertex \( v \in V(G) \).

Theorem 11. Let \( G \in \text{VPT} \) and let \( h \geq 3 \). If \( G \) is a minimal non-[h, 2, 1] graph, then \( G \in [h + 1, 2, 1] \).

Proof. Let \( C \in \mathcal{C}(G) \) and \( v \not\in C \). We know that \( G - v \not\in [h, 2, 1] \) then, by Theorem 7, \( \chi(B((G - v)/C)) \leq h \). By Claim 4, \( \chi(B((G - v)/C)) \leq \chi(B(G/C)) \leq \chi(B(G/C) - v) \geq \chi(B(G/C)) \). Thus \( \chi(B(G/C)) - 1 \leq h \) and hence \( \chi(B(G/C)) \leq h + 1 \). Then, by Theorem 7, \( G \in [h + 1, 2, 1] \). \( \square \)

Theorem 12. Let \( K \) be a principal clique of a VPT minimal non-[h, 2, 1] graph \( G \), with \( h \geq 3 \). Then: (i) \( V(B(K)) = V(G) - K \); (ii) if \( v \in V(G) - K \) then \( |N(v) \cap K| > 1 \); (iii) \( B(K) \) is \((h + 1)\)-vertex critical; (iv) if \( s_1, s_2 \in V(G) - K \) then \( N(s_1) \cap K \neq N(s_2) \cap K \).
**Proof.** By Theorem 11, $G \in [h + 1, 2, 1]$. Then, by Theorem 7, since $K$ is a principal clique of $G$, we have that
\[
\chi(B(G/K)) = h + 1. \tag{1}
\]

(i) It is clear that $V(B(G/K)) \subseteq V(G) - K$. Suppose there exists $v \in V(G) - K$ such that $v \notin V(B(G/K))$. Thus, by Claim 4, $B((G - v)/K) = B(G/K)$. Since $G$ is a minimal non-$[h, 2, 1]$ graph, $G - v \in [h, 2, 1]$ and, by Theorem 7, $B((G - v)/K) = h$-colorable. Thus $B(G/K)$ is $h$-colorable which contradicts the condition (1).

(ii) By item (i), we know that $v \in V(B(K))$; then $|N(v) \cap K| \geq 1$. If $|N(v) \cap K| = 1$, $v$ would be an isolated vertex of $B(G/K)$ and $\chi(B(G/K)) = \chi(B(G/K) - v)$. But, by Claim 4 and Theorem 7, $\chi(B(G/K) - v) = \chi(B(G - v)/K) = h$, which also contradicts the condition (1).

(iii) By the condition (1), $\chi(B(G/K)) = h + 1$; assume, in order to obtain a contradiction, that $B(G/K)$ is not $(h + 1)$-vertex critical; then there exists $v \in V(B(G/K))$ such that $\chi(B(G/K) - v) = h + 1$. Then, since $v \in V(B(G/K))$, by Claim 4, $\chi(B(G - v)/K) = \chi(B(G/K) - v) = h + 1$, which contradicts the fact that $G$ is a minimal non-$[h, 2, 1]$ graph.

(iv) We will see that if $N(s_1) \cap K = N(s_2) \cap K$ then $s_1s_2 \notin E(B(K))$ and $N_{B(G/K)}(s_1) = N_{B(G/K)}(s_2)$, which will contradict the fact that $B(G/K)$ is $(h + 1)$-vertex critical. Indeed, if $N(s_1) \cap K = N(s_2) \cap K$ then $s_1s_2 \notin E(B(K))$ by the definition of branch graph. Moreover, if $s_3 \in N_{B(G/K)}(s_1)$ then there exist $k_1, k_2, k_3 \in K$ such that $k_1s_1 \in E(G), k_1s_2 \in E(G), k_2s_3 \notin E(G), k_3s_3 \in E(G)$. And, since $N(s_1) \cap K = N(s_2) \cap K$, it follows that $k_1s_2 \in E(G), k_2s_3 \in E(G), k_3s_3 \notin E(G)$. In addition, $s_1s_2 \notin E(G)$ because in other case would be an induced $4$-cycle $[s_2, k_3, s_3, s]$ in $G$, contradicting the fact that $G \in VPT$ (see Fig. 2). Hence $s_3 \in N_{B(G/K)}(s_2)$; we have proved that $N_{B(G/K)}(s_1) \subseteq N_{B(G/K)}(s_2)$. By symmetry, it is easy to see that $N_{B(G/K)}(s_2) \subseteq N_{B(G/K)}(s_1)$. \(\square\)

The following lemma and definition are used in the proof of Theorem 15 which states that any VPT minimal non-$[h, 2, 1]$ graph is split and has no dominated stable vertices.

