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Indefinite least-squares problems and pseudo-regularity

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ABSTRACT

Given two Krein spaces $\mathcal{H}$ and $\mathcal{K}$, a (bounded) closed-range operator $C : \mathcal{H} \to \mathcal{K}$ and a vector $y \in \mathcal{K}$, the indefinite least-squares problem consists in finding those vectors $u \in \mathcal{H}$ such that

$$[Cu - y, Cu - y] = \min_{x\in\mathcal{H}} [Cx - y, Cx - y].$$

The indefinite least-squares problem has been thoroughly studied before under the assumption that the range of C is a uniformly J-positive subspace of $\mathcal{K}$. Along this article the range of C is only supposed to be a J-nonnegative pseudo-regular subspace of $\mathcal{K}$. This work is devoted to present a description for the set of solutions of this abstract problem in terms of the family of J-normal projections onto the range of C.

1. Introduction

In signal processing applications it is frequently assumed that the mathematical model, describing the physical phenomena under study, satisfies the following equation:

$$z = Hx + \eta,$$

where $H \in \mathbb{R}^{m\times n}$ is known and $x \in \mathbb{R}^n$ is a parameter that needs to be determined. Sometimes, due to physical restrictions, it is not possible to measure x, and it is necessary to estimate this vector based on

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the measurement z, which is corrupted by noise η. According to the characteristics of the noise, different techniques may be used to estimate x. For instance, when no statistical information about the noise measurement is available, the $\mathcal{H}^\infty$-estimation technique has been proved to be an appropriate approach for several engineering problems. Given $\gamma > 0$, the $\mathcal{H}^\infty$-estimation technique in $\mathbb{R}^n$ consists in finding an estimation $\hat{x}$ of the vector x, such that:

$$\max_{x\in\mathbb{R}^n} \frac{\|x-\hat{x}\|^2}{\|z-Hx\|^2} \leq \gamma^2, \tag{1.1}$$

or equivalently,

$$\min_{x\in\mathbb{R}^n} \left(\|z-Hx\|^2 - \frac{1}{\gamma^2}\|x-\hat{x}\|^2 \right) \geq 0. \tag{1.2}$$

Note that the left hand side of (1.2) can be modeled as the minimization of an indefinite inner product on an affine manifold. In fact, $\mathbb{R}^{m+n}$ can be endowed with the indefinite inner product $[\,x,y\,] := x^T Jy$, $x,y \in \mathbb{R}^{m+n}$, where $J \in L(\mathbb{R}^{m+n})$ is the fundamental symmetry given by $J = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$. Then, considering $C := \begin{pmatrix} H \\ \gamma^{-1} I_n \end{pmatrix} \in L(\mathbb{R}^n, \mathbb{R}^{m+n})$ and $y := \begin{pmatrix} z \\ \gamma^{-1}\hat{x} \end{pmatrix} \in \mathbb{R}^{m+n}$, the $\mathcal{H}^\infty$-estimation problem is equivalent to finding a vector y (which depends on z) such that the following indefinite least-squares problem (ILSP) admits a solution:

$$\min_{x\in\mathbb{R}^n} [\,y-Cx, y-Cx\,], \tag{1.3}$$

and to show that this minimum is nonnegative, see [8].

This work is devoted to studying an abstract ILSP: Given arbitrary Krein spaces $\mathcal{H}$ and $\mathcal{K}$, a closed-range operator $C \in L(\mathcal{H},\mathcal{K})$ and a vector $y \in \mathcal{K}$, find the vectors $u \in \mathcal{H}$ such that

$$[\,y-Cu, y-Cu\,] = \min_{x\in\mathcal{H}} [\,y-Cx, y-Cx\,].$$

In finite-dimensional spaces, the ILSP has been exhaustively studied see e.g. [13,14,21,8,15,20,7]. In these papers, if J is the fundamental symmetry of $\mathcal{K}$, it is assumed that $C^T JC$ is a positive-definite matrix, which is a sufficient condition for the existence of a unique solution for the ILSP. This is equivalent to assuming that C is injective and the range of C (hereafter denoted by $R(C)$) is a uniformly J-positive subspace of $\mathcal{K}$. Then, the regularity of $R(C)$ plays an essential role, since it guarantees the existence of a J-selfadjoint projection onto $R(C)$, which determines the unique solution of the ILS problem (1.3).

