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ASTEROIDAL QUADRUPLES IN NON ROOTED PATH GRAPHS

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Abstract

A directed path graph is the intersection graph of a family of directed subpaths of a directed tree. A rooted path graph is the intersection graph of a family of directed subpaths of a rooted tree. Rooted path graphs are directed path graphs. Several characterizations are known for directed path graphs: one by forbidden induced subgraphs and one by forbidden asteroids. It is an open problem to find such characterizations for rooted path graphs. For this purpose, we are studying in this paper directed path graphs that are non rooted path graphs. We prove that such graphs always contain an asteroidal quadruple.

Keywords: clique trees, rooted path graphs, asteroidal quadruples.

2010 Mathematics Subject Classification: 05C75.

1. Introduction

A graph is *chordal* if it contains no cycle of length at least four as an induced subgraph. A classical result [5] states that a graph is chordal if and only if it is the (vertex) intersection graph of a family of subtrees of a tree.

Natural subclass of chordal graphs are path graphs, directed path graphs, rooted directed path graphs and interval graphs. A graph is a *path graph* if it is the intersection graph of a family of subpaths of a tree. A graph is a *directed path graph* if it is the intersection graph of a family of directed subpaths of a directed tree. A graph is a *rooted path graph* if it is the intersection graph of a family of directed subpaths of a rooted tree. A graph is an *interval graph* if it is the intersection graph of a family of subpaths of a path.

By definition we have the following inclusions between the different considered classes and these inclusion are strict as showed in Figure 1.

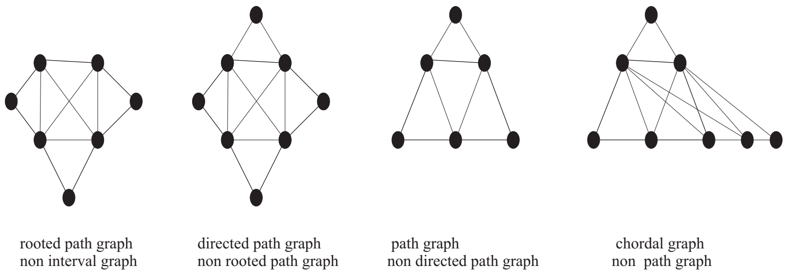

Figure 1. interval $\subset$ rooted path $\subset$ directed path $\subset$ path $\subset$ chordal.

Lekkerkerler and Boland [6] proved that a chordal graph is an interval graph if and only if it contains no asteroidal triple. As byproduct, they found a characterization of interval graphs by forbidden induced subgraphs.

Panda [10] found the characterization of directed path graph by forbidden induced subgraphs and then Cameron, Hoáng and Lévêque [3] gave a characterization of this class in terms of forbidden asteroidal triples.

Lévêque, Maffray and Preissman [7] found the characterization of path graphs by forbidden induced subgraphs but there is still no nice characterization in terms of forbidden asteroids for this class.

Characterizing rooted path graph by forbidden induced subgraphs or forbidden asteroids are open problems. It is certainly too difficult to characterizing rooted path graphs by forbidden induced subgraphs as there are too many (families of) graphs to exclude but Cameron, Hoáng and Lévêque [2] suggest that directed path graphs could be characterized by forbidding some particular type of asteroidal quadruples.

An *asteroidal triple* in a graph G is a set of three non-adjacent vertices such that for any two of them there exists a path between them that does not intersect the neighborhood of the third. The graph in Figure 2 is an example of a graph that contains an asteroidal triple. The vertices forming the asteroidal triple are circled.

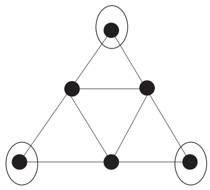

Figure 2. 3-sun containing an asteroidal triple.

An *asteroidal quadruple* is a set of four non-adjacent vertices such that any three of them is an asteroidal triple.

In this paper we prove the following result.

Theorem 1. *If G is a directed path graph that is not a rooted path graph, then G has an asteroidal quadruple.*

Note that there are graphs containing asteroidal quadruples which are rooted path graphs; for example, the graph in Figure 3.

