LAGRANGIAN REDUCTION OF DISCRETE MECHANICAL SYSTEMS BY STAGES

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ABSTRACT. In this work we introduce a category of discrete Lagrange–Poincaré systems $\mathfrak{LP}_d$ and study some of its properties. In particular, we show that the discrete mechanical systems and the discrete mechanical systems obtained by the Lagrangian reduction of symmetric discrete mechanical systems are objects in $\mathfrak{LP}_d$. We introduce a notion of symmetry groups for objects of $\mathfrak{LP}_d$ and introduce a reduction procedure that is closed in the category $\mathfrak{LP}_d$. Furthermore, under some conditions, we show that the reduction in two steps (first by a closed normal subgroup of the symmetry group and then by the residual symmetry group) is isomorphic in $\mathfrak{LP}_d$ to the reduction by the full symmetry group.

1. INTRODUCTION

The study of mechanical systems with symmetries is a classical subject. A standard technique used in the area is the construction of a certain dynamical system —the *reduced system*— where some or all of the original symmetries have been eliminated and whose trajectories can be used to obtain the trajectories of the original system. This general idea has been developed and used in many different contexts. In the Lagrangian formulation of Classical Mechanics, one such approach is given by E. Routh [24], although it was implicit in Lagrange's original ideas. A modern treatment, including nonholonomic constraints, is given by H. Cendra, J. Marsden and T. Ratiu in [4]. In the modern Hamiltonian case, there are the original works of V. Arnold [1], S. Smale [25, 26], K. Meyer [22], J. Marsden and A. Weinstein [18] and, among the recent literature, J. Marsden et al. [19]. There is also a field-theoretic version as explained, for instance, by M. Castrillón Lopez and T. Ratiu in [2]. In the case of discrete mechanical systems (DMS), different versions of reduction theory have been considered, among others, by S. Jalnapurkar et al. in [12], R. McLachlan and M. Perlmutter in [21] and the authors in [7]. Reduction theory has also been developed in the context of Lie groupoids and Lie algebroids as discussed by J. C. Marrero et al in [16] and by D. Iglesias et al in [11].

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In some cases, if G is a symmetry group of a mechanical system, it may be convenient to consider a partial reduction, that is, the reduction of the system by a subgroup $H \subset G$ and, eventually, as a second step, the reduction of any remaining symmetries in the associated reduced system. This process is called *reduction by (two) stages*. A problem is that, in general, the reduced system associated to a symmetric mechanical system is a dynamical system that is not a mechanical system. Therefore, a second reduction cannot be performed in the framework of mechanical systems. For continuous time systems, the solution found by different authors consists of enlarging the class of systems considered beyond the mechanical ones, developing a reduction theory for those generalized systems that extends the original reduction of mechanical systems and, eventually, considering the reduction by stages in this generalized framework. This is the case, for instance, in the Lagrangian context, of [5], and, for Lagrangian systems with nonholonomic constraints, of [3, 6]. In the Hamiltonian case, it is extensively analyzed in [19]. It should be remarked that in the Lie algebroid or Lie groupoid contexts the problem described in this paragraph does not arise as the reduction of an object in one of these categories lies within the same category.

The purpose of the present work is to introduce a generalized framework to study the reduction of DMS by stages. In this sense, it parallels [5] for discrete time mechanical systems. More precisely, a category $\mathfrak{LP}_d$ of *discrete Lagrange–Poincaré systems* (DLPS) is introduced and it is shown that DMSs are among its objects in a natural way. Also, the reduced systems associated to symmetric DMSs are in $\mathfrak{LP}_d$. The dynamics of a DLPS is defined via a variational principle that, for a DMS, reduces to the discrete Hamilton Principle. Then, a reduction theory for symmetric DLPSs is developed. It is shown that this theory, when applied to DMS, coincides with the one defined in [7]. At this stage, we can prove the main result of the paper, Theorem 7.6, showing that, under appropriate conditions, the reduction in two stages is feasible and isomorphic in $\mathfrak{LP}_d$ to the full reduction in one step.

The construction of reduced systems considered in [5], [3, 6], [15], [7] and here, all require additional data: a connection or a *discrete connection* on a certain principal bundle. We prove that, even though the reduced DLPSs depend on the specific discrete connection used, any two choices lead to DLPSs that are isomorphic in $\mathfrak{LP}_d$. Last, we prove that under fairly general conditions discrete connections on the appropriate principal bundles satisfying the conditions required by the reduction by stages results exist.

It is well known and very useful that DMSs carry natural symplectic structures. But, in general, their associated reduced systems are not symplectic; instead, they carry Poisson structures. In contrast, general DLPSs do not carry a natural symplectic or Poisson structure; we prove that, when a Poisson structure is added to a DLPS, then it descends to any reduced system associated to it. A consequence of this fact is that all DLPSs obtained by a finite number of reductions from a symmetric DMS, have a "natural" Poisson structure coming from the symplectic structure of the original DMS.

The plan for the paper is as follows. Section 2 reviews the notion of discrete connection on a principal bundle and some of its basic properties. Section 3 introduces discrete Lagrange–Poincaré systems, their dynamics and explores some examples. Section 4 defines a category whose objects are the DLPSs. Section 5 introduces the notion of symmetry group of a DLPS and, then, constructs a "reduced" DLPS associated to any symmetric DLPS and discrete connection; it also compares the dynamics of the reduced DLPS and that of the original DLPS, proving that the trajectories of one system can be obtained from those of the other. An example of reduction process is analyzed in Section 6. Section 7 considers the reduction of DLPSs in two stages. Section 8 studies some aspects of Poisson structures on DLPSs. The paper closes with Section 9, where we list some basic and general results on group actions on manifolds and on principal bundles; most of this material is standard and it is included to have a unified notation and reference point.

Finally, we wish to thank Hernán Cendra for his continuous interest in this work and many very useful discussions.

2. General recollections

Let G be a Lie group acting on the left on the manifold Q by l^Q in such a way that the quotient map $\pi^{Q,G} : Q \to Q/G$ be a principal G-bundle; we also consider the induced diagonal G-action $l^{Q\times Q}$ on $Q \times Q$. Leok, Marsden and Weinstein introduced in [15] a notion of discrete connection on principal G-bundles. The following definition comes from [8], which the reader should refer to for further details on discrete connections.

Definition 2.1. Let $Hor \subset Q \times Q$ be an $l^{Q\times Q}$-invariant submanifold containing the diagonal $\Delta_Q \subset Q \times Q$. We say that Hor defines the *discrete connection* $\mathcal{A}_d$ on the principal bundle $\pi^{Q,G} : Q \to Q/G$ if $(id_Q \times \pi^{Q,G})|_{Hor} : Hor \to Q \times (Q/G)$ is an injective local diffeomorphism. We denote Hor by $Hor_{\mathcal{A}_d}$.

When $Hor_{\mathcal{A}_d}$ is a discrete connection on $\pi^{Q,G} : Q \to Q/G$, it is easy to see that for any $(q_0, q_1) \in Q \times Q$, there is a unique $g \in G$ such that $(q_0, l^Q_{g^{-1}}(q_1)) \in Hor_{\mathcal{A}_d}$, where $l^Q_g(q) := l^Q(g, q)$. In this case, the *discrete connection form* $\mathcal{A}_d : Q \times Q \to G$ is defined by $\mathcal{A}_d(q_0, q_1) := g$.

Remark 2.2. It is well known that when the principal G-bundle is not trivial, the existence of g stated above cannot be assured in general. It is true, though, when (q_0, q_1) is in a certain open subset of $Q \times Q$ containing the diagonal, known as the *domain* of the discrete connection. Still, in what follows, we omit this technical detail in order to keep the notation simple.

As in the case of connections on a principal G-bundle, discrete connections define a notion of discrete horizontal lift, that we introduce below.

Definition 2.3. Let $\mathcal{A}_d$ be a discrete connection on the principal G-bundle $\pi^{Q,G} : Q \to Q/G$. The *discrete horizontal lift* $h_d : Q \times (Q/G) \to Q \times Q$ is the inverse map of the injective local diffeomorphism $(id_Q \times \pi)|_{Hor_{\mathcal{A}_d}} : Hor_{\mathcal{A}_d} \to Q \times (Q/G)$. Explicitly

$$h_d^{q_0}(r_1) = h_d(q_0, r_1) := (q_0, q_1) \quad \Leftrightarrow \quad (q_0, q_1) \in Hor_{\mathcal{A}_d} \text{ and } \pi^{Q,G}(q_1) = r_1.$$

We define $\overline{h_d} := p_2 \circ h_d$ and $\overline{h_d^{q_0}} := p_2 \circ h_d^{q_0}$, where $p_2 : Q \times Q \to Q$ is the projection on the second variable. More generally, $p_j : X_1 \times \cdots \times X_k \to X_j$ is the projection from the Cartesian product onto the j-th component.

Remark 2.4. In the same spirit of Remark 2.2, h_d may not be defined on all of $Q \times (Q/G)$ but only on an open subset. We ignore this fact in what follows.

Discrete connections and their lifts satisfy a number of properties. The next result reviews some of them.

Proposition 2.5. *Let $\mathcal{A}_d$ be a discrete connection on the principal G-bundle $\pi^{Q,G} : Q \to Q/G$. Then,*

(1) the discrete connection form $\mathcal{A}_d$ and the discrete horizontal lift h_d are smooth functions and,

(2) if we consider the left G-actions on G and $Q \times (Q/G)$ given by

$$l_g^G(g') := gg'g^{-1} \quad \text{and} \quad l_g^{Q\times(Q/G)}(q_0, r_1) := (l_g^Q(q_0), r_1)$$

as well as the diagonal action on $Q \times Q$ then $\mathcal{A}_d$, h_d and $\overline{h_d}$ are G-equivariant.

(3) More generally, for any $g_0, g_1 \in G$,

$$\mathcal{A}_d(l_{g_0}^Q(q_0), l_{g_1}^Q(q_1)) = g_1 \mathcal{A}_d(q_0, q_1) g_0^{-1} \quad \text{for all} \quad q_0, q_1 \in Q.$$

Proof. See Lemma 3.2 and Theorems 3.4 and 4.4 in [8]. □

In what follows we use several notions that are reviewed in the Appendix (Section 9). For instance, fiber bundles and their maps are introduced in Definitions 9.6 and 9.13, while the action of a Lie group on a fiber bundle is introduced in Definition 9.14.

When a Lie group G acts on the fiber bundle (E, M, ϕ, F) and F_2 is a right G-manifold, it is possible to construct an *associated bundle* on M/G with total space $(E \times F_2)/G$ and fiber $F \times F_2$. The special case when $F_2 = G$ acting on itself by $r_g(h) := g^{-1}hg$ is known as the *conjugate bundle* and is denoted by $\widetilde{G}_E$ (see Example 9.17).

Proposition 2.6. *Let G be a Lie group that acts on the fiber bundle (E, M, ϕ, F) and $\mathcal{A}_d$ be a discrete connection on the principal G-bundle $\pi^{M,G} : M \to M/G$. Define*[1] *$\widetilde{\Phi}_{\mathcal{A}_d} : E \times M \to E \times G \times (M/G)$ and $\widetilde{\Psi}_{\mathcal{A}_d} : E \times G \times (M/G) \to E \times M$ by*

$$\widetilde{\Phi}_{\mathcal{A}_d}(\epsilon, m) := (\epsilon, \mathcal{A}_d(\phi(\epsilon), m), \pi^{M,G}(m)),$$
$$\widetilde{\Psi}_{\mathcal{A}_d}(\epsilon, w, r) := (\epsilon, l_w^M(\overline{h_d^{\phi(\epsilon)}}(r))).$$

Then, $\widetilde{\Phi}_{\mathcal{A}_d}$ and $\widetilde{\Psi}_{\mathcal{A}_d}$ are smooth functions, inverses of each other. If we view $E \times M$ and $E \times G \times (M/G)$ as fiber bundles over M via $\phi \circ p_1$, then $\widetilde{\Phi}_{\mathcal{A}_d}$ and $\widetilde{\Psi}_{\mathcal{A}_d}$ are bundle maps (over the identity). In addition, if we consider the left G-actions $l^{E\times M}$ and $l^{E\times G\times(M/G)}$ defined by

$$l_g^{E\times M}(\epsilon, m) := (l_g^E(\epsilon), l_g^M(m)) \quad \text{and} \quad l_g^{E\times G\times(M/G)}(\epsilon, w, r) := (l_g^E(\epsilon), l_g^G(w), r),$$

then, $\widetilde{\Phi}_{\mathcal{A}_d}$ and $\widetilde{\Psi}_{\mathcal{A}_d}$ are G-equivariant, so they induce diffeomorphisms $\Phi_{\mathcal{A}_d} : (E \times M)/G \to \widetilde{G}_E \times (M/G)$ and $\Psi_{\mathcal{A}_d} : \widetilde{G}_E \times (M/G) \to (E \times M)/G$.

Proof. Being composition of smooth functions (see part 1 in Proposition 2.5), $\widetilde{\Phi}_{\mathcal{A}_d}$ and $\widetilde{\Psi}_{\mathcal{A}_d}$ are smooth; direct computations involving part 3 of Proposition 2.5 show that $\widetilde{\Phi}_{\mathcal{A}_d}$ and $\widetilde{\Psi}_{\mathcal{A}_d}$ are inverses of each other. Using part 2 from Proposition 2.5 it is

[1]As we mentioned in Remark 2.2, the discrete connection $\mathcal{A}_d$ need not be defined on $M \times M$ but, rather, on an open subset. This restricts the domain of $\widetilde{\Phi}_{\mathcal{A}_d}$ and $\widetilde{\Psi}_{\mathcal{A}_d}$ to appropriate open subsets, where the results of Proposition 2.6 hold. Still, we ignore this point and keep working as if $\mathcal{A}_d$ were globally defined in order to avoid a more involved notation.

easy to verify the G-equivariance of $\widetilde{\Phi}_{\mathcal{A}_d}$. Being $\widetilde{\Psi}_{\mathcal{A}_d} = (\widetilde{\Phi}_{\mathcal{A}_d})^{-1}$, its G-equivariance follows. The last part of the statement is derived from Corollary 9.4. $\square$

Remark 2.7. Notice that $(E \times M)/G$ can be seen as a fiber bundle over M/G via $\phi \,\check{\circ}\, p_1$, corresponding to the associated fiber bundle $\widetilde{M}_E$ constructed in Example 9.17. Similarly $\widetilde{G}_E \times (M/G)$ is a fiber bundle over M/G via $\phi \,\check{\circ}\, p_1$. In this context, $\Phi_{\mathcal{A}_d}$ and $\Psi_{\mathcal{A}_d}$ are bundle maps (over the identity).

After Proposition 2.6, we have the commutative diagram

$$\begin{tikzcd}
E \times M \arrow[r, "\widetilde{\Phi_{\mathcal{A}_d}}", "\sim"'] \arrow[d, "\pi^{E\times M,G}"'] \arrow[dr, "\Upsilon_{\mathcal{A}_d}"] & (E \times G) \times (M/G) \arrow[d, "(\pi^{E\times G,G}\circ p_1)\times p_2"] \\
(E \times M)/G \arrow[r, "\sim", "\Phi_{\mathcal{A}_d}"'] & \widetilde{G}_E \times (M/G)
\end{tikzcd} \tag{2.1}$$

consisting of manifolds and smooth maps (or bundle maps, understanding that the top row are bundles over M and the bottom row are bundles over M/G). In (2.1), we have defined

$$\Upsilon_{\mathcal{A}_d} := \Phi_{\mathcal{A}_d} \circ \pi^{E\times M,G} = ((\pi^{E\times G,G} \circ p_1) \times p_2) \circ \widetilde{\Phi_{\mathcal{A}_d}}. \tag{2.2}$$

Lemma 2.8. *Let G act on the fiber bundle (E, M, ϕ, F) and $\mathcal{A}_d$ be a discrete connection on the principal G-bundle $\pi^{M,G} : M \to M/G$. Then, $\Upsilon_{\mathcal{A}_d} : E \times M \to (\widetilde{G}_E \times (M/G))$ defined by* (2.2) *is a principal G-bundle.*

Proof. Since $\Phi_{\mathcal{A}_d}$ is a diffeomorphism and $\pi^{E\times M,G}$ is a surjective submersion, $\Upsilon_{\mathcal{A}_d}$ is also a surjective submersion. Also, as $\Upsilon^{-1}_{\mathcal{A}_d}(\Upsilon_{\mathcal{A}_d}(\epsilon_0, m_1)) = l^{E\times M}_G(\epsilon_0, m_1)$, by Theorem 9.8, we conclude that $(E \times M, \widetilde{G}_E \times (M/G), \Upsilon_{\mathcal{A}_d}, G)$ is a principal G-bundle. $\square$

When G acts on the fiber bundle (E, M, ϕ, F) and $\mathcal{A}_d$ is a discrete connection on the principal G-bundle $\pi^{M,G} : M \to M/G$, we have the following commutative diagram involving the conjugate bundle $\widetilde{G}_E$.

$$\begin{tikzcd}
E \arrow[d, "\pi^{M,G}\circ\phi"'] & E \times M \arrow[l, "p_1"'] \arrow[r, "\widetilde{\Phi_{\mathcal{A}_d}}"] \arrow[dr, "\Upsilon_{\mathcal{A}_d}"] & (E \times G) \times (M/G) \arrow[d, "(\pi^{E\times G,G}\circ p_1)\times p_2"] \\
M/G & & \widetilde{G}_E \times (M/G) \arrow[ll, "p^{M/G}\circ p_1"]
\end{tikzcd} \tag{2.3}$$

3. Discrete Lagrange–Poincaré systems

A discrete mechanical system (DMS) as in [17] is a pair (Q, L_d) where Q is a finite dimensional manifold known as the *configuration space* and $L_d : Q \times Q \to \mathbb{R}$ is a smooth function called the *discrete lagrangian.* Trajectories of such a system are critical points of an action function determined by L_d.

In this section we introduce an extended notion of DMS as a dynamical system whose dynamics arises from a variational principle. In addition, we find the corresponding equations of motion. In Section 4, we formulate a categorical framework that contains the extended systems.

3.1. **Discrete Lagrange–Poincaré systems and dynamics.** The reduction procedure introduced in [7] and reviewed in the unconstrained situation later, in Section 3.2, has a shortcoming in that, in most cases, when applied to a DMS, the resulting dynamical system is not a DMS. The main objective of this paper is to overcome this problem by considering a larger class of discrete mechanical systems that is closed by the reduction procedure. In order to define the larger class of DMSs we will consider more general "discrete velocity" phase spaces than $Q \times Q$; concretely, we will consider spaces of the form $E \times M$, where $\phi : E \to M$ is a fiber bundle. Furthermore, we will consider discrete time dynamical systems on such spaces, whose dynamics will be defined using a lagrangian function and a variational principle. In this section we study the extended discrete velocity phase spaces and discrete lagrangian systems on them.

The motivation for the notion of extended discrete velocity phase space that we consider in this paper comes from the type of space obtained by the reduction process introduced in [7]. There, the reduced space associated to a discrete system on $Q \times Q$ with symmetry group G is the space $(Q/G) \times (Q/G) \times_{Q/G} \widetilde{G}$, that is a fibered product of the pair bundle $(Q/G) \times (Q/G)$ and the fiber bundle $\widetilde{G} \to Q/G$, where $\widetilde{G} = \widetilde{G}_E$ for E the fiber bundle $id_Q : Q \to Q$ (see Example 9.17). This space is not a standard space for a DMS due to the presence of $\widetilde{G}$. Therefore, it seems reasonable to enlarge the class of spaces to be considered by looking at spaces that are the fibered product of a pair bundle $M \times M$ and a fiber bundle $E \to M$. In fact, for continuous mechanical systems, this is the approach of [5], where their extended velocity phase space is of the form $TQ \oplus V$ and V is a vector bundle over Q. Yet, we will consider a minor variation of the preceding idea: instead of $(M \times M) \times_M E$, we will consider $E \times M$ that, as fiber bundles over M (by $\phi \circ p_1$ in the second space) are isomorphic. The technical advantage of using this last space is that it is easier to work with a product manifold rather than with a fibered product.

Given a fiber bundle $\phi : E \to M$ we will denote $C'(E) := E \times M$, seen as a fiber bundle over M by $\phi \circ p_1$. Similarly, we define the *discrete second order manifold* $C''(E) := (E \times M) \times_{p_2,(\phi \circ p_1)} (E \times M)$ that we view as a fiber bundle over M via the map induced by p_2.

Remark 3.1. Given a fiber bundle $\phi : E \to M$, the second order manifold $C''(E) \to M$ is isomorphic as a fiber bundle to the fiber bundle $\phi \circ p_2 : E \times E \times M \to M$ via $F_E((\epsilon_0, m_1), (\epsilon_1, m_2)) := (\epsilon_0, \epsilon_1, m_2)$.

Definition 3.2. Let $\phi : E \to M$ be a fiber bundle. A *discrete path* in $C'(E)$ is a collection $(\epsilon_\cdot, m_\cdot) = ((\epsilon_0, m_1), \ldots, (\epsilon_{N-1}, m_N))$ where $((\epsilon_k, m_{k+1}), (\epsilon_{k+1}, m_{k+2})) \in C''(E)$ for $k = 0, \ldots, N-2$.