Even for the general setting it is known that the ILSP admits a solution if and only if $R(C)$ is J-nonnegative and $y \in R(C) + R(C)^{[\perp]}$, see e.g. [6, Thm. 8.4]. Then, the ILSP is well-posed only for the vectors y in the (not necessarily closed) subspace $R(C)+R(C)^{[\perp]}$. Moreover, given $y \in R(C)+R(C)^{[\perp]}$, $u \in \mathcal{H}$ is a solution of the ILSP if and only if $y - Cu \in R(C)^{[\perp]}$ (see Lemma 3.1), i.e. if u is a solution of the normal equation associated to $Cx = y$:

$$C^\#(Cx - y) = 0,$$

where $C^\#$ stands for the J-adjoint operator of C.

The assumption that $R(C)$ is a uniformly J-positive subspace of $\mathcal{K}$ implies that the ILSP is properly defined for every $y \in \mathcal{K}$, but this is a quite restrictive condition. Along this article (most of the time) it is assumed that $R(C)$ is a J-nonnegative pseudo-regular subspace of $\mathcal{K}$. Thus, the ILSP admits solutions for every vector in the (proper) closed subspace $R(C) + R(C)^{[\perp]}$. The pseudo-regularity of $R(C)$ is equivalent to the closedness of $R(C^{\#}C)$, see Lemma 3.4. Hence, under this assumption, the Moore–Penrose inverse $(C^{\#}C)^{\dagger}$ of $C^{\#}C$ is bounded and the solutions of the normal equation, and therefore of the ILSP, are exactly those

$$u \in u_y + N(C^{\#}C),$$

where $u_y = (C^{\#}C)^{\dagger}C^{\#}y$ is the unique solution in $N(C^{\#}C)^{\perp}$.

It is also worthy to mention that if $\mathcal{K}$ is a Pontryagin space (i.e. $\kappa := \min\{\dim \mathcal{K}_+, \dim \mathcal{K}_-\} < \infty$ for any fundamental decomposition $\mathcal{K} = \mathcal{K}_+[\dotplus]\mathcal{K}_-$) then every closed subspace turns out to be pseudo-regular. Therefore, in this case the assumption reduces to assume that $R(C)$ is just J-nonnegative.

Another advantage of considering an operator C with pseudo-regular range is that there is a family of J-normal projections onto $R(C)$. These projections, which have been previously studied in [19], are the main technical tool used along this work in order to characterize the set of solutions of the ILSP.

The article is organized as follows: Section 2 introduces the notation and terminology used along. It also contains some preliminaries on Krein spaces, mainly on pseudo-regularity and J-normal projections.

The indefinite least-squares problem is described in Section 3. After a brief reminder of the state of the art of the problem, it is studied under the assumption that the range of C is a J-nonnegative pseudo-regular subspace of $\mathcal{K}$. Also, some considerations are made in order to compare the ILSP associated to $Cx = y$ and the ILSP associated to another equation $C'x = y$, where C' is a closed-range operator such that $R(C')$ is a uniformly J-positive subspace of $R(C)$.

Until this point the Krein space structure of $\mathcal{H}$, the domain of C, was unnecessary. However, Section 4 is devoted to consider a minimization problem among the indefinite least-squares solutions of $Cx = y$. A minimal least-squares solution (MILSS) of $Cx = y$ is a vector $w \in u_y + N(C^{\#}C)$ such that

$$[w, w] = \min_{u \in u_y + N(C^{\#}C)} [u, u].$$

If the ILSP associated to $Cx = y$ admits solutions, in order to guarantee the existence of a MILSS of $Cx = y$ it is necessary and sufficient that $N(C^{\#}C)$ is J-nonnegative and that the affine manifold $u_y + N(C^{\#}C)$ intersects $N(C^{\#}C)^{[\perp]}$, see Proposition 4.1. If it is also assumed that $N(C^{\#}C)$ and $R(C)$ are pseudo-regular subspaces of $\mathcal{H}$ and $\mathcal{K}$, respectively, then the set of MILSS can be computed in terms of the J-normal projections onto these subspaces and the Moore–Penrose inverse of C, see Theorem 4.3.

Finally, in Section 5 the operators used in Theorem 4.3 to describe the MILSS of $Cx = y$ are shown to be a family of generalized inverses of a fixed operator C' with regular range.

2. Preliminaries

Along this work $\mathcal{H}$ denotes a complex (separable) Hilbert space. If $\mathcal{K}$ is another Hilbert space then $L(\mathcal{H}, \mathcal{K})$ is the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$.

If $T \in L(\mathcal{H}, \mathcal{K})$ then $R(T)$ stands for its range and $N(T)$ for its nullspace.