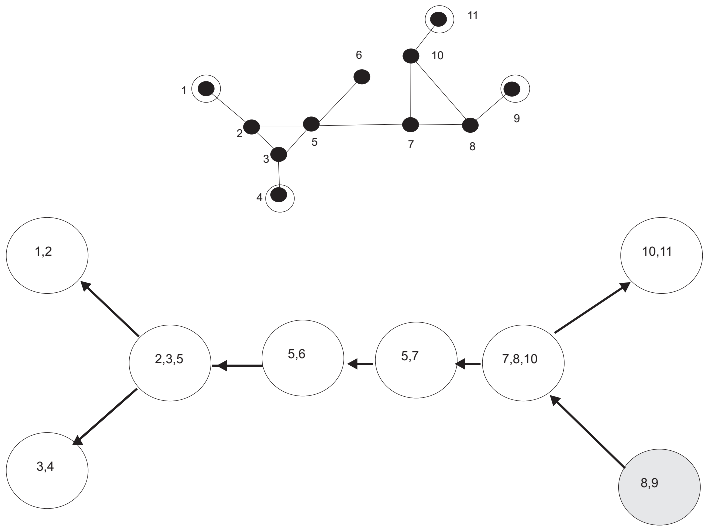

Figure 3. Rooted path graph with an asteroidal quadruple.

The paper is organized as follows: in Section 2, we give some definitions and notation. In Section 3, we study the intersection models of directed path graphs that are minimally non rooted path graphs. Finally, in Section 4, we give a proof of Theorem 1.

2. Definitions and Notation

A *clique* in a graph G is a maximal set of pairwise adjacent vertices. Let $\boldsymbol{C}(G)$ be the *set of all cliques* of G.

The *neighborhood* of a vertex x is the set $N(x)$ of vertices adjacent to x and the *closed neighborhood* of x is the set $N[x] = \{x\} \cup N(x)$.

A *clique tree* T of a graph G is a tree whose vertices are the elements of $\boldsymbol{C}(G)$ and such that for each vertex x of G, those elements of $\boldsymbol{C}(G)$ that contain x induce a subtree of T, which we will denote by T_x.

Note that G is the intersection graph of the subtrees $(T_x)_{x \in V(G)}$. Gavril [5] proved that a graph is chordal if and only if it has a clique tree. Clique trees are called *models* of the graph.

In [9], Monma and Wei introduced the notation UV, DV and RDV to refer to the classes of path graphs, directed path graphs and rooted path graphs respectively. They also prove the following clique tree characterizations for these classes. A graph is a *path graph* or a *UV graph* if it admits a *UV-model*, i.e., a clique tree T such that T_x is a subpath of T for every $x \in V(G)$. A graph is a *directed path graph* or a *DV graph* if it admits a *DV-model*, i.e., a clique tree T whose edges can be directed such that T_x is a directed subpath of T for every $x \in V(G)$. A graph is a *rooted path graph* or an *RDV graph*, if it admits an *RDV-model*, i.e., a clique tree T that can be rooted and whose edges are directed from the root toward the leaves such that T_x is a directed subpath of T for every $x \in V(G)$.

It is clear that a graph is an *interval graph* if it admits a clique tree T that is a path such that T_x is a subpath of T for every $x \in V(G)$.

It was proved in [4] that if G is a DV graph, then any UV-model of G can be directed to obtain a DV-model of G. We say that a DV-model T of a DV graph G can be *rooted* if T can be rooted at a vertex such that it becomes an RDV-model of G.

Let T be a clique tree. We often use capital letters to denote the vertices of a clique tree as these vertices correspond to cliques of G. In order to simplify the notation, we often write $X \in T$ instead of $X \in V(T)$, and $e \in T$ instead of $e \in E(T)$. If T' is a subtree of T, then $G_{T'}$ denotes the subgraph of G that is induced by the vertices of $\bigcup_{X \in V(T')} X$.

If G is a graph and $V' \subseteq V(G)$, then $G \setminus V'$ denotes the subgraph of G induced by $V(G) \setminus V'$. If $E' \subseteq E(G)$, then $G - E'$ denotes the subgraph of G induced by $E(G) \setminus E'$. If G, G' are two graphs, then $G + G'$ denotes the graph whose vertices are $V(G) \cup V(G')$ and edges are $E(G) \cup E(G')$. Note that if T, T' are two trees such that $|V(T) \cap V(T')| \leq 1$, then $T + T'$ is a forest.

