A NOTE ON PATH DOMINATION

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Abstract

We study domination between different types of walks connecting two non-adjacent vertices $u$ and $v$ of a graph (shortest paths, induced paths, paths, tolled walks). We succeeded in characterizing those graphs in which every $uv$-walk of one particular kind dominates every $uv$-walk of other specific kind. We thereby obtained new characterizations of standard graph classes like chordal, interval and superfragile graphs.

Keywords: domination, paths, geodesics, chordal graphs, interval graphs.

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1. Introduction

An interval representation of a graph $G$ is a family $(I_w)_{w \in V(G)}$ of intervals of the real line satisfying that two vertices of $G$ are adjacent if and only if the corresponding intervals have nonempty intersection. Graphs admitting an interval representation are called interval graphs [2, 9, 16]. A simple idea arising from the topology of the line is that if $P$ and $P'$ are induced paths between two non-adjacent vertices of an interval graph, then every internal vertex of $P$ is adjacent to some internal vertex of $P'$, and vice versa. This property is not enough to characterize interval graphs, a counterexample is the graph $F_2$ in Figure 1 which is not an interval graph. We wonder if interval graphs can be characterized in terms of domination between paths. In a wider sense, we are interested in understanding the structure of those graphs in which for every pair of non-adjacent vertices $u$ and $v$, and every pair of $uv$-walks $W$ and $W'$, each internal vertex of $W'$ is adjacent to some internal vertex of $W$. 
Inspired by such ideas we studied domination between different types of walks connecting two non-adjacent vertices $u$ and $v$ of a graph $G$, not necessarily interval. We succeeded in characterizing the graphs in which every $uv$-walk of one particular kind inside tolled walks (which are introduced in the present work), paths, induced paths or shortest paths dominates every $uv$-walk of other specific kind. We thereby obtained new characterizations of standard graph classes like chordal, interval and superfragile graphs [2].

In the context of convexity theory, several graph convexity spaces arise when intervals are defined using different types of walks: geodesic convexity, monophonic convexity [14], all-paths convexity [3], triangle-path convexity [4], longest-path convexity [5], and others [10]. As a by-product, we prove that every geodesic interval (monophonic interval) of a graph $G$ is chordal if and only if in $G$ there exists domination between shortest paths (induced paths).

The main results are stated and proved in Section 3. Conclusions and some remarks on related topics that may be motivating for future works are developed in Section 4.

2. Definitions and Basic Results

Let $G$ be a finite, simple and connected graph. The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. We write $N(v)$ for the set of neighbors of the vertex $v$ and $N[v]$ for the closed neighborhood. A clique is a subset of pairwise adjacent vertices. A vertex $v$ is simplicial if $N(v)$ is a clique. The subgraph induced in $G$ by a subset $S \subseteq V(G)$ is denoted by $G[S]$.

A walk in $G$ is a sequence $W : v_1, v_2, \ldots, v_k$ whose terms are vertices of $G$, not necessarily distinct, such that $v_i$ is adjacent to $v_{i+1}$ for $i \in \{1, 2, \ldots, k-1\}$. If $v_1 = u$ and $v_k = v$, we say that $W$ connects $u$ to $v$ and refer to $W$ as an $uv$-walk. The vertices $u$ and $v$ are called the ends of the walk; the vertices $v_2, v_3, \ldots, v_{k-1}$ are its internal vertices. The integer $k - 1$ is the length of the walk. The distance $d(u, v)$ between the vertices $u$ and $v$ is the length of a shortest $uv$-walk.
An *uv-tolled-walk* is an *uv*-walk satisfying that the only internal vertex adjacent to \(v_1\) is \(v_2\) and the only internal vertex adjacent to \(v_k\) is \(v_{k-1}\). An *uv-path* is an *uv*-walk with all its vertices distinct. An *uv-induced-path* (or *chordless path*, or *monophonic path*) is an *uv*-path such that two of its vertices are adjacent if and only if they are consecutive. The chordless path of length \(k\) is denoted by \(P_k\). An *uv-shortest-path* (or *geodesic*) is an *uv*-path of length \(d(u, v)\).

Notice that every shortest-path is an induced-path and every induced-path is a tolled-walk. However, paths and tolled-walks are incomparable: the sequence \(u, 4, 2, 3, v\) of vertices of \(F_2\) in Figure 4 is an *uv*-path which is not an *uv*-tolled-walk. In the same graph, the sequence \(u, 1, 2, w, 2, 3, v\) is an *uv*-tolled-walk that is not an *uv*-path.