Given two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of a Hilbert space $\mathcal{H}$, $\mathcal{S} \dotplus \mathcal{T}$ denotes the direct sum of them. Moreover, $\mathcal{S} \oplus \mathcal{T}$ stands for their (direct) orthogonal sum and $\mathcal{S} \ominus \mathcal{T} := \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^{\perp}$.

If $\mathcal{H} = \mathcal{S} \dotplus \mathcal{T}$, $P_{\mathcal{S}//\mathcal{T}}$ denotes the (unique, bounded) projection onto $\mathcal{S}$ along $\mathcal{T}$. In the particular case of $\mathcal{T} = \mathcal{S}^{\perp}$, the *orthogonal projection onto* $\mathcal{S}$ is denoted by $P_{\mathcal{S}}$.

In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject see [6,2,1].

Given a Krein space $(\mathcal{H}, [\ ,\ ])$ with a *fundamental decomposition* $\mathcal{H} = \mathcal{H}_+ \dotplus \mathcal{H}_-$, the direct (orthogonal) sum of the Hilbert spaces $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ is denoted by $(\mathcal{H}, \langle\ ,\ \rangle)$.

Observe that the inner products of $\mathcal{H}$ are related by means of a *fundamental symmetry*, i.e. a unitary selfadjoint operator $J \in L(\mathcal{H})$ which satisfies:

$$[\,x, y\,] = \langle\, Jx, y\,\rangle\,, \qquad x, y \in \mathcal{H}.$$

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces, $L(\mathcal{H}, \mathcal{K})$ stands for the vector space of linear transformations which are bounded with respect to the associated Hilbert spaces $(\mathcal{H}, \langle\ ,\ \rangle_{\mathcal{H}})$ and $(\mathcal{K}, \langle\ ,\ \rangle_{\mathcal{K}})$. Given $T \in L(\mathcal{H}, \mathcal{K})$, the J-adjoint operator of T is defined by $T^{\#} = J_{\mathcal{H}} T^* J_{\mathcal{K}}$, where $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$ are the fundamental symmetries associated to $\mathcal{H}$ and $\mathcal{K}$, respectively. An operator $T \in L(\mathcal{H})$ is J-selfadjoint if $T = T^{\#}$.

A vector $x \in \mathcal{H}$ is *J-positive* if $[\,x, x\,] > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *J-positive* if every $x \in \mathcal{S}$, $x \neq 0$, is a J-positive vector. J-nonnegative, J-neutral, J-negative and J-nonpositive vectors and subspaces are defined analogously.

Given a subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$, the *J-orthogonal subspace* to $\mathcal{S}$ is defined by

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} : [\,x, s\,] = 0, \text{ for every } s \in \mathcal{S}\}.$$

The isotropic part of $\mathcal{S}$, $\mathcal{S}^\circ := \mathcal{S} \cap \mathcal{S}^{[\perp]}$ can be a non-trivial subspace. It holds that

$$\mathcal{H} = \overline{\mathcal{S} + \mathcal{S}^{[\perp]}} \oplus J(\mathcal{S}^\circ),$$

see [2, Prop. 1.7.6]. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *J-non-degenerated* if $\mathcal{S} \cap \mathcal{S}^{[\perp]} = \{0\}$. Otherwise, it is a J-degenerated subspace of $\mathcal{H}$.

A (closed) subspace $\mathcal{S}$ of $\mathcal{H}$ is *regular* if $\mathcal{S} \dotplus \mathcal{S}^{[\perp]} = \mathcal{H}$. Equivalently, $\mathcal{S}$ is regular if and only if there exists a (unique) J-selfadjoint projection E onto $\mathcal{S}$, see e.g. [2, Thm. 1.7.16].

On the other hand, a closed subspace $\mathcal{S}$ of $\mathcal{H}$ is called *pseudo-regular* if the algebraic sum $\mathcal{S} + \mathcal{S}^{[\perp]}$ is closed. Equivalently, $\mathcal{S}$ is pseudo-regular if there exists a regular subspace $\mathcal{M}$ such that $\mathcal{S} = \mathcal{S}^\circ [\dotplus] \mathcal{M}$, where $[\dotplus]$ stands for the J-orthogonal direct sum of the subspaces, see [9].

The importance of pseudo-regular subspaces lies in the fact that they enable to generalize some Pontryagin spaces arguments to general Krein spaces. They have also been used as a technical tool for the study of spectral functions (and distributions) for particular classes of operators in Krein spaces [10,11,17,18,22] and to extend the Beurling–Lax theorem for shifts in indefinite metric spaces [3,4].