Let T be a tree. For $V' \subseteq V(T)$, let $T[V']$ be the minimal subtree of T containing V'. Then for $X, Y \in V(T)$, $T[X, Y]$ is the subpath of T between X

and Y. Let $T[X,Y) = T[X,Y] \setminus Y$, $T(X,Y] = T[X,Y] \setminus X$ and $T(X,Y) = T[X,Y] \setminus \{X,Y\}$. Note that some of these paths may be empty or reduced to a single vertex when X and Y are equal or adjacent. If $X \in V(T)$ and $e \in E(T)$, with $e = AB$ and $A \in T[X,B]$, then let $T[X,e] = T[X,B]$, $T[X,e) = T[X,A]$, $T(X,e] = T(X,B]$ and $T(X,e) = T(X,A]$. Given a vertex $X \in V(T(Y,Z))$, we say that there is a *vertex crossing* X *in* $T[Y,Z]$ if $X' \cap X'' \neq \emptyset$ where X' and X'' are the two neighbors of X in $T[Y,Z]$.

In a clique tree T, the *label* of an edge AB of T is defined as $lab(AB) = A \cap B$. We say that $X \in V(T)$ *dominates* $e \in E(T)$ if $lab(e) \subseteq X$. We say that an edge e satisfying a given property P is *maximally farthest from a vertex* C if there is no edge e', distinct from e, satisfying this property and such that e is between C and e' i.e., e is an edge in $T[C,e']$.

Let T be a clique tree of G. Let $\mathbf{D}(T)$ (or $\mathbf{D}$ for short) be the vertices of T of degree at least three. Observe that if T is not a path and H is a leaf of T then there exists $C \in T[\mathbf{D}]$ such that $T[H,C] \cap T[\mathbf{D}] = \{C\}$. In this case we say that $T[H,C]$ is a *branch* of T incident to C. Let C be a vertex of $\mathbf{D}$. The union of all the branches incident to C forms a subtree of T called the *bouquet* of T incident to C and denoted $Bouq(C)$. Note that if C is a leaf of $T[\mathbf{D}]$ then there are at least two branches in $Bouq(C)$ (otherwise $Bouq(C)$ can be empty or reduced to one branch).

Let T be a tree, we denote by $ln(T)$ the number of leaves of T. The *leafage* of a chordal graph G is the minimum integer ℓ such that G admits a model T with $ln(T) = \ell$.

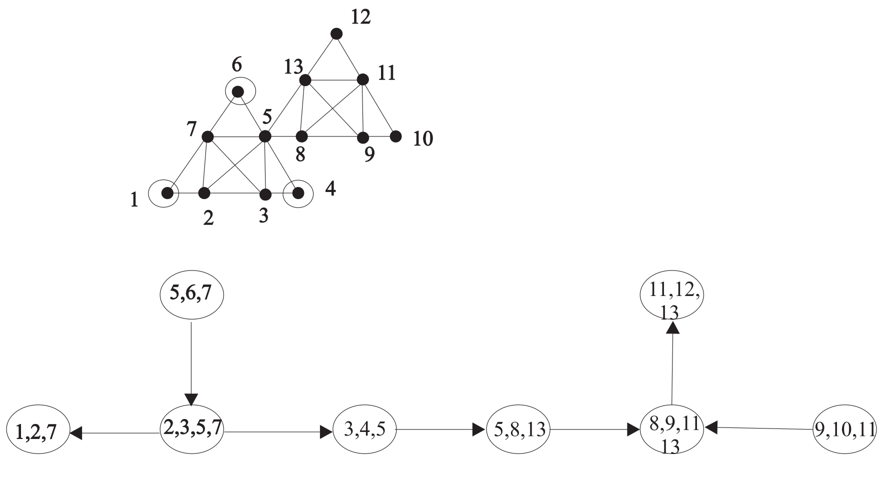

Figure 4. A DV graph with leafage four and asteroidal number three.

An *asteroidal set* A in a graph G is a set of non-adjacent vertices such that for any $v \in A$ the vertices of $A \setminus \{v\}$ appears in the same component of $G \setminus N[v]$, see [1]. Note that this definition is compatible with the definition of asteroidal triple and quadruple already given. The *asteroidal number* of a graph G is the maximum integer a such that G admits an asteroidal set of cardinality a. If G is a chordal graph containing an asteroidal set A of size k, then in any model T of G, T has at least k leaves [8]. Thus the asteroidal number of a chordal graph is less or equal to its leafage, and this inequality can be strict as shown by the example of the DV graph in Figure 4.