**Lemma 13.** Let $K$ be a principal clique of a VPT minimal non-$[h, 2, 1]$ graph $G$, with $h \geq 3$. Then, $K - \{k\} \in \mathcal{E}(G - k)$ for all $k \in K$.

**Proof.** Let $(\mathcal{P}, T)$ be an $(h + 1, 2, 1)$-representation of $G$ and let $q \in V(T)$ such that $K = C_q$. We claim that $(\mathcal{P}, T)$ is full at $q$. Indeed, suppose, for a contradiction, that $(\mathcal{P}, T)$ is not full at $q$. We can assume, without loss of generality, that if $x$ is an end vertex of a path $P_x \in \mathcal{P}$ then there exists a path $P_y \in \mathcal{P}$ intersecting $P_x$ only in $x$, in other case the vertex $x$ can be removed from $P_x$. This implies that any path of $\mathcal{P}$ linking two branches intersects paths contained in those branches. Hence since $(\mathcal{P}, T)$ is not full at $q$, there exist branches $T_x$ and $T_y$ of $T$ at $q$ which are linked by no path of $\mathcal{P}$. Then, by Theorem 2, we can obtain a new VPT representation $(\mathcal{P}', T')$ of $G$ with $d_T(q) \leq h$. Thus, by Lemma 6, $B(G/C_q)$ is $h$-colorable which contradicts the fact that $C_q$ is a principal clique of $G$.

Hence since $(\mathcal{P}, T)$ is full at $q$, every pair of branches of $T$ at $q$ are linked by a path of $\mathcal{P}$. If there exists $k \in C_q$ such that $C_q - \{k\}$ is not a clique of $G - k$, there must exists $v \in V(G) - C_q$ such that $v$ is adjacent to all the vertices of $C_q - \{k\}$. Let $T_1, T_2, \ldots, T_h$ be the branches of $T$ at $q$. Assume, without loss of generality, that $P_k$ is contained in $[q] \cup T_1 \cup T_2$. Since $v \in V(G) - C_q$, there exists $i$ such that $P_i$ is contained in $T_i$. And, since $h \geq 3$, there exists a branch $T_i$, with $s \neq 1, 2, i$. Let $P_a$ be the path of $\mathcal{P}$ linking $T_i$ and $T_t$, with $r \neq i$. It is clear that $u \in C_q$ and $v$ is not adjacent to $u$, which contradicts the fact that $v$ is adjacent to all the vertices of $C_q - \{k\}$. Thus $C_q - \{k\} \in \mathcal{E}(G - k)$. \(\square\)

**Definition 14.** A canonical VPT representation of $G$ is a pair $(\mathcal{P}, T)$ where $T$ is a tree whose vertices are the members of $\mathcal{E}(G)$, $\mathcal{P}$ is the family $(P_v)_{v \in V(G)}$ with $P_v = \{v \in \mathcal{E}(G) \mid v \in C\}$ and $P_v$ is a subpath of $T$ for all $v \in V(G)$.

In [13] it was proved that every VPT graph admits a canonical VPT representation.

**Theorem 15.** Let $G$ be a VPT graph and let $h \geq 3$. If $G$ is a minimal non-$[h, 2, 1]$ graph, then $G \in \mathcal{S}^\prime$ without dominated stable vertices.