Also, $\mathcal{S}$ is pseudo-regular if and only if $\mathcal{S}$ is the range of a J-normal projection, i.e. if there exists a projection $Q \in L(\mathcal{H})$ with $R(Q) = \mathcal{S}$ such that $QQ^{\#} = Q^{\#}Q$, see [19, Thm. 4.3]. In particular, given a pseudo-regular subspace $\mathcal{S}$, $Q_0 = P_{\mathcal{S}//\mathcal{S}^{[\perp]} \ominus \mathcal{S}^\circ + J(\mathcal{S}^\circ)}$ is a J-normal projection onto $\mathcal{S}$. However, if $\mathcal{S}^\circ \neq \{0\}$ then there are infinitely many J-normal projections Q satisfying $R(Q) = \mathcal{S}$. In what follows, $\mathcal{Q}_{\mathcal{S}}$ stands for the set of J-normal projections onto the pseudo-regular subspace $\mathcal{S}$, i.e.

$$\mathcal{Q}_{\mathcal{S}} = \{Q \in L(\mathcal{H}) :\ Q^2 = Q,\ QQ^{\#} = Q^{\#}Q \text{ and } R(P) = \mathcal{S}\ \}.$$

The next is a technical remark that will be frequently used along this work. It shows that, given a vector $y \in \mathcal{S} + \mathcal{S}^{[\perp]}$, the J-normal projections onto $\mathcal{S}$ provide the different decompositions of y as a sum of a vector in $\mathcal{S}$ and a vector in $\mathcal{S}^{[\perp]}$, i.e. if $Q \in \mathcal{Q}_{\mathcal{S}}$ then

$$y = Qy + (I - Q)y, \quad \text{where} \quad Qy \in \mathcal{S} \quad \text{and} \quad (I - Q)y \in \mathcal{S}^{[\perp]}.$$

Remark 2.1. If $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$ and $y \in \mathcal{S} + \mathcal{S}^{[\perp]}$, given any $Q \in \mathcal{Q}_{\mathcal{S}}$, then

$$Q^{\#}(I - Q)y = 0.$$

Indeed, if $P = Q(I - Q)^\#$ then $R(P) = \mathcal{S} \cap N(Q^\#) = \mathcal{S} \cap \mathcal{S}^{[\perp]} = \mathcal{S}^\circ$ and $N(P^\#) = R(P)^{[\perp]} = (\mathcal{S}^\circ)^{[\perp]} = \mathcal{S} + \mathcal{S}^{[\perp]}$. Therefore, if $y \in \mathcal{S} + \mathcal{S}^{[\perp]}$ then $Q^\#(I - Q)y = P^\# y = 0$. In particular, $(I - Q)y \in N(Q^\#) = R(Q)^{[\perp]} = \mathcal{S}^{[\perp]}$.

The following results belong to [19]. Their statements are included in order to make the paper self-contained.

Proposition 2.2. *A bounded projection Q acting on $\mathcal{H}$ is J-normal if and only if there exist a J-selfadjoint projection $E \in L(\mathcal{H})$ and a projection $P \in L(\mathcal{H})$ satisfying $PP^\# = P^\# P = 0$ such that*

$$Q = E + P.$$

The projections E and P are uniquely determined by Q. More precisely, $E = QQ^\#$ and $P = Q(I - Q^\#)$.

Projections $P \in L(\mathcal{H})$ satisfying $PP^\# = P^\# P = 0$ were previously considered in [17,11], in connection with neutral dual companions. If $\mathcal{S}$ is a fixed (closed) J-neutral subspace of $\mathcal{H}$, a *neutral dual companion of* $\mathcal{S}$ is another (closed) J-neutral subspace $\mathcal{T}$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{S} \dotplus \mathcal{T}^{[\perp]}$ holds. If $\mathcal{T}$ is a neutral dual companion of $\mathcal{S}$ then also $\mathcal{H} = \mathcal{T} \dotplus \mathcal{S}^{[\perp]}$ holds. So, the pair of subspaces $(\mathcal{S}, \mathcal{T})$ is called a *neutral dual pair*. Note that in this case $\mathcal{S} \dotplus \mathcal{T}$ is a regular subspace of $\mathcal{H}$.

A J-neutral subspace $\mathcal{N}$ of $\mathcal{H}$ is said to be a *hypermaximal J-neutral* subspace if it is simultaneously both maximal J-nonnegative and maximal J-nonpositive. Equivalently, $\mathcal{N}$ is a hypermaximal J-neutral subspace if and only if $\mathcal{N} = \mathcