

# Exact partition function in $U(2) \times U(2)$ ABJM theory deformed by mass and Fayet-Iliopoulos terms

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**ABSTRACT:** We exactly compute the partition function for  $U(2)_k \times U(2)_{-k}$  ABJM theory on  $S^3$  deformed by mass  $m$  and Fayet-Iliopoulos parameter  $\zeta$ . For  $k = 1, 2$ , the partition function has an infinite number of Lee-Yang zeros. For general  $k$ , in the decompactification limit the theory exhibits a quantum (first-order) phase transition at  $m = 2\zeta$ .

**KEYWORDS:** Matrix Models, Supersymmetric gauge theory, AdS-CFT Correspondence, Chern-Simons Theories

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## 1 Introduction

The dynamics of two coincident M2 branes on the orbifold  $\mathbb{R}^8/\mathbb{Z}_k$  is described by ABJM theory, three-dimensional  $U(2)_k \times U(2)_{-k}$  supersymmetric Chern-Simons theory with bi-fundamental matter [1]. For this particular gauge group, the ABJM theory has  $\mathcal{N} = 8$  superconformal symmetry and is in fact equivalent to Gustavsson-Bagger-Lambert theory [2, 3]. The partition function for the theory on  $\mathbb{S}^3$  can be computed by supersymmetric localization [4, 5]. This theory can be deformed, preserving  $\mathcal{N} = 4$  supersymmetry, by adding mass and Fayet-Iliopoulos (FI) parameters  $m, \zeta$ , and the localization technique then reduces the full supersymmetric functional integral to the matrix integral [5]

$$Z = \frac{1}{4} \int \frac{d^2\mu}{(2\pi)^2} \frac{d^2\nu}{(2\pi)^2} \frac{\sinh^2 \frac{\mu_1 - \mu_2}{2} \sinh^2 \frac{\nu_1 - \nu_2}{2}}{\prod_{i,j} \cosh \left( \frac{\mu_i - \nu_j + m}{2} \right) \cosh \left( \frac{\mu_i - \nu_j - m}{2} \right)} e^{\frac{ik}{4\pi} \sum_i (\mu_i^2 - \nu_i^2) - \frac{ik}{2\pi} \zeta \sum_i (\mu_i + \nu_i)} \quad (1.1)$$

where  $i, j = 1, 2$ . The parameter  $\zeta$  represents a Fayet-Iliopoulos parameter for the diagonal  $U(1)$  subgroup, whereas  $m$  corresponds to a mass for the chiral multiplets. The partition function should be understood as a function  $Z(2\zeta, m; k)$ , but for ease of presentation we will omit its arguments unless needed. For  $k = 1$ , the theory is mirror dual to  $\mathcal{N} = 4$  supersymmetric super Yang-Mills theory with gauge group  $U(2)$  coupled to a single fundamental hypermultiplet and a single adjoint hypermultiplet [5].

By shifting the integration variables,  $x \equiv \mu - \zeta$ ,  $y \equiv \nu + \zeta$ , the partition function becomes

$$Z = \frac{1}{4} \int \frac{d^2x}{(2\pi)^2} \frac{d^2y}{(2\pi)^2} \frac{\sinh^2 \frac{x_1 - x_2}{2} \sinh^2 \frac{y_1 - y_2}{2}}{\prod_{i,j} \cosh \frac{x_i - y_j + m_1}{2} \cosh \frac{x_i - y_j - m_2}{2}} e^{\frac{ik}{4\pi} \sum_i (x_i^2 - y_i^2)}, \quad (1.2)$$

where  $m_1, m_2$  are

$$m_1 = m + 2\zeta \quad \text{and} \quad m_2 = m - 2\zeta. \quad (1.3)$$

Note that  $\zeta$  has dimension of mass. We are using units where the radius  $R$  of the three-sphere is  $R = 1$ .

The purpose of this note is to explicitly carry out the integration in (1.2). In the  $m = \zeta = 0$  case, the integral was computed in [6] (a discussion of the partition function in the more general ABJ case can be found in [7]). On the other hand, the  $m, \zeta$ -deformed ABJM theory was studied in [8] using the Fermi-gas formulation [9] and at large  $N$  for the  $U(N)_k \times U(N)_{-k}$  gauge group in [10] (with  $\zeta = 0$ ) and in [11] (with general  $m, \zeta \neq 0$ ), where phase transitions in the complex parameter space generated by  $m_1, m_2$  and  $g = 2\pi i/k$  were investigated. Our explicit formula will uncover some interesting physical properties of the mass-deformed system with gauge group  $U(2)_k \times U(2)_{-k}$ .