Definition 3.3. Let $\phi : E \to M$ be a fiber bundle. An *infinitesimal variation chaining map* $\mathcal{P}$ on E is a homomorphism of vector bundles over $\widetilde{p_1}$, according to the following commutative diagram (of vector bundles)

$$\begin{tikzcd}
TE \arrow[d] & \widetilde{p_3}^*(TE) \arrow[l] \arrow[r, "\mathcal{P}"] \arrow[d] & \ker(d\phi) \arrow[r, hook] \arrow[d] & TE \arrow[dl] \\
E & C''(E) \arrow[l, "\widetilde{p_3}"'] \arrow[r, "\widetilde{p_1}"'] & E &
\end{tikzcd}$$

where $\widetilde{p_1}((\epsilon_0, m_1), (\epsilon_1, m_2)) := \epsilon_0$ and $\widetilde{p_3}((\epsilon_0, m_1), (\epsilon_1, m_2)) := \epsilon_1$. Notice that since $\phi : E \to M$ is a fiber bundle, $\ker(d\phi)$ has constant rank, so it defines a vector subbundle of $TE \to E$.

Definition 3.4. Let $\phi : E \to M$ be a fiber bundle. A *discrete Lagrange–Poincaré system* (DLPS) over E is a triple $\mathcal{M} := (E, L_d, \mathcal{P})$ where $L_d : C'(E) \to \mathbb{R}$ is a smooth function and $\mathcal{P}$ is an infinitesimal variation chaining map on E.

Definition 3.5. Let $(E, L_d, \mathcal{P})$ be a DLPS and $(\epsilon_\cdot, m_\cdot) = ((\epsilon_0, m_1), \ldots, (\epsilon_{N-1}, m_N))$ be a discrete path in $C'(E)$. An *infinitesimal variation* on $(\epsilon_\cdot, m_\cdot)$ is a tangent vector $(\delta\epsilon_\cdot, \delta m_\cdot) = ((\delta\epsilon_0, \delta m_1), \ldots, (\delta\epsilon_{N-1}, \delta m_N)) \in T_{(\epsilon_\cdot, m_\cdot)} C'(E)^N$ such that

$$\delta m_k = d\phi(\epsilon_k)(\delta\epsilon_k) \quad \text{for} \quad k = 1, \ldots, N-1 \tag{3.1}$$

or, equivalently, that $((\delta\epsilon_{k-1}, \delta m_k), (\delta\epsilon_k, \delta m_{k+1})) \in TC''(E)$ for $k = 1, \ldots, N-1$. An *infinitesimal variation on* $(\epsilon_\cdot, m_\cdot)$ *with fixed endpoints* is an infinitesimal variation $(\delta\epsilon_\cdot, \delta m_\cdot)$ on $(\epsilon_\cdot, m_\cdot)$ such that

$$\begin{aligned}
\delta m_N =& 0, \\
\delta\epsilon_{N-1} =& \widetilde{\delta\epsilon_{N-1}}, \\
\delta\epsilon_k =& \widetilde{\delta\epsilon_k} + \mathcal{P}((\epsilon_k, m_{k+1}), (\epsilon_{k+1}, m_{k+2}))(\widetilde{\delta\epsilon_{k+1}}), \quad \text{if} \quad k = 1, \ldots, N-2, \\
\delta\epsilon_0 =& \mathcal{P}((\epsilon_0, m_1), (\epsilon_1, m_2))(\widetilde{\delta\epsilon_1}),
\end{aligned} \tag{3.2}$$

where $\widetilde{\delta\epsilon_k} \in T_{\epsilon_k} E$ is arbitrary for $k = 1, \ldots, N-1$.

Remark 3.6. The name "infinitesimal variation with fixed endpoints" is not entirely accurate in Definition 3.5. Certainly, $\delta m_N = 0$ means that m_N remains fixed. On the other hand, $\delta\epsilon_0$ does not necessarily vanish, but neither it is arbitrary, as it is determined by $\widetilde{\delta\epsilon_1}$ through $\mathcal{P}$. As the $\widetilde{\delta\epsilon_k}$ are arbitrary for $k = 1, \ldots, N-1$, given $\delta\epsilon_k$ for $k = 1, \ldots, N-1$, it is possible to find $\widetilde{\delta\epsilon_k}$ for $k = 1, \ldots, N-1$ such that $\delta\epsilon_{N-1} = \widetilde{\delta\epsilon_{N-1}}$ and $\delta\epsilon_k = \widetilde{\delta\epsilon_k} + \mathcal{P}((\epsilon_k, m_{k+1}), (\epsilon_{k+1}, m_{k+2}))(\widetilde{\delta\epsilon_{k+1}})$ for all $k = 1, \ldots, N-2$. In this case, $\delta\epsilon_0$ turns out to be a function of all $\delta\epsilon_1, \ldots, \delta\epsilon_{N-1}$.

Definition 3.7. Let $\mathcal{M} = (E, L_d, \mathcal{P})$ be a DLPS. The *discrete action* of $\mathcal{M}$ is a function from the space of all discrete curves on $C'(E)$ to $\mathbb{R}$ defined by $S_d(\epsilon_\cdot, m_\cdot) := \sum_{k=0}^{N-1} L_d(\epsilon_k, m_{k+1})$. A *trajectory* of $\mathcal{M}$ is a discrete curve $(\epsilon_\cdot, m_\cdot)$ in $C'(E)$ such that

$$dS_d(\epsilon_\cdot, m_\cdot)(\delta\epsilon_\cdot, \delta m_\cdot) = 0 \tag{3.3}$$

for all infinitesimal variations $(\delta\epsilon_\cdot, \delta m_\cdot)$ on $(\epsilon_\cdot, m_\cdot)$ with fixed endpoints, that is, satisfying (3.1) and (3.2).

The following Proposition characterizes the trajectories of a DLPS in terms of (algebraic) equations.

Proposition 3.8. *Let $\mathcal{M} = (E, L_d, \mathcal{P})$ be a DLPS and $(\epsilon_\cdot, m_\cdot)$ be a discrete path in $C'(E)$. Then, $(\epsilon_\cdot, m_\cdot)$ is a trajectory of $\mathcal{M}$ if and only if for all $k = 1, \ldots, N-1$,*

$$\begin{aligned}
D_1 L_d(\epsilon_k, m_{k+1}) +& D_2 L_d(\epsilon_{k-1}, m_k) \circ d\phi(\epsilon_k) \\
&+ D_1 L_d(\epsilon_{k-1}, m_k) \circ \mathcal{P}((\epsilon_{k-1}, m_k), (\epsilon_k, m_{k+1})) = 0
\end{aligned} \tag{3.4}$$

in $T^*_{\epsilon_k} E$, where D_j denotes the restriction to the j-th component of the exterior differential on a Cartesian product.

Proof. Equation (3.4) is obtained from the standard variational computation of $dS_d(\epsilon_\cdot, m_\cdot)(\delta\epsilon_\cdot, \delta m_\cdot)$, taking into account that the fixed endpoint infinitesimal variations $(\delta\epsilon_\cdot, \delta m_\cdot)$ over $(\epsilon_\cdot, m_\cdot)$ satisfy (3.2) for arbitrary $\widetilde{\delta\epsilon_k} \in T_{\epsilon_k}E$. □

Next, we introduce sufficient conditions for the existence of a flow on a DLPS $\mathcal{M} = (E, L_d, \mathcal{P})$. Consider the commutative diagram (of smooth maps)

$$\begin{tikzcd} & T^*E \arrow[d] \\ E\times E\times M \arrow[ur, "\mathcal{E}"] \arrow[r, "p_2"'] & E \end{tikzcd}$$

where

$$\begin{aligned}\mathcal{E}(\epsilon_0, \epsilon_1, m_2) :=& D_1L_d(\epsilon_1, m_2) + D_1L_d(\epsilon_0, \phi(\epsilon_1)) \circ \mathcal{P}((\epsilon_0, \phi(\epsilon_1)), (\epsilon_1, m_2)) \\ &+ D_2L_d(\epsilon_0, \phi(\epsilon_1)) \circ d\phi(\epsilon_1).\end{aligned}$$

Notice that all trajectories $((\epsilon_0, m_1), (\epsilon_1, m_2))$ of $\mathcal{M}$ satisfy $\mathcal{E}(\epsilon_0, \epsilon_1, m_2) = 0_{\epsilon_1} \in T^*_{\epsilon_1}E$. Conversely, given $(\epsilon_0, \epsilon_1, m_2) \in E \times E \times M$ such that $\mathcal{E}(\epsilon_0, \epsilon_1, m_2) = 0_{\epsilon_1}$, then $((\epsilon_0, \phi(\epsilon_1)), (\epsilon_1, m_2))$ is a trajectory of $\mathcal{M}$.

Let $\mathcal{Z} \subset T^*E$ be the image of the zero section of the canonical projection $T^*E \to E$. It is easy to check that $\mathcal{Z} \subset T^*E$ is an embedded submanifold.

Proposition 3.9. *Let $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{Z}$ be as above.*

(1) *Assume that $(\epsilon_0, \epsilon_1, m_2) \in E \times E \times M$ is such that $\mathcal{E}(\epsilon_0, \epsilon_1, m_2) = 0_{\epsilon_1}$ and that $\operatorname{Im}(d\mathcal{E}(\epsilon_0, \epsilon_1, m_2)) + T_{0_{\epsilon_1}}\mathcal{Z} = T_{0_{\epsilon_1}}T^*E$. Then, there is an open subset $U \subset E \times E \times M$ with $(\epsilon_0, \epsilon_1, m_2) \in U$ and such that $\mathcal{E}_U := U \cap \mathcal{E}^{-1}(\mathcal{Z})$ is an embedded submanifold of $E\times E\times M$ with $\dim(\mathcal{E}_U) = \dim(E)+\dim(M)$.*

(2) *Consider the smooth map $p_1 \times (\phi \circ p_2) : E \times E \times M \to C'(E)$. In addition to what was assumed in part 1, suppose that*
   - *(i) $d(p_1 \times (\phi \circ p_2))|_{\mathcal{E}_U}(\epsilon_0, \epsilon_1, m_2) \in \hom(T_{(\epsilon_0,\epsilon_1,m_2)}\mathcal{E}_U, T_{(\epsilon_0,\phi(\epsilon_1))}C'(E))$ is injective and*
   - *(ii) $d(p_2 \times p_3)|_{\mathcal{E}_U}(\epsilon_0, \epsilon_1, m_2) \in \hom(T_{(\epsilon_0,\epsilon_1,m_2)}\mathcal{E}_U, T_{(\epsilon_1,m_2)}C'(E))$ is injective.*

   Then, there are open sets $V_1, V_2 \subset C'(E)$ such that $(\epsilon_0, \phi(\epsilon_1)) \in V_1$ and $(\epsilon_1, m_2) \in V_2$ and a diffeomorphism $F_{\mathcal{M}} : V_1 \to V_2$ such that $F_{\mathcal{M}}(\epsilon_0, \phi(\epsilon_1)) = (\epsilon_1, m_2)$ and, for all $(\epsilon_0', m_1') \in V_1$, $((\epsilon_0', m_1'), F_{\mathcal{M}}(\epsilon_0', m_1'))$ is a trajectory of $\mathcal{M}$.

Proof. Part 1 follows immediately from the transversality argument on page 28 of [9], applied to the point $(\epsilon_0, \epsilon_1, m_2)$. Notice that, as $(\epsilon_0, \epsilon_1, m_2) \in \mathcal{E}_U$, it is not the empty set.

Let $P := (p_1 \times (\phi \circ p_2))|_{\mathcal{E}_U} : \mathcal{E}_U \to C'(E)$. As $\dim(\mathcal{E}_U) = \dim(C'(E))$, condition 2i implies that $dP(\epsilon_0, \epsilon_1, m_2)$ is an isomorphism and, consequently, P is a local diffeomorphism at $(\epsilon_0, \epsilon_1, m_2)$. Hence, there are open sets $V_1 \subset C'(E)$ and $V_2' \subset \mathcal{E}_U$ such that $(\epsilon_0, \phi(\epsilon_1)) \in V_1$ and $(\epsilon_0, \epsilon_1, m_2) \in V_2'$ where $P|_{V_2'}$ is a diffeomorphism onto V_1. In addition, as $\dim(\mathcal{E}_U) = \dim(C'(E))$, condition 2ii implies that $d(p_2 \times p_3)|_{\mathcal{E}_U}(\epsilon_0, \epsilon_1, m_2)$ is a local diffeomorphism at $(\epsilon_0, \epsilon_1, m_2)$ so that (eventually shrinking V_2') $V_2 := (p_2 \times p_3)(V_2') \subset C'(E)$ is open and $(p_2 \times p_3)|_{V_2'} : V_2' \to V_2$ is a diffeomorphism.

Let $F_{\mathcal{M}} : V_1 \to V_2$ be the diffeomorphism $F_{\mathcal{M}} := (p_2 \times p_3) \circ (P|_{V_2'})^{-1}$. By construction, $F_{\mathcal{M}}(\epsilon_0, \phi(\epsilon_1)) = (\epsilon_1, m_2)$. Furthermore, for $(\epsilon_0', m_1') \in V_1$, if we let

$(\epsilon'_1, m'_2) := F_{\mathcal{M}}(\epsilon'_0, m'_1)$, it follows readily that $((\epsilon'_0, m'_1), (\epsilon'_1, m'_2)) \in C''(E)$ is a trajectory of $\mathcal{M}$. $\square$

Definition 3.10. The function $F_{\mathcal{M}}$ that appears in part 2 of Proposition 3.9 is the *discrete Lagrangian flow* of $\mathcal{M}$.

Remark 3.11. If $(\epsilon_., m_.) = ((\epsilon_0, m_1), \ldots, (\epsilon_{N-1}, m_N))$ is a trajectory of the DLPS $\mathcal{M}$, then it satisfies (3.4) for $k = 0, \ldots, N-1$. But then, if $j = 0, \ldots, N-2$, $((\epsilon_j, m_{j+1}), (\epsilon_{j+1}, m_{j+2}))$ also satisfies (3.4) (for $k = j, j+1$) and, by Proposition 3.8, is also a trajectory of $\mathcal{M}$. That is, contiguous points of a trajectory of $\mathcal{M}$, form a trajectory of $\mathcal{M}$.

The following example shows how a DMS can be seen as a DLPS.

Example 3.12. Let (Q, L_d) be a DMS. Define the fiber bundle $\phi : E \to M$ by $id_Q : Q \to Q$, so that L_d defines a lagrangian function on $C'(id_Q : Q \to Q) = Q \times Q$. Next, let $\mathcal{P}((q_{k-1}, q_k), (q_k, q_{k+1}))(\delta q_k) := 0$ for all $\delta q_k \in T_{q_k} Q$. We define the DLPS $\mathcal{M} := (E, L_d, \mathcal{P})$.

Discrete paths $(\epsilon_., m_.)$ of $\mathcal{M}$ are, in the current context, the same as discrete paths $q_.$ in Q^2. Such discrete paths are trajectories of $\mathcal{M}$ if and only if they satisfy (3.4) that, in this case, becomes

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0 \tag{3.5}$$

for all k, that is the usual discrete Euler–Lagrange equation (see equation (1.3.3) in [17]) that characterizes the trajectories of (Q, L_d). Hence, all DMSs can be seen as DLPSs whose dynamics coincide with those of the original systems.

Remark 3.13. As, by Example 3.12, all DMSs are DLPSs, we can specialize Proposition 3.9 to the case of a DMS (Q, L_d). A simple analysis provides the following statement. Let $(q_0, q_1, q_2) \in Q \times Q \times Q$ be a solution of (3.5) (for $k = 1$) such that L_d is regular[3] at (q_0, q_1) and (q_1, q_2). Then there are open sets $V_1, V_2 \subset Q \times Q$ with $(q_0, q_1) \in V_1$ and $(q_1, q_2) \in V_2$ and a diffeomorphism $F_{L_d} : V_1 \to V_2$ such that $F_{L_d}(q_0, q_1) = (q_1, q_2)$ and that $(q'_0, F_{L_d}(q'_0, q'_1))$ is a solution of (3.5) (for $k = 1$) for all $(q'_0, q'_1) \in V_1$. We emphasize that the existence of a trajectory (q_0, q_1, q_2) as a staring point cannot be avoided. For example, when $Q = \mathbb{R}$ and $L_d(q_0, q_1) := \frac{1}{2}(q_1 - q_0)^2 - \eta(q_0 + q_1)^3$ for $\eta > 0$, we have that L_d is regular at (q_0, q_1) and (q_1, q_2) for $q_0, q_1, q_2 < -\frac{1}{6\eta}$. But, it is easy to check that, if $q_1 < -\frac{1}{24\eta}$, there is no trajectory of the form (q_0, q_1, q_2).

The dynamical system obtained by the reduction process of a symmetric DMS can be seen as a DLPS, as we describe in the following section.

3.2. **Reduced system associated to a symmetric discrete mechanical system.** We say that the Lie group G is a symmetry group of the DMS (Q, L_d) if G acts on Q in such a way that the quotient mapping $\pi^{Q,G} : Q \to Q/G$ is a principal G-bundle and $L_d \circ l_g^{Q \times Q} = L_d$ for all $g \in G$. Given such a system we can construct a discrete time dynamical system called the *reduced system* whose dynamics captures

[2] A discrete path $x_.$ in a manifold X is an element of the Cartesian product X^N, for some $N \in \mathbb{N}$.

[3] Regularity at (q_0, q_1) means that, with respect to local coordinates q_j^a (for $j = 0, 1$ and $a = 1, \ldots, n := \dim(Q)$) neat q_0 and q_1, the matrix $\frac{\partial^2 L_d(q_0, q_1)}{\partial q_0^j \partial q_1^k} \in \mathbb{R}^{n \times n}$ be invertible.

the essential behavior of the original dynamics. First we review the construction of the reduced system and, then, compare the dynamics of the reduced to that of the unreduced system. After that, we prove that the reduced system can be seen as a DLPS with the same trajectories.

Given a discrete connection $\mathcal{A}_d$ on the principal G-bundle $\pi^{Q,G} : Q \to Q/G$, we can specialize the commutative diagram (2.1) to the case where $\phi : E \to M$ is $id_Q : Q \to Q$:

$$\begin{tikzcd}
Q \times Q \arrow[r, "\widetilde{\Phi_{\mathcal{A}_d}}"] \arrow[d, "\pi^{Q\times Q,G}"'] \arrow[dr, "\Upsilon_{\mathcal{A}_d}"] & (Q \times G) \times (Q/G) \arrow[d, "(\pi^{Q\times G,G} \circ p_1)\times p_2"] \\
(Q \times Q)/G \arrow[r, "\Phi_{\mathcal{A}_d}"'] & \widetilde{G} \times (Q/G)
\end{tikzcd}$$

where $\widetilde{G} = (Q \times G)/G$ with G acting on Q by l^Q and on G by conjugation and, explicitly,

$$\Upsilon_{\mathcal{A}_d}(q_0, q_1) = (\pi^{Q\times G,G}(q_0, \mathcal{A}_d(q_0, q_1)), \pi^{Q,G}(q_1)). \tag{3.6}$$

By the G-invariance of L_d, there is a well defined map $\check{L}_d : \widetilde{G} \times (Q/G) \to \mathbb{R}$ such that $\check{L}_d(v_0, r_1) = L_d(q_0, q_1)$ whenever $(q_0, q_1) \in Q \times Q$ satisfies $(v_0, r_1) = \Upsilon_{\mathcal{A}_d}(q_0, q_1)$. The action associated to $\check{L}_d$ is $\check{S}_d(v_\cdot, r_\cdot) := \sum_k \check{L}_d(v_k, r_{k+1})$.

The following result from [7][4] relates the dynamics of the original system to a variational principle for a system on $\widetilde{G} \times (Q/G)$.

Theorem 3.14. *Let G be a symmetry group of the DMS (Q, L_d). Fix a discrete connection $\mathcal{A}_d$ on the principal G-bundle $\pi^{Q,G} : Q \to Q/G$. Let $q_\cdot$ be a discrete path in Q, $r_k := \pi^{Q,G}(q_k)$, $w_k := \mathcal{A}_d(q_k, q_{k+1})$ and $v_k := \pi^{Q\times G,G}(q_k, w_k)$ be the corresponding discrete paths in Q/G, G and $\widetilde{G}$ (see footnote 2). Then, the following statements are equivalent.*

(1) $q_\cdot$ satisfies the variational principle $dS_d(q_\cdot)(\delta q_\cdot) = 0$ for all vanishing end-points variations $\delta q_\cdot$ over $q_\cdot$.

(2) $d\check{S}_d(r_\cdot, v_\cdot)(\delta r_\cdot, \delta v_\cdot) = 0$ for all variations $(\delta v_\cdot, \delta r_\cdot)$ such that

$$(\delta v_k, \delta r_{k+1}) := d\Upsilon_{\mathcal{A}_d}(q_k, q_{k+1})(\delta q_k, \delta q_{k+1}) \tag{3.7}$$

for $k = 0, \ldots, N-1$ and where $\delta q_\cdot$ is a fixed endpoints variation over $q_\cdot$.

Remark 3.15. The more general Theorem 5.11 in [7] requires the additional data of a connection $\mathcal{A}$ on the principal G-bundle $\pi^{Q,G} : Q \to Q/G$. With this additional information the variations $\delta q_\cdot$ are decomposed in $\mathcal{A}$-horizontal and $\mathcal{A}$-vertical parts.