3. DV Non-RDV Graphs and Their Models

Lemma 2. *Let G be a DV graph. Every DV-model of G with at most* 3 *leaves can be rooted.*

Proof. The lemma is clear if the DV-model T has at most 2 leaves, as then G is an interval graph and T can be rooted at either one of its extremities. Suppose now that T is a DV-model of G with $ln(T) = 3$. Let C be the only vertex of T whose degree is 3 and C_1, C_2, C_3 be the neighbors of C in T. As G is a DV graph, it does not contain the 3-sun as induced subgraph so in T it is not possible that for each $i \neq j$, $i, j \in \{1, 2, 3\}$ there exists a vertex crossing C in $T[C_i, C_j]$. Suppose by symmetry that there is no vertex crossing C in $T[C_1, C_3]$ and let H_2 be the leaf of T such that $C_2 \in T[H_2, C]$. Then T can be rooted at H_2. ■

A consequence of Lemma 2 is that DV non-RDV graphs have leafage at least four. The graph of Figure 4 is a DV graph with leafage four and asteroidal number three. Thus Lemma 2 does not directly imply Theorem 1. This is the purpose of the rest of this paper.

A DV graph G that is minimally non-RDV is connected by minimality. Thus for any clique tree T of G and any edge e of T, the label of e is non empty and one can choose a vertex of G in $lab(e)$. This property will often be used in the proofs.

Lemma 3. *Let G be a DV graph, minimally non-RDV, T a DV-model of G with at least four leaves, H a leaf of T and $T[H, C]$ a branch of T. Then*

(1) *Every edge of T dominated by H is in $T[H, C]$ or $T[\mathbf{D}]$.*

(2) *If there is an edge of T dominated by H in $T(H, C]$, then there is one in $T[\mathbf{D}]$.*

Proof. (1) Suppose by contradiction that there exists an edge e dominated by H in a branch $T[H', C']$, with $H' \neq H$ but maybe $C = C'$. Let $e = AB$ with $B \in T[A, C']$. Let $T' = T - E(T[H', B])$. The subgraph T' is a proper subtree of T. Then $G_{T'}$ is an induced subgraph of G. Hence by minimality of G, $G_{T'}$ is an RDV graph. Let T'' be an RDV-model of $G_{T'}$, rooted at a vertex R. All vertices of $lab(e)$ are twins in $G_{T'}$, they have the same neighbor, as for every $x \in lab(e)$, $T'_x = T'[H,B]$. Thus it is possible to build an RDV-model of G with T'' and $T[H',A]$ by the following. Let $x \in lab(e)$ and $Z,W \in T''$ such that $T''_x = T''[Z,W]$ and $W \in T''[Z,R]$. Then $T'' + ZA + T[A,H']$ rooted at R is an RDV-model of G, a contradiction.

(2) Suppose by contradiction that there is an edge dominated by H in $T(H,C]$ but none in $T[\mathbf{D}]$. Let e be an edge dominated by H that is maximally farthest from H. By (1) e is in $T[H,C]$ or $T[\mathbf{D}]$. By assumption e is not in $T[\mathbf{D}]$, and so it is in $T(H,C]$. Let $e = AB$ with $B \in T[A,C]$. Let $T' = T - E(T[H,A])$. The subgraph T' is a proper subtree of T as $A \neq H$. By the choice of e, A is a leaf of every clique tree of $G_{T'}$. By minimality of G, $G_{T'}$ is an RDV graph. Let T'' be an RDV-model of $G_{T'}$ rooted at R. Then $T'' + T[H,A]$ is an RDV-model of G when rooted at R or H, a contradiction. ■

Let T be a DV-model of G, and T' a (non necessarily proper) subtree of T. We say that a vertex H of T is *good in* T' if there is no edge of $T' \setminus H$ dominated by H, otherwise H is *bad in* T'. We just say that a leaf H of T is *good* (resp. *bad*) if it is good in T (resp. bad in T).

Lemma 3 shows that if H is a bad leaf, then there is an edge dominated by H in $T[\mathbf{D}]$ and there is no edge dominated by H in a branch different from the one containing H.

Good leaves are related to existence of asteroidal sets by the following lemma.