The partition function (1.2) manifests the  $m_1 \leftrightarrow m_2$  symmetry or, equivalently,  $\zeta \rightarrow -\zeta$ . A less obvious symmetry is  $m_2 \rightarrow -m_2$ , or [8, 11]

$$Z(2\zeta, m; k) = Z(m, 2\zeta; k). \tag{1.4}$$

For the  $k = 1$  case, this symmetry already appeared in [5], where it was also explained by the fact that the corresponding brane configuration is self-mirror. The symmetry implies, in particular, that a FI-deformation  $\zeta$  on the massless theory is equivalent to a mass-deformation  $m = 2\zeta$  in the theory with vanishing FI-parameter. The case  $m = 2\zeta$  — representing a fixed point of this symmetry — is special, as we shall shortly see. In the dual  $\mathcal{N} = 4$  supersymmetric super Yang-Mills theory,  $m_2 = 0$  corresponds to coupling the theory to a massless adjoint hypermultiplet.

## 2 Residue integration

The partition function for the  $m, \zeta$ -deformed ABJM theory with  $U(N)_k \times U(N)_{-k}$  gauge group can be written in the following form [5, 11]

$$Z(2\zeta, m; k) = \sum_{\rho} (-1)^{\rho} \frac{1}{N!} \int d^N \tau \frac{e^{-ikm_2 \sum_i \tau_i}}{\prod_i \cosh(k\pi\tau_i) \cosh(\pi(\tau_i - \tau_{\rho(i)}) - \frac{m_1}{2})}, \tag{2.1}$$

where the sum goes over permutations. The derivation uses a trigonometric identity, Fourier integrations and only holds for opposite Chern-Simons levels (see section 2 in [11] for details). For  $N = 2$ , the formula (2.1) then leads to the following expression

$$Z = \frac{1}{2}(Z_1 - Z_2), \tag{2.2}$$

with

$$Z_1 = \int d\tau_1 d\tau_2 \frac{e^{-ikm_2(\tau_1 + \tau_2)}}{\cosh(\pi k\tau_1) \cosh(\pi k\tau_2) \cosh^2(\frac{m_1}{2})}, \tag{2.3}$$

and

$$Z_2 = \int d\tau_1 d\tau_2 \frac{e^{-ikm_2(\tau_1 + \tau_2)}}{\cosh(\pi k\tau_1) \cosh(\pi k\tau_2) \cosh(\pi(\tau_1 - \tau_2) - \frac{m_1}{2}) \cosh(\pi(\tau_1 - \tau_2) + \frac{m_1}{2})}, \tag{2.4}$$

Using the identity

$$\frac{1}{\cosh^2 \frac{m_1}{2}} - \frac{1}{\cosh(\pi\tau - \frac{m_1}{2}) \cosh(\pi\tau + \frac{m_1}{2})} = \frac{\operatorname{sech}^2 \frac{m_1}{2} \sinh^2 \pi\tau}{\cosh(\pi\tau - \frac{m_1}{2}) \cosh(\pi\tau + \frac{m_1}{2})} \tag{2.5}$$

and the formula for the Fourier transform [11]

$$\int du \frac{e^{-ikm_2u}}{\cosh\left(\frac{\pi k}{2}(u+v)\right)\cosh\left(\frac{\pi k}{2}(u-v)\right)} = \frac{4\sin(km_2v)}{k\sinh(\pi kv)\sinh m_2}, \quad (2.6)$$

we obtain

$$Z = \frac{1}{k^2\sinh(m_2)\cosh^2\frac{m_1}{2}} \int du \frac{\sin(m_2u)\sinh^2\frac{\pi u}{k}}{\sinh(\pi u)\cosh\left(\frac{\pi u}{k}-\frac{m_1}{2}\right)\cosh\left(\frac{\pi u}{k}+\frac{m_1}{2}\right)}. \quad (2.7)$$

In the limit  $m_2 \rightarrow 0$ , the partition function becomes

$$Z|_{m_2=0} = \frac{1}{k^2\cosh^2\frac{m_1}{2}} \int du \frac{u\sinh^2\frac{\pi u}{k}}{\sinh(\pi u)\cosh\left(\frac{\pi u}{k}-\frac{m_1}{2}\right)\cosh\left(\frac{\pi u}{k}+\frac{m_1}{2}\right)}. \quad (2.8)$$

In the following, we compute the integrals (2.7), (2.8) by residue integration.