**Proof.** Case (1): Suppose that $G \in \mathcal{S}^\prime$ with split partition $(S, K)$, and $G$ has dominated stable vertices. Let $(\mathcal{P}, T)$ be a canonical VPT representation of $G$, and let $q \in V(T)$ such that $K = C_q$. Assume that $N_T(q) = \{q_1, q_2, \ldots, q_k\}$, with $k > h$, and call $T_1, T_2, \ldots, T_h$ to the branches of $T$ at $q$ containing the vertices $q_1, q_2, \ldots, q_k$ respectively. It is clear that for each $q_i$ with $1 \leq i \leq k$, there exists $P_{w_i} \in \mathcal{P}$ such that $q_i \in V(P_{w_i})$ and $q \notin V(P_{w_i})$. Notice that every $w_i \in S$.

Suppose that $S = \{w_1, w_2, \ldots, w_k\}$. Since $G$ has dominated stable vertices, by item (iv) of Theorem 12 we can assume, without loss of generality, that $N_{\mathcal{P}}(w_1) \subseteq N_{\mathcal{P}}(w_2)$. This means that $w_1$ and $w_2$ are not adjacent in $B(G/C_q)$; thus, by item (iii) of Theorem 12, $N_{B(G/C_q)}(w_1) \subseteq N_{B(G/C_q)}(w_2)$. Hence there exists $l \in V(B(G/C_q)) - \{w_1, w_2\}$, such that $l \notin N_{B(G/C_q)}(w_1) - N_{B(G/C_q)}(w_2)$. Since $V(B(G/C_q)) = S$ we can assume that $l = w_3$. Then, by the definition of branch graph, there exists $z \in C_q$ such that $zW_1 \in E(G), zW_2 \in E(G)$ and, since $N(w_1) \subseteq N(w_2), zW_2 \in E(G)$, which implies that $P_z$ contains the vertices $q_1, q_2, q_3$. Then $P_z$ is not a path. This contradicts the fact that $(\mathcal{P}, T)$ is a VPT representation of $G$.

We conclude that $S' = S - \{w_1, w_2, \ldots, w_k\} \neq \emptyset$. Let $G' = G - S'$. Notice that $C_q \in \mathcal{E}(G')$ and $V(B(G'/C_q)) = \{w_1, w_2, \ldots, w_k\}$. Since $G$ is a minimal non-$[h, 2, 1]$ graph, then $G' \in [h, 2, 1]$ and $\chi(B(G'/C_q)) \leq h$.

We claim that there exists an $h$-coloring of $B(G'/C_q)$ such that if there exists $x \in C_q$ and $w_j \in \{w_1, w_2, \ldots, w_k\}$ with $w_1 \in E(G), w_2 \in E(G)$ then $w_i$ and $w_j$ have different colors in $B(G'/C_q)$. \(\square\)
Indeed, if $w_i$ and $w_j$ have the same color in $B(G'/C_q)$ then $w_i,w_j \not\in E(B(G'/C_q))$. Then we can assume that $N(w_i) \subseteq N(w_j)$, since, by hypothesis, there exists $x \in C_q$ such that $xw_i \in E(G)$ and $xw_j \in E(G)$; thus, for any $s \neq j$, no vertex of $C_q$ is adjacent to $w_i$ and $w_j$. This implies that $w_j$ is an isolated vertex of $B(G'/C_q)$. Therefore, we can change the color of $w_i$ to either of the $h-1$ remaining colors. This process can be repeated until we have the desired $h$-coloring of $B(G'/C_q)$.

Hence we consider an $h$-coloring, denoted $c$, of $B(G'/C_q)$ satisfying the condition (2).

Now, we give an $h$-coloring, denoted $c$, of $B(G/C_q)$ as follows: given $v \in V(B(G/C_q))$, by Lemma 5, there exists $1 \leq i \leq k$ such that $P_{w_i}$ is contained in $T_q$, we define $c(v) = c^i(w_i)$. Notice that, in particular, $c(w_i) = c^i(w_i)$.

We will see that $c$ is a proper coloring of $B(G/C_q)$. That is, we have to see that if $uv \in E(B(G/C_q))$ then $c(u) \neq c(v)$. Since $uv \in E(B(G/C_q))$, by Lemma 5, $P_v$ and $P_u$ are in different branches of $T$ at $q$ say $T_1$ and $T_j$. Moreover, there exists $x \in C_q$ such that $xu \in E(G)$ and $xv \in E(G)$, but this implies that $xw_i \in E(G)$ and $xw_j \in E(G)$ since the coloring satisfies the condition $2$, $c^i(w_i) \neq c^i(w_j)$. Thus $c(u) \neq c(v)$. Therefore, our coloring is proper.