Lemma 4. *Let T be a clique tree of a graph G and $V' \subseteq V(T)$. If for every $X \in V'$, X is a leaf of $T[V']$ that is good in $T[V']$, then G has an asteroidal set A of size $|V'|$ where each set $X \in V'$ contains exactly one distinct vertex of A.*

Proof. Let $V' = \{X_1,\ldots,X_a\}$. For $i = 1,\ldots,a$, X_i is a leaf of $T[V']$. Let $T_i = T[V'] \setminus X_i$. Let X'_i be the neighbor of X_i in T_i, and $x_i \in X_i \setminus X'_i$. As T is a model of G, x_i does not appear in a vertex of T_i. Let $A = \{x_1,\ldots,x_a\}$. As X_i is good in $T[V']$ we can choose in every edge e of T_i a vertex $v_e \in (lab(e) \setminus X_i) \subseteq (V(G) \setminus N[x_i])$. Let $V_i = \bigcup_{e \in E(T_i)} v_e$. The set V_i connects the vertices of $A \setminus x_i$. So the vertices of $A \setminus x_i$ are in the same connected component of $G \setminus N[x_i]$ and A forms an asteroidal set. ■

Lemma 5 [2]. *If G contains an asteroidal quadruple $x_1,\ldots,x_4$ with x_1,x_2 having a common neighbor and x_3,x_4 also having a common neighbor, then G is not RDV.*

Lemma 6. *Let G be a DV graph, minimally non-RDV. Let T be a DV-model of G minimizing the number of leaves of T and then maximizing the number of vertices of $T[\mathbf{D}]$. Then, in each bouquet of T, there is at most one bad leaf.*

Proof. Suppose by contradiction that $C \in T[\mathbf{D}]$ with $Bouq(C)$ having two bad leaves H_1, H_2, i.e., there are edges in $T \setminus H_1$ and $T \setminus H_2$, dominated by H_1, H_2 respectively.

By Lemma 3, there is an edge e_1 in $T[\mathbf{D}]$ dominated by H_1. We assume that e_1 is chosen such that there is no edge dominated by H_1 between e_1 and C. We choose e_2 analogously.

Suppose that e_i is between C and e_j for some $j \neq i$. Then there is a vertex v in $lab(e_j)$ and H_j. Also v is in $lab(e_i)$ and in H_i. Then v is in H_1, H_2 and $lab(e_j)$, so T_v is not a path, contradicting T being a DV-model. So e_i is not between C and e_j for $j \neq i$.

Let $e_i = A_iB_i$ with $B_i \in T[A_i, C]$ for $i = 1, 2$. Since T is a clique tree with minimum number of leaves, the degree of B_i must be 2, otherwise $T - e_i + A_iH_i$ is a DV-model with fewer leaves than T. So $B_i \neq C$ and B_i is not in $T[C, B_j]$ for $i \neq j$.

Claim 7. *B_i is good in $T[H_1, H_2, B_j]$, for $\{i, j\} = \{1, 2\}$.*

Proof. Suppose by contradiction that there exists an edge $e = AB$ of $T[H_1, C]$, with $B \in T[A, C]$ and $lab(e) \subseteq B_1$. Then $T_0 = T - \{e, e_1\} + B_1A + A_1H_1$ is a DV-model of G. If $B \neq C$, then T_0 has the same number of leaves as T and $T_0[\mathbf{D}(T_0)]$ has bigger size than $T[\mathbf{D}(T)]$, a contradiction. So $B = C$ and then T_0 has fewer leaves than T, a contradiction. So there is no edge of $T[H_1, C]$ dominated by B_1.

Suppose now that that there exists an edge e of $T[H_2, B_2]$ with $lab(e) \subseteq B_1$. Then there is a vertex v in $lab(e_2)$ and H_2. Then v is also in $lab(e)$ and so in B_1. So v is in H_2, B_1, B_2, contradicting T being a DV-model.

By the choice of e_1, there is no edge of $T[C, B_1]$ dominated by B_1. So finally B_1 is good in $T[H_1, H_2, B_2]$. Analogously, B_2 is good in $T[H_1, H_2, B_1]$. □

Claim 8. *H_i is good in $T[H_j, B_1, B_2]$, for $\{i, j\} = \{1, 2\}$.*

Proof. Suppose by contradiction that there exists an edge e of $T[H_2, B_2]$ with $lab(e) \subseteq H_1$. Then there is a vertex v in $lab(e_2)$ and H_2. Then v is also in $lab(e)$ and so in H_1. So v is in H_1, H_2, B_2, contradicting T being a DV-model. The existence of an edge e of $T[C, B_1]$ with $lab(e) \subseteq H_1$ contradicts the choice of e_1. Hence H_1 is good in $T[H_2, B_1, B_2]$. Analogously, H_2 is good in $T[H_1, B_1, B_2]$. □