To compute (2.7) we follow the ideas in [6], where the partition function was computed in the case  $m = \zeta = 0$ .

Thus we start by writing the integrand as the product of two even functions  $f, g$

$$Z = \frac{1}{k^2\sinh(m_2)\cosh^2\frac{m_1}{2}} \int du f(u)g(u), \quad (2.9)$$

with

$$f(u) = \frac{\sin m_2u}{\sinh \pi u}, \quad g(u) = \frac{\sinh^2\frac{\pi u}{k}}{\cosh\left(\frac{\pi u}{k}-\frac{m_1}{2}\right)\cosh\left(\frac{\pi u}{k}+\frac{m_1}{2}\right)}. \quad (2.10)$$

Under the shift  $u \rightarrow u + ik$  these functions transform as

$$\begin{aligned} f(u) &\rightarrow (-)^k \cosh(m_2k)f(u) + \text{odd function}, \\ g(u) &\rightarrow g(u) \end{aligned} \quad (2.11)$$

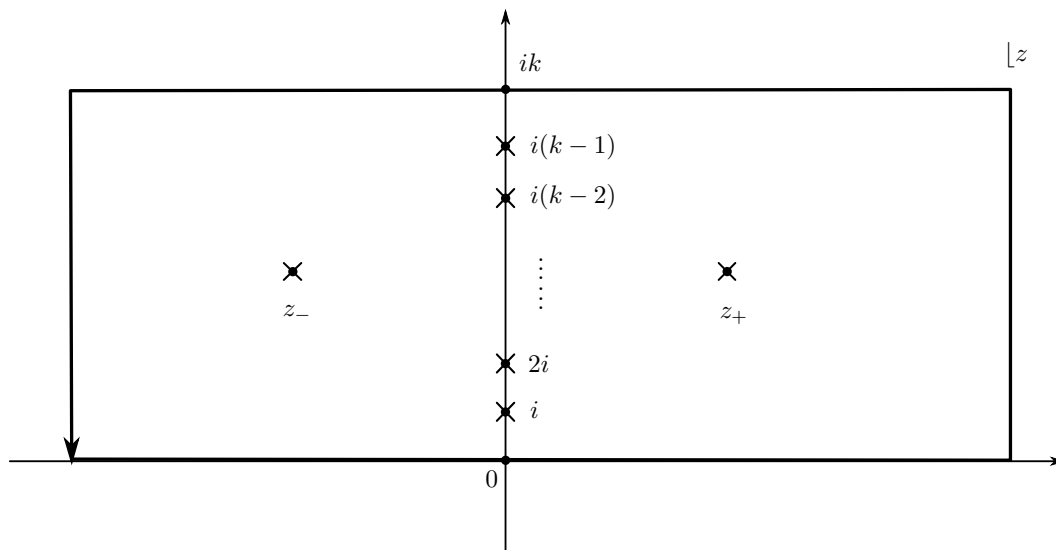
These properties imply that the integral in (2.9) along the curve  $u = x + ik$  with  $x \in \mathbb{R}$  will differ from the integration along the real axis by the factor  $(-)^k \cosh(m_2k)$ . Therefore, the rectangular contour composed by the real axis, two vertical segments and the displaced real axis  $u = x + ik$  becomes appropriate for residue computation in the case  $m_2 \neq 0$  (see figure 1).<sup>1</sup>

The residues encircled by the contour comprise the ones arising from the poles of  $f(z)$  located at  $z = in$  with  $n = 1, \dots, k$  and those of  $g(z)$  located at  $z_{\pm} = \pm \frac{m_1k}{2\pi} + i\frac{k}{2}$ . The pole located at  $z = ik$  does not contribute due to a double zero in the numerator of  $g(z)$ . Calling  $C$  the closed rectangular contour described above and  $\mathcal{F}(z) = f(z)g(z)$  one finds

$$\begin{aligned} \oint_C dz \mathcal{F}(z) &= (1 - (-)^k \cosh(m_2k)) \int du \mathcal{F}(u) \\ &= 2\pi i \left[ \sum_{n=1}^{k-1} \text{Res}_{z=in} \mathcal{F}(z) + \text{Res}_{z=z_{\pm}} \mathcal{F}(z) \right] \end{aligned}$$

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<sup>1</sup>It is easily seen that the vertical contours do not contribute when we push them to infinity.