The reduced system associated to (Q, L_d) is the discrete dynamical system on $\widetilde{G} \times (Q/G)$ whose trajectories are the discrete paths that satisfy the variational principle stated in point 2 of Theorem 3.14. A DLPS $\mathcal{M} := (E, \check{L}_d, \mathcal{P})$ is associated to this reduced system; we prove later that the trajectories of both systems coincide. Define the fiber bundle $\phi : E \to M$ as the *conjugate bundle* $p^{Q/G} : \widetilde{G} \to Q/G$, where $p^{Q/G}(\pi^{Q\times G,G}(q, w)) := \pi^{Q,G}(q)$. The reduced Lagrangian $\check{L}_d : \widetilde{G} \times (Q/G) \to \mathbb{R}$ defines a real valued function on $C'(E) = E \times M$.

[4]In fact, Theorem 3.14 here is part of Theorem 5.11 in [7], specialized to the unconstrained case, and where we have adapted the notation slightly to match the one used in the present paper.

In order to define the infinitesimal variation chaining function, we consider $\Upsilon_{\mathcal{A}_d} : Q \times Q \to \widetilde{G} \times (Q/G)$ defined by (3.6). Then define $\mathcal{P} \in \hom(p_3^*(T\widetilde{G}), \ker(dp^{Q/G}))$ by

$$\mathcal{P}((v_0, r_1), (v_1, r_2))(\delta v_1) := D_2(p_1 \circ \Upsilon_{\mathcal{A}_d})(q_0, q_1)(\delta q_1) \in T_{v_0}\widetilde{G} \tag{3.8}$$

where (q_0, q_1, q_2) are such that $(v_0, r_1) = \Upsilon_{\mathcal{A}_d}(q_0, q_1)$ and $(v_1, r_2) = \Upsilon_{\mathcal{A}_d}(q_1, q_2)$, and $\delta q_1 \in T_{q_1}Q$ is such that $D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(q_1, q_2)(\delta q_1) = \delta v_1$. Lemma 3.16 proves that $\mathcal{P}$ is well defined.

Lemma 3.16. *Let Q, $\mathcal{A}_d$ and $\Upsilon_{\mathcal{A}_d}$ be as before. Then, the following assertions are true.*

1. *For $(q_0, q_1) \in Q \times Q$, $D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(q_0, q_1) : T_{(q_0,q_1)}(Q \times \{q_1\}) \to T_{\Upsilon_{\mathcal{A}_d}(q_0,q_1)}\widetilde{G}$ is an isomorphism of vector spaces.*
2. *For $((v_0, r_1), (v_1, r_2)) \in C''(E)$ and $\delta v_1 \in T_{v_1}\widetilde{G}$ define $\mathcal{P}((v_0, r_1), (v_1, r_2))(\delta v_1)$ using* (3.8)*. Then, $\mathcal{P}$ is well defined. In addition, $\mathcal{P}$ is linear in δv_1.*
3. *For $((v_0, r_1), (v_1, r_2)) \in C''(E)$ and $\delta v_1 \in T_{v_1}\widetilde{G}$ we have*
$$dp^{Q/G}(v_0)(\mathcal{P}((v_0, r_1), (v_1, r_2))(\delta v_1)) = 0.$$

We skip the proof of Lemma 3.16 as we will be proving more general statements later: see point 2 in Lemma 5.1 for point 1 and Lemma 5.10 for points 2 and 3.

Next, we compare discrete trajectories of $\mathcal{M}$ with the reduced trajectories given by part 2 of Theorem 3.14. We denote points in $E = \widetilde{G}$ with v and in $M = Q/G$ with r. The following result proves that all discrete paths in $C'(E)$ arise from discrete paths in Q.

Lemma 3.17. *Let $(v_\cdot, r_\cdot)$ be a discrete path in $C'(E)$ and $q_0 \in Q$ such that $p^{Q/G}(v_0) = \pi^{Q,G}(q_0)$. Then, there exists a unique discrete path in $C'(id_Q : Q \to Q)$ such that $\Upsilon_{\mathcal{A}_d}(q_k, q_{k+1}) = (v_k, r_{k+1})$ for all $k = 0, \ldots, N-1$.*

Proof. See Proposition 5.2, that is the same result, in a more general context. □

A trajectory $(v_\cdot, r_\cdot) = ((v_0, r_1), \ldots, (v_{N-1}, r_N))$ of $\mathcal{M}$ is a pair of discrete paths $v_\cdot$ and $r_\cdot$ such that $\phi(v_k) = p^{Q/G}(v_k) = r_k$ for $k = 1, \ldots, N-1$, and satisfies $dS_d(v_\cdot, r_\cdot)(\delta v_\cdot, \delta r_\cdot) = 0$ for all infinitesimal variations $(\delta v_\cdot, \delta r_\cdot)$ on $(v_\cdot, r_\cdot)$ with fixed endpoints. Those infinitesimal variations are given by (3.1) and (3.2).

In what follows, we fix discrete paths $(v_\cdot, r_\cdot)$ in $\mathcal{M}$ and $q_\cdot$ in Q such that $(v_k, r_{k+1}) = \Upsilon_{\mathcal{A}_d}(q_k, q_{k+1})$ for all k. The following result compares the infinitesimal variations over $(v_\cdot, r_\cdot)$ in $\mathcal{M}$ to those coming from (3.7).

Proposition 3.18. *With the notation as above, the following statements are true.*

1. *Given a fixed endpoint variation $\delta q_\cdot$ over the discrete path $q_\cdot$ in Q, the infinitesimal variation $(\delta v_\cdot, \delta r_\cdot)$ defined by* (3.7) *is an infinitesimal variation with fixed endpoints over $(v_\cdot, r_\cdot)$ in $\mathcal{M}$.*
2. *Given a discrete variation $(\delta v_\cdot, \delta r_\cdot)$ over $(v_\cdot, r_\cdot)$ with fixed endpoints, there is a fixed endpoints variation $\delta q_\cdot$ over the discrete path $q_\cdot$ such that* (3.7) *holds for all k.*

Proof. (1) Let $(\delta v_\cdot, \delta r_\cdot)$ be the variation defined by (3.7) in terms of $\delta q_\cdot$. Let $\widetilde{\delta v_k} := D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(q_k, q_{k+1})(\delta q_k) \in T_{v_k}\widetilde{G}$ for $k = 0, \ldots, N-1$. Direct computations using (3.7) prove that $(\delta v_\cdot, \delta r_\cdot)$ satisfies (3.1) and (3.2). Thus, it is an infinitesimal variation with fixed endpoints in $\mathcal{M}$ on $(v_\cdot, r_\cdot)$.

(2) Write $(\delta v_\cdot, \delta r_\cdot)$ according to (3.1) and (3.2) for some vectors $\widetilde{\delta v_k} \in T_{v_k}\widetilde{G}$ and $k = 1, \ldots, N-1$. Let $\delta q_N := 0 \in T_{q_N}Q$, $\delta q_0 := 0 \in T_{q_0}Q$ and, for each $k = 1, \ldots, N-1$, using point 1 in Lemma 3.16, let $\delta q_k \in T_{q_k}Q$ be such that $D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(q_k, q_{k+1})(\delta q_k) = \widetilde{\delta v_k}$. Straightforward computations using (3.1) and (3.2) now show that $\delta q_\cdot$ as constructed is an infinitesimal variation over $q_\cdot$ with fixed endpoints and that (3.7) holds.

□

Corollary 3.19. *A discrete path $(v_\cdot, r_\cdot)$ is a trajectory of $\mathcal{M}$ if and only if it is a trajectory of the reduced system according to point 2 in Theorem 3.14.*

Hence, the family of DLPSs contains in a natural way all DMSs as well as all the dynamical systems obtained by reduction of symmetric DMSs.

4. Categorical formulation

In many circumstances it is useful to be able to consider "maps" between mechanical systems. One example in the area of interest of this paper is the reduction process, seen as a map from a symmetric system to a reduced one. Another example is the comparison of different reductions of the same symmetric system. More generally, a symmetry could be seen as a map from a system to itself. A common framework for considering spaces together with their maps is provided by constructing a category (see, for instance, [5]). In this section we study the basic properties of DLPSs and their morphisms in this categorical context.

Definition 4.1. We define the *category of discrete Lagrange–Poincaré systems* as the category $\mathfrak{LP}_d$ whose objects are DLPSs. Given $\mathcal{M}, \mathcal{M}' \in \mathrm{ob}_{\mathfrak{LP}_d}$ with $\mathcal{M} = (E, L_d, \mathcal{P})$ and $\mathcal{M}' = (E', L'_d, \mathcal{P}')$, a map $\Upsilon : C'(E) \to C'(E')$ is in $\mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}')$ if

(1) Υ is a surjective submersion,
(2) $D_1(p_1 \circ \Upsilon)(\epsilon_0, m_1) : T_{(\epsilon_0, m_1)}(E \times \{m_1\}) \to T_{p_1(\Upsilon(\epsilon_0, m_1))}E'$ is onto for all $(\epsilon_0, m_1) \in C'(E)$,
(3) $D_1(p_2 \circ \Upsilon)(\epsilon_0, m_1) = 0$ for all $(\epsilon_0, m_1) \in C'(E)$
(4) as maps from $C''(E)$ to M',
$$p_2^{C'(E'),M'} \circ \Upsilon \circ p_1^{C''(E),C'(E)} = \phi' \circ p_1^{C'(E'),E'} \circ \Upsilon \circ p_2^{C''(E),C'(E)}, \tag{4.1}$$
where $p_j^{A,B} : A \to B$ are the maps induced by the canonical projections of a Cartesian product onto its factors,
(5) $L_d = L'_d \circ \Upsilon$,
(6) For all $((\epsilon_0, m_1), (\epsilon_1, m_2), \delta\epsilon_1) \in p_3^*(TE)$,
$$\begin{split}\mathcal{P}'(\Upsilon^{(2)}((\epsilon_0, m_1),(\epsilon_1, m_2)))(D_1(p_1 \circ \Upsilon)(\epsilon_1, m_2)(\delta\epsilon_1)) = \\ d(p_1 \circ \Upsilon)(\epsilon_0, m_1)(\mathcal{P}((\epsilon_0, m_1), (\epsilon_1, m_2))(\delta\epsilon_1), d\phi(\epsilon_1)(\delta\epsilon_1))\end{split} \tag{4.2}$$
(see Remark 4.2 below).

Remark 4.2. If $\Upsilon \in \mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}')$, by point 4, $\Upsilon \times \Upsilon$ defines a map $\Upsilon^{(2)} : C''(E) \to C''(E')$, which is used in point 6.

Lemma 4.3. *Let $\Upsilon \in \mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}')$, $((\epsilon_0, m_1), (\epsilon_1, m_2)) \in C''(E)$ and $(\epsilon'_0, m'_1) := \Upsilon(\epsilon_0, m_1)$. The following assertions are true.*

(1) Given $\delta\epsilon_0' \in T_{\epsilon_0'}E'$*, if* $D_1(p_1 \circ \Upsilon)(\epsilon_0, m_1)(\delta\epsilon_0) = \delta\epsilon_0'$ *for some* $\delta\epsilon_0 \in T_{\epsilon_0}E$*, then* $d\Upsilon(\epsilon_0, m_1)(\delta\epsilon_0, 0) = (\delta\epsilon_0', 0)$.

(2) If $\delta\epsilon_1 \in T_{\epsilon_1}E$,

$$D_2(p_2 \circ \Upsilon)(\epsilon_0, m_1)(d\phi(\epsilon_1)(\delta\epsilon_1)) = d\phi'(\epsilon_1')(D_1(p_1 \circ \Upsilon)(\epsilon_1, m_2)(\delta\epsilon_1)). \tag{4.3}$$

Proof. Point 1 follows from morphism's condition 3, satisfied by Υ. Point 2 follows by noticing that $(0, d\phi(\epsilon_1)(\delta\epsilon_1), \delta\epsilon_1, 0) \in T_{((\epsilon_0,m_1),(\epsilon_1,m_2))}C''(E)$ and, then, using (4.1). $\square$

Proposition 4.4. $\mathfrak{LP}_d$ *is a category considering the standard composition of functions and identity mappings.*

Proof. In order to prove that the given data defines a category one has to check that the composition mapping is associative and the identities are left and right identities for the composition mapping. The composition of functions and the identity mappings meet those requirements, so the only thing left to prove is that $\circ$ is well defined in $\mathfrak{LP}_d$, that is, that $\circ : \mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}', \mathcal{M}'') \times \mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}') \to \mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}'')$ and that $id_{\mathcal{M}} := id_{C'(E)} \in \mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M})$. Both properties follow in a lengthy but straightforward manner. $\square$

Lemma 4.5. *Let* $\Upsilon' \in \mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}')$ *and* $\Upsilon'' \in \mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}'')$ *where* $\mathcal{M} = (E, L_d, \mathcal{P})$, $\mathcal{M}' = (E', L_d', \mathcal{P}')$ *and* $\mathcal{M}'' = (E'', L_d', \mathcal{P}'')$*. If* $F : C'(E') \to C'(E'')$ *is a smooth map such that the diagram*

$$\begin{tikzcd}
& C'(E) \arrow[dl, "\Upsilon'"'] \arrow[dr, "\Upsilon''"] & \\
C'(E') \arrow[rr, "F"'] & & C'(E'')
\end{tikzcd}$$

is commutative, then $F \in \mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}', \mathcal{M}'')$*. Furthermore, if* F *is a diffeomorphism, then* F *is an isomorphism in* $\mathfrak{LP}_d$.

Proof. That F satisfies morphism's conditions 1 and 2 follows easily using the corresponding property of the morphism Υ'' to lift the data (point or tangent vector) to $C'(E)$ and, then, using Υ' to push down to $C'(E')$.

Given $(\epsilon_0', m_1') \in C'(E')$ and $\delta\epsilon_0' \in T_{\epsilon_0'}E'$, let $(\epsilon_0, m_1) \in C'(E)$ and $\delta\epsilon_0 \in T_{\epsilon_0}E$ such that $\Upsilon'(\epsilon_0, m_1) = (\epsilon_0', m_1')$ and $D_1(p_1 \circ \Upsilon')(\epsilon_0, m_1)(\delta\epsilon_0) = \delta\epsilon_0'$, by point 1 in Lemma 4.3, $d\Upsilon'(\epsilon_0, m_1)(\delta\epsilon_0, 0) = (\delta\epsilon_0', 0)$. As $p_2 \circ \Upsilon'' = p_2 \circ F \circ \Upsilon'$, taking differentials and evaluating at (ϵ_0, m_1) we get

$$\begin{aligned}
D_1(p_2 \circ F)(\epsilon_0', m_1')(\delta\epsilon_0') =& d(p_2 \circ F)(\epsilon_0', m_1')(\delta\epsilon_0', 0) = d(p_2 \circ \Upsilon'')(\epsilon_0, m_1)(\delta\epsilon_0, 0) \\
=& D_1(p_2 \circ \Upsilon'')(\epsilon_0, m_1)(\delta\epsilon_0) = 0,
\end{aligned}$$

where the last identity holds because $\Upsilon'' \in \mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}'')$. Thus, F satisfies morphism's condition 3.

The remaining conditions follow in a similar fashion, and we conclude that $F \in \mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}', \mathcal{M}'')$.

The last assertion of the statement follows easily as the first part of the Lemma proves that F^{-1} is a morphism in $\mathfrak{LP}_d$ and since, as functions, F and F^{-1} are mutually inverses, they have the same property as morphisms in $\mathfrak{LP}_d$. $\square$

Lemma 4.6. *Let $\Upsilon \in \operatorname{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}')$ for $\mathcal{M} = (E, L_d, \mathcal{P})$ and $\mathcal{M}' = (E', L'_d, \mathcal{P}')$ such that $\Upsilon : C'(E) \to C'(E')$ is a diffeomorphism. Then Υ is an isomorphism of $\mathfrak{LP}_d$.*

Proof. As $\Upsilon \in \operatorname{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}')$ and, by Proposition 4.4, $id_{C'(E)} \in \operatorname{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M})$, the result follows from Lemma 4.5 with $\mathcal{M}'' := \mathcal{M}$ and $F := \Upsilon^{-1}$. $\square$

The following result exposes the relation between trajectories of a DLPS and their images under a morphism in $\mathfrak{LP}_d$.

Theorem 4.7. *Given $\Upsilon \in \operatorname{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}')$ with $\mathcal{M} = (E, L_d, \mathcal{P})$ and $\mathcal{M}' = (E', L'_d, \mathcal{P}')$, let $(\epsilon_\cdot, m_\cdot) = ((\epsilon_0, m_1), \ldots, (\epsilon_{N-1}, m_N))$ be a discrete path in $C'(E)$ and define $(\epsilon'_k, m'_{k+1}) := \Upsilon(\epsilon_k, m_{k+1})$ for $k = 0, \ldots, N-1$. Then, $(\epsilon_\cdot, m_\cdot)$ is a trajectory of $\mathcal{M}$ if and only if $(\epsilon'_\cdot, m'_\cdot)$ is a trajectory of $\mathcal{M}'$.*

Proof. Assume that $(\delta\epsilon_\cdot, \delta m_\cdot)$ is an infinitesimal variation in $\mathcal{M}$ over $(\epsilon_\cdot, m_\cdot)$ and that $(\delta\epsilon'_\cdot, \delta m'_\cdot)$ is an infinitesimal variation in $\mathcal{M}'$ over $(\epsilon'_\cdot, m'_\cdot)$ satisfying

$$d\Upsilon(\epsilon_k, m_{k+1})(\delta\epsilon_k, \delta m_{k+1}) = (\delta\epsilon'_k, \delta m'_{k+1}) \quad \text{for} \quad k = 0, \ldots, N-1. \tag{4.4}$$

Then, using the chain rule, we see that

$$dS_d(\epsilon_\cdot, m_\cdot)(\delta\epsilon_\cdot, \delta m_\cdot) = dS'_d(\epsilon'_\cdot, m'_\cdot)(\delta\epsilon'_\cdot, \delta m'_\cdot). \tag{4.5}$$

Next we prove the equivalence of the assertions in the statement.

Assume that $(\epsilon_\cdot, m_\cdot)$ is a trajectory of $\mathcal{M}$. Let $(\delta\epsilon'_\cdot, \delta m'_\cdot)$ be an infinitesimal variation with fixed endpoints in $\mathcal{M}'$ over the path $(\epsilon'_\cdot, m'_\cdot)$. That is, there are $\widetilde{\delta\epsilon'_k} \in T_{\epsilon'_k}E'$ for $k = 1, \ldots, N-1$ such that (3.1) and (3.2) hold with $\delta\epsilon'_k$ and $\widetilde{\delta\epsilon'_k}$ instead of $\delta\epsilon_k$ and $\widetilde{\delta\epsilon_k}$.

By morphism's property 2 applied to Υ, there exist $\widetilde{\delta\epsilon_k} \in T_{\epsilon_k}E$ such that $D_1(p_1 \circ \Upsilon)(\epsilon_k, m_{k+1})(\widetilde{\delta\epsilon_k}) = \widetilde{\delta\epsilon'_k}$ for $k = 1, \ldots, N-1$; we fix one such vector for each k. Next apply (3.1) and (3.2) to define an infinitesimal variation $(\delta\epsilon_\cdot, \delta m_\cdot)$ on $(\epsilon_\cdot, m_\cdot)$ with fixed endpoints based on the $\widetilde{\delta\epsilon_\cdot}$ constructed above.

Direct computations using the morphism properties of Υ show that condition (4.4) holds for these variations. Then, using (4.5),

$$dS'_d(\epsilon'_\cdot, m'_\cdot)(\delta\epsilon'_\cdot, \delta m'_\cdot) = dS_d(\epsilon_\cdot, m_\cdot)(\delta\epsilon_\cdot, \delta m_\cdot) = 0,$$

where the last equality holds because $(\delta\epsilon_\cdot, \delta m_\cdot)$ is an infinitesimal variation with fixed endpoints in $\mathcal{M}$ over $(\epsilon_\cdot, m_\cdot)$, that is a trajectory of $\mathcal{M}$. Finally, as $(\delta\epsilon'_\cdot, \delta m'_\cdot)$ was an arbitrary infinitesimal variation with fixed endpoints in $\mathcal{M}'$ over the path $(\epsilon'_\cdot, m'_\cdot)$, we conclude that $(\epsilon'_\cdot, m'_\cdot)$ is a trajectory of $\mathcal{M}'$.

A similar argument shows that if $(\epsilon'_\cdot, m'_\cdot)$ is a trajectory of $\mathcal{M}'$, then $(\epsilon_\cdot, m_\cdot)$ is a trajectory of $\mathcal{M}$. $\square$

5. Reduction of discrete Lagrange–Poincaré systems

The purpose of this section is to define what is meant by a group of symmetries of a DLPS. Also, a reduction result is studied.

5.1. **Symmetry groups of discrete Lagrange–Poincaré systems.** Recall that a G-action on a fiber bundle consists of a pair of G-actions l^E and l^M, satisfying a number of conditions (Definition 9.14). We can use these actions to define "diagonal" G-actions on the fiber bundles $C'(E)$ and $C''(E)$ by

$$\begin{aligned} l_g^{C'(E)}(\epsilon_0, m_1) :=& (l_g^E(\epsilon_0), l_g^M(m_1)) \\ l_g^{C''(E)}((\epsilon_0, m_1), (\epsilon_1, m_2)) :=& (l_g^{C'(E)}(\epsilon_0, m_1), l_g^{C'(E)}(\epsilon_1, m_2)). \end{aligned} \tag{5.1}$$

These actions are smooth and free because l^E and l^M have those properties. In addition, the bundle projection maps of $C'(E)$ and $C''(E)$ on M are G-equivariant and $\pi^{M,G} : M \to M/G$ is a principal G-bundle. In fact, it is easy to check that G acts on the fiber bundles $\phi \circ p_1 : C'(E) \to M$ and $\phi \circ p_3 : C''(E) \to M$.