Let $T' = T[H_1, H_2, B_1, B_2]$. Note that T' is a proper subtree of T since the edges e_1 and e_2 are not in T'. By Lemma 4, there is an asteroidal quadruple h_1, h_2, b_1, b_2 in $G_{T'}$ with $h_i \in H_i$ and $b_i \in B_i$ for $i = 1, 2$. Let $x_1 \in lab(e_1)$ and $x_2 \in lab(e_2)$. We have $x_1 \in H_1 \cap B_1$ and $x_2 \in H_2 \cap B_2$. So $G_{T'}$ is not RDV by Lemma 5. This contradicts the fact that G is minimally not RDV. ■

Lemma 9. *Let G be a DV graph, minimally non-RDV. Suppose T is a DV-model of G minimizing the number of leaves of T and then maximizing the number of vertices of $T[\mathbf{D}]$, C is a vertex of degree 3 of T that is a leaf of $T[\mathbf{D}]$, H is a bad leaf of $Bouq(C)$, and e is an edge of $T[\mathbf{D}]$ dominated by H and maximally farthest from C. Then*

(1) *There is a vertex of $\mathbf{D}$ in $T(C, e)$.*

(2) *If there is no vertex of $\mathbf{D}$ in $T(C, e)$ that has degree at least 3 in $T[\mathbf{D}]$, then for each vertex C' of $\mathbf{D}$ in $T(C, e)$, all the leaves of T that are in $Bouq(C')$ are good.*

Proof. (1) Let $e = AB$ with $B \in T[A, C]$ and suppose by contradiction that there is no vertex of $\mathbf{D}$ in $T(C, B]$.

Let T_1 and T_2 be the components of $T - e$ containing B and A respectively. Define $T'_1 = T_1 + e$ and $T'_2 = T_2 + e$. Observe that $ln(T'_1) = 3$ and by Lemma 2, T'_1 is a DV-model of $G_{T'_1}$ that can be rooted at a vertex R_1, with $R_1 \in \{H, A\}$.

By Lemma 3, there is no edge dominated by H in a branch of T distinct from $T[H, C]$. So by the choice of e, there is no edge of T_2 dominated by H. Thus B is a leaf of every clique tree of $G_{T'_2}$. By minimality of G, its proper subgraph $G_{T'_2}$ is RDV. Let T''_2 be an RDV-model of $G_{T'_2}$ rooted at a vertex R_2. Let $\widetilde{A}$ be the neighbor of B in T''_2. Note that $lab(e) = A \cap B = \widetilde{A} \cap B = \widetilde{A} \cap H$.

- If $R_1 = H$ and $R_2 = B$, then $(T'_1 \setminus A) + T''_2$ is an RDV-model of G rooted at H, a contradiction.
- If $R_1 = A$ and $R_2 = B$, then $(T'_1 \setminus A) + (T''_2 \setminus B) + H\widetilde{A}$ is an RDV-model of G rooted at B, a contradiction.
- If $R_1 = H$ and $R_2 \neq B$, then $(T'_1 \setminus A) + (T''_2 \setminus B) + H\widetilde{A}$ is an RDV-model of G rooted at R_2, a contradiction.
- If $R_1 = A$ and $R_2 \neq B$, then $(T'_1 \setminus A) + T''_2$ is an RDV-model of G rooted at R_2, a contradiction.

So there is a vertex of $\mathbf{D}$ in $T(C, e)$.

(2) Let C' be a vertex of $\mathbf{D}$ in $T(C, e)$ that has degree 2 in $T[\mathbf{D}]$, and suppose by contradiction that H' is a leaf of T in $Bouq(C')$ that is bad. Then by Lemma 3, there is $e' \in T[\mathbf{D}]$ dominated by H'. Suppose that $e' \in T[C, e]$. Then $lab(e) \subseteq lab(e')$. Then every vertex of $lab(e)$ is in H, H', A, contradicting T being a DV-model of G. So $e' \notin T(C, e]$. As there is no vertex of $\mathbf{D}$ in $T(C, e)$ that has degree at least 3 in $T[\mathbf{D}]$, the edge e is in $T[C', e']$. So $lab(e') \subseteq lab(e)$. Then every vertex of $lab(e')$ is in H, H', A, contradicting T being a DV-model of G. Therefore, each leaf of T in $Bouq(C')$ is good. ∎

4. Proof of the Main Theorem

***Proof of Theorem* 1.** Let G be a DV graph, minimally non-RDV. Let T be a DV-model of G minimizing the number of leaves of T and then maximizing the number of vertices of $T[\mathbf{D}]$. By Lemma 2, T has at least 4 leaves. We consider different cases corresponding to the number of leaves of $T[\mathbf{D}]$.