The walk \(W : v_1, v_2, \ldots, v_k\) contains the walk \(W' : v'_1, v'_2, \ldots, v'_\ell\) if there exists an strict increasing function \(\Phi : \{1, 2, \ldots, \ell\} \to \{1, 2, \ldots, k\}\) such that \(v'_i = v_{\Phi(i)}\) for \(1 \leq i \leq \ell\). Notice that this is a transitive relation between walks.

It is well known that every *uv*-walk contains some *uv*-path and every *uv*-path contains some *uv*-induced-path [17]. However, not every *uv*-induced-path contains some *uv*-shortest-path.

**Definition.** The *uv*-walk \(W : v_1, v_2, \ldots, v_k\) dominates the *uv*-walk \(W' : v'_1, v'_2, \ldots, v'_\ell\) if every internal vertex of \(W'\) is adjacent to some internal vertex of \(W\) or belongs to \(W\), i.e., for every \(i \in \{2, \ldots, \ell - 1\}\) there exists \(j \in \{2, \ldots, k - 1\}\) such that either \(v'_i\) is adjacent to \(v_j\) or \(v'_i = v_j\).

In order to simplify the statement of the main results in the next section, we introduce the following notation.

\[
W_1(u, v) = \{W : W \text{ is an } \text{uv-shortest-path}\},
\]
\[
W_2(u, v) = \{W : W \text{ is an } \text{uv-induced-path}\},
\]
\[
W_3(u, v) = \{W : W \text{ is an } \text{uv-path}\},
\]
\[ \tilde{W}_3(u, v) = \{ W : W \text{ is an } uv\text{-toll-walk} \}, \]
\[ W_4(u, v) = \{ W : W \text{ is an } uv\text{-walk} \}. \]

The following two remarks summarize the relation between the different types of walks we have considered.

**Remark 1.**
\[ W_1(u, v) \subseteq W_2(u, v) \subseteq W_3(u, v) \subseteq W_4(u, v). \]

**Remark 2.** If \( W \in W_4(u, v) \), then \( W \) contains some \( W' \in W_2(u, v) \).

A cycle of length \( k \) in a graph \( G \) is a path \( C : v_1, v_2, \ldots, v_k \) plus an edge between \( v_1 \) and \( v_k \). The edges \( v_i v_{i+1} \) for \( i \in \{1, 2, \ldots, k-1\} \) and \( v_kv_1 \) are the edges of the cycle; any other edge of \( G \) between two vertices of \( C \) is called a chord. The cycle of length \( k \) without chords is denoted by \( C_k \).

Chordal graphs, defined as those graphs in which every cycle of length greater than three has a chord, have been widely studied and admit different characterizations. As intersection graphs, chordal graphs are described as the graphs admitting a representation by subtrees of a tree. Thus, clearly, every interval graph is chordal. In terms of vertex elimination orders, chordal graphs are seen as those graphs whose vertices can be totally ordered \( v_1, v_2, \ldots, v_n \) in such a way that every \( v_i \) is a simplicial vertex of \( G[\{v_i, v_{i+1}, \ldots, v_n\}] \). See \([2, 9]\) for more on interval graphs, chordal graphs and related classes of graphs.

A distance-hereditary graph is a graph in which every induced path is a geodesic \([6]\). A graph is Ptolemaic if for every four vertices \( v_1, v_2, v_3 \) and \( v_4 \),
\[ d(v_1, v_2) \cdot d(v_3, v_4) \leq d(v_1, v_3) \cdot d(v_2, v_4) + d(v_1, v_4) \cdot d(v_2, v_3). \]

In \([7]\), it was proved that Ptolemaic graphs are exactly the distance-hereditary chordal graphs. The graph \( F_1 \) in Figure 1 is Ptolemaic but it is not interval. The graph \( gem \) in Figure 2 is an interval graph but it is not Ptolemaic.

A graph is superfragile \([2, 15]\) if it has a vertex elimination order with respect to the two rules below, such that at each stage every vertex is eligible for elimination. Recall that \( P_3 \) denotes the induced path with three vertices.