Thus we have an $h$-coloring of $B(G/C_q)$ which contradicts the fact that $C_q$ is a principal clique of $G$. We conclude that, if $G \in \text{Split}$ then $G$ has no dominated stable vertices.

Case (2): Suppose that $G \not\in \text{Split}$. Since $G$ is a minimal non-$[h,2,1]$ graph, by Theorem 11, $G \in [h+1,2,1]$. Let $(\mathcal{P},T)$ be an $(h+1,2,1)$-representation of $G$ and let $q \in V(T)$ such that $C_q$ is a principal clique of $G$. We obtain from $G$ a new graph $\tilde{G}$ with $V(\tilde{G}) = V(G)$, as follows.

$\tilde{G}$ has the $(h+1,2,1)$-representation $(\mathcal{P}',T), \mathcal{P}' = (P'_v)_{v \in V(G)}$, given by:

$$P'_v = \begin{cases} P_v & \text{if } v \in C_q, \\ \{q_v\} & \text{if } v \in V(G) - C_q, \end{cases}$$

where $q_v$ is the vertex of $P_v$ closest to $q$.

We will prove that $\tilde{G}$ is a split graph, $\tilde{G} \not\in [h,2,1]$ and $\tilde{G}$ has dominated stable vertices.

It is clear that $C_q$ is a clique of $\tilde{G}$. We claim that $(V(G) - C_q, C_q)$ is a split partition of $\tilde{G}$. Indeed, if $x, y \in V(G) - C_q$ and $xy \in E(\tilde{G})$ then $q_x = q_y$. Thus $N_G(x) \cap C_q = N_G(y) \cap C_q$ which contradicts item (iv) of Theorem 12.

By Lemma 9, $C_q$ is a principal clique of $\tilde{G}$. Then, to see that $\tilde{G} \not\in [h,2,1]$ it is enough to see that $\text{B}(\tilde{G}/C_q) = B(G/C_q)$ because in such case $\chi(\tilde{G}/C_q) = \chi(B(G/C_q)) = h + 1$, and it follows by Theorem 7.

It is straightforward to show that $V(B(\tilde{G}/C_q)) = V(B(G/C_q))$ and that $E(B(G/C_q)) \subseteq E(\tilde{B}(G/C_q))$. On the other hand, notice that if $xy \in E(B(\tilde{G}/C_q))$ and $xy \not\in E(B(G/C_q))$ then $xy \in E(G)$. Thus, without loss of generality, we can assume that $N_G(x) \cap C_q \subseteq N_G(y) \cap C_q$ which contradicts the fact that $x$ and $y$ are adjacent in $B(G/C_q)$. We conclude that $\text{B}(\tilde{G}/C_q) = B(G/C_q)$.

Now, to see that $\tilde{G}$ has dominated stable vertices, let $x$ and $y$ be vertices of $V(\tilde{G}) - C_q$ adjacent in $G$, they exist since $G \not\in \text{Split}$. Since $xy \in E(G)$, we can assume, without loss of generality, that $q_x$ lies on the path of $T$ between $q$ and $q_y$, which implies that $x$ dominates $y$ in $\tilde{G}$.

Then, by Case (1), $\tilde{G}$ is a minimal non-$[h,2,1]$ graph. Thus there exists $v \in V(\tilde{G})$ such that $(\tilde{G} - v) \not\in [h+1,2,1]$. If $v \in V(B(\tilde{G}/C_q))$, then $\chi(B((\tilde{G} - v)/C_q)) = h + 1$. Moreover, by Claim 4 and since $B(\tilde{G}/C_q) = B(G/C_q)$, we have that $\text{B}(\tilde{G}/C_q) = B((\tilde{G} - v)/C_q) - v = B(G/C_q) - v = B(G - v)/C_q$. Hence $\chi(B((\tilde{G} - v)/C_q)) = h + 1$ which contradicts the fact that $G$ is a minimal non-$[h,2,1]$ graph.