Case 1: $ln(T[\mathbf{D}]) = 1$. By Lemma 3, every leaf of T is good and since $ln(T) \geq 4$, G has an asteroidal quadruple by Lemma 4.

Case 2: $ln(T[\mathbf{D}]) = 2$. Let C_1, C_2 be the leaves of $T[\mathbf{D}]$. We have $deg(C_i) \geq 3$. If there are at least four good leaves of T among the leaves of $Bouq(C_1)$ or $Bouq(C_2)$, then G has an asteroidal quadruple by Lemma 4. So we can assume that there are no four good leaves of T among the leaves of $Bouq(C_1)$ or $Bouq(C_2)$. By Lemma 6, in each bouquet there is at most one bad leaf. So we have $deg(C_1)+deg(C_2) \leq 7$. Moreover if $deg(C_1) = deg(C_2) = 3$, then at least one of $Bouq(C_1)$, $Bouq(C_2)$ contains a bad leaf, and if $\{deg(C_1), deg(C_2)\} = \{3, 4\}$, then both of $Bouq(C_1)$, $Bouq(C_2)$ contain a bad leaf. Thus in any case we can assume, by symmetry, that $deg(C_1) = 3$ and $Bouq(C_1)$ contains a bad leaf of T.

Let H_1, H_2 be the leaves of T that are in $Bouq(C_1)$ such that H_1 is bad and H_2 is good. Then, by Lemma 3, there is at least one edge of $T[\mathbf{D}]$ dominated by H_1. Let e be such an edge maximally farthest from H_1. Let $e = AB$ with $B \in T[C_1, A]$. Since H_1 is bad, by Lemma 9, (1), there is $C_3 \in \mathbf{D}$ in $T(C_1, e)$. Since $T[\mathbf{D}]$ has only 2 leaves, there is no vertex of $\mathbf{D}$ in $T(C_1, C_2)$ that has degree at least 3 in $T[\mathbf{D}]$. Thus, by Lemma 9, (2), every leaf of T that is in $Bouq(C_3)$ is good. There is at least one such leaf H as C_3 has degree 2 in $T[\mathbf{D}]$.

If there are at least two good leaves of T in $Bouq(C_2)$, then T has four good leaves and G has an asteroidal quadruple by Lemma 4. So we can assume that $Bouq(C_2)$ does not contain two good leaves of T. Then, by Lemma 6, $deg(C_2) = 3$ and $Bouq(C_2)$ contains a bad leaf of T. Let H_3, H_4 be the leaves of T that are in $Bouq(C_2)$ such that H_3 is bad and H_4 is good. Then, by Lemma 3, there is at least one edge of $T[\mathbf{D}]$ dominated by H_3. Let e' be such an edge maximally farthest from H_3. Let $e' = A'B'$ with $B' \in T[C_2, A']$. Since H_3 is bad, by Lemma 9 (1), there is $C_4 \in \mathbf{D}$ in $T(C_2, e')$ (maybe $C_4 = C_3$). Thus, by Lemma 9 (2), every leaf of T that is in $Bouq(C_4)$ is good. There is at least one such leaf H' as C_4 has degree 2 in $T[\mathbf{D}]$. Thus if $C_4 \neq C_3$, then $H \neq H'$ and T has four good leaves, H_2, H_4, H, H', and G has an asteroidal quadruple by Lemma 4. So we can assume that $C_4 = C_3$. Then $C_1, A', B', C_3, B, A, C_2$ appear in this order along $T[C_1, C_2]$ and $lab(e) = lab(e')$.