We can also define G-actions on $\ker(d\phi)$ and $p_3^*(TE)$ by

$$\begin{aligned} l_g^{TE}(\epsilon_0, \delta\epsilon_0) :=& dl_g^E(\epsilon_0)(\delta\epsilon_0), \\ l_g^{p_3^*(TE)}((\epsilon_0, m_1), (\epsilon_1, m_2), \delta\epsilon_1) :=& (l_g^{C''(E)}((\epsilon_0, m_1), (\epsilon_1, m_2)), dl_g^E(\epsilon_1)(\delta\epsilon_1)). \end{aligned} \tag{5.2}$$

We denote the G-action on $\ker(d\phi)$ by l^{TE} because it is the restriction of the natural G-action on TE. The action l^{TE} is well defined by the G-equivariance of ϕ.

Lemma 5.1. *Let G be a Lie group acting on the fiber bundle $\phi : E \to M$ and $\mathcal{A}_d$ be a discrete connection on the principal G-bundle $\pi^{M,G} : M \to M/G$. Define $\Upsilon^{(2)}_{\mathcal{A}_d} : C''(E) \to C''(\widetilde{G}_E)$ as the restriction of $(\Upsilon_{\mathcal{A}_d} \circ p_1) \times (\Upsilon_{\mathcal{A}_d} \circ p_2) : C'(E) \times C'(E) \to C'(\widetilde{G}_E) \times C'(\widetilde{G}_E)$ to the corresponding spaces, where $\Upsilon_{\mathcal{A}_d}$ is the surjective submersion defined in* (2.2). *Then*

1. *$\Upsilon^{(2)}_{\mathcal{A}_d}$ is well defined.*
2. *$D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0, m_1) : T_{(\epsilon_0, m_1)}(E \times \{m_1\}) \to T_{(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0, m_1)}\widetilde{G}_E$ is an isomorphism of vector spaces for every $(\epsilon_0, m_1) \in C'(E)$.*
3. *$\Upsilon^{(2)}_{\mathcal{A}_d} : C''(E) \to C''(\widetilde{G}_E)$ is a principal G-bundle with structure group G. In particular, $C''(E)/G \simeq C''(\widetilde{G}_E)$.*
4. *For $((v_0, r_1), (v_1, r_2)) \in C''(\widetilde{G}_E)$ and $(\epsilon_0, m_1) \in C'(E)$ such that $\Upsilon_{\mathcal{A}_d}(\epsilon_0, m_1) = (v_0, r_1)$, there is a unique $(\epsilon_1, m_2) \in C'(E)$ such that $((\epsilon_0, m_1), (\epsilon_1, m_2)) \in C''(E)$ and $\Upsilon^{(2)}_{\mathcal{A}_d}((\epsilon_0, m_1), (\epsilon_1, m_2)) = ((v_0, r_1), (v_1, r_2))$.*

Proof. A simple computation shows that, for $((\epsilon_0, m_1), (\epsilon_1, m_2)) \in C''(E)$, we have $p_2(\Upsilon_{\mathcal{A}_d}(\epsilon_0, m_1)) = p^{M/G}(p_1(\Upsilon_{\mathcal{A}_d}(\epsilon_1, m_2)))$, proving point 1.

Let $(v_0, r_1) := \Upsilon_{\mathcal{A}_d}(\epsilon_0, m_1) = (\pi^{E\times G,G}(\epsilon_0, \mathcal{A}_d(\phi(\epsilon_0), m_1)), \pi^{M,G}(m_1))$. It is easy to check that if $\delta\epsilon_0 \in \ker(D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0, m_1))$, then $(\delta\epsilon_0, 0) \in \ker(d\Upsilon_{\mathcal{A}_d}(\epsilon_0, m_1)) = \{(\xi_E(\epsilon_0), \xi_M(m_1)) \in T_{(\epsilon_0, m_1)}(E \times M) : \xi \in \mathfrak{g}\}$. But, being $\pi^{M,G} : M \to M/G$ a principal G-bundle, $\xi_M(m_1) = 0$ implies that $\xi = 0$, and we conclude that $\delta\epsilon_0 = 0$, so that $D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0, m_1) : T_{(\epsilon_0, m_1)}(E \times \{m_1\}) \to T_{v_0}\widetilde{G}_E$ is one to one. As, in addition, $\dim(T_{(\epsilon_0, m_1)}(E \times \{m_1\})) = \dim(T_{v_0}\widetilde{G}_E)$, we conclude that point 2 is true.

Consider the commutative diagram

$$\begin{tikzcd} C''(E) \arrow[r, "F_E"] \arrow[d, "\Upsilon^{(2)}_{\mathcal{A}_d}"'] & E \times E \times M \arrow[d, "\widetilde{\Upsilon}^{(2)}_{\mathcal{A}_d}"] \\ C''(\widetilde{G}_E) \arrow[r, "F_{\widetilde{G}_E}"'] & \widetilde{G}_E \times \widetilde{G}_E \times (M/G) \end{tikzcd}$$

where F_E and $F_{\widetilde{G}_E}$ are the diffeomorphisms introduced in Remark 3.1 and

$$\widetilde{\Upsilon}^{(2)}_{\mathcal{A}_d}(\epsilon_0,\epsilon_1,m_2) := ((p_1\circ\Upsilon_{\mathcal{A}_d})(\epsilon_0,\phi(\epsilon_1)),\Upsilon_{\mathcal{A}_d}(\epsilon_1,m_2)).$$

It is clear that $\widetilde{\Upsilon}^{(2)}_{\mathcal{A}_d}$ is smooth. Furthermore, as the projection to its last two components is simply $\Upsilon_{\mathcal{A}_d}: E\times M\to\widetilde{G}_E\times(M/G)$, that is known to be a surjective submersion and applying point 2 to the first component, we conclude that $\widetilde{\Upsilon}^{(2)}_{\mathcal{A}_d}$ is a submersion. We check explicitly that $\widetilde{\Upsilon}^{(2)}_{\mathcal{A}_d}$ is surjective. Let $(v_0,v_1,r_2)\in\widetilde{G}_E\times\widetilde{G}_E\times(M/G)$. Then, by definition of $\widetilde{G}_E$, there are $(\epsilon_0,m_1)\in E\times M$ such that $\Upsilon_{\mathcal{A}_d}(\epsilon_0,m_1)=(v_0,p^{M/G}(v_1))$. Next, choose $(\epsilon_1',m_2')\in E\times M$ such that $\Upsilon_{\mathcal{A}_d}(\epsilon_1',m_2')=(v_1,r_2)$. Notice that, using diagram (2.3), $\pi^{M,G}(\phi(\epsilon_1'))=p^{M/G}(v_1)=\pi^{M,G}(m_1)$. Hence, as $\pi^{M,G}:M\to M/G$ is a principal G-bundle, there is $g'\in G$ such that $l^M_{g'}(\phi(\epsilon_1'))=m_1$. We define $(\epsilon_1,m_2):=l^{E\times M}_{g'}(\epsilon_1',m_2')$. By construction, $\Upsilon_{\mathcal{A}_d}(\epsilon_1,m_2)=(v_1,r_2)$ and $\phi(\epsilon_1)=m_1$. All together, $\widetilde{\Upsilon}^{(2)}_{\mathcal{A}_d}(\epsilon_0,\epsilon_1,m_2)=(v_0,v_1,r_2)$, showing that $\widetilde{\Upsilon}^{(2)}_{\mathcal{A}_d}$ is onto. Using that $\Upsilon_{\mathcal{A}_d}$ is a principal G-bundle, it follows easily that $(\widetilde{\Upsilon}^{(2)}_{\mathcal{A}_d})^{-1}(v_0,v_1,r_2)=l^{E\times E\times M}_G\{(\epsilon_0,\epsilon_1,m_2)\}$, showing that $(\widetilde{\Upsilon}^{(2)}_{\mathcal{A}_d})^{-1}(v_0,v_1,r_2)$ coincides with the orbit of the free "diagonal" action of G on $E\times E\times M$. Theorem 9.8 proves that $\widetilde{\Upsilon}^{(2)}_{\mathcal{A}_d}: E\times E\times M\to\widetilde{G}_E\times\widetilde{G}_E\times(M/G)$ is a principal G-bundle. Finally, since the diffeomorphism F_E is G-equivariant (when considering the G-actions $l^{C''(E)}$ and $l^{E\times E\times M}$), we conclude that point 3 holds.

Notice that in the first step of the previous construction, we picked $(\epsilon_0,m_1)\in C'(E)$ such that $\Upsilon_{\mathcal{A}_d}(\epsilon_0,m_1)=(v_0,p^{M/G}(v_1))$. In the context of point 4, such (ϵ_0,m_1) is given. Hence, the rest of the construction produces (ϵ_1,m_2) so that $((\epsilon_0,m_1),(\epsilon_1,m_2))\in C''(E)$ and $\Upsilon^{(2)}_{\mathcal{A}_d}((\epsilon_0,m_1),(\epsilon_1,m_2))=((v_0,r_1),(v_1,r_2))$. The uniqueness of that pair follows from the fact this is the only element in the G-orbit that has (ϵ_0,m_1) as the first component. Hence, point 4 is valid. □

Proposition 5.2. *Let G be a Lie group acting on the fiber bundle $\phi:E\to M$ and $\mathcal{A}_d$ a discrete connection on the principal G-bundle $\pi^{M,G}:M\to M/G$. Given a discrete path $(v_\cdot,r_\cdot)=((v_0,r_1),\ldots,(v_{N-1},r_N))$ in $C'(\widetilde{G}_E)$ and $(\widetilde{\epsilon}_0,\widetilde{m}_1)\in C'(E)$ such that $\Upsilon_{\mathcal{A}_d}(\widetilde{\epsilon}_0,\widetilde{m}_1)=(v_0,r_1)$, there is a unique discrete path $(\epsilon_\cdot,m_\cdot)$ in $C'(E)$ such that $(\epsilon_0,m_1)=(\widetilde{\epsilon}_0,\widetilde{m}_1)$ and $\Upsilon_{\mathcal{A}_d}(\epsilon_k,m_{k+1})=(v_k,r_{k+1})$ for all k.*

Proof. The proof is by induction in the length of the reduced discrete path, N. If $N=0$, taking $(\epsilon_0,m_1):=(\widetilde{\epsilon}_0,\widetilde{m}_1)$ solves the problem. Otherwise, assume that the result holds for all lengths $<N$ and $(v_\cdot,r_\cdot)=((v_0,r_1),\ldots,(v_{N-1},r_N))$. Then, there is a discrete path $((\epsilon_0,m_1),\ldots,(\epsilon_{N-2},m_{N-1}))$ in $C'(E)$ that lifts $((v_0,r_1),\ldots,(v_{N-2},r_{N-1}))$ starting at $(\widetilde{\epsilon}_0,\widetilde{m}_1)$. In particular, $\Upsilon_{\mathcal{A}_d}((\epsilon_{N-2},m_{N-1}))=(v_{N-2},r_{N-1})$. As, in addition, $((v_{N-2},r_{N-1}),(v_{N-1},r_N))\in C''(\widetilde{G}_E)$, by point 4 in Lemma 5.1, there is $(\epsilon_{N-1},m_N)\in C'(E)$ such that $((\epsilon_{N-2},m_{N-1}),(\epsilon_{N-1},m_N))\in C''(E)$ and $\Upsilon^{(2)}_{\mathcal{A}_d}((\epsilon_{N-2},m_{N-1}),(\epsilon_{N-1},m_N))=((v_{N-2},r_{N-1}),(v_{N-1},r_N))$. This proves that $((\epsilon_0,m_1),\ldots,(\epsilon_{N-2},m_{N-1}),(\epsilon_{N-1},m_N))$ is a discrete path in $C'(E)$ starting at $(\widetilde{\epsilon}_0,\widetilde{m}_1)$ and that lifts $((v_0,r_1),\ldots,(v_{N-1},r_N))$. This proves that the statement holds for discrete paths of length N so that, by the induction principle, it holds for arbitrary lengths. □

Definition 5.3. A Lie group G is a *symmetry group* of the DLPS $\mathcal{M}=(E,L_d,\mathcal{P})$ if

(1) G acts on the fiber bundle $\phi : E \to M$ (Definition 9.14),
(2) considering the "diagonal action" of G on $C'(E)$, $l^{C'(E)}$ defined in (5.1), L_d is G-invariant, and
(3) $\mathcal{P}$ is a G-equivariant element of $\hom(p_3^*(TE), \ker(d\phi))$ for the G-actions l^{TE} and $l^{p_3^*(TE)}$ defined in (5.2). In other words,

$$\mathcal{P} \circ l_g^{p_3^*(TE)} = l_g^{TE} \circ \mathcal{P} = dl_g^E \circ \mathcal{P} \quad \text{for all} \quad g \in G. \tag{5.3}$$

Example 5.4. Let (Q, L_d) be a DMS and $\mathcal{M} := (id_Q : Q \to Q, L_d, 0)$ the DLPS associated to (Q, L_d) in Example 3.12. If G is a symmetry group of (Q, L_d) as in Section 3.2, then G acts on the fiber bundle $id_Q : Q \to Q$ and L_d is G-invariant. Also, as $\mathcal{P} = 0$, condition (5.3) is trivially satisfied. Hence, G is a symmetry group of $\mathcal{M}$.

Lemma 5.5. *Let $\mathcal{M} = (E, L_d, \mathcal{P}) \in \text{ob}_{\mathfrak{LP}_d}$ and G be a Lie group. Then, for $g \in G$,* (5.3) *holds if and only if* (4.2) *holds for $\Upsilon := l_g^{C'(E)}$ and $\mathcal{M}' = \mathcal{M}$.*

Proof. Unravel the definitions. □

Proposition 5.6. *Let $\mathcal{M} = (E, L_d, \mathcal{P}) \in \text{ob}_{\mathfrak{LP}_d}$ and G a Lie group. Then G is a symmetry group of $\mathcal{M}$ if and only if G acts on the fiber bundle $\phi : E \to M$ and $l_g^{C'(E)} \in \text{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M})$ for all $g \in G$.*

Proof. Assume that G is a symmetry group of $\mathcal{M}$. Then, by definition, G acts on the fiber bundle $\phi : E \to M$. We have to prove that $l_g^{C'(E)} \in \text{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M})$. It is immediate that $l_g^{C'(E)}$ is a diffeomorphism, so it has morphism's property 1. As $p_1 \circ l_g^{C'(E)} = l_g^E \circ p_1$, we have $D_1(p_1 \circ l_g^{C'(E)}) = dl_g^E$, that is an isomorphism; hence, $l_g^{C'(E)}$ has morphism's property 2. As $p_2 \circ l_g^{C'(E)} = l_g^M \circ p_2$, $D_1(p_2 \circ l_g^{C'(E)}) = D_1(l_g^M \circ p_2) = 0$, it follows that $l_g^{C'(E)}$ has morphism's property 3. Also, as on $C''(E)$ we have that

$$\begin{aligned} p_2^{C'(E),M} \circ l_g^{C'(E)} \circ p_1^{C''(E),C'(E)} =& l_g^M \circ p_2^{C'(E),M} \circ p_1^{C''(E),C'(E)} \\ =& \phi \circ p_1^{C'(E),E} \circ l_g^{C'(E)} \circ p_2^{C''(E),C'(E)}, \end{aligned}$$

we see that $l_g^{C'(E)}$ has morphism's property 4. As $L_d \circ l_g^{C'(E)} = L_d$, $l_g^{C'(E)}$ has morphism's property 5 and Lemma 5.5 shows that morphism's property 6 is valid for $l_g^{C'(E)}$. We conclude that $l_g^{C'(E)} \in \text{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M})$.

Conversely, if G acts on the fiber bundle $\phi : E \to M$ and $l_g^{C'(E)} \in \text{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M})$, the first condition for being a symmetry group is met. The other two follow from morphism's properties 5 and 6, together with Lemma 5.5. □

Later on we will be interested in subgroups of a symmetry group of a DLPS. The following results establish that closed subgroups of a symmetry group of a system $\mathcal{M}$ are symmetry groups of $\mathcal{M}$.

Lemma 5.7. *Let G act on the fiber bundle (E, M, ϕ, F) and $H \subset G$ be a closed Lie subgroup. Then H acts on the fiber bundle (E, M, ϕ, F).*

Proof. We consider the H-actions on E, M and F obtained by restricting the G-actions l^E, l^M and r^F to H. Hence, all are smooth and the first two are free; also, ϕ is H-equivariant. As $\pi^{M,G} : M \to M/G$ is a principal G-bundle, by Lemma 9.11, the G-action l^M is proper and, being $H \subset G$ closed, the H-action l^M obtained by restriction is proper. Then applying Corollary 9.10 to the H-action l^M, we see that $\pi^{M,H} : M \to M/H$ is a principal H-bundle. Given $m \in M$, there is a trivializing chart (U, Φ_U) with $m \in U$, an open G-invariant subset of M, and Φ_U G-equivariant. Thus, U is H-invariant and Φ_U is H-equivariant, so that (U, Φ_U) is the type of trivializing chart required in point 3 of Definition 9.14 to conclude that H acts on the fiber bundle (E, M, ϕ, F). □

Proposition 5.8. *Let G be a symmetry group of $\mathcal{M} \in \text{ob}_{\mathfrak{LP}_d}$. If $H \subset G$ is a closed Lie subgroup, then H is a symmetry group of $\mathcal{M}$.*

Proof. Since G is a symmetry group of $\mathcal{M} = (E, L_d, \mathcal{P})$, G acts on the fiber bundle (E, M, ϕ, F) and, by Lemma 5.7, the same happens to the closed subgroup H, when acting via the restricted G-actions. That L_d is H-invariant and $\mathcal{P}$ is H-equivariant, follow from the fact that they have those properties for G, and that H acts by the restriction of the corresponding G-actions. Thus, H is a symmetry group of $\mathcal{M}$. □

Remark 5.9. When G is a symmetry group of $\mathcal{M} = (E, L_d, \mathcal{P}) \in \text{ob}_{\mathfrak{LP}_d}$ there are functions $J_d : C'(E) \to \mathfrak{g}^*$ and $(J_d)_\xi : C'(E) \to \mathbb{R}$ defined as follows. $J_d(\epsilon_0, m_1)(\xi) := -D_1 L_d(\epsilon_0, m_1)(\xi_E(\epsilon_0))$ for $(\epsilon_0, m_1) \in C'(E)$ and $\xi \in \mathfrak{g}$, where ξ_E is the infinitesimal generator associated to ξ by the G-action on E. Then, $(J_d)_\xi(\epsilon_0, m_1) := J_d(\epsilon_0, m_1)(\xi)$. In some sense, these functions resemble the momentum mappings that appear in the context of DMS. It is easy to check that when $(\epsilon_\cdot, m_\cdot)$ is a trajectory of $\mathcal{M}$, for any $\xi \in \mathfrak{g}$,

$$
\begin{aligned}
(J_d)_\xi(\epsilon_k, m_{k+1}) =& (J_d)_\xi(\epsilon_{k-1}, m_k) \\
&+ D_1 L_d(\epsilon_{k-1}, m_k) \circ \mathcal{P}((\epsilon_{k-1}, m_k), (\epsilon_k, m_{k+1}))(\xi_E(\epsilon_k))
\end{aligned}
\tag{5.4}
$$

for all $k = 0, \ldots, N-1$. This last expression shows how J_d evolves on a given trajectory of $\mathcal{M}$. In particular, when the image of $\mathcal{P}$ is contained in $\ker(D_1 L_d)$, the momentum is conserved along the trajectories; this is the case of a DLPS arising from a discrete mechanical system (see Example 3.12). Equation (5.4) can also be compared with the momentum evolution equation in the nonholonomic case: equation (35) in [7].

5.2. **Reduced discrete Lagrange–Poincaré system.** Let G be a symmetry group of $\mathcal{M} = (E, L_d, \mathcal{P}) \in \text{ob}_{\mathfrak{LP}_d}$. We want to construct a new DLPS that, as will be shown later, will play the role of the reduced system of $\mathcal{M}$. First of all, since G acts on (E, M, ϕ, F), the conjugate bundle $(\widetilde{G}_E, M/G, p^{M/G}, F \times G)$, introduced in Example 9.17, is a fiber bundle.

Fix a discrete connection $\mathcal{A}_d$ on the principal G-bundle $\pi^{M,G} : M \to M/G$ and let $\Upsilon_{\mathcal{A}_d} : E \times M \to \widetilde{G}_E \times (M/G)$ be the map introduced in (2.2) that, by Lemma 2.8, is a principal G-bundle. Define $\check{L}_d : \widetilde{G}_E \times (M/G) \to \mathbb{R}$ by $\check{L}_d(v_0, r_1) := L_d(\epsilon_0, m_1)$ for any $(\epsilon_0, m_1) \in \Upsilon_{\mathcal{A}_d}^{-1}(v_0, r_1)$; $\check{L}_d$ is well defined by the G-invariance of L_d.