If $v \in C_q$, then, by Lemma 13, $C_q - v \in E(\tilde{G} - v)$; therefore $C_q - v \in E(\tilde{G})$. Thus $\tilde{G} - v \in \text{Split}$ with split partition $(V(G) - C_q, C_q - v)$. Then, by Lemma 9, $C_q - v$ is a principal clique of $G - v$. Hence $\chi(B((\tilde{G} - v)/(C_q - v))) = h + 1$. Moreover, it is easy to see that $B((\tilde{G} - v)/(C_q - v)) = B((G - v)/(C_q - v))$; thus $\chi(B((\tilde{G} - v)/(C_q - v))) = \chi(B((G - v)/(C_q - v))) = h + 1$ which contradicts the fact that $G$ is a minimal non-$[h,2,1]$ graph.

We conclude that $G \in \text{Split}$.

In Theorem 12 we gave some necessary conditions on the branch graph with respect to a principal clique of a minimal non-$[h,2,1]$ graph. In what follows, in Theorem 16, using the fact that all minimal non-$[h,2,1]$ graphs are split without dominated stable vertices and the fact that the central clique of a split graph is principal, we will give more necessary conditions for minimal non-$[h,2,1]$ graphs.

**Theorem 16.** Let $G$ be a VPT graph and let $h \geq 3$. If $G$ is a minimal non-$[h,2,1]$ graph with split partition $(S,K)$ then: (i) for all $k \in K, |N(k) \cap S| = 2$; (ii) $|E(B(G/K))| = |K|$; (iii) $B(G/K)$ is $(h+1)$-critical.

**Proof.** By Theorem 15, $G \in \text{Split}$ without dominated stable vertices. Let $(S,K)$ be the split partition of $G$. By Lemma 9 $K$ is a principal clique of $G$, by Theorem 11 $\chi(B(G/K)) = h + 1$ and by item (i) of Theorem 12 $V(B(G/K)) = 5$.

(i) Since $G \in \text{VPT} \cap \text{Split}$ without dominated stable vertices, $|N(k) \cap S| \leq 2$ for all $k \in K$. Suppose there exists $k' \in K$ such that $|N(k') \cap S| < 2$.

By Theorem 11, $G \in [h+1,2,1]$. Let $(\mathcal{P},T)$ be an $(h+1,2,1)$-representation of $G$ and let $q \in V(T)$ such that $K = C_q$.

By Lemma 13, $C_q - \{k'\} \in \text{C}(G - K)$.

If $|N(k') \cap S| = 0$ then $B((G - k')/(C_q - \{k'\})) = B(G/C_q)$. Thus $\chi(B((G - k')/(C_q - \{k'\}))) = \chi(B(G/C_q)) = h + 1$, which contradicts the fact that $G$ is a minimal non-$[h,2,1]$ graph.
2. If $|N(K') \cap S| = 1$: We will see that $B((G - K)/(C_q - \{k\})) = B(G/C_q)$. It is clear, by item (ii) of Theorem 12, that $V(B((G - K)/(C_q - \{k\}))) = V(B(G/C_q))$ and $E(B((G - K)/(C_q - \{k\}))) \subseteq E(B(G/C_q))$. Let $uv \in E(B(G/C_q))$ such that $uv \notin E(B((G - K)/(C_q - \{k\})))$. Since $|N(K') \cap S| = 1$ we can assume, without loss of generality, that $(N(u) \cap C_q) - (N(u) \cap C_q) = \{k\}$. Therefore, we have that $N_{B(G/C_q)}(u) = \{u\}$, because if there is $\neq u$ such that $w \in N_{B(G/C_q)}(v)$ then there is $\bar{v} \in C_q$ such that $\bar{w} \in E(G)$. And, so $\bar{v}$ is not $\{k\}$ since $|N(K') \cap S| = 1$; thus $u \in E(G)$, because $(N(u) \cap C_q) - (N(u) \cap C_q) = \{k\}$. Then but $u, v, w \subseteq N(\bar{v})$, which contradicts the fact that $|N(K') \cap S| \leq 2$ for all $k \in K$.

Hence $d_{B(G/C_q)}(v) = 1$, which contradicts the fact that $H$ is $(h + 1)$-vertex critical.