Suppose that T has a leaf H'' distinct from H_1, H_2, H_3, H_4, H. Then H'' is either in $Bouq(C_1)$, $Bouq(C_2)$ or in $Bouq(C''')$ with C''' distinct from C_1, C_2. As $deg(C_1) = deg(C_2) = 3$ we have H'' is in $Bouq(C''')$ with C''' distinct from C_1, C_2. Then by Lemma 9 (2) applied either to C_1, H_1, e or to C_2, H_3, e', we have

H'' is good. Thus T has four good leaves and G has an asteroidal quadruple by Lemma 4. So we can assume that the only leaves of T are H_1, H_2, H_3, H_4, H.

Let $v \in lab(e)$. Vertex v is a vertex crossing C_1 in $T[H_1, C_2]$, C_3 in $T[C_1, C_2]$, C_2 in $T[C_1, H_3]$. As T is a DV-model, there is no vertex crossing C_3 in $T[H, C_1]$ or in $T[H, C_2]$. Suppose by symmetry that there is no vertex crossing C_3 in $T[H, C_1]$. As T is a DV-model, there is no vertex crossing C_1 in $T[H_2, H_1]$ or in $T[H_2, C_3]$. If there is no vertex crossing C_1 in $T[H_2, H_1]$, let $T' = T$, otherwise, there is no vertex crossing C_1 in $T[H_2, C_3]$ and we let $T' = T - e' + B'H_1$. As T is a DV-model there is no vertex crossing C_2 in $T[H_4, H_3]$ or in $T[H_4, C_3]$. If there is no vertex crossing C_2 in $T[H_4, C_3]$, then T' can be rooted at H_3, otherwise there is no vertex crossing C_2 in $T[H_4, H_3]$, and $T' - e + BH_3$ can be rooted at A, a contradiction

Case 3: $ln(T[\mathbf{D}]) = 3$. Let C_1, C_2 and C_3 be leaves of $T[\mathbf{D}]$ and C the vertex of degree 3 in $T[\mathbf{D}]$. Each $Bouq(C_i)$ contains at least two leaves of T. By Lemma 6, at most one leaf of each $Bouq(C_i)$ is bad, so there exists a leaf H_i of T in $Bouq(C_i)$ that is good for $i = 1, 2, 3$. If there is a leaf of T distinct from H_1, H_2, H_3 that is good, then G has an asteroidal quadruple by Lemma 4. So we can assume that $Bouq(C_i)$ has exactly two leaves, one that is good, already denoted H_i, and one that is bad, denoted H'_i, and that the only bouquets of T with at least two leaves are $Bouq(C_1)$, $Bouq(C_2)$ and $Bouq(C_3)$. Let e_i be an edge dominated by H'_i that is maximally farthest from H'_i. By Lemma 3, $e_i \in T[\mathbf{D}]$.

Suppose by contradiction that, for all $i \in \{1, 2, 3\}$, $e_i \notin T[C_i, C]$. Then, we can assume by symmetry that $e_1 \in T[C_2, C]$. Since $e_3 \notin T[C, C_3]$, we have $e_3 \in T[C_1, C_2]$. If $e_3 \in T[C_1, e_1]$ then $lab(e_1) \subseteq lab(e_3)$ and every vertex of $lab(e_1)$ is in H'_1, H'_3, e_1, contradicting T being a DV-model of G. If $e_3 \in T[e_1, C_2]$ then $lab(e_3) \subseteq lab(e_1)$ and every vertex of $lab(e_3)$ is in H'_1, H'_3, e_3, contradicting T being a DV-model of G. Thus there exists $i \in \{1, 2, 3\}$ such that $e_i \in T[C_i, C]$ and we can assume by symmetry that $e_1 \in T[C_1, C]$. Then by Lemma 9 (1), there is a vertex C' of $\mathbf{D}$ in $T(C_1, e_1)$. Since $ln(T[\mathbf{D}]) = 3$, there is no vertex of $\mathbf{D}$ in $T(C, e_1)$ that has degree at least 3 in $T[\mathbf{D}]$. Thus, by Lemma 9 (2), every leaf of T that is in $Bouq(C')$ is good. There is at least one such leaf because C' has degree 2 in $T[\mathbf{D}]$, so G has an asteroidal quadruple.

Case 4: $ln(T[\mathbf{D}]) \geq 4$. Then there are at least four bouquets having at least two branches each. By Lemma 6 in each of these four bouquets there is at most one bad leaf and thus at least one good leaf. Hence T has at least four good leaves and G has an asteroidal quadruple by Lemma 4. ∎

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