**Rule 1.** If \( v \) does not appear as an end vertex in an induced \( P_3 \), then \( v \) may be removed.

**Rule 2.** If \( v \) does not appear as an internal vertex in an induced \( P_3 \), then \( v \) may be removed.

Chordal, interval, Ptolemaic and superfragile graphs have been characterized by forbidden induced subgraphs.
Theorem 3. A graph is chordal if and only if it does not contain a chordless cycle $C_k$ with $k \geq 4$ as induced subgraph.

Theorem 4 [8]. A graph is interval if and only if it is chordal and it does not contain any one of the graphs $F_1, F_2, F_{3,k}$ or $F_{4,k}$ in Figure 1 as induced subgraphs.

Theorem 5 [7]. A graph is Ptolemaic if and only if it is chordal and it does not contain the graph gem in Figure 2 as induced subgraph.

Theorem 6 [15]. A graph is superfragile if and only if it contains none of the graphs $C_4$, $P_4$ or dart in Figure 2 as induced subgraph.

Denote by Chordal, Interval, and Superfragile to the classes of chordal, interval and superfragile graphs, respectively.

3. Main Results

Let $G$ be any graph and $i, j \in \{1, 2, 3, 4\}$. We say that $G \in W_i/W_j$ if for every pair of non-adjacent vertices $u$ and $v$ of $G$, every $W \in W_i(u, v)$ dominates every $W' \in W_j(u, v)$, i.e.,

$$W \in W_i(u, v) \text{ and } W' \in W_j(u, v) \implies W \text{ dominates } W'.$$

In an analogous way, we define $W_i/W_3$ and $W_3/W_j$.

The aim of the present paper is to describe the graph classes $W_i/W_j$. Our main results are summarized in Table 1.

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<tr>
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<th>$W_1$</th>
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<td>$W_1$</td>
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<td>Ptolemaic</td>
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<td>Superfragile</td>
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<td>Chordal</td>
<td>Ptolemaic</td>
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Table 1. With $W_i$ in the first column and $W_j$ in the first row, the table describe each one of the graph classes $W_i/W_j$ except $W_1/W_3$. Recall that $W_1$: shortest-paths; $W_2$: induced-paths; $W_3$: paths; $W_3$: tolled-walks; $W_4$: walks. The classes Ptolemaic and g-Chordal are defined in 3 and 3. Theorem 15 provides a partial characterization of $W_1/W_3$. The classes $W_1/W_1$ and $W_4/W_3$ are not closed under taking induced subgraphs.
Lemma 7. For every $i, j \in \{1, 2, 3, 4\}$, the following statements hold.

1. $W_i/W_1 \supseteq W_i/W_2 \supseteq W_i/W_3 \supseteq W_i/W_4$.
2. $W_i/W_1 \supseteq W_i/W_2 \supseteq W_i/W_3 \supseteq W_i/W_4$.
3. $\tilde{W}_3/W_1 \supseteq \tilde{W}_3/W_2 \supseteq \tilde{W}_3/W_3 \supseteq \tilde{W}_3/W_4$.
4. $\tilde{W}_3/W_1 \supseteq \tilde{W}_3/W_2 \supseteq \tilde{W}_3/W_3 \supseteq \tilde{W}_3/W_4$.
5. $W_4/W_j = \tilde{W}_3/W_j = W_3/W_j = W_2/W_j \subseteq W_1/W_j$.
6. $W_4/\tilde{W}_3 = \tilde{W}_3/\tilde{W}_3 = W_3/\tilde{W}_3 = W_2/\tilde{W}_3 \subseteq W_1/\tilde{W}_3$.

Proof. Statements 1, 2, 3 and 4 follow in a straightforward way from Remark 1.

Also by Remark 1, we have $W_4/W_j \subseteq W_3/W_j \subseteq W_2/W_j \subseteq W_1/W_j$. And by Remark 2, $W_2/W_j \subseteq W_3/W_j \subseteq W_4/W_j$. Thus, $W_4/W_j = W_3/W_j = W_2/W_j \subseteq W_1/W_j$.

In an analogous way, we have $W_4/W_j = \tilde{W}_3/W_j = W_2/W_j \subseteq W_1/W_j$, which completes the proof of statement 5.

Statement 6 has an identical proof to that of statement 5 replacing $W_j$ by $\tilde{W}_3$.

Notice that Lemma 7 implies that the last four rows of Table 1 must be the same. The following theorems addresses the characterization of the classes in the first two columns.

Theorem 8. $W_2/W_1 = W_2/W_2 = W_1/W_2 = \text{Chordal}$.

Proof. Let $G$ be a chordal graph and assume, in order to derive a contradiction, that $G \not\subset W_2/W_2$. Then there exist two non-adjacent vertices $u$ and $v$, and two $uv$-induced-paths $W : w_1, \ldots, w_m$ and $W' : w'_1, \ldots, w'_\ell$ such that $W$ does not dominate $W'$. It follows that there is some internal vertex $w'_k$ of $W'$, which is neither a vertex of $W$ nor adjacent to an internal vertex of $W$. Denote by $r$ the greatest $i < k$ such that $w'_i$ is adjacent to some vertex of $W$. Notice that $1 \leq r < k$. Denote by $s$ the smallest $i > k$ such that $w'_i$ is adjacent to some vertex of $W$. Observe that $k < s \leq \ell$. We can choose the nearest vertices $w_r'$ and $w_s'$ of $W$ adjacent to $w'_r$ and $w'_s$ respectively (by nearest we mean minimizing $|r' - s'|$).

Notice it could be $r' = s'$. By the concatenation of the subpath of $W'$ between $w'_r$ and $w'_s$ and the subpath of $W$ between $w_s'$ and $w_r'$, we obtain the induced cycle $w'_r, \ldots, w'_k, \ldots, w'_s, w'_s, w'_r, \ldots, w'_r$ with at least four vertices, which contradicts Theorem 3.

On the other hand, if a graph $G$ has an induced cycle $C_k$ with $k \geq 4$, then any two vertices $u$ and $v$ of the cycle at distance 2 determine on the cycle an $uv$-shortest-path and an $uv$-induced-path such that neither of them dominates the other. Thus $W_1/W_2$ and $W_2/W_1$ are contained in $\text{Chordal}$. Lemma 7 completes the proof.
**Definition.** The class of Ptolemaic graphs which contain none of the graphs co-chair or $D$ in Figure 2 as induced subgraph is denoted by $\text{Ptolemaic}^-$. In other words,

$$\text{Ptolemaic}^- = \text{Ptolemaic} \cap \{\text{co-chair, D}\}-\text{free} = \text{Chordal} \cap \{\text{gem, co-chair, D}\}-\text{free}.$$ 

**Theorem 9.** $W_2/W_3 = W_1/W_3 = \text{Ptolemaic}^-$. 

**Proof.** Let $G$ be a chordal graph with no induced subgraph isomorphic to a gem, a co-chair or the graph $D$ in Figure 2. Assume, in order to derive a contradiction, that $G \notin W_2/W_3$. Then there exist two non-adjacent vertices $u$ and $v$, an $uv$-induced-path $W : w_1, \ldots, w_m$ and an $uv$-path $W' : w'_1, \ldots, w'_t$ satisfying that $W$ does not dominate $W'$. Thus, there is some internal vertex $w'_k$ of $W'$ that is neither a vertex of $W$ nor adjacent to an internal vertex of $W$.

Without loss of generality, we can assume that $W'$ is an $uv$-path with minimum length between the ones that are not dominated by $W$. This implies that the subpaths $w'_1, \ldots, w'_k$ and $w'_k, \ldots, w'_t$ of $W'$ are induced-paths. Since $G$ is chordal, by Theorem 8, $W'$ is not an induced-path, thus there is some $w_i$ with $i < k$ which is adjacent to some $w'_j$ with $j > k$; moreover, $w'_{k-1}$ must be adjacent to $w'_{k+1}$.

Notice that $w'_{k-1}$ and $w'_{k+1}$ are not internal vertices of $W$, and the $uv$-path $w'_1, \ldots, w'_{k-1}, w'_{k+1}, \ldots, w'_t$ is dominated by $W$ since it is shorter than $W'$.

We will deal with two cases. First assume $w'_{k-1} \neq u$ and $w'_{k+1} \neq v$. Then both vertices, $w'_{k-1}$ and $w'_{k+1}$, are adjacent to some internal vertex $w_h$ of $W$.

We claim that $w_{h-1}$ is non-adjacent to $w'_k$. Indeed, if it were then $w_{h-1}$ would not be an internal vertex of $W$, thus $w_{h-1} = w_1 = w'_1 = u$, which contradicts the fact that $w'_1, \ldots, w'_k, w'_k$ is an induced-path. Therefore, $w_{h-1}$ must be adjacent to both vertices $w'_{k-1}$ and $w'_{k+1}$ because in other case the vertices $w_{h-1}$, $w_h$, $w'_{k-1}$, $w'_k$ and $w'_{k+1}$ induce a subgraph isomorphic to a gem or to a co-chair.

In an analogous way, we prove that $w_{h+1}$ is non-adjacent to $w'_k$ and adjacent to $w'_{k-1}$ and $w'_{k+1}$.

It follows that the vertices $w_{h-1}$, $w_h$, $w_{h+1}$, $w'_{k-1}$, $w'_k$ and $w'_{k+1}$ induce a subgraph isomorphic to the graph $D$ in Figure 2, in contradiction with the hypothesis.

Now we consider the case $w'_{k-1} = u$ or $w'_{k+1} = v$. By symmetry considerations, it is sufficient to address the case $w'_{k-1} = u$. Observe that $w'_{k-1} = u$ implies $w'_{k+1}$ is adjacent to $u$ and to $w_2$. Since $w'_k$ is adjacent to no interval vertex of $W$ and it is not adjacent to $v$ because, as we said previously, $w'_1, \ldots, w'_t$ is a chordless path, it follows that $w'_k$ is adjacent neither to $w_2$ nor to $w_3$. Therefore, if $w_3$ is non-adjacent to $w'_{k+1}$, then there is an induced co-chair, and if $w_3$ is adjacent to $w'_{k+1}$, then there is an induced gem, both cases contradict our assumptions.

On the other hand, by Lemma 7 and Theorem 8, $W_1/W_3 \subseteq W_1/W_2 =$ Chordal. Moreover, as it is shown in Figure 3, each induced forbidden subgraph
Figure 3. In each graph above, the vertex labelled \( w \) belongs to an \( uv \)-path and it is adjacent to no internal vertex of the bold \( uv \)-shortest-path.

for the class \( \text{Ptolemaic}^- \) (gem, co-chair and \( D \)) has a pair of non-adjacent vertices \( u \) and \( v \), and an \( uv \)-path which is not dominated by an \( uv \)-shortest-path. Thus the class \( W_1/W_3 \) is contained in \( \text{Ptolemaic}^- \). Lemma 7 completes the proof.

**Theorem 10.** \( W_2/W_3 = \text{Interval} \).

**Proof.** Let \( G \) be an interval graph and assume, in order to derive a contradiction, that \( G \not\in W_2/W_3 \). Then there exist two non-adjacent vertices \( u \) and \( v \), an \( uv \)-induced-path \( W : w_1, \ldots, w_m \) and an \( uv \)-toll-walk \( W' : w'_1, \ldots, w'_\ell \) such that \( W \) does not dominate \( W' \). It follows that there is some internal vertex \( w'_k \) of \( W' \), which is neither a vertex of \( W \) nor adjacent to an internal vertex of \( W \).

Let \( I_u = [x_u, y_u] \) and \( I_v = [x_v, y_v] \) with \( x_u < y_u < x_v < y_v \) be the intervals corresponding to vertices \( u \) and \( v \) in a given interval representation of \( G \). It is clear that the segment of line \([y_u, x_v]\) is contained in the union of the intervals corresponding to the internal vertices of \( W \), then we can assume that the interval \( I_{w'_k} \) is contained in \((-\infty, y_u)\). This implies that there is a vertex \( w'_i \) with \( i > k \geq 2 \) adjacent to \( u \), which contradicts the fact that \( W' \) is an \( uv \)-toll-walk.

On the other hand, by Lemma 7 and Theorem 8, \( W_2/W_3 \subseteq W_2/W_2 = \text{Chordal} \). Moreover, as it is shown in Figure 4, each induced forbidden subgraph for the class \( \text{Interval} \) has a pair of non-adjacent vertices \( u \) and \( v \), and an \( uv \)-toll-walk which is not dominated by an \( uv \)-induced-path. Thus the class \( W_2/W_3 \) is contained in \( \text{Interval} \). Lemma 7 completes the proof.

**Theorem 11.** \( W_1/W_4 = W_2/W_4 = \text{Superfragile} \).

**Proof.** Let \( G \) be superfragile and assume, in order to derive a contradiction, that \( G \not\in W_2/W_4 \). Then there exist two non-adjacent vertices \( u \) and \( v \), an \( uv \)-induced-paths \( W : w_1, \ldots, w_m \) and an \( uv \)-walk \( W' : w'_1, \ldots, w'_\ell \) such that \( W \) does not dominate \( W' \). It follows that there is some internal vertex \( w'_k \) of \( W' \), which is neither a vertex of \( W \) nor adjacent to an internal vertex of \( W \).
Figure 4. In each graph above, the vertex labelled \( w \) belongs to an \( uv \)-tolled-walk \( W' \) and it is adjacent to no internal vertex of the bold \( uv \)-induced-path. For \( F_1 \) take \( W' : u, 1, 2, 4, w, 4, 2, 3, v \); for \( F_2 \), \( W' : u, 1, 2, w, 2, 3, v \); for \( F_{3,k} \), \( W' : u, 1, 2, \ldots, k, w, k, y, v \); and for \( F_{4,k} \) take \( W' : u, 1, 2, \ldots, k, w, y, v \).

Since \( G \) has no induced \( P_4 \), we have that \( W \) must be a \( P_3 \) : \( u, w, v \) for some vertex \( w \) with \( w \neq w_k' \) and \( w \) non-adjacent to \( w_k' \), and \( d(w_k', u) \leq 2 \).

If \( d(w_k', u) = 1 \) and \( w_k' \) is adjacent to \( v \), then there is an induced \( C_4 \). If \( d(w_k', u) = 1 \) and \( w_k' \) is non-adjacent to \( v \), then there is an induced \( P_4 \). Both cases contradict Theorem 6.

If \( d(w_k', u) = 2 \), let \( w_k', x, u \) be a shortest path. Notice that \( x \neq w \) and \( x \neq v \). In addition, \( w_k' \) is not adjacent to \( v \) because otherwise there will be an induced \( P_4 : w_k', v, w, u \). Moreover, \( x \) is adjacent to \( w \) because otherwise there will be an induced \( P_4 : w_k', x, u, w \).

Thus, either \( x \) is adjacent to \( v \) and there is an induced subgraph isomorphic to the graph dart in Figure 2, or \( x \) is non-adjacent to \( v \) and there is an induced \( P_4 : w_k', x, u, v \). Both cases contradict again Theorem 6.

On the other hand, it is easy to see that each induced forbidden subgraph for the class Superfragile (\( C_4, P_4 \), dart) in Figure 2 has a pair of non-adjacent vertices \( u \) and \( v \), and an \( uv \)-walk which is not dominated by an \( uv \)-shortest-path. Notice that in the case of \( P_4 : v_1, v_2, v_3, v_4 \), we can considerer \( u = v_1, v = v_3 \) and the \( uv \)-walk \( u = v_1, v_2, v_3, v_4, v_3 = v \) which is not dominated by the \( uv \)-shortest-path \( u = v_1, v_2, v_3 = v \).

It follows that class \( W_1/W_4 \) is contained in Superfragile. Lemma 7 completes the proof.

Unlike the graph classes described by the preceding theorems, we will see that \( W_1/W_1 \) and \( W_1/W_3 \) are not hereditary classes of graphs, i.e., they are not closed under taking induced subgraphs.

Indeed, it is easy to see that every cycle \( C_{2k} \) with \( k > 2 \) plus an universal vertex belongs to \( W_1/W_1 \), but the cycle \( C_{2k} \) does not. Notice that the class of \( C_4 \)-free diameter 2 graphs is contained in \( W_1/W_1 \) and that

**Remark 12.** If \( G \in W_1/W_1 \), then \( G \) is \( C_4 \)-free.
Since the class $W_1/W_1$ is not hereditary, it cannot be characterized by forbidden induced subgraphs. Instead, we present in Theorem 14 a characterization based on geodesic intervals.

The closed geodesic interval $I_g[u,v]$ for two vertices $u$ and $v$ of a graph $G$ is the set of all vertices lying on some $uv$-shortest-path of $G$. Geodesic intervals were studied and characterized by Nebeský [12, 13] and play an important role in the study of metric and convexity properties of graphs [2, 11].

The vertices of $I_g[u,v]$ can be partitioned into level sets $L_i$ for $i \in \{0, 1, \ldots, d(u,v)\}$ according to the distance to $u$ by doing

$$L_i = \{ x \in I_g[u,v] : d(u,x) = i \}.$$

**Lemma 13.** If $L_i$ is a level set of a closed geodesic interval $I_g[u,v]$ of a graph $G \in W_1/W_1$, then $L_i$ is a clique.