(ii) First we will prove that $|E(B(G/K))| \leq |K|$. Let $e = uv \in E(B(G/K))$. By the definition of branch graph, there exists $k \in K$ such that $e = uv \in E(G)$. Thus for each $e \in E(B(G/K))$ there exists $k \in K$. Hence, by item (i), $|E(B(G/K))| \leq |K|$. Now we will see that $|K| \leq |E(B(G/K))|$. Let $k \in K$. By item (i), $|N(k) \cap S| = 2$. Suppose that $N(k) \cap S = \{u, v\}$, hence $N(u) \cap N(v) \neq \emptyset$. Since there are no dominated stable vertices, $N(u) \nsubseteq N(v), N(v) \nsubseteq N(u)$. Thus $uv \in E(B(G/K))$. Hence for each $k \in K$ there exist $u, v \in S$ such that $uv \in E(B(G/K))$. Observe that if $k \in K$ such that $\bar{k} \neq k$, then $N(\bar{k}) \cap S \neq N(k) \cap S$. Because if $N(\bar{k}) \cap S = N(k) \cap S$, then $\bar{k}$ and $k$ are true twins in $G$ which contradicts the fact that $G$ is a minimal non-[h, 2, 1] graph. Therefore, $|K| \leq |E(B(G/K))|$. (iii) Let $e = uv$ by any edge of $B(G/K)$, we will prove that $\chi(B(G/K) - e) < \chi(B(G/K))$. Notice it is enough to show that there exists $k \in K$ such that $B(G/K) - e = B((G - k)/(K - \{k\}))$; in fact, since $G$ is a minimal non-[h, 2, 1] graph, we have that $G - k \in [h, 2, 1]$, and so $\chi(B((G - k)/(K - \{k\}))) \leq h < h + 1 = \chi(B(G/K))$.

Let $k$ be a vertex of $K$ adjacent to both $u$ and $v$. Observe that, by Lemma 13, $K - \{k\}$ is a clique of $G - k$.

Since every vertex of $S$ is adjacent to at least two vertices of $K$, it follows that $V(B((G - k)/(K - \{k\}))) = V(B(G/K)) = V(B((G - k)/(K - \{k\}))) \leq h < h + 1 = \chi(B(G/K))$.

Let $xy$ be an edge of $B(G/K)$, and let $k_1, k_2, k_3$ be vertices of $K$ such that $k_1x \in E(G), k_2y \in E(G), k_3x \notin E(G), k_3y \notin E(G)$. Notice that $k_2 \neq k$ because every vertex of $K$ has exactly two neighbors in $S$ and $e \neq xy$. Therefore, if $k_2 \neq k$ and $k_3 \neq k$, we have that $xy \in E(B((G - k)/(K - \{k\})))$ and the proof is completed. We can assume without loss of generality that $k_2 = k$ and, since the only neighbors of $k$ in $S$ are $u$ and $v$, also, without loss of generality, we can assume that $x = u$.

Since, by item (iii) of Theorem 12, $B(G/K)$ is $(h + 1)$-vertex critical, then any vertex of $B(G/K)$ has degree at least $h$. Thus there exists $k_4 \in K$ such that $k_4x \in E(G)$ and $k_4 \neq k, k_4 \neq k_3$. If $k_4y \notin E(G)$, then $N(k_4) = N(k_4) = \{x, y\} \cup K$ which means that $k_3$ and $k_4$ are true twins contradicting the minimality of $G$. Hence $k_4y \notin E(G)$. The existence of vertices $k_1, k_2, k_3, k_4$ implies that $xy \in E(B((G - k)/(K - \{k\})))$. □

4. Building minimal non-[h,2,1] graphs

The construction presented here is similar to that done in [1], and a generalization of that used in [4]. Given a graph $H$ with $V(H) = \{v_1, \ldots, v_n\}$, let $G_H$ be the graph with vertices:

$$V(G_H) = \begin{cases} v_i & \text{for each } 1 \leq i \leq n; \\
v_j & \text{for each } 1 \leq i < j \leq n \text{ such that } v_i v_j \in E(H); \\
v_i & \text{for each } 1 \leq i \leq n \text{ with } d_H(v_i) = 1; \\
\end{cases}$$

and the cliques of $G_H$ are $K_H$ and $C_i$, for $1 \leq i \leq n$, where:

$$K_H = \{v_i | 1 \leq i < j \leq n\} \cup \{\bar{v}_i | 1 \leq i \leq n \text{ and } d_H(v_i) = 1\},$$

$$C_i = \{v_i\} \cup \{v_j | v_j \in N_H(v_i)\} \cup \{\bar{v}_i | d_H(v_i) = 1\}.$$ 

In Fig. 1 we offer an example.