**Figure 1.** Rectangular contour for residue computation. The poles on the imaginary axis  $z = in$  with  $n = 1, \dots, k - 1$  arise from the  $f$  function, while those at  $z_{\pm} = \pm \frac{m_1 k}{2\pi} + i \frac{k}{2}$  follow from the  $g$  function.

which gives

$$\int du \mathcal{F}(u) = \frac{2\pi i}{1 - (-)^k \cosh(m_2 k)} \left[ -\frac{i}{\pi} \sum_{n=1}^{k-1} (-)^n \frac{\sin^2\left(\frac{n\pi}{k}\right) \sinh(m_2 n)}{\cosh\left(\frac{m_1}{2} - \frac{in\pi}{k}\right) \cosh\left(\frac{m_1}{2} + \frac{in\pi}{k}\right)} + R_k \right] \tag{2.12}$$

where

$$R_k = \begin{cases} (-)^{\frac{k}{2}} \frac{ik}{\pi} \frac{\coth \frac{m_1}{2} \sinh \frac{km_2}{2}}{\sinh \frac{km_1}{2}} \cos \frac{km_1 m_2}{2\pi}, & k \text{ even} \\ (-)^{\frac{k+1}{2}} \frac{ik}{\pi} \frac{\coth \frac{m_1}{2} \cosh \frac{km_2}{2}}{\cosh \frac{km_1}{2}} \sin \frac{km_1 m_2}{2\pi}, & k \text{ odd} \end{cases} \tag{2.13}$$

*Case  $m_2 = 0, k$  odd.* It is evident from (2.12) that the  $m_2 \rightarrow 0$  limit of (2.9) is smooth, the result is

$$Z|_{m_2=0} = \frac{1}{k^2 \cosh^2 m} \left[ \sum_{n=1}^{k-1} (-)^n \frac{n \sin^2\left(\frac{n\pi}{k}\right)}{\cosh\left(m - \frac{in\pi}{k}\right) \cosh\left(m + \frac{in\pi}{k}\right)} - (-)^{\frac{k+1}{2}} \frac{k^2 m \coth m}{\pi \cosh km} \right], \quad k \text{ odd} \tag{2.14}$$

where we have used  $m_1 = 2m$ .

*Case  $m_2 = 0, k$  even.* The factor multiplying the bracket in (2.12) prevents taking  $m_2 \rightarrow 0$  in the even  $k$  case. To compute the integral in (2.8) we consider

$$I = \int du \tilde{f}(u) g(u), \tag{2.15}$$

with  $g(u)$  as in (2.10) and

$$\tilde{f}(u) = \frac{i}{k} \frac{(u - ik/2)^2}{\sinh \pi u}.$$

Upon integration, the odd piece in  $\tilde{f}$  vanishes against  $g(u)$  and therefore the partition function (2.8) can be written as

$$Z|_{m_2=0} = \frac{1}{k^2 \cosh^2 m} I \tag{2.16}$$

The shift  $u \rightarrow u + ik$  in  $\tilde{f}(u)$  gives

$$\tilde{f}(u) \rightarrow (-)^{k+1} \tilde{f}(-u).$$

As discussed below (2.11), this property makes the rectangular contour in figure 1 appropriate for computing  $I$  by residues.

For the residues analysis we should now consider the pole in  $\tilde{f}(z)$  at the origin  $z = 0$  but a zero in  $g(z)$  eliminates it; along the same lines the residue from  $z = ik/2$  is absent since a zero appears for  $\tilde{f}$ . Calling  $\tilde{\mathcal{F}}(z) = \tilde{f}(z)g(z)$  one finds

$$\oint_C dz \tilde{\mathcal{F}}(z) = 2I,$$

on the other hand

$$\begin{aligned} \oint_C dz \tilde{\mathcal{F}}(z) &= 2\pi i \left[ \sum_{n=0}^{k-1} \text{Res}_{z=in} \tilde{\mathcal{F}}(z) + \text{Res}_{z=z_{\pm}} \tilde{\mathcal{F}}(z) \right] \\ &= 2\pi i \left[ \frac{i}{k\pi} \sum_{n=1}^{k-1} (-)^n \left( \frac{k}{2} - n \right)^2 \frac{\sin^2 \left( \frac{n\pi}{k} \right)}{\cosh \left( m - \frac{in\pi}{k} \right) \cosh \left( m + \frac{in\pi}{k} \right)} + \tilde{\mathbf{R}}_k \right]. \end{aligned} \tag{2.17}$$

where

$$\tilde{\mathbf{R}}_k = (-)^{\frac{k}{2}} \frac{2i(mk)^2 \coth(m) \sinh mk}{\pi^3 \cosh(2mk) - 1}$$

The  $n = \frac{k}{2}$  term in the sum vanishes as expected. The final result is

$$\begin{aligned} Z|_{m_2=0} &= -\frac{1}{k \cosh^2 m} \\ &\quad \left[ \sum_{n=1}^{k-1} (-)^n \left( \frac{n}{k} - \frac{1}{2} \right)^2 \frac{\sin^2 \left( \frac{n\pi}{k} \right)}{\cosh \left( m - \frac{in\pi}{k} \right) \cosh \left( m + \frac{in\pi}{k} \right)} + (-)^{\frac{k}{2}} \frac{2m^2 k \coth(m) \sinh mk}{\pi^2 \cosh(2mk) - 1} \right] \end{aligned} \tag{2.18}$$

### 3 Summary of results and limits

Thus we have obtained

$$Z = \frac{2}{k^2 \sinh(m_2)} \frac{1}{1 - (-1)^k \cosh(m_2 k)} (J_1 - J_2) \tag{3.1}$$

where

$$J_1 = \frac{1}{\cosh^2 \left( \frac{m_1}{2} \right)} \sum_{n=1}^{k-1} (-1)^n \frac{\sin^2 \left( \frac{n\pi}{k} \right) \sinh(m_2 n)}{\cosh \left( \frac{m_1}{2} - \frac{in\pi}{k} \right) \cosh \left( \frac{m_1}{2} + \frac{in\pi}{k} \right)} \tag{3.2}$$

and

$$J_2 = \begin{cases} (-)^{\frac{k}{2}} \frac{2k \sinh \frac{km_2}{2}}{\sinh(m_1) \sinh \frac{km_1}{2}} \cos \frac{km_1 m_2}{2\pi}, & k \text{ even} \\ (-)^{\frac{k+1}{2}} \frac{2k \cosh \frac{km_2}{2}}{\sinh(m_1) \cosh \frac{km_1}{2}} \sin \frac{km_1 m_2}{2\pi}, & k \text{ odd} \end{cases} \quad (3.3)$$

Using

$$\frac{2}{1 + \cosh \alpha} = \frac{1}{\cosh^2 \left(\frac{\alpha}{2}\right)}, \quad \frac{2}{1 - \cosh \alpha} = -\frac{1}{\sinh^2 \left(\frac{\alpha}{2}\right)}, \quad (3.4)$$

we can finally put the partition function in the form

$$Z|_{k \text{ even}} = -\frac{1}{k^2 \sinh(m_2) \sinh^2 \left(\frac{km_2}{2}\right)} (J_1 - J_2) \quad (3.5)$$

$$Z|_{k \text{ odd}} = \frac{1}{k^2 \sinh(m_2) \cosh^2 \left(\frac{km_2}{2}\right)} (J_1 - J_2) \quad (3.6)$$

In the formulas (3.5)–(3.6), the symmetry  $m_1 \leftrightarrow m_2$  — which is manifest in the integral form (1.2) — is hidden. Interestingly, this symmetry is only recovered upon summation over  $n$ . On the other hand, the symmetry  $m_2 \rightarrow -m_2$  is manifest.

Note that  $Z$  is real. While this is expected in a unitary theory, it is not generally the case in Chern-Simons theories (for a discussion, see [12]). In the present case, it is related to the fact the theory is a combination of two Chern-Simons theory with opposite levels.<sup>2</sup>

Consider, as particular examples, the important cases  $k = 1, 2$ . The partition functions take the form

$$Z|_{k=1} = \frac{2}{\sinh(m_1) \sinh(m_2) \cosh \left(\frac{m_1}{2}\right) \cosh \left(\frac{m_2}{2}\right)} \sin \left(\frac{m_1 m_2}{2\pi}\right), \quad (3.7)$$

$$Z|_{k=2} = \frac{2}{\sinh^2(m_1) \sinh^2(m_2)} \sin^2 \left(\frac{m_1 m_2}{2\pi}\right). \quad (3.8)$$

Now the symmetry  $m_1 \leftrightarrow m_2$  has become manifest.

Note that the partition functions for  $k = 1, 2$  have zeros. Restoring the  $R$  dependence, the zeros are located at

$$m_1 m_2 R^2 = 2\pi^2 n, \quad n = \pm 1, \pm 2, \dots \quad (3.9)$$

They represent Lee-Yang zeros (see, for example, [13]). In the infinite volume,  $R \rightarrow \infty$ , the zeros condense in a certain line, and a phase transition should emerge. The fact that the partition function has zeros seems to be related to the fact that the coupling,  $g = 2\pi i/k$ , is imaginary for real  $k$ . Indeed, from the general expressions (3.2)–(3.3) we see that the arguments of the sine and cosine functions in (3.7), (3.8) contain a factor  $\pi/k$ . If the coupling  $g$  is (unphysically) continued to the real line by taking  $k \rightarrow ik$ , the partition function zeros would then lie on the imaginary  $g$ -axis, in accordance with the Lee-Yang theorem (see [11] for a related discussion).

For the undeformed ABJM theory, the  $k = 1$  case is of special interest, since it is conjectured to describe the dynamics of two M2 branes in eleven-dimensional Minkowski

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<sup>2</sup>We thank Miguel Tierz for comments on this point.

spacetime. An interesting question is what is the origin of these Lee-Yang singularities in the brane realization.

The partition function  $Z(2\zeta, m; k)$  does not have any zeros for  $k > 2$ . For higher values of  $k$ , the partition function becomes more involved, below we quote explicitly the  $k = 3$  and  $k = 4$  cases

$$Z|_{k=3} = \frac{2}{3} \frac{2 - \sin\left(\frac{3m_1m_2}{2\pi}\right) \operatorname{csch}\left(\frac{m_1}{2}\right) \operatorname{csch}\left(\frac{m_2}{2}\right)}{(\cosh m_1 + \cosh 2m_1)(\cosh m_2 + \cosh 2m_2)} \quad (3.10)$$

$$Z|_{k=4} = \frac{1 - \operatorname{sech}(m_1) - \operatorname{sech}(m_2) + \cos\left(\frac{2m_1m_2}{\pi}\right) \operatorname{sech}(m_2) \operatorname{sech}(m_1)}{8 \sinh^2 m_1 \sinh^2 m_2} \quad (3.11)$$

Note that the symmetry under the exchange  $m_1 \leftrightarrow m_2$  is manifest.

**Asymptotic formulas.** Let us consider the limit of a large sphere,  $mR \gg 1$ , at fixed  $k$ . Assuming  $m_1 > 0$ ,  $m_2 > 0$  and restoring the  $R$  dependence, we find

$$Z|_{k=1} \sim 32 e^{-\frac{3}{2}(m_1+m_2)R} \sin\left(\frac{m_1m_2R^2}{2\pi}\right), \quad (3.12)$$

$$Z|_{k=2} \sim 32 e^{-2(m_1+m_2)R} \sin^2\left(\frac{m_1m_2R^2}{2\pi}\right), \quad (3.13)$$

$$Z|_{k>2} \sim \frac{64}{k^2} e^{-2(m_1+m_2)R} \sin^2\left(\frac{\pi}{k}\right). \quad (3.14)$$

The general asymptotic formula with arbitrary sign for  $m_2$  and  $m_2 \neq 0$ , is obtained by replacing  $m_2$  by  $|m_2|$ .

The absolute value implies a discontinuity in the first derivative of  $F = -\ln Z$ . This indicates a first-order phase transition in the parameter  $m_2$  at  $m_2 = 0$ , i.e., when the two mass scales  $m, 2\zeta$  cross. Explicitly, at large  $R$ , we have

$$F = 2(|m_1| + |m_2|)R + O(1), \quad k > 1. \quad (3.15)$$

Hence

$$\left. \frac{d\Delta F}{dm_2} \right|_{m_2=0} = 4R, \quad \Delta F \equiv F_{m_2>0} - F_{m_2<0}. \quad (3.16)$$

For  $k = 1$  the discontinuity in the first derivative of  $\Delta F$  is equal to  $3R$ , as can be seen from (3.12).