**Proof.** First we will prove the proposition for $i = 1$. Assume, in order to obtain a contradiction, that there exist two non-adjacent vertices $v_1$ and $v'_1$ in the level set $L_1$ of $I_g[u,v]$. Let $P : u, v_1, v_2, \ldots, v$ and $P' : u, v'_1, v'_2, \ldots, v$ be two $uv$-shortest-paths. Since $v'_1$ must be adjacent to some internal vertex of $P$, we have that $v'_1$ is adjacent to $v_2$, which implies the existence of the induced $C_4 : u, v_1, v_2, v'_1$ in contradiction with Remark 12.

Now let $i > 1$ and assume, in order to obtain a contradiction, that $v_i$ and $v'_i$ are two non-adjacent vertices in $L_i$. As before, let $P : u, v_1, \ldots, v_i, \ldots, v$ and $P' : u, v'_1, \ldots, v'_i, \ldots, v$ be $uv$-shortest-paths. Since $v'_i$ must be adjacent to some internal vertex of $P$, we have that $v'_i$ is adjacent to $v_{i-1}$ or to $v_{i+1}$; without loosing generality let $v'_i$ be adjacent to $v_{i-1}$. It follows that $v_i$ and $v'_i$ belong to the level set $L_1$ of $I_g[v_{i-1}, v]$, thus $v_i$ and $v'_i$ are adjacent, in contradiction with our assumption. \hfill \blacksquare

**Definition.** We let $g$-Chordal denote the class of graphs $G$ in which any closed geodesic interval induces a chordal subgraph.

Notice that the class of geodetic graphs (for every pair of its vertices there is a unique shortest path between them [2]) is contained in $g$-Chordal.

**Theorem 14.** $W_1/W_1 = g$-Chordal.

**Proof.** Let $u$ and $v$ be vertices of a graph $G \in W_1/W_1$. In order to obtain a contradiction, assume that $k \geq 4$ and $C_k : v_1, v_2, \ldots, v_k$ is an induced cycle in the closed geodesic interval $I_g[u,v]$. For every $j \in \{1, \ldots, k\}$, let $L_{j_i}$ be the level set of $I_g[u,v]$ containing $v_i$. We can clearly assume that $j_1 \leq j_i$ for $2 \leq i \leq k$ and $j_2 = j_1 + 1$. By Lemma 13, two vertices of $C_k$ in a same level are adjacent; thus $j_k \neq j_2$. This implies $j_k = j_1$ and $j_{k-1} = j_2$; so $v_{k-1}$ and $v_2$ are adjacent. Therefore, $k = 4$ in contradiction with Remark 12.
Now let $P$ and $P'$ be $uv$-shortest-paths with $u$ and $v$ two non-adjacent vertices of a graph $G \in g$-Chordal. We have to prove that $P$ dominates $P'$. By definition of $g$-Chordal, the subgraph $G[I_g[u,v]]$ induced by $I_g[u,v]$ is chordal. Since $P$ and $P'$ are induced paths in $G[I_g[u,v]]$, by Theorem 8, $P$ dominates $P'$.

The rest of the paper is devoted to the study of the more intricate class $W_1/W_3$. Observe that the graph $F_1$ in Figure 4 does not belong to $W_1/W_3$ (the bold $uv$-induced-path is also an $uv$-shortest-path). However, $F_1$ plus an universal vertex (i.e., a vertex adjacent to every vertex of $F_1$) belongs to $W_1/W_3$ since it is $C_4$-free and has diameter 2. Consequently $W_1/W_3$ is not hereditary. Theorem 15 presents a partial characterization of the graphs in this class.

**Definition.** A graph $G$ belongs to the class $\text{Interval}^+$ if $G$ is chordal, contains none of the graphs $F_2$ or $F_{4,k}$ in Figure 1 as induced subgraph and satisfies the following condition.

If $G$ has an induced subgraph $H$ isomorphic to $F_1$ ($F_{3,k}$), then the distance in $G$ between the vertices of $F_1$ ($F_{3,k}$) labelled $u$ and $v$ in Figure 4 is 2, and any vertex of $G$ adjacent to both $u$ and $v$ is universal to $F_1$ ($F_{3,k}$).

Notice that every interval graph belongs to $\text{Interval}^+$.