**Fig. 1.** A graph $H$ and the graph $G_H$. 
Theorem 19. \( H \ra G \ra H \) is a critical graph if and only if \( H \) is \((h+1)\)-critical.

Lemma 17. (i) \( G \) is a VPT \( \cap \) Split graph without dominated stable vertices; (ii) \( B(G/H, K_1) = H \).

Theorem 18. Let \( h \geq 3 \). The graph \( G \ra H \) is a minimal non-[\( h, 2, 1 \)] graph if and only if \( H \) is \((h + 1)\)-critical.

Proof. Assume that \( G \ra H \) is a minimal non-[\( h, 2, 1 \)] graph. By item (ii) of Lemma 17, \( B(G/H, K_1) = H \). Hence, by item (iii) of Theorem 16, \( H \) is \((h + 1)\)-critical.

Now, let \( H \) be an \((h + 1)\)-critical graph with \( V(H) = \{v_1, v_2, \ldots, v_n\} \). By Lemmas 17 and 9, \( \max_{c \in E(G)}(\chi(B(G/H, K_1))) = 1 \). Hence, by Theorem 7, \( G \ra H \) are true twins in \( H \). Let us see that \( G \ra H \) \& \( H \ra v \) for all \( v \in V(G) \). First, if \( v = v_j \in V(H) \), using Claim 4 and item (ii) ofLemma 17, we have that \( B((G \ra H) \ra v_j) = B(G/H, K_1) \ra v_j = H - v_i \). Thus since \( H \) is \((h + 1)\)-critical, \( B(G \ra H, K_1) = h \). Since \( G \ra H \ra v_j \) is a split graph with central clique \( K_1 \), by Lemma 9 and Theorem 7, we have that \( G \ra H \ra v_i \in [h, 2, 1] \). Secondly, if \( v = v_j \) where \( e = v_j \in E(H) \), as a direct consequence of the way in which the graph \( G \ra H \) is obtained, we have that \( B(G \ra H, v_j) = B(G/H, K_1) \ra v_j = H - e \). Then \( B((G \ra H, v_j) / (K_1 - \{v_j\})) = H - e \). And, \( H - e = h \) because \( H \) is \((h + 1)\)-critical. Hence \( G \ra H \ra v_i \in [h, 2, 1] \). Since \( H \) is a critical graph it has no degree 1 vertices, and so \( G \ra H \) has no more vertices. \( \square \)

5. Characterization of minimal non-[\( h, 2, 1 \)] graphs

In this section, we give a characterization of VPT minimal non-[\( h, 2, 1 \)] graphs, with \( h \geq 3 \). The main result of this section is Theorem 19 which states that the only VPT minimal non-[\( h, 2, 1 \)] graphs are the graphs \( G \ra H \) constructed from \((h + 1)\)-critical graphs \( H \).

Moreover, in Theorem 20, we show that the family of graphs constructed from \((h + 1)\)-critical graphs together with the family of minimal forbidden induced subgraphs for VPT [12,15], is the family of minimal forbidden induced subgraphs for \([h, 2, 1]\), with \( h \geq 3 \).

Theorem 19. Let \( h \geq 3 \) and let \( G \) be a VPT graph. \( G \) is a minimal non-[\( h, 2, 1 \)] graph if and only if there exists an \((h + 1)\)-critical graph \( H \) such that \( G \cong G \ra H \).

Proof. The reciprocal implication follows directly applying Theorem 18.