Next we define $\check{\mathcal{P}} \in \hom(p_3^*(T(\widetilde{G}_E)), \ker(dp^{M/G}))$. By point 3 of Lemma 5.1, given any $((v_0, r_1), (v_1, r_2)) \in C''(\widetilde{G}_E)$, there are elements $((\epsilon_0, m_1), (\epsilon_1, m_2)) \in C''(E)$ such that $\Upsilon_{\mathcal{A}_d}^{(2)}((\epsilon_0, m_1), (\epsilon_1, m_2)) = ((v_0, r_1), (v_1, r_2))$. In fact, those elements form a G-orbit in $C''(E)$; we fix one element in the orbit. Also, by point 2 of Lemma 5.1, $D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_1, m_2) : T_{(\epsilon_1, m_2)}(E \times \{m_2\}) \to T_{v_1} \widetilde{G}_E$ is an isomorphism of vector spaces. Consequently, every element $((v_0, r_1), (v_1, r_2), \delta v_1) \in p_3^*(T(\widetilde{G}_E))$

is $\delta v_1 = D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_1, m_2)(\delta\epsilon_1)$ for a unique $\delta\epsilon_1 \in T_{(\epsilon_1,m_2)}(E \times \{m_2\})$. Let

$$\begin{aligned}\check{\mathcal{P}}((v_0,r_1),(v_1,r_2))(\delta v_1) :=&D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0,m_1)(\mathcal{P}((\epsilon_0,m_1),(\epsilon_1,m_2))(\delta\epsilon_1))\\ &+ D_2(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0,m_1)(d\phi(\epsilon_1)(\delta\epsilon_1)).\end{aligned} \tag{5.5}$$

Lemma 5.10. *Under the previous conditions, the map defined by* (5.5) *is a well defined homomorphism* $\check{\mathcal{P}} \in \hom(p_3^*(T(\widetilde{G}_E)), \ker(dp^{M/G}))$.

Proof. Two things have to be checked: that the image of $\check{\mathcal{P}}$ is contained in $\ker(dp^{M/G})$ and that the definition is independent of any choices involved in lifting the input data to $p_3^*(TE)$. Since the points $((\epsilon_0,m_1),(\epsilon_1,m_2))$ lying over $((v_0,r_1),(v_1,r_2))$ form a G-orbit, any other such point would be of the form $((\epsilon_0',m_1'),(\epsilon_1',m_2')) = l_g^{C''(E)}((\epsilon_0,m_1),(\epsilon_1,m_2))$ for some $g \in G$. It follows from the G-invariance of $\Upsilon_{\mathcal{A}_d}$ that

$$D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_1,m_2)(\delta\epsilon_1) = D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(l_g^{C'(E)}(\epsilon_1,m_2))(dl_g^E(\epsilon_1)(\delta\epsilon_1)).$$

Hence, a variation $((v_0,r_1),(v_1,r_2),\delta v_1) \in p_3^*(T(\widetilde{G}_E))$ lifts to the (unique for each g) variation $(l_g^{C''(E)}((\epsilon_0,m_1),(\epsilon_1,m_2)), dl_g^E(\epsilon_1)(\delta\epsilon_1))$ for arbitrary $g \in G$. Then, for a given $g \in G$, using the G-equivariance of $\mathcal{P}$ and the G-invariance of $\Upsilon_{\mathcal{A}_d}$, we see that replacing $((\epsilon_0,m_1),(\epsilon_1,m_2))$ and $\delta\epsilon_1$ by $l_g^{C''(E)}((\epsilon_0,m_1),(\epsilon_1,m_2))$ and $(l_g^{C''(E)}((\epsilon_0,m_1),(\epsilon_1,m_2)), dl_g^E(\epsilon_1)(\delta\epsilon_1))$ does not alter the value of the left side of (5.5). This proves $\check{\mathcal{P}}$ is independent of the choices made.

Direct computations show that the image of $\check{\mathcal{P}}$ is contained in $\ker(dp^{M/G})$. $\square$

Definition 5.11. Let G be a symmetry group of $\mathcal{M} = (E, L_d, \mathcal{P}) \in \mathrm{ob}_{\mathfrak{LP}_d}$ and $\mathcal{A}_d$ a discrete connection on the principal G-bundle $\pi^{M,G} : M \to M/G$. The DLPS $(\widetilde{G}_E, \check{L}_d, \check{\mathcal{P}}) \in \mathrm{ob}_{\mathfrak{LP}_d}$ defined above is called the *reduced discrete Lagrange–Poincaré system* obtained as the reduction of $\mathcal{M}$ by the symmetry group G using the discrete connection $\mathcal{A}_d$. We denote this system by $\mathcal{M}/G$ or $\mathcal{M}/(G, \mathcal{A}_d)$.

Example 5.12. Given a DMS (Q, L_d), let $\mathcal{M} := (Q, L_d, 0)$ be the DLPS constructed in Example 3.12. Let G be a symmetry group of (Q, L_d). By Example 5.4, G is a symmetry group of $\mathcal{M}$. Fix a discrete connection $\mathcal{A}_d$ on the principal G-bundle $\pi^{Q,G} : Q \to Q/G$. The reduced DLPS $\mathcal{M}/(G, \mathcal{A}_d)$ is $(\widetilde{G}_E, \check{L}_d, \check{\mathcal{P}})$ where the fiber bundle $\phi : \widetilde{G}_E \to M/G$ is $p^{Q/G} : \widetilde{G} \to Q/G$, the lagrangian is determined by $\check{L}_d \circ \Upsilon_{\mathcal{A}_d} = L_d$ and, according to (5.5),

$$\check{\mathcal{P}}((v_{k-1},r_k),(v_k,r_{k+1}))(\delta v_k) = D_2(p_1 \circ \Upsilon_{\mathcal{A}_d})(q_{k-1},q_k)(\delta q_k),$$

where $(v_{k-1},r_k) = \Upsilon_{\mathcal{A}_d}(q_{k-1},q_k)$, $(v_k,r_{k+1}) = \Upsilon_{\mathcal{A}_d}(q_k,q_{k+1})$ and $\delta v_k = D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(q_k,q_{k+1})(\delta q_k)$. Notice that this DLPS coincides with the DLPS associated to the reduction of (Q, L_d) in Section 3.2. In other words, the reduced system $\mathcal{M}/(G, \mathcal{A}_d)$ extends the reduction construction of DMSs introduced in [7].

Proposition 5.13. *Let G be a symmetry group of $\mathcal{M} = (E, L_d, \mathcal{P}) \in \mathrm{ob}_{\mathfrak{LP}_d}$ and $\mathcal{A}_d$ a discrete connection on the principal G-bundle $\pi^{M,G} : M \to M/G$. Then $\Upsilon_{\mathcal{A}_d}$ defined by* (2.2) *is in* $\mathrm{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}/(G, \mathcal{A}_d))$.

Proof. We have already noticed that $\Upsilon_{\mathcal{A}_d} : C'(E) \to \widetilde{G}_E$ is a surjective submersion, so that morphism's property 1 holds. By point 2 of Lemma 5.1, morphism's property 2 holds. As $p_2 \circ \Upsilon_{\mathcal{A}_d} = \pi^{M,G} \circ p_2$, if $i_1 : TE \to T(C'(E)) = TE \oplus TM$

is the natural inclusion, we have that $D_1(p_2 \circ \Upsilon_{\mathcal{A}_d}) = d\pi^{M,G} \circ dp_2 \circ i_1 = 0$, as $\operatorname{Im}(i_1) \subset \ker(dp_2)$, so that morphism's property 3 is valid. As

$$(p_2^{C'(\widetilde{G}_E),M/G} \circ \Upsilon_{\mathcal{A}_d} \circ p_1^{C''(E),C'(E)})((\epsilon_0, m_1), (\epsilon_1, m_2)) = \pi^{M,G}(m_1)$$

and

$$\begin{aligned}(p^{M,G} \circ p_1^{C'(\widetilde{G}_E),\widetilde{G}_E} \circ \Upsilon_{\mathcal{A}_d} \circ p_2^{C''(E),C'(E)})&((\epsilon_0, m_1), (\epsilon_1, m_2)) \\ &= p^{M,G}(\pi^{E\times G,G}(\epsilon_1, \mathcal{A}_d(\phi(\epsilon_1), m_2))) = \pi^{M,G}(\phi(\epsilon_1)) = \pi^{M,G}(m_1),\end{aligned}$$

morphism's property 4 is satisfied. Morphism's property 5 is satisfied by G being a symmetry group of $\mathcal{M}$ and, by definition of $\check{\mathcal{P}}$ (5.5), we see that (4.2) holds, proving that morphism's property 6 holds for $\Upsilon_{\mathcal{A}_d}$. □

When a DLPS is symmetric, constructing the associated reduced system requires the choice of a discrete connection. The following result proves that all reduced DLPSs obtained from a DLPS by this procedure are isomorphic in $\mathfrak{LP}_d$, independently of the discrete connection chosen.

Proposition 5.14. *Let G be a symmetry group of the $\mathcal{M} = (E, L_d, \mathcal{P}) \in \operatorname{ob}_{\mathfrak{LP}_d}$ and $\mathcal{A}_d^1, \mathcal{A}_d^2$ two discrete connections on the underlying principal G-bundle $\pi^{M,G} : M \to M/G$. Then, the reduced systems $\mathcal{M}/(G, \mathcal{A}_d^1)$ and $\mathcal{M}/(G, \mathcal{A}_d^2)$ are isomorphic in $\mathfrak{LP}_d$.*

Proof. By Lemma 2.8, $\Upsilon_{\mathcal{A}_d^1}, \Upsilon_{\mathcal{A}_d^2} : C'(E) \to C'(\widetilde{G}_E)$ are principal G-bundles. Then, we have the following commutative diagrams of smooth maps, where the horizontal arrows are diffeomorphisms (see Proposition 2.6)

$$\begin{tikzcd} & C'(E) \arrow[dl, "\Upsilon_{\mathcal{A}_d^1}"'] \arrow[d, "\pi^{C'(E),G}"] \\ C'(\widetilde{G}_E) \arrow[r, "\Phi^{-1}_{\mathcal{A}_d^1}"'] & C'(E)/G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C'(E) \arrow[d, "\pi^{C'(E),G}"'] \arrow[dr, "\Upsilon_{\mathcal{A}_d^2}"] & \\ C'(E)/G \arrow[r, "\Phi_{\mathcal{A}_d^2}"'] & C'(\widetilde{G}_E) \end{tikzcd}$$

Joining the two diagrams we obtain the commutative diagram of smooth maps

$$\begin{tikzcd} & C'(E) \arrow[dl, "\Upsilon_{\mathcal{A}_d^1}"'] \arrow[dr, "\Upsilon_{\mathcal{A}_d^2}"] & \\ C'(\widetilde{G}_E) \arrow[rr, "\Phi_{\mathcal{A}_d^2} \circ \Phi^{-1}_{\mathcal{A}_d^1}"'] & & C'(\widetilde{G}_E) \end{tikzcd}$$

The result then follows from Lemma 4.5 because the horizontal arrow is a diffeomorphism and, by Proposition 5.13, the non-horizontal arrows are morphisms in $\mathfrak{LP}_d$. □

5.3. **Dynamics of the reduced discrete Lagrange–Poincaré system.** The following result compares the dynamics of a reduced DLPS to that of the original symmetric system.

Theorem 5.15. *Let G be a symmetry group of the DLPS $\mathcal{M} = (E, L_d, \mathcal{P})$, $\mathcal{A}_d$ a discrete connection on the principal G-bundle $\pi^{M,G} : M \to M/G$ and $\mathcal{M}/(G, \mathcal{A}_d) = (\widetilde{G}_E, \check{L}_d, \check{\mathcal{P}})$ the corresponding reduced DLPS. If $(\epsilon_\cdot, m_\cdot) = ((\epsilon_0, m_1), \ldots, (\epsilon_{N-1}, m_N))$ is a discrete path in $C'(E)$, we define a discrete path $(v_\cdot, r_\cdot)$ in $C'(\widetilde{G}_E)$ by $(v_k, r_{k+1}) :=$*

$\Upsilon_{\mathcal{A}_d}(\epsilon_k, m_{k+1})$ *for* $k = 0, \dots, N-1$. *Then,* $(\epsilon., m.)$ *is a trajectory of* $\mathcal{M}$ *if and only if* $(v., r.)$ *is a trajectory of* $\mathcal{M}/(G, \mathcal{A}_d)$.

Proof. As, by Proposition 5.13, $\Upsilon_{\mathcal{A}_d} \in \operatorname{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}/(G, \mathcal{A}_d))$, the result follows from Theorem 4.7. □

Corollary 5.16. *In the same setting of Theorem 5.15, the following assertions are equivalent.*

(1) $(\epsilon., m.)$ *is a trajectory of* $\mathcal{M}$.
(2) Equation (3.4) *is satisfied.*
(3) $(v., r.)$ *is a trajectory of* $\mathcal{M}/(G, \mathcal{A}_d)$.
(4) For all $k = 0, \dots, N-1$,

$$\begin{aligned} D_1\check{L}_d(v_k, r_{k+1}) + D_1\check{L}_d(v_{k-1}, r_k)\check{\mathcal{P}}((v_{k-1}, r_k), (v_k, r_{k+1})) \\ + D_2\check{L}_d(v_{k-1}, r_k)dp^{M/G}(v_k) = 0 \end{aligned} \tag{5.6}$$

Proof. Use Theorem 5.15 and Proposition 3.8, applied to $\mathcal{M}$ and $\mathcal{M}/(G, \mathcal{A}_d)$. □

The following reconstruction result shows how, knowing the discrete trajectories of a reduced system, the trajectories of the original system can be recovered.

Theorem 5.17. *Let* G *be a symmetry group of the DLPS* $\mathcal{M} = (E, L_d, \mathcal{P})$, $\mathcal{A}_d$ *a discrete connection on the principal* G*-bundle* $\pi^{M,G} : M \to M/G$ *and* $\mathcal{M}/(G, \mathcal{A}_d)$ *the corresponding reduced DLPS. Let* $(v., r.)$ *be a trajectory of* $\mathcal{M}/(G, \mathcal{A}_d)$ *and* $(\widetilde{\epsilon}_0, \widetilde{m}_1) \in C'(E)$ *such that* $\Upsilon_{\mathcal{A}_d}(\widetilde{\epsilon}_0, \widetilde{m}_1) = (v_0, r_1)$. *Then, there exists a unique trajectory* $(\epsilon., m.)$ *of* $\mathcal{M}$ *such that* $(\epsilon_0, m_1) = (\widetilde{\epsilon}_0, \widetilde{m}_1)$ *and* $\Upsilon_{\mathcal{A}_d}(\epsilon_k, m_{k+1}) = (v_k, r_{k+1})$ *for all* k.

Proof. By Proposition 5.2, the discrete path $(v., r.)$ lifts to a unique discrete path $(\epsilon., m.)$ in $C'(E)$, starting at $(\widetilde{\epsilon}_0, \widetilde{m}_1)$. Then, $(\epsilon_0, m_1) = (\widetilde{\epsilon}_0, \widetilde{m}_1)$ and $(v_k, r_{k+1}) = \Upsilon_{\mathcal{A}_d}(\epsilon_k, m_{k+1})$ for all k. By Theorem 5.15, $(\epsilon., m.)$ is a trajectory of $\mathcal{M}$. □

Remark 5.18. Theorem 5.17 asserts that all trajectories of a reduced DLPS $\mathcal{M}/(G, \mathcal{A}_d)$ come from trajectories of the original system $\mathcal{M}$. It is possible to give a direct description of the reconstruction process in terms of lifting discrete paths (see Lemma 5.1 and Proposition 5.2). This process is inductive, so it suffices to describe the initial step, as we do next.

Given a discrete path $\rho := ((v_0, r_1), (v_1, r_2)) \in C''(\widetilde{G}_E)$ and $(\epsilon_0, m_1) \in C'(E)$ such that $\Upsilon_{\mathcal{A}_d}(\epsilon_0, m_1) = (v_0, r_1)$, the discrete lift of ρ starting at (ϵ_0, m_1) is $\hat{\rho} := ((\epsilon_0, m_1), (l_g^E(\epsilon_1'), l_g^M(m_2')))$ where $(\epsilon_1', m_2') \in \Upsilon_{\mathcal{A}_d}^{-1}(v_1, r_2)$ is arbitrary —as $\Upsilon_{\mathcal{A}_d}$ is onto, it is always possible to find such pairs (ϵ_1', m_2')— and $g \in G$ is the unique element making $l_g^M(\phi(\epsilon_1')) = m_1$.

Using (2.3) we see that $\pi^{M,G}(\phi(\epsilon_1')) = p^{M/G}(p_1(\Upsilon_{\mathcal{A}_d}(\epsilon_1', m_2'))) = p^{M/G}(v_1) = r_1 = \pi^{M,G}(m_1)$, so that m_1 and $\phi(\epsilon_1')$ are in the same G-orbit and g is well defined. Furthermore, as $\phi(l_g^E(\epsilon_1')) = l_g^M(\phi(\epsilon_1')) = m_1$, we have that $\hat{\rho} \in C''(E)$. Finally, as $\Upsilon_{\mathcal{A}_d}$ is G-invariant $\Upsilon_{\mathcal{A}_d}(l_g^E(\epsilon_1'), l_g^M(m_2')) = \Upsilon_{\mathcal{A}_d}(l_g^{C'(E)}(\epsilon_1', m_2')) = \Upsilon_{\mathcal{A}_d}(\epsilon_1', m_2') = (v_1, r_2)$. Hence $\Upsilon_{\mathcal{A}_d}^{(2)}(\hat{\rho}) = \rho$, and $\hat{\rho}$ is, indeed, the corresponding lifted path.

6. Example

In this section we illustrate the reduction techniques introduced so far with the reduction of an explicit symmetric DLPS and give a description of the resulting system.

6.1. **The system and a symmetry group.** The starting point is the DMS (Q, L_d), where $Q := \mathbb{C}^2 - \Delta_{xy}$, for Δ_{xy} the diagonal in $\mathbb{C}^2$ and

$$L_d(q_0, q_1) := \frac{1}{2h}(\|q_1^x - q_0^x\|^2 + \|q_1^y - q_0^y\|^2) - \frac{h}{2}V(\|(q_0^y - q_0^x)\|^2) \tag{6.1}$$

where $h \neq 0$ is a real constant. This DMS arises as a simple discretization of the mechanical system consisting of two distinct unit-mass particles in the plane that interact via a potential V, that only depends on the distance between the particles.

Following Example 3.12, we associate a DLPS $\mathcal{M}$ to (Q, L_d). Take the fiber bundle $\phi : E \to M$ to be $id_Q : Q \to Q$, the Lagrangian function L_d and $\mathcal{P} := 0$. Define the DLPS $\mathcal{M} := (Q, L_d, \mathcal{P})$.

Recall that $SE(2)$ is the group of special Euclidean symmetries of $\mathbb{R}^2 \simeq \mathbb{C}$. We can identify $SE(2)$ with $\{(A, v) \in \mathbb{C}^2 : |A| = 1\} = U(1)\times\mathbb{C}$. The product operation is $(A_1, v_1)\cdot(A_2, v_2) = (A_1A_2, A_1v_2+v_1)$ with null element $e_{SE(2)} = (1, 0)$ and inverse $(A, v)^{-1} = (A^{-1}, -A^{-1}v)$. The subset $T_2 := \{(1, v) \in U(1)\times\mathbb{C}\} \subset SE(2)$ is a closed normal subgroup that is isomorphic (as a group) to $\mathbb{C}$.

$SE(2)$ acts naturally on $\mathbb{C}$ by $l^{\mathbb{C}}_{(A,v)}(z) := Az+v$. This action induces the diagonal action of $SE(2)$ on $Q\times Q$ by $l^{\mathbb{C}^2}_{(A,v)}(q) := (l^{\mathbb{C}}_{(A,v)}(q^x), l^{\mathbb{C}}_{(A,v)}(q^y)) = (Aq^x+v, Aq^y+v)$. Since Q is preserved by $l^{\mathbb{C}^2}$, $SE(2)$ acts smoothly on Q by the restricted action, that we denote by l^Q.

It is immediate that l^Q is a free action. Being $U(1)$ compact, l^Q is a proper action. Then, by Corollary 9.10, $\pi^{Q,SE(2)} : Q \to Q/SE(2)$ is a principal $SE(2)$-bundle.

From the previous discussion and the fact that $\phi = id_Q$ is an $SE(2)$-invariant trivialization of $\phi : E \to M$, we conclude that $SE(2)$ acts on the fiber bundle $\phi : E \to M$. As $L_d \circ l^{Q\times Q}_{(A,v)} = L_d$ for all $(A, v) \in SE(2)$ and $\mathcal{P} := 0$ is $SE(2)$-equivariant, we conclude that $SE(2)$ is a symmetry group of $\mathcal{M}$. Being $T_2 \subset SE(2)$ a closed subgroup, it is also a symmetry group of $\mathcal{M}$ by Proposition 5.8.

6.2. **A discrete connection.** In this section we use the canonical real inner products in $\mathbb{C}^2$ and $\mathbb{C}$ to produce a discrete connection $\mathcal{A}_d^{T_2}$ on the principal T_2-bundle $\pi^{Q,T_2} : Q \to Q/T_2$, following the construction given in Section 5 of [8]. The idea of that construction (in the current setting) is as follows. As T_2 acts by isometries on $\mathbb{C}$ (with the canonical real product), T_2 acts by isometries on $\mathbb{C}^2$ via the diagonal action (with the canonical real inner product on $\mathbb{C}^2$). This last inner product induces a T_2-invariant riemannian metric on Q. The horizontal subspace for the discrete connection is an open subset of the set of pairs $(q_0, q_1) \in Q \times Q$ such that $q_1 = \exp^Q(v)$ for some $v \in T_{q_0}Q$ that is orthogonal to $T_{q_0}\mathcal{V}(q_0) = \ker(d\pi^{Q,T_2}(q_0))$, the tangent space to the l^Q-orbit through q_0.