For the general theory with gauge group  $U(N)_k \times U(N)_{-k}$ , large  $N$  phase transitions in the complex parameter  $Ng = 2\pi iN/k$  were studied in [10, 11]. These phase transitions require taking infinite volume and, at the same time, a strong coupling limit with fixed  $kR$  — a limit that already appeared in the context of supersymmetric  $U(N)$  Chern-Simons theory with massive fundamental matter in [14, 15]. It should be noted that such decompactification limit is different from the present (more physical) limit of large  $R$  at fixed  $k$ .

Another interesting aspect of (3.14) is that it is in a form suitable for a weak coupling expansion in powers of  $1/k$ :

$$Z|_{k>2} \sim -\frac{32}{k^2} e^{-2(m_1+m_2)R} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{2\pi}{k}\right)^{2n}. \quad (3.17)$$



The perturbative expansion has an infinite radius of convergence. However, the original theory on the three-sphere of *finite* radius  $R$  has an asymptotic perturbative expansion, with  $2n!$  asymptotic behavior for the  $1/k^{2n}$  term. This can be seen by using the integral form (2.7) and generalizing the study of [16, 17] on the resurgence properties of the perturbation series of ABJM theory. Now, expanding the integrand in (2.7), one finds a series with finite radius of convergence determined by the poles of  $\text{sech}(\pi u/k \pm m_1/2)$  in the complex  $u$ -plane. The integral over  $u$  then adds an extra  $(2n)!$ , leading to an asymptotic (but Borel summable) perturbation series.

#### 4 The special case $m_2 = 0$

The  $m_2 = 0$  case is special and must be considered separately. In particular, it represents the critical point in the phase transitions that arise in the decompactification limit. In section 2 we have obtained the following formulas:

*Odd  $k$ :*

$$Z|_{m_2=0} = \frac{1}{k^2 \cosh^2 m} \sum_{n=1}^{k-1} (-)^n \frac{n \sin^2 \frac{\pi n}{k}}{\cosh\left(m + \frac{i\pi n}{k}\right) \cosh\left(m - \frac{i\pi n}{k}\right)} + \frac{(-)^{\frac{k-1}{2}} 2m}{\pi \cosh(km) \sinh(2m)}. \tag{4.1}$$

*Even  $k$ :*

$$Z|_{m_2=0} = \frac{1}{k \cosh^2 m} \sum_{n=1}^{k-1} (-)^{n+1} \left(\frac{n}{k} - \frac{1}{2}\right)^2 \frac{\sin^2\left(\frac{n\pi}{k}\right)}{\cosh\left(m - \frac{i n \pi}{k}\right) \cosh\left(m + \frac{i n \pi}{k}\right)} + (-)^{\frac{k}{2}+1} \frac{4m^2}{\pi^2} \frac{\sinh mk}{\sinh(2m)(\cosh(2mk) - 1)} \tag{4.2}$$

In particular,

$$\begin{aligned} Z|_{k=1} &= \frac{2m}{\pi \cosh(m) \sinh(2m)}, \\ Z|_{k=2} &= \frac{2m^2}{\pi^2 \sinh^2(2m)}. \end{aligned} \tag{4.3}$$

Note that the partition function does not have zeros in this case.

**Asymptotic formulas  $m_2 = 0$ .** Consider again the limit of a large sphere,  $mR \gg 1$ , at fixed  $k$ , but now with  $m_2 = 0$ . We find

$$Z|_{k=1} \sim \frac{8mR}{\pi} e^{-3mR}, \tag{4.4}$$

$$Z|_{k=2} \sim \frac{8}{\pi^2} m^2 R^2 e^{-4mR}, \tag{4.5}$$

$$Z|_{k>2} \sim \frac{4}{k^2} e^{-4mR} \tan^2 \frac{\pi}{k}. \tag{4.6}$$

Note that these formulas differ from the asymptotic formulas (3.12)–(3.14) given above for  $Z(m_1, m_2)$  at  $m_2 = 0$ . This is expected, since the latter were obtained by assuming  $|m_1 R|, |m_2 R| \rightarrow \infty$ .

Unlike the  $m_2 \neq 0$  case, the perturbation series for this flat-theory limit has now finite radius of convergence  $|\pi/k| < \pi/2$ , therefore perturbation series is convergent for all  $k > 2$ , where the formula applies. On the other hand, just like the general  $m_2 \neq 0$  case, the theory on a finite-radius  $S^3$  has an asymptotic perturbation series with  $2n!$  asymptotic behavior.

Finally, it would be interesting to study supersymmetric Wilson loops in the present mass/FI deformed theory, along the lines of [18].

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