Let \( G \) be a minimal non-[\( h, 2, 1 \)] graph. By Theorem 15, we know that \( G \in \text{Split without dominated stable vertices.} \)

Let \((S, K)\) be a split partition of \( G \). By Theorem 11, \( G \in [h + 1, 2, 1] \). Let \( H = B(G/K) \). By item (iii) of Theorem 16, \( H \) is an \((h + 1)\)-critical graph. Let us see that \( G \cong G \ra H \). Let \( G \ra H = (S \ra H, K_1) \). By item (ii) of Lemma 17, \( B(G/H, K_1) = H \); then \( B(G/H, K_1) = B(G/K) \). So, since \( V(B(G/H, K_1)) = V(B(G/K)) \), \( S \ra H = S \). Moreover, since \( E(B(G/H, K_1)) = E(B(G/K)) \), by item (ii) of Theorem 16, \( |K_1| = |K| \) and, by item (i) of Theorem 16, \( |N(k) \cap S| = 2 \) for all \( k \in K \). Suppose that \( N(k) \cap S = \{v_1, v_2\} \); we will see that \( v_1 \in E(H) \). It is clear that \( v_1 \in E(G) \) and \( v_2 \in E(G) \). Moreover, by item (ii) of Theorem 12, there exist \( k', k'' \in K \) such that \( k' \in E(G) \), \( k'' \in E(G) \). If \( k' = k'' \) then, since \( N(k) \cap S = 2 \) for all \( k \in K \), we have that \( k' \) and \( k'' \) are true twins in \( G \), which contradicts the fact that \( G \ra H \ra v_i \) is a non-[\( h, 2, 1 \)] graph. Hence \( k' \neq k'' \). Thus \( k' \in E(G) \) and \( k'' \in V(G) \). Therefore, \( v_1 \in E(H) \).

Hence we can define a function \( f \) that assigns to each vertex \( k \in K \) an edge \( v_1 \in E(H) \), that is, an element of \( K_1 \). Note that in \( G \ra H \) the vertex \( v_i \in K_1 \) is adjacent exactly to \( v_1 \). Hence the function \( f \) can be extended to a new function \( \tilde{f} \) from \( K \cup S \) to \( K_1 \cup S \), being the identity function from \( S \) to \( S \). Moreover, \( \tilde{f} \) is an isomorphism between \( G \) and \( G \ra H \). \( \square \)

Theorem 20. Let \( h \geq 3 \). A graph \( G \) is a minimal non-[\( h, 2, 1 \)] if and only if \( G \) is one of the members of \( F_0, F_1, \ldots, F_{16} \) or \( G \cong G \ra H \), where \( H \) is an \((h + 1)\)-critical graph.

Proof. By Theorem 19, if \( G \cong G \ra H \) where \( H \) is an \((h + 1)\)-critical graph, then \( G \) is a minimal non-[\( h, 2, 1 \)] graph.

If \( G \) is any of the members of \( F_0, \ldots, F_{16} \), then \( G \not\in \text{VPT} \) and \( G - v \in \text{VPT} \) for all \( v \in V(G) \). Moreover, in [4] it was proved that \( G - v \in \text{EPT} \) for all \( v \in V(G) \). Thus \( G - v \in \text{VPT} \cap \text{EPT} = [3, 2, 1] [8] \), which implies that \( G - v \in [h, 2, 1] \). Hence \( G \) is a minimal non-[\( h, 2, 1 \)] graph.

Let \( h \geq 3 \) and let \( G \) be a minimal non-[\( h, 2, 1 \)] graph.

Case (1): \( G \not\in \text{VPT} \). Since \( G \) is a minimal non-[\( h, 2, 1 \)] graph, then \( G - v \in [h, 2, 1] \) for all \( v \in V(G) \). Thus \( G - v \in \text{VPT} \) for all \( v \in V(G) \). Then, \( G \) is a minimal forbidden induced subgraph for VPT. Hence \( G \) is one of the members of \( F_0, F_1, \ldots, F_{16} \).

Case (2): \( G \in \text{VPT} \). Then, by Theorem 19, \( G \cong G \ra H \), where \( H \) is an \((h + 1)\)-critical graph.

Notice that, since every \( G \ra H \) is VPT, no member of \( F_0, F_1, \ldots, F_{16} \) is an induced subgraph of \( G \ra H \). \( \square \)
Fig. 2. Minimal forbidden induced subgraphs for VPT graphs (the vertices in the cycle marked by bold edges form a clique). See [12,15] for more details.

References