**Theorem 15.** $W_1/W_3 \subseteq \text{Interval}^+$.

**Proof.** Let $G \in W_1/W_3$. By Lemma 7 and Theorem 8, $W_1/W_3 \subseteq W_1/W_2 = \text{Chordal}$, therefore $G$ is chordal.

Assume, in order to obtain a contradiction, that $G$ has an induced subgraph isomorphic to the graph $F_2$ in Figure 1. In Figure 4, we show two non-adjacent vertices $u$ and $v$ of $F_2$ and an $uv$-induced-path $W$ which does not dominate an $uv$-toggled-walk $W'$. Since the length of $W$ is 2 and $u$ and $v$ are non-adjacent, we have that such $W$ is also an $uv$-shortest-path in $G$. It contradicts the fact that every shortest-path dominates every toggled-walk.

In analogous way it can be proved that $G$ has no induced subgraph isomorphic to the graph $F_{4,k}$.

Notice that using the same argument we can prove that if $G$ has an induced subgraph isomorphic to $F_{3,k}$, then the distance in $G$ between the vertices of $F_{3,k}$ labelled $u$ and $v$ in Figure 4 cannot be 3, then it must be 2. Let $x$ be the internal vertex of some $uv$-shortest-path; clearly $x$ is not a vertex of $F_{3,k}$ and is adjacent to $u$ and $v$. Since $G \in W_1/W_3$ and $W' : u, 1, 2, \ldots, k, w, k, y, v$ is an $uv$-toggled-walk (see Figure 4), it follows that $x$ is adjacent to every vertex of $F_1$.

A reasoning analogous to the one applied in the case of $F_{3,k}$ shows that if $G$ has an induced subgraph isomorphic to $F_1$, then the distance in $G$ between
the vertices of $F_1$ labelled $u$ and $v$ in Figure 4 is at most 3, and also resolves the case $d(u, v) = 2$. We claim that $d(u, v) = 3$ leads to a contradiction. Indeed, assume that $u, x, z, v$ is an $uv$-shortest-path. Notice that neither $x$ nor $z$ may be the vertex of $T_1$ labelled $w$ in Figure 4. In addition, since $w$ is an internal vertex of an $uv$-tolled-walk, $x$ or $z$ must be adjacent to $w$. Without loss of generality, let $x$ be adjacent to $w$. Thus $u, x, w$ is an $uw$-shortest-path. Since $G \in W_1/W_3$, and $u, 1, 2, 3, v, 3, 2, 4, w$ is an $uw$-tolled-walk, we have that $x$ must be adjacent to $v$, which contradicts that $u, x, z, v$ is an $uv$-shortest-path.

\section{Conclusions}

We have obtained characterization of the graphs in which, for every pair of non-adjacent vertices $u$ and $v$, every $uv$-walk, tolled-walk, path, induced-path or shortest-path dominates every $uv$-walk, tolled-walk, path, induced-path or shortest-path, with the exception of those in which every $uv$-shortest-path dominates every $uv$-tolled-walk. We let open the problem of determining if such graphs are exactly the ones in $\text{Interval}^+$.  

\textbf{Conjecture 16.} $\text{Interval}^+ \subseteq W_1/W_3$.

We have proved that the classes $W_1/W_1$ and $W_1/W_3$ are not hereditary (closed under taking induced subgraphs), but $W_1/W_2$, $W_1/W_3$ and $W_1/W_4$ are.

Regarding to convexity theory, we propose the study of the convexity space obtained by considering tolled-walk intervals.

We have proved that $W_1/W_1$ is the class of graphs in which every geodesic interval is chordal, while $W_1/W_2$ is the class of graphs in which every monophonic interval is chordal, we wonder what other graph classes can be characterized using this approach.

Finally, we observe the following property of $g$-\textbf{Chordal}. According to [11], the interval function of a graph $G$ is the mapping $f : V(G) \times V(G) \rightarrow 2^{V(G)}$ given by $f(u, v) = I_g[u, v]$ (the closed geodesic interval). It is clear that any hereditary class of graphs is closed under the interval function, in the sense that the subgraph induced by $f(u, v)$ also belongs to the class. However, this is not necessarily true for non-hereditary graph classes. We observe that the class $g$-\textbf{Chordal} is closed for the interval function. Other non-hereditary graph class for which this property holds is the class of median graphs [2, 10, 11].

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**Note added in proof:** Reference [2] in [1] is the present paper; the notion of tolled-walk introduced in the current work was used there to develop the toll convexity.

**References**


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