The previous construction gives that

$$Hor_{\mathcal{A}_d^{T_2}} := \{(q_0, q_1) \in Q \times Q : q_0^x + q_0^y = q_1^x + q_1^y\}$$

is a discrete connection on the principal T_2-bundle $\pi^{Q,T_2} : Q \to Q/T_2$. Straightforward computations show that the discrete connection form is

$$\mathcal{A}_d^{T_2}(q_0, q_1) = \left(1, \frac{1}{2}\big((q_1^x + q_1^y) - (q_0^x + q_0^y)\big)\right). \tag{6.2}$$

Remark 6.1. Other discrete connections can be considered on the principal T_2-bundle $\pi^{Q,T_2} : Q \to Q/T_2$. For instance, one can define an affine discrete connection whose horizontal space consists of a level manifold of the discrete momentum function (see Remark 5.9), This would lead to the reduction procedure considered in Section 11 of [7] for DMS with horizontal symmetries.

6.3. **The reduced system.** Using the discrete connection $\mathcal{A}_d^{T_2}$ we construct the reduced system $\mathcal{M}/(T_2, \mathcal{A}_d^{T_2}) = (\widetilde{(T_2)_Q}, \check{L}_d, \check{\mathcal{P}})$, in such a way that

$$\Upsilon_{\mathcal{A}_d^{T_2}}(q_0, q_1) = (\pi^{Q\times T_2, T_2}(q_0, \mathcal{A}_d^{T_2}(q_0, q_1)), \pi^{Q,T_2}(q_1))$$

is in $\operatorname{mor}_{\mathcal{M}}(\mathcal{M}, \mathcal{M}/(T_2, \mathcal{A}_d^{T_2}))$. Below we give an explicit DLPS $\mathcal{M}'$, isomorphic to $\mathcal{M}/(T_2, \mathcal{A}_d^{T_2})$.

6.3.1. *An alternative model.* Define $\phi' : E' \to M'$ by $p_1 : \mathbb{C}^* \times T_2 \to \mathbb{C}^*$, so that (E', M', ϕ', T_2) is a trivial fiber bundle. Define the map $\Upsilon : C'(E) \to C'(E')$ by

$$\Upsilon(q_0, q_1) := \left(\left(\frac{1}{\sqrt{2}}(q_0^x - q_0^y), \left(1, \frac{1}{2}((q_1^x + q_1^y) - (q_0^x + q_0^y))\right)\right), \frac{1}{\sqrt{2}}(q_1^x - q_1^y) \right).$$

Clearly Υ is smooth and T_2-invariant.

We intend to define a DLPS structure on $\phi' : E' \to M'$ in such a way that Υ is a morphism. This forces us to define

$$L_d{}'((r_0, z_0), r_1) = \frac{1}{2h}\left(2\left\|z_0^s\right\|^2 + \left\|r_1 - r_0\right\|^2\right) - \frac{h}{2} V\left(2\left\|r_0\right\|^2\right) \tag{6.3}$$

and

$$\mathcal{P}'\left(((r_0, z_0), r_1), ((r_1, z_1), r_2)\right)\left(b\frac{\partial}{\partial r_1} + c\frac{\partial}{\partial z_1}\right) = -c\frac{\partial}{\partial z_0}, \tag{6.4}$$

where $(r, z) \in \mathbb{C}^* \times \mathbb{C}$. A number of computations confirm that $\mathcal{M}' := (E', L_d{}', \mathcal{P}')$ is a DLPS and that $\Upsilon \in \operatorname{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}')$.

By the T_2-invariance of Υ, there is a smooth map $\widetilde{\Upsilon}$ such that the diagram

$$\begin{tikzcd}
C'(E') & Q \times Q \arrow[l, "\Upsilon"'] \arrow[d, "\pi^{Q\times Q, T_2}"] \arrow[dr, bend left, "\Upsilon_{\mathcal{A}_d^{T_2}}"] & \\
& (Q\times Q)/T_2 \arrow[ul, "\widetilde{\Upsilon}"] & \widetilde{(T_2)_Q} \arrow[l, "\sim", "\Psi_{\mathcal{A}_d^{T_2}}"']
\end{tikzcd}$$

is commutative. Define the smooth map $\check{\Upsilon} := \widetilde{\Upsilon} \circ \Psi_{\mathcal{A}_d^{T_2}}$. As Υ is onto and satisfies $\Upsilon^{-1}(\Upsilon(q_0, q_1)) = l_{T_2}^{Q\times Q}(q_0, q_1)$ for all $(q_0, q_1) \in Q \times Q$, which easily implies that $\widetilde{\Upsilon}$ is one to one, $\check{\Upsilon}$ is a diffeomorphism. By Lemma 4.5, $\check{\Upsilon}$ is an isomorphism in $\mathfrak{LP}_d$.

All together, $\mathcal{M}'$ is an explicit model for the reduced DLPS $\mathcal{M}/(T_2, \mathcal{A}_d^{T_2})$.

6.3.2. *Equations of motion.* Trajectories in $\mathcal{M}'$ are found using (3.4). Evaluating the left side of (3.4) on an arbitrary tangent vector $b\frac{\partial}{\partial r_1} + c\frac{\partial}{\partial z_1} \in T_{(r_1, z_1)}E'$ and computing the corresponding derivatives we obtain the equations

$$\operatorname{Re}((z_1 - z_0)\overline{c}) = 0 \quad \text{and} \quad \operatorname{Re}(((r_1 - r_0) - (r_2 - r_1) - 2h^2 V'(2\left\|r_1\right\|^2)r_1)\overline{b}) = 0$$

which, due to de arbitrariness of $b, c \in \mathbb{C}$, lead to

$$z_1 = z_0 \quad \text{and} \quad r_2 = 2r_1 - r_0 - 2h^2 V'(2\left\|r_1\right\|^2)r_1.$$

It should be noticed that the z_k are (proportional to) the velocity of the center of mass of the original system, which explains the fact that z_k is constant for a trajectory, while r_k gives the position of one particle relative to the other.

Remark 6.2. There is a $U(1)$ action on $\mathcal{M}'$ given by $l^{E'}_{(A,0)}(r,z) := (Az, Az)$. This action is a "residue" of the original $SE(2)$ action on $\mathcal{M}$. This action is, indeed, a symmetry of $\mathcal{M}'$ and can be reduced using the same techniques. During the rest of the paper we will show that, under appropriate conditions, this second reduction produces a system that is isomorphic to $\mathcal{M}/SE(2)$.

7. Reduction in two stages

Let G be a symmetry group of the DLPS $\mathcal{M}$ and $H \subset G$ a normal closed subgroup. In this section we apply the reduction theory of $\mathcal{M}$ by H and, provided that G/H is a symmetry group of $\mathcal{M}/H$, perform a second reduction. Last, we compare the two step reduction with the reduction $\mathcal{M}/G$ performed in one step.

7.1. **Residual symmetry group.**

Lemma 7.1. *Let G act on the fiber bundle (E, M, ϕ, F) and $H \subset G$ be a closed normal subgroup. Define mappings*

$$\begin{aligned} l^{\widetilde{H}_E}_{\pi^{G,H}(g)}(\pi^{E\times H,H}(\epsilon, w)) :=& \pi^{E\times H,H}(l^E_g(\epsilon), l^G_g(w)),\\ l^{M/H}_{\pi^{G,H}(g)}(\pi^{M,H}(m)) :=& \pi^{M,H}(l^M_g(m)). \end{aligned} \tag{7.1}$$

Then $l^{\widetilde{H}_E}$, $l^{M/H}$ and the trivial right action on $F \times H$ define a G/H-action on the fiber bundle $(\widetilde{H}_E, M/H, p^{M/H}, F \times H)$.

Proof. By Lemma 5.7, H acts on (E, M, ϕ, F). Hence, by Example 9.17, the conjugate bundle $(\widetilde{H}_E, M/H, p^{M/H}, F \times H)$ is a fiber bundle. Consider the G-action on $E \times H$ defined by

$$l^{E\times H}_g(\epsilon, w) := (l^E_g(\epsilon), l^G_g(w)), \tag{7.2}$$

where $l^G_g(w) = gwg^{-1}$. Being a product of smooth actions, it is a smooth action. Also, as l^E is a free and proper G-action (see Lemma 9.11), the same is true for $l^{E\times H}$. Therefore, by Lemma 9.12, the function $l^{\widetilde{H}_E}$ given in (7.1) defines a smooth, free and proper G/H-action on $\widetilde{H}_E = (E \times H)/H$.

Analogously, as the G-action l^M on M is smooth, free and proper (this last fact by Lemma 9.11), Lemma 9.12 proves that $l^{M/H}$ defined in (7.1) is a smooth, free and proper G/H-action on M/H. Then, Corollary 9.10 proves that the quotient map $\pi^{M/H,G/H} : M/H \to (M/H)/(G/H)$ is a principal G/H-bundle.

It follows from the G-equivariance of $\phi : E \to M$ that $p^{M/H} : \widetilde{H}_E \to M/H$ is G/H-equivariant for the G/H-actions defined above.

For the rest of the proof we construct G/H-equivariant trivializing charts of the fiber bundle $(\widetilde{H}_E, M/H, p^{M/H}, F\times H)$. Since this is a local problem (in the base of the bundle) we will assume that $\phi : E \to M$ is $p_1 : M \times F \to M$ with the G action on E given by $l^E_g(m, f) = (l^M_g(m), r^F_{g^{-1}}(f))$. Even more, as $\pi^{M,G} : M \to M/G$ is a principal G-bundle, by shrinking M further we may assume that $\pi^{M,G}$ is a trivial principal G-bundle, that is, $M = (M/G) \times G$, where the G-action is given by left multiplication on G. All together, we have that $\phi : E \to M$ is given by the projection on the first two components $p_{12} : (M/G) \times G \times F \to (M/G) \times G$ and the G-actions are $l_g^E(\pi^{M,G}(m), g', f) = (\pi^{M,G}(m), gg', r_{g^{-1}}^F(f))$ and $l_g^M(\pi^{M,G}(m), g') = (\pi^{M,G}(m), gg')$.

Define the map

$$\sigma : \underbrace{(M/G) \times G \times F}_{=E} \times H \to \underbrace{(M/G) \times G}_{=M} \times F \times H$$

by

$$\sigma(\pi^{M,G}(m), g', f, h) := (\pi^{M,G}(m), g', r_{g'}^F(f), (g')^{-1} h g').$$

A quick check shows that σ is a G-equivariant diffeomorphism for the left actions $l^{E \times H}$ and $l_g^{M \times F \times H}(m, f, h) := (l_g^M(m), f, h)$, for all $g \in G$. Restricting those actions to $H \subset G$ and applying Corollary 9.4, σ induces a map $\Phi^{\widetilde{H}_E} : \widetilde{H}_E \to (M/H) \times F \times H$. It is easy to see that $\Phi^{\widetilde{H}_E}$ provides a (local) G/H-equivariant trivialization of $p^{M/H} : \widetilde{H}_E \to M/H$, concluding the proof of the fact that G/H acts on the fiber bundle $(\widetilde{H}_E, M/H, p^{M/H}, F \times H)$. □

Lemma 7.2. *Let G be a Lie group acting on Q by the action l^Q in such a way that $\pi^{Q,G} : Q \to Q/G$ is a principal G-bundle. Assume that $H \subset G$ is a closed and normal subgroup and that $\mathcal{A}_d$ is a discrete connection on the principal H-bundle $\pi^{Q,H} : Q \to Q/H$ whose domain is G-invariant for the diagonal G-action $l^{Q \times Q}$. Then, the following assertions are equivalent.*

(1) *For each $g \in G$ and (q_0, q_1) in the domain of $\mathcal{A}_d$,*

$$\mathcal{A}_d(l_g^Q(q_0), l_g^Q(q_1)) = g \mathcal{A}_d(q_0, q_1) g^{-1}. \tag{7.3}$$

(2) *The submanifold $Hor_{\mathcal{A}_d} \subset Q \times Q$ is G-invariant for the G-action $l^{Q \times Q}$.*

Proof. Recall that (q_0, q_1) in the domain of $\mathcal{A}_d$ is in $Hor_{\mathcal{A}_d}$ if and only if $\mathcal{A}_d(q_0, q_1) = e$. Assume that (7.3) holds, for each $g \in G$. Let $(q_0, q_1) \in Hor_{\mathcal{A}_d}$. Then, for any $g \in G$,

$$\mathcal{A}_d(l_g^Q(q_0), l_g^Q(q_1)) = g \mathcal{A}_d(q_0, q_1) g^{-1} = g g^{-1} = e,$$

showing that $l_g^{Q \times Q}(q_0, q_1) = (l_g^Q(q_0), l_g^Q(q_1)) \in Hor_{\mathcal{A}_d}$.

Conversely, if (q_0, q_1) is in the domain of $\mathcal{A}_d$, which is G-invariant, we have that $(l_g^Q(q_0), l_g^Q(q_1))$ is also in the domain of $\mathcal{A}_d$. Then $\mathcal{A}_d(l_g^Q(q_0), l_g^Q(q_1)) = h \in H$ if and only if

$$(l_g^Q(q_0), l_{h^{-1}}^Q(l_g^Q(q_1))) \in Hor_{\mathcal{A}_d}. \tag{7.4}$$

But, as $(q_0, l_{\mathcal{A}_d(q_0,q_1)^{-1}}^Q(q_1)) \in Hor_{\mathcal{A}_d}$ and $Hor_{\mathcal{A}_d}$ is G-invariant, we have that

$$(l_g^Q(q_0), l_{g\mathcal{A}_d(q_0,q_1)^{-1}g^{-1}}^Q(l_g^Q(q_1))) = l_g^{Q \times Q}(q_0, l_{\mathcal{A}_d(q_0,q_1)^{-1}}^Q(q_1)) \in Hor_{\mathcal{A}_d}$$

so that $h := g \mathcal{A}_d(q_0, q_1)^{-1} g^{-1}$ satisfies (7.4). As the element of G with this property is unique, we conclude that (7.3) holds. □

Proposition 7.3. *Let G be a symmetry group of $\mathcal{M} = (E, L_d, \mathcal{P}) \in \mathrm{ob}_{\mathfrak{L}\mathfrak{P}_d}$ and $H \subset G$ be a closed and normal subgroup. Choose a discrete connection $\mathcal{A}_d$ of the principal H-bundle $\pi^{Q,H} : Q \to Q/H$ such that either one of the conditions in Lemma 7.2 holds. Then G/H is a symmetry group of the DLPS $\mathcal{M}^H := \mathcal{M}/(H, \mathcal{A}_d) = (\widetilde{H}_E, \check{L}_d, \check{\mathcal{P}})$ obtained by the reduction of $\mathcal{M}$ using $\mathcal{A}_d$.*

Proof. By Lemma 7.1, G/H acts on the fiber bundle $(\widetilde{H}_E, M/H, p^{M/H}, F\times H)$. Recall that $\Upsilon_{\mathcal{A}_d}: C'(E)\to C'(\widetilde{H}_E)$ is defined by

$$\Upsilon_{\mathcal{A}_d}(\epsilon_0,m_1) := (\pi^{E\times H,H}(\epsilon_0,\mathcal{A}_d(\phi(\epsilon_0),m_1)),\pi^{M,H}(m_1)).$$

Unraveling the definitions and taking (7.3) into account, we have that

$$\Upsilon_{\mathcal{A}_d}\circ l_g^{C'(E)} = l_{\pi^{G,H}(g)}^{\widetilde{H}_E\times(M/H)}\circ \Upsilon_{\mathcal{A}_d} \quad \text{for all} \quad g\in G. \tag{7.5}$$

Then, as $\check{L}_d$ satisfies $\check{L}_d\circ\Upsilon_{\mathcal{A}_d} = L_d$, and L_d is G-invariant for the G-action $l^{E\times M}$, we have that

$$\check{L}_d\circ l_{\pi^{G,H}(g)}^{\widetilde{H}_E\times(M/H)}\circ\Upsilon_{\mathcal{A}_d} = \check{L}_d\circ\Upsilon_{\mathcal{A}_d}\circ l_g^{E\times M} = L_d\circ l_g^{E\times M} = L_d = \check{L}_d\circ\Upsilon_{\mathcal{A}_d}.$$

Hence, as $\Upsilon_{\mathcal{A}_d}$ is onto, $\check{L}_d$ is G/H-invariant for the G/H-action $l^{\widetilde{H}_E\times(M/H)}$.

Differentiating the first component of (7.5) we see that, for $v_0=\Upsilon_{\mathcal{A}_d}(\epsilon_0,m_1)$,

$$D_1(p_1\circ\Upsilon_{\mathcal{A}_d})(l_g^{C'(E)}(\epsilon_0,m_1))dl_g^E(\epsilon_0)(\delta\epsilon_0) = dl_{\pi^{G,H}(g)}^{\widetilde{H}_E}(v_0)D_1(p_1\circ\Upsilon_{\mathcal{A}_d})(\epsilon_0,m_1)(\delta\epsilon_0) \tag{7.6}$$

and

$$D_2(p_1\circ\Upsilon_{\mathcal{A}_d})(l_g^{C'(E)}(\epsilon_0,m_1))dl_g^M(m_1)(\delta m_1) = dl_{\pi^{G,H}(g)}^{\widetilde{H}_E}(v_0)D_2(p_1\circ\Upsilon_{\mathcal{A}_d})(\epsilon_0,m_1)(\delta m_1). \tag{7.7}$$

Now, fix $\nu = ((v_0,r_1),(v_1,r_2))\in C''(\widetilde{H}_E)$ and take $\eta=((\epsilon_0,m_1),(\epsilon_1,m_2))\in C''(E)$ such that $\Upsilon_{\mathcal{A}_d}^{(2)}(\eta)=\nu$. Then, by point 2 of Lemma 5.1, any $\delta v_1\in T_{v_1}(\widetilde{H}_E)$ is of the form $\delta v_1 = D_1(p_1\circ\Upsilon_{\mathcal{A}_d})(\epsilon_1,m_2)(\delta\epsilon_1)$ for a unique $\delta\epsilon_1\in T_{\epsilon_1}E$. Then, using (7.6), we obtain

$$D_1(p_1\circ\Upsilon_{\mathcal{A}_d})(l_g^{C'(E)}(\epsilon_1,m_2))dl_g^E(\epsilon_1)(\delta\epsilon_1) = dl_{\pi^{G,H}(g)}^{\widetilde{H}_E}(v_1)(\delta v_1),$$

which shows that the unique element of $T_{l_g^E(\epsilon_1)}E$ that represents $dl_{\pi^{G,H}(g)}^{\widetilde{H}_E}(v_1)(\delta v_1)\in T_{l_{\pi^{G,H}(g)}^{\widetilde{H}_E}(v_1)}\widetilde{H}_E$ is $dl_g^E(\epsilon_1)(\delta\epsilon_1)$. Also, notice that using (7.5) we obtain

$$\Upsilon_{\mathcal{A}_d}^{(2)}(l_g^{C''(E)}(\eta)) = l_{\pi^{G,H}(g)}^{C''(\widetilde{H}_E)}(\Upsilon_{\mathcal{A}_d}^{(2)}(\eta)) = l_{\pi^{G,H}(g)}^{C''(\widetilde{H}_E)}(\nu).$$

We use this information to compute $\check{\mathcal{P}}\circ l_{\pi^{G,H}(g)}^{p_3^*T(\widetilde{H}_E)}$. For any $g\in G$,

$$\begin{aligned}(\check{\mathcal{P}}\circ l_{\pi^{G,H}(g)}^{p_3^*T(\widetilde{H}_E)})(\nu,\delta v_1) &= \check{\mathcal{P}}(l_{\pi^{G,H}(g)}^{C''(\widetilde{H}_E)}(\nu))(dl_{\pi^{G,H}(g)}^{\widetilde{H}_E}(v_1)(\delta v_1))\\ &= D_1(p_1\circ\Upsilon_{\mathcal{A}_d})(l_g^{C'(E)}(\epsilon_0,m_1))(\mathcal{P}(l_g^{C''(E)}(\eta))(dl_g^E(\epsilon_1)(\delta\epsilon_1)))\\ &\quad + D_2(p_1\circ\Upsilon_{\mathcal{A}_d})(l_g^{C'(E)}(\epsilon_0,m_1))(d\phi(l_g^E(\epsilon_1))(dl_g^E(\epsilon_1)(\delta\epsilon_1))).\end{aligned} \tag{7.8}$$

Using the G-equivariance of $\mathcal{P}$ and (7.6), we have

$$\begin{aligned}D_1(p_1\circ\Upsilon_{\mathcal{A}_d})&(l_g^{C'(E)}(\epsilon_0,m_1))(\mathcal{P}(l_g^{C''(E)}(\eta)(dl_g^E(\epsilon_1)(\delta\epsilon_1))))\\ &= D_1(p_1\circ\Upsilon_{\mathcal{A}_d})(l_g^{C'(E)}(\epsilon_0,m_1))(l_g^{TE}(\epsilon_0)(\mathcal{P}(\eta)(\delta\epsilon_1)))\\ &= D_1(p_1\circ\Upsilon_{\mathcal{A}_d})(l_g^{C'(E)}(\epsilon_0,m_1))(dl_g^E(\epsilon_0)(\mathcal{P}(\eta)(\delta\epsilon_1)))\\ &= dl_{\pi^{G,H}(g)}^{\widetilde{H}_E}(v_0)D_1(p_1\circ\Upsilon_{\mathcal{A}_d})(\epsilon_0,m_1)(\mathcal{P}(\eta)(\delta\epsilon_1))\end{aligned}$$

and, using (7.7),

$$\begin{aligned}
D_2(p_1 \circ \Upsilon_{\mathcal{A}_d})&(l_g^{C'(E)}(\epsilon_0, m_1))(d\phi(l_g^E(\epsilon_1))(dl_g^E(\epsilon_1)(\delta\epsilon_1))) \\
&= D_2(p_1 \circ \Upsilon_{\mathcal{A}_d})(l_g^{C'(E)}(\epsilon_0, m_1))(dl_g^M(m_1)(d\phi(\epsilon_1)(\delta\epsilon_1))) \\
&= dl^{\widetilde{H}_E}_{\pi^{G,H}(g)}(v_0) D_2(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0, m_1)(d\phi(\epsilon_1)(\delta\epsilon_1)).
\end{aligned}$$

Going back to (7.8), we obtain

$$\begin{aligned}
\check{\mathcal{P}} \circ l^{p_3^* T(\widetilde{H}_E)}_{\pi^{G,H}(g)}(\nu, \delta v_1) &= \check{\mathcal{P}}(l^{C''(\widetilde{H}_E)}_{\pi^{G,H}(g)}(\nu))(dl^{\widetilde{H}_E}_{\pi^{G,H}(g)}(v_1)(\delta v_1)) \\
&= dl^{\widetilde{H}_E}_{\pi^{G,H}(g)}(v_0)\big(D_1(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0, m_1)(\mathcal{P}(\eta)(\delta\epsilon_1)) \\
&\qquad\qquad + D_2(p_1 \circ \Upsilon_{\mathcal{A}_d})(\epsilon_0, m_1)(d\phi(\epsilon_1)(\delta\epsilon_1))\big) \\
&= dl^{\widetilde{H}_E}_{\pi^{G,H}(g)}(v_0)\big(\check{\mathcal{P}}(\nu)(\delta v_1)\big),
\end{aligned}$$

showing that $\check{\mathcal{P}}$ is G/H-equivariant. Hence, G/H is a symmetry group of $\mathcal{M}^H$. □

Example 7.4. In Section 6.1 we introduced a DLPS $\mathcal{M}$ and saw that $SE(2)$ was one of its symmetry groups. As $T_2 \subset SE(2)$ is a closed normal subgroup, it was also a symmetry group of $\mathcal{M}$. A simple verification shows that the discrete connection form $\mathcal{A}_d^{T_2}$ defined in (6.2) satisfies (7.3) for $G := SE(2)$ and $H := T_2$ so that, by Proposition 7.3, $SE(2)/T_2$ is a symmetry group of $\mathcal{M}/(T_2, \mathcal{A}_d^{T_2}) \simeq \mathcal{M}'$. As $SE(2)/T_2 \simeq U(1)$, we see that this fact is already suggested in Remark 6.2.

7.2. **Comparison with reduction by the full symmetry group.** Here we consider a symmetry group G of $\mathcal{M} = (E, L_d, \mathcal{P}) \in \mathrm{ob}_{\mathfrak{LP}_d}$. Fixing a discrete connection $\mathcal{A}_d^G$ on the principal G-bundle $\pi^{M,G} : M \to M/G$ we have the reduced system $\mathcal{M}^G := \mathcal{M}/(G, \mathcal{A}_d^G)$. When $H \subset G$ is a closed and normal subgroup, by Proposition 5.8, H is a symmetry group of $\mathcal{M}$ and, when $\mathcal{A}_d^H$ is a discrete connection on the principal H-bundle $\pi^{M,H} : M \to M/H$ we have the reduced system $\mathcal{M}^H := \mathcal{M}/(H, \mathcal{A}_d^H)$. Furthermore, when $\mathcal{A}_d^H$ satisfies any one of the conditions in Lemma 7.2, by Proposition 7.3, G/H is a symmetry group of $\mathcal{M}^H$. Fixing a discrete connection $\mathcal{A}_d^{G/H}$ on the principal G/H-bundle $\pi^{M/H,G/H} : M/H \to \frac{M/H}{G/H}$, we have the reduced system $\mathcal{M}^{G/H} := \mathcal{M}^H/(G/H, \mathcal{A}_d^{G/H})$. The following diagram depicts the relation between the different DLPSs and morphisms.

$$\begin{tikzcd}
 & \mathcal{M} \arrow[ddl, "\Upsilon_{\mathcal{A}_d^G}"'] \arrow[dr, "\Upsilon_{\mathcal{A}_d^H}"] & & \\
 & & \mathcal{M}^H \arrow[dr, "\Upsilon_{\mathcal{A}_d^{G/H}}"] & \\
\mathcal{M}^G & & & \mathcal{M}^{G/H}
\end{tikzcd}$$

At the "geometric level", the corresponding spaces and smooth maps are

$$\begin{tikzcd}
 & C'(E) \arrow[dl, "\Upsilon_{\mathcal{A}_d^G}"'] \arrow[d, "\Upsilon_{\mathcal{A}_d^H}"] & \\
C'(\widetilde{G}_E) & C'(\widetilde{H}_E) \arrow[r, "\Upsilon_{\mathcal{A}_d^{G/H}}"] & C'(\widetilde{G/H}_{\widetilde{H}_E})
\end{tikzcd}$$

We can enlarge the previous diagram by adding the different diffeomorphisms $\Phi_{\mathcal{A}_d}$ introduced in Proposition 2.6 and by taking into account the commutative diagram (2.1). The resulting diagram follows.

$$\begin{tikzcd}
 & & C'(E) \arrow[ddll, bend right, "\Upsilon_{\mathcal{A}_d^G}"'] \arrow[ddl, "\pi^{C'(E),G}"'] \arrow[d, "\pi^{C'(E),H}"] \arrow[dr, bend left, "\Upsilon_{\mathcal{A}_d^H}"] & & \\
 & & \frac{C'(E)}{H} \arrow[r, "\Phi_{\mathcal{A}_d^H}", "\sim"'] \arrow[dl, "F_1"'] \arrow[d, "\pi^{\frac{C'(E)}{H},G/H}"'] & C'(\widetilde{H}_E) \arrow[d, "\pi^{C'(\widetilde{H}_E),G/H}"'] \arrow[dr, "\Upsilon_{\mathcal{A}_d^{G/H}}"] & \\
C'(\widetilde{G}_E) & \frac{C'(E)}{G} \arrow[l, "\sim", "\Phi_{\mathcal{A}_d^G}"'] & \frac{\frac{C'(E)}{H}}{G/H} \arrow[l, "\sim", "F_2"'] \arrow[r, "\sim", "\check{\Phi}_{\mathcal{A}_d^H}"'] & \frac{C'(\widetilde{H}_E)}{G/H} \arrow[r, "\sim", "\Phi_{\mathcal{A}_d^{G/H}}"'] & C'(\widetilde{G/H}_{\widetilde{H}_E})
\end{tikzcd} \tag{7.9}$$

The following result introduces the new functions that appear in diagram (7.9) and explores their basic properties.

Lemma 7.5. *Under the previous conditions,*

1. $\Phi_{\mathcal{A}_d^H} : \frac{C'(E)}{H} \to C'(\widetilde{H}_E)$ *is a* G/H*-equivariant diffeomorphism. Hence it induces a smooth diffeomorphism* $\check{\Phi}_{\mathcal{A}_d^H} : \frac{\frac{C'(E)}{H}}{G/H} \to \frac{C'(\widetilde{H}_E)}{G/H}$.
2. $\pi^{C'(E),G} : C'(E) \to \frac{C'(E)}{G}$ *is a smooth* H*-invariant map, hence it induces a smooth map* $F_1 : \frac{C'(E)}{H} \to \frac{C'(E)}{G}$.
3. $F_1 : \frac{C'(E)}{H} \to \frac{C'(E)}{G}$ *is a smooth* G/H*-invariant map, hence it induces a smooth map* $F_2 : \frac{\frac{C'(E)}{H}}{G/H} \to \frac{C'(E)}{G}$*. Furthermore,* F_2 *is a diffeomorphism.*
4. *The diagram* (7.9) *is commutative.*

Proof. Unraveling the definitions and recalling that $\mathcal{A}_d^H$ satisfies (7.3), we see that $\Phi_{\mathcal{A}_d^H}$ is G/H-equivariant. As, by Proposition 2.6, $\Phi_{\mathcal{A}_d^H}$ is smooth, we conclude from Corollary 9.4 that the induced map $\check{\Phi}_{\mathcal{A}_d^H}$ is smooth. Furthermore, as $\Phi_{\mathcal{A}_d^H}$ is also a diffeomorphism by Proposition 2.6, its inverse is also G/H-equivariant, so that $\check{\Phi}_{\mathcal{A}_d^H}$ is a diffeomorphism. Hence point 1 in the statement is proved. By construction the square in diagram (7.9) is commutative.

Point 2 follows immediately using the H-invariance of $\pi^{C'(E),G}$ and Corollary 9.5. Furthermore, it is immediate that F_1 is G/H-invariant, and the same argument proves that F_2 is a well defined smooth map. It is easy to check that the map $\pi^{\frac{C'(E)}{H},G/H} \circ \pi^{C'(E),H} : C'(E) \to \frac{\frac{C'(E)}{H}}{G/H}$ is smooth and G-invariant, so it induces a smooth inverse of F_2, showing that F_2 is a diffeomorphism. This proves point 3. By definition, the two triangles involving F_1 in diagram (7.9) are commutative.

The commutativity of the three remaining triangles in diagram (7.9) is due to the commutativity of diagram (2.1). $\square$

Theorem 7.6. *Consider the data given at the beginning of this section. Let $F : C'(\widetilde{G/H}_{\widetilde{H}_E}) \to C'(\widetilde{G}_E)$ be defined by the bottom row of diagram (7.9), that is, $F := \Phi_{\mathcal{A}_d^G} \circ F_2 \circ (\Phi_{\check{\mathcal{A}}_d^H})^{-1} \circ (\Phi_{\mathcal{A}_d^{G/H}})^{-1}$. Then, the following statements are true.*

(1) F is a diffeomorphism.
(2) $F \in \operatorname{mor}_{\mathfrak{LP}_d}(\mathcal{M}^G, \mathcal{M}^{G/H})$.
(3) F is an isomorphism in $\mathfrak{LP}_d$.

Proof. By Proposition 2.6, $\Phi_{\mathcal{A}_d^G}$ and $\Phi_{\mathcal{A}_d^{G/H}}$ are diffeomorphisms and, by Lemma 7.5, the same happens to F_2 and $\Phi_{\check{\mathcal{A}}_d^H}$. Hence, F is a diffeomorphism, proving point 1.

Next, as $\Upsilon_{\mathcal{A}_d^{G/H}} \in \operatorname{mor}_{\mathfrak{LP}_d}(\mathcal{M}^H, \mathcal{M}^{G/H})$ and $\Upsilon_{\mathcal{A}_d^H} \in \operatorname{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}^H)$, we have that $\Upsilon_{\mathcal{A}_d^{G/H}} \circ \Upsilon_{\mathcal{A}_d^H} \in \operatorname{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}^{G/H})$. Also, $\Upsilon_{\mathcal{A}_d^G} \in \operatorname{mor}_{\mathfrak{LP}_d}(\mathcal{M}, \mathcal{M}^G)$, F is smooth and $F \circ \Upsilon_{\mathcal{A}_d^G} = \Upsilon_{\mathcal{A}_d^{G/H}} \circ \Upsilon_{\mathcal{A}_d^H}$, so that, by Lemma 4.5 point 2 is true. Using point 1 and Lemma 4.6, point 3 follows. $\square$

Theorem 7.7. *Consider the data given at the beginning of this section.*

(1) Let $(\epsilon_\cdot, m_\cdot) = ((\epsilon_0, m_1), \ldots, (\epsilon_{N-1}, m_N))$ be a discrete path in $C'(E)$. For $k = 0, \ldots, N-1$ define the discrete paths $(v_k^H, r_{k+1}^H) := \Upsilon_{\mathcal{A}_d^H}(\epsilon_k, m_{k+1})$, $(v_k^G, r_{k+1}^G) := \Upsilon_{\mathcal{A}_d^G}(\epsilon_k, m_{k+1})$ and $(v_k^{G/H}, r_{k+1}^{G/H}) := \Upsilon_{\mathcal{A}_d^H}(v_k^H, r_{k+1}^H)$ in $C'(\widetilde{H}_E)$, $C'(\widetilde{G}_E)$ and $C'(\widetilde{G/H}_{\widetilde{H}_E})$ respectively. Then, the following assertions are equivalent.
   (a) $(\epsilon_\cdot, m_\cdot)$ is a trajectory of $\mathcal{M}$.
   (b) $(v_\cdot^G, r_\cdot^G)$ is a trajectory of $\mathcal{M}^G$.
   (c) $(v_\cdot^H, r_\cdot^H)$ is a trajectory of $\mathcal{M}^H$.
   (d) $(v_\cdot^{G/H}, r_\cdot^{G/H})$ is a trajectory of $\mathcal{M}^{G/H}$.

(2) Let $F : C'(\widetilde{G/H}_{\widetilde{H}_E}) \to C'(\widetilde{G}_E)$ be the diffeomorphism defined in Theorem 7.6. Then $F(v_k^{G/H}, r_{k+1}^{G/H}) = (v_k^G, r_{k+1}^G)$ for all k.

(3) The DLPSs $\mathcal{M}^G$ and $\mathcal{M}^{G/H}$ are isomorphic in $\mathfrak{LP}_d$.

Proof. By Proposition 5.13 $\Upsilon_{\mathcal{A}_d^G}$, $\Upsilon_{\mathcal{A}_d^H}$ and $\Upsilon_{\mathcal{A}_d^{G/H}}$ are morphisms in $\mathfrak{LP}_d$. Then, point 1 follows from Theorem 4.7.

Point 2 is true by the following computation.

$$
\begin{aligned}
(v_k^G, r_{k+1}^G) =& \Upsilon_{\mathcal{A}_d^G}(\epsilon_k, m_{k+1}) = (F \circ \Upsilon_{\mathcal{A}_d^{G/H}} \circ \Upsilon_{\mathcal{A}_d^H})(\epsilon_k, m_{k+1}) \\
=& (F \circ \Upsilon_{\mathcal{A}_d^{G/H}})(v_k^H, r_{k+1}^H) = F(v_k^{G/H}, r_{k+1}^{G/H}).
\end{aligned}
$$

Point 3 is immediate from point 3 in Theorem 7.6. $\square$

7.3. **Discrete connections derived from a Riemannian metric.** The conditions stated at the beginning of Section 7.2 require the choice of three discrete connections $\mathcal{A}_d^H$, $\mathcal{A}_d^G$ and $\mathcal{A}_d^{G/H}$ on the corresponding principal bundles. One case where such connections are known to exist is when the total space of the corresponding principal bundle carries a Riemannian metric and the structure group acts by isometries; this is the content of Theorem 5.2 in [8]. In addition, $\mathcal{A}_d^H$ is required to satisfy either one of the conditions in Lemma 7.2. In this Section we prove that when the total space Q of a principal G-bundle $\pi^{Q,G}: Q \to Q/G$ is equipped with a G-invariant Riemannian metric, it is possible to apply Theorem 5.2 in [8] to construct discrete connections $\mathcal{A}_d^H$ satisfying the conditions in Lemma 7.2 on the principal H-bundle $\pi^{Q,H}: Q \to Q/H$ for any closed and normal subgroup $H \subset G$.

The construction analyzed in Theorem 5.2 in [8] is as follows. When Q is a Riemannian manifold and a Lie group H acts on Q by isometries, the *vertical bundle* $\mathcal{V}^H \subset TQ$ defined by $\mathcal{V}_q^H := T_q(l_H^Q(q)) \subset T_qQ$ has an orthogonal complement, the *horizontal bundle* $\mathcal{H}^H$. This horizontal bundle determines a connection $\mathcal{A}^H$ on the principal H-bundle $\pi^{Q,H} : Q \to Q/H$. In addition, there is a unique Riemannian metric on Q/H that makes Q/H a Riemannian manifold and $\pi^{Q,H}$ a Riemannian submersion. Standard results of Riemannian Geometry show that, for any $r \in Q/H$, there are open sets $W_r \subset Q/H$ containing r and such that any two points in W_r can be joined by a unique length-minimizing geodesic that, also, is contained in W_r (see Theorem 3.6 on page 166 of [13]); we call these sets *geodesically convex*. Using such a collection $\{W_r : r \in Q/H\}$, the open set

$$\mathcal{U} := \cup_{r\in Q/H} \left((\pi^{Q,H})^{-1}(W_r) \times (\pi^{Q,H})^{-1}(W_r)\right) \subset Q \times Q \tag{7.10}$$

is defined. Then, a function $\mathcal{A}_d^H : \mathcal{U} \to H$ is constructed as follows. Given $(q_0, q_1) \in \mathcal{U}$, there is $r \in Q/H$ such that $\pi^{Q,H}(q_0), \pi^{Q,H}(q_1) \in W_r$. Let $\gamma : [0,1] \to Q/H$ be the unique length-minimizing geodesic contained in W_r and joining $\pi^{Q,H}(q_0)$ to $\pi^{Q,H}(q_1)$. Let $\widetilde{\gamma}$ be the $\mathcal{A}^H$-horizontal lift of γ to Q, starting at q_0. Finally, let

$$\mathcal{A}_d^H(q_0, q_1) := \kappa_{q_1}(\widetilde{\gamma(1)}, q_1), \tag{7.11}$$

where $\kappa_{q_1} : Q_{\pi^{Q,H}(q_1)} \to H$ is the smooth map defined by $\kappa_{q_1}(l_h^Q(q_1), q_1) := h$. Theorem 5.2 in [8] asserts that there is a discrete connection $\mathcal{A}_d^H$ on the principal H-bundle $\pi^{Q,H} : Q \to Q/H$ whose domain is $\mathcal{U}$ and whose associated discrete form is given by (7.11).

Below, we consider the case where G is a Lie group and $H \subset G$ is a closed normal subgroup. G acts on the Riemannian manifold Q by isometries and in such a way such that $\pi^{Q,G} : Q \to Q/G$ is a principal G-bundle. Then, by restricting the G-action to an H-action, H acts by isometries on Q and $\pi^{Q,H} : Q \to Q/H$ is a principal H-bundle. But, still, G/H acts on Q/H by isometries and making $\pi^{Q/H,G/H} : Q/H \to (Q/H)/(G/H)$ a principal G/H-bundle.

Lemma 7.8. *Under the previous conditions, there is a collection of open subsets $\{W_r \subset Q/H : r \in Q/H\}$ that are geodesically convex as above that, in addition, satisfies*

$$l^{Q/H}_{\pi^{G,H}(g)}(W_{\pi^{Q,H}(q)}) = W_{\pi^{Q,H}(l_g^Q(q))} \quad \text{for all} \quad q \in Q \text{ and } g \in G. \tag{7.12}$$

Proof. In the current context, we have the commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\pi^{Q,H}} & Q/H \\ {\scriptstyle \pi^{Q,G}}\downarrow & & \downarrow{\scriptstyle \pi^{Q/H,G/H}} \\ Q/G & \xrightarrow[\phi]{\sim} & \frac{Q/H}{G/H} \end{array}$$

where all the π-mappings are principal bundles and ϕ is a diffeomorphism. Let $\widetilde{\sigma}$ be a section of $\pi^{Q/H,G/H}$ that may be discontinuous, and define $\sigma : Q \to Q/H$ by $\sigma := \widetilde{\sigma} \circ \phi \circ \pi^{Q,G}$. It is easy to see that σ is G-invariant, that its image intersects each G/H-orbit in Q/H in exactly one point and that, for each $q \in Q$, $\sigma(q)$ and $\pi^{Q,H}(q)$ are on the same G/H-orbit.

For each $\sigma(q) \in Q/H$, let $W_{\sigma(q)} \subset Q/H$ be any geodesically convex open subset. Using the G/H-action $l^{Q/H}$, for each $q \in Q$, we define

$$W_{\pi^{Q,H}(q)} := l^{Q/H}_{\pi^{G,H}(g)}(W_{\sigma(q)}) \quad \text{where} \quad \pi^{G,H}(g) := \kappa(\sigma(q), \pi^{Q,H}(q)).$$

Since $l^{Q/H}_{\pi^{G,H}(g)}$ is an isometry in Q/H, the open sets $W_{\pi^{Q,H}(q)}$ are also geodesically convex. A direct computation shows that the collection $\{W_{\pi^{Q,H}(q)} : q \in Q\}$ satisfies (7.12). □

Proposition 7.9. *With the same conditions as above, let $\mathcal{U}$ be defined by* (7.10), *for a collection of geodesically convex open subsets $\{W_r \subset Q/H : r \in Q/H\}$ satisfying* (7.12). *Then*

(1) $\mathcal{U}$ is G-invariant for the diagonal G-action $l^{Q\times Q}$.
(2) The discrete connection with domain $\mathcal{U}$ and discrete connection form $\mathcal{A}_d^H$ defined above satisfies condition (7.3) *in Lemma 7.2.*

Proof. Let $(q_0, q_1) \in \mathcal{U}$ and $g \in G$. By definition of $\mathcal{U}$, there is $\pi^{Q,H}(q) \in Q/H$ such that $\pi^{Q,H}(q_0), \pi^{Q,H}(q_1) \in W_{\pi^{Q,H}(q)}$. Hence, for $j = 0, 1$,

$$\pi^{Q,H}(l_g^Q(q_j)) = l^{Q/H}_{\pi^{G,H}(g)}(\underbrace{\pi^{Q,H}(q_j)}_{\in W_{\pi^{Q,H}(q)}}) \in l^{Q/H}_{\pi^{G,H}(g)}(W_{\pi^{Q,H}(q)}) = W_{\pi^{Q,H}(l_g^Q(q))},$$

Hence $l_g^{Q\times Q}(q_0, q_1) = (l_g^Q(q_0), l_g^Q(q_1)) \in \mathcal{U}$, proving part 1.

Given $(q_0, q_1) \in \mathcal{U}$ and $\pi^{Q,H}(q)$ as above, let γ_0 and γ_1 be the unique length-minimizing geodesics contained in $W_{\pi^{Q,H}(q)}$ and $W_{\pi^{Q,H}(l_g^Q(q))}$ going from $\pi^{Q,H}(q_0)$ to $\pi^{Q,H}(q_1)$ and from $\pi^{Q,H}(l_g^Q(q_0))$ to $\pi^{Q,H}(l_g^Q(q_1))$. Let $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ be the $\mathcal{A}^H$-horizontal lifts starting at q_0 and $l_g^Q(q_0)$ respectively.

Notice that, by the uniqueness of the length-minimizing geodesics in $W_{\pi^{Q,H}(l_g^Q(q))}$ and since $l^{Q/H}_{\pi^{Q,H}(g)}$ is an isometry in Q/H, we have that $\gamma_1 = l^{Q/H}_{\pi^{Q,H}(g)} \circ \gamma_0$.

Let $\rho := l_g^Q \circ \widetilde{\gamma}_0$. It is easy to check that ρ is a lift of γ_1 starting at $l_g^Q(q_0)$. It is also $\mathcal{A}^H$-horizontal, a fact that follows from the G-invariance of $\mathcal{H}^H$, that is, from $dl_g^Q(q')(\mathcal{H}^H_{q'}) \subset \mathcal{H}^H_{l_g^Q(q')}$ for all $q' \in Q$. By the uniqueness of the horizontal lifts, we conclude that $\widetilde{\gamma}_1 = \rho$.

Finally, using (7.11), we have

$$\begin{aligned}\mathcal{A}_d^H(l_g^Q(q_0), l_g^Q(q_1)) =& \kappa_{l_g^Q(q_1)}(\widetilde{\gamma}_1(1), l_g^Q(q_1)) = \kappa_{l_g^Q(q_1)}(l_g^Q(\widetilde{\gamma}_0(1)), l_g^Q(q_1)) \\ =& g\kappa_{q_1}(\widetilde{\gamma}_0(1), q_1)g^{-1} = g\mathcal{A}_d^H(q_0, q_1)g^{-1},\end{aligned}$$

that is, identity (7.3) holds, concluding the proof of part 2. □

8. Poisson structures

It is a well known and used fact that if (Q, L_d) is a regular DMS, there is a symplectic structure ω_{L_d} defined in (an open subset containing the diagonal of) $Q \times Q$. Furthermore, the discrete Lagrangian flow F_{L_d} is symplectic for ω_{L_d}. This structure is important both for the theoretical as well as the numerical applications of DMSs. Still, the dynamical system obtained as the reduction of a DMS may not carry a symplectic structure: an obvious reason could be that $\dim(\widetilde{G}\times(Q/G)) = 2\dim(Q)-\dim(G)$ could be odd, making it impossible for the reduced space $\widetilde{G}\times(Q/G)$ to be a symplectic manifold.

The purpose of this section is to show that when a symmetric DLPS has a Poisson structure, in a sense to be defined below, and the symmetry group acts by Poisson maps, then its reduction also carries a Poisson structure and the reduction morphism is a Poisson map. In principle, these structures could be uninteresting — for instance, the trivial Poisson structure is always a Poisson structure for a DLPS. Still, when a DLPS has an interesting structure, as is the case of those DLPSs obtained from DMSs, the natural Poisson structure arising from the symplectic structure is inherited by all reductions, as we see below.

Definition 8.1. Let $\mathcal{M}=(E,L_d,\mathcal{P})$ be a DLPS. We say that a Poisson structure $\{,\}_{C'(E)}$ on $C'(E)$ is a *Poisson structure of* $\mathcal{M}$ if the flow map $F_{\mathcal{M}}$ is a Poisson map for $\{,\}_{C'(E)}$.

Proposition 8.2. *Let $\{,\}_{C'(E)}$ be a Poisson structure of $\mathcal{M}=(E,L_d,\mathcal{P})$. If G is a symmetry group of $\mathcal{M}$ that preserves $\{,\}_{C'(E)}$ and $\mathcal{A}_d$ is a discrete connection on $\pi^{M,G}: M\to M/G$, then there is a Poisson structure $\{,\}_{C'(\widetilde{G}_E)}$ of the reduced system $\mathcal{M}/(G,\mathcal{A}_d)$ such that the reduction morphism $\Upsilon_{\mathcal{A}_d}$ is a Poisson map, i.e.,*

$$\Upsilon^*_{\mathcal{A}_d}(\{f_1,f_2\}_{C'(\widetilde{G}_E)}) = \{\Upsilon^*_{\mathcal{A}_d}(f_1),\Upsilon^*_{\mathcal{A}_d}(f_2)\}_{C'(E)} \quad \text{for all} \quad f_1,f_2\in C^\infty(\widetilde{G}_E). \tag{8.1}$$

Proof. Being G a symmetry group of $\mathcal{M}$, by Lemma 2.8, $\Upsilon_{\mathcal{A}_d}: C'(E)\to C'(\widetilde{G}_E)$ is a principal G-bundle. Then, the G-action on $C'(E)$ is free and proper and $\Upsilon_{\mathcal{A}_d}$ is a surjective submersion. As G acts on $C'(E)$ by Poisson maps, it follows from Theorem 10.5.1 in [20] that there is a unique Poisson structure $\{,\}_{C'(\widetilde{G}_E)}$ on $C'(\widetilde{G}_E)$ such that $\Upsilon_{\mathcal{A}_d}$ becomes a Poisson map, hence (8.1) holds. By Theorem 5.15, we have the commutative diagram of manifolds and smooth maps:

$$\begin{tikzcd} C'(E) \arrow[r, "F_{\mathcal{M}}"] \arrow[d, "\Upsilon_{\mathcal{A}_d}"'] & C'(E) \arrow[d, "\Upsilon_{\mathcal{A}_d}"] \\ C'(\widetilde{G}_E) \arrow[r, "F_{\mathcal{M}/G}"'] & C'(\widetilde{G}_E) \end{tikzcd}$$

where $F_{\mathcal{M}/G}$ is the flow of the reduced system. As $\Upsilon_{\mathcal{A}_d}$ and $F_{\mathcal{M}}$ are Poisson maps, with $\Upsilon_{\mathcal{A}_d}$ onto, it follows from Lemma 8.3 below that, $F_{\mathcal{M}/G}$ is a Poisson map. All together, we have seen that $\{,\}_{C'(\widetilde{G}_E)}$ is a Poisson structure of $\mathcal{M}/(G,\mathcal{A}_d)$. □

Lemma 8.3. *Let $\phi_1: M\to M_1$ and $\phi_2: M\to M_2$ be Poisson maps and assume that ϕ_1 is onto. If $f: M_1\to M_2$ is a smooth map such that $f\circ\phi_1=\phi_2$, then f is a Poisson map.*

Proof. As $f\circ\phi_1=\phi_2$, a direct computation shows that, for $h_1,h_2\in C^\infty(M_2)$, $\phi_1^*(f^*(\{h_1,h_2\}_{M_2}))=\phi_1^*(\{f^*(h_1),f^*(h_2)\}_{M_1})$. The result follows by noticing that, as ϕ_1 is onto, ϕ_1^* is one to one. □

Recall that a regular DMS (Q, L_d) carries a natural closed 2-form ω_{L_d} that is symplectic in, at least, an open subset of $Q \times Q$ containing the diagonal Δ_Q. We have the following result.

Lemma 8.4. *Let G be a symmetry group of the regular discrete mechanical system (Q, L_d). Then, the diagonal G-action on $Q \times Q$ is symplectic for the symplectic form ω_{L_d}.*

Proof. See the argument at the beginning of page 375 in [17]. □

In particular, a DLPS $\mathcal{M} = (Q, L_d, \mathcal{P})$ that comes from a DMS (Q, L_d) as in Example 3.12, carries a natural Poisson structure $\{,\}_{Q\times Q}$ arising from the symplectic structure ω_{L_d} on $Q \times Q$. It is well known that $F^*_{\mathcal{M}}(\omega_{L_d}) = \omega_{L_d}$ (see Section 1.3.2 in [17]). Hence $F_{\mathcal{M}}$ is a Poisson map and, consequently, $\{,\}_{Q\times Q}$ is a Poisson structure of $\mathcal{M}$.

When G is a symmetry group of (Q, L_d), it is a symmetry group of $\mathcal{M}$ and, by Lemma 8.4, it acts on $C'(Q) = Q \times Q$ by Poisson maps for $\{,\}_{Q\times Q}$. Fixing a discrete connection $\mathcal{A}_d$ on $\pi^{Q,G} : Q \to Q/G$, by Proposition 8.2, the reduced system $\mathcal{M}/(G, \mathcal{A}_d)$ has a natural Poisson structure induced by $\{,\}_{Q\times Q}$ and $\Upsilon_{\mathcal{A}_d}$ is a Poisson map.

We conclude that all DLPSs obtained from a DMS by a finite number of reductions have natural Poisson structures that make the corresponding reduction morphisms Poisson maps.

9. Appendix

The purpose of this Appendix is to review some basic definitions and standard results, using a notation that is compatible with the rest of the paper. Sections 9.1 and 9.2 contain well known material. Section 9.3 contains some nonstandard material.

9.1. **Group actions on manifolds.** A continuous map $f : X \to Y$ between topological spaces is *proper* if $f^{-1}(K)$ is compact for every $K \subset Y$ compact. A G-action l^M of a Lie group G on a manifold M is *proper* if the map $L^M : G \times M \to M \times M$ defined by $L^M(g, m) := (l^M_g(m), m)$ is proper. The following result gives a characterization of properness in terms of sequences that is very convenient in practice.

Proposition 9.1. *Let M be a manifold and G be a Lie group acting on M by l^M. Assume that l^M has the property that for any convergent sequence $(m_j)_{j\in\mathbb{N}}$ in M and sequence $(g_j)_{j\in\mathbb{N}}$ in G such that the sequence $(l^M_{g_j}(m_j))_{j\in\mathbb{N}}$ is convergent, there exists a convergent subsequence of $(g_j)_{j\in\mathbb{N}}$. Then l^M is a proper action. Conversely, if the action l^M is proper, then the property holds.*

Proof. See Proposition 9.13 in [14]. □

Theorem 9.2. *Let l^M be a smooth, free and proper action of the Lie group G on M. Then, the quotient space M/G is a topological manifold of dimension* $\dim(M)-\dim(G)$*. In addition, M/G has a unique smooth structure with the property that the quotient map $\pi^{M,G} : M \to M/G$ is a smooth submersion. Furthermore, $\pi^{M,G} : M \to M/G$ is a principal G-bundle (Definition 9.7).*

Proof. See Theorem 9.16 in [14]. □

Proposition 9.3. *Let G be a Lie group acting smoothly on the manifolds M and N in such a way that $\pi^{M,G} : M \to M/G$ and $\pi^{N,G} : N \to N/G$ are smooth submersions (in particular, M/G and N/G are smooth manifolds). If $f : M \to N$ is a smooth G-equivariant map, then there is a unique smooth map $\check{f} : M/G \to N/G$ such that $\pi^{N,G} \circ f = \check{f} \circ \pi^{M,G}$.*

Proof. An application of the local description of submersions. □

Corollary 9.4. *Let G be a Lie group acting smoothly, freely and properly on the manifolds M and N. If $f : M \to N$ is a smooth G-equivariant map, then there is a unique smooth map $\check{f} : M/G \to N/G$ such that $\pi^{N,G} \circ f = \check{f} \circ \pi^{M,G}$.*

Corollary 9.5. *Let G be a Lie group acting smoothly, freely and properly on the manifold M. If $f : M \to N$ is a smooth G-invariant map, then there is a unique smooth map $\check{f} : M/G \to N$ such that $f = \check{f} \circ \pi^{M,G}$.*

9.2. **Bundles.**

Definition 9.6. A *fiber bundle* is a quadruple (E, M, ϕ, F) where E, M and F are smooth manifolds and $\phi : E \to M$ is a smooth map such that each $m \in M$ has a neighborhood $U \subset M$ and a diffeomorphism $\Phi_U : \phi^{-1}(U) \to U \times F$ that makes the following diagram commutative.

$$\begin{tikzcd} \phi^{-1}(U) \arrow[r, "\Phi_U"] \arrow[d, "\phi"'] & U \times F \arrow[dl, "p_1"] \\ U & \end{tikzcd} \tag{9.1}$$

In this case, E, M and F are called the *total space*, *base space* and *fiber* of the fiber bundle. A pair (U, Φ_U) as above is called a *trivializing chart* of the bundle. It is convenient to denote a fiber bundle (E, M, ϕ, F) by E or ϕ.

If (E, M, ϕ, F) is a fiber bundle, given two of its trivializing charts (U_α, Φ_α) and (U_β, Φ_β) such that $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, we can write $(\Phi_\alpha \circ \Phi_\beta^{-1})(m, f) = (m, \Phi_{\alpha\beta}(m)(f))$ for all $m \in U_{\alpha\beta}$ and $f \in F$, for a smooth map $\Phi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Diff}(F)$ known as a *transition function* of the bundle. The fiber bundle is called a *G-bundle* for a Lie group G if there is a right G-action on F denoted by r^F such that all transition functions are of the form $\Phi_{\alpha\beta}(m) = r^F_{\chi_{\alpha\beta}(m)}$ for a family of smooth functions $\chi_{\alpha\beta} : U_{\alpha\beta} \to G$ that satisfy $\chi_{\beta\gamma}(m)\chi_{\alpha\beta}(m) = \chi_{\alpha\gamma}(m)$ for all $m \in U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

Definition 9.7. Let (E, M, ϕ, G) be a G-bundle such that G acts on the fiber G by right multiplication. Then, E is called a *principal G-bundle* over M.

Theorem 9.8. *Let (E, M, ϕ, G) be a principal G-bundle. Then ϕ is a surjective submersion, G acts freely on the left on E and the G-orbits for this action are of the form $\phi^{-1}(m)$ for $m \in M$. Conversely, if $\phi : E \to M$ is a surjective submersion and the Lie group G acts freely on the left on E in such a way that the G orbits are of the form $\phi^{-1}(m)$ for $m \in M$, then (E, M, ϕ, G) is a principal G-bundle.*

Proof. The first part is direct computation. See Lemma 18.3 in [23] for the converse (in the right action case). □

Remark 9.9. All principal G-bundles are left G-spaces by Theorem 9.8. This follows, eventually, from the fact that our G-spaces have right G-actions on the fibers. The opposite choices are common in most of the fiber bundle literature (see [10]). Our choice is the standard one in Geometric Mechanics (see [5]).

Corollary 9.10. *In the context of Theorem 9.2, $\pi^{M,G} : M \to M/G$ is a principal G-bundle.*

Lemma 9.11. *Let $\psi : M \to R$ be a principal G-bundle. Then the G-action on M is proper. If, furthermore, (E, M, ϕ, F) is a fiber bundle and G acts on E by l^E making ϕ equivariant, then l^E is proper.*

Proof. The first statement follows from Proposition 9.1, using the local triviality of the bundles. The second repeats the same argument building on the properness of the G-action on M. □

Lemma 9.12. *Let G be a Lie group acting smoothly, properly and freely on the manifold Q. Let $H \subset G$ be a closed and normal Lie subgroup. The G-action on Q induces an H-action on Q. Then*

(1) *G/H acts on Q/H by the induced action $l^{Q/H}_{\pi^{G,H}(g)}(\pi^{Q,H}(q)) := \pi^{Q,H}(l^Q_g(q))$.*
(2) *The G/H-action $l^{Q/H}$ is free and proper.*

Definition 9.13. Let (E_j, M_j, ϕ_j, F_j) be fiber bundles for $j = 1, 2$. A *bundle map* from E_1 to E_2 is a pair (Ψ, ψ) of smooth maps $\Psi : E_1 \to E_2$ and $\psi : M_1 \to M_2$ such that the following diagram is commutative.

$$\begin{tikzcd} E_1 \arrow[r, "\Psi"] \arrow[d, "\phi_1"'] & E_2 \arrow[d, "\phi_2"] \\ M_1 \arrow[r, "\psi"'] & M_2 \end{tikzcd}$$

9.3. **Group actions on bundles.** The following definition introduces what we mean by the action of a Lie group on a fiber bundle. We warn the reader that it may not be completely standard.

Definition 9.14. Let G be a Lie group and (E, M, ϕ, F) a fiber bundle. We say that G *acts on the fiber bundle E* if there are free left G-actions l^E and l^M and a right G-action r^F on F such that

(1) l^M induces a principal G-bundle structure $\pi^{M,G} : M \to M/G$,
(2) ϕ is a G-equivariant map for the given actions,
(3) for every $m \in M$ there is a trivializing chart (U, Φ_U) of E such that U is G-invariant, $m \in U$ and, when considering the left G-action $l^{U\times F}$ on $U \times F$ given by $l^{U\times F}_g(m, f) := (l^M_g(m), r^F_{g^{-1}}(f))$, the map Φ_U is G-equivariant.

Example 9.15. Let G act on the fiber bundle (E, M, ϕ, F_1) by the left actions l^E and l^M and the right action r^{F_1} on F_1, and let F_2 be a right G-manifold for the action r^{F_2}. Consider the left G-action $l^{E\times F_2}$ on the fiber bundle $(E \times F_2, M, \phi \circ p_1, F_1 \times F_2)$ defined by $l^{E\times F_2}_g(\epsilon, f_2) := (l^E_g(\epsilon), r^{F_2}_{g^{-1}}(f_2))$. Then G acts on the fiber bundle $E \times F_2$. The only part of the verification that requires some work is the existence of local G-equivariant trivializations. This is done by taking the (Cartesian) product of G-equivariant trivialization of E and the identity mapping on F_2. Using the right G-action $r_g^{F_1\times F_2} := r_g^{F_1} \times r_g^{F_2}$ on $F_1 \times F_2$ makes the resulting mapping G-equivariant, in the sense of point 3 of Definition 9.14.

Proposition 9.16. *Let G be a Lie group that acts on the fiber bundle (E, M, ϕ, F). Then ϕ induces a smooth map $\check{\phi} : E/G \to M/G$ such that $(E/G, M/G, \check{\phi}, F)$ is a fiber bundle.*

Proof. Since the G-actions on E and M are free and proper and ϕ is equivariant, by Theorem 9.2 and Corollary 9.4, we have that E/G and M/G are manifolds, the quotient mappings $\pi^{E,G} : E \to E/G$ and $\pi^{M,G} : M \to M/G$ are smooth submersions and $\check{\phi}$ is smooth.

An outline of the proof of the local triviality of $(E/G, M/G, \check{\phi}, F)$ goes as follows. Since the existence of local trivializations is a local matter, we can assume that $\pi^{M,G} : M \to M/G$ is a trivial G-principal bundle, that is, it is $p_1 : R \times G \to R$ for some manifold R and the G-action on M is $l_g^{R\times G}(r, g') := (r, gg')$. Similarly, we can assume that $\phi : E \to M$ is $p_1 : (R \times G) \times F \to R \times G$ and the G-action on E is $l_g^{(R\times G)\times F}((r, g'), f) := ((r, gg'), r_{g^{-1}}^F(f))$.

Using Corollary 9.4, p_1 induces a map $\check{p_1} : ((R \times G) \times F)/G \to (R \times G)/G = R$. In addition, define $\sigma : (R \times G) \times F \to R \times F$ by $\sigma(r, g, f) := (r, r_g^F(h))$. As σ is smooth and G-invariant, it induces a smooth map $\check{\sigma} : ((R \times G) \times F)/G \to R \times F$. In fact, $\check{\sigma}$ is a diffeomorphism and satisfies $p_1 \circ \check{\sigma} = \check{p_1}$. Thus, we have the following commutative diagram

$$\begin{tikzcd}
E/G = ((R\times G)\times F) \arrow[r, "\check{\sigma}"] \arrow[d, "\check{\phi}=\check{p_1}"'] & R\times F \arrow[dl, "p_1"] \\
M/G = R &
\end{tikzcd}$$

showing the (local) triviality of the bundle $(E/G, M/G, \check{\phi}, F)$, ending the proof. □

Example 9.17. Applying Proposition 9.16 to the setting of Example 9.15 we conclude that if G acts on the fiber bundle (E, M, ϕ, F_1) and F_2 is a right G-manifold, then $((E \times F_2)/G, M/G, \check{\phi \circ p_1}, F_1 \times F_2)$ is a fiber bundle that we call the *associated bundle* and denote by $\widetilde{F_2}_E$. A special case of this construction is the so called *conjugate bundle*, denoted by $\widetilde{G}_E$, that corresponds to the case when $F_2 = G$ and the right action is $r_g^{F_2}(h) := l_{g^{-1}}^G(h) = g^{-1}hg$. For the conjugate bundle, we define $p^{M/G} := \check{\phi \circ p_1}$.

When the Lie group G acts on a manifold Q in such a way that $\pi^{Q,G} : Q \to Q/G$ is a principal G-bundle, $(Q, Q, id_Q, \{pt\})$ is a fiber bundle with a G-action. The conjugate bundle in this case, $\widetilde{G}_Q$ coincides with the conjugate bundle $p^{Q/G} : \widetilde{G} \to Q/G$ considered in Section 3.2 and in [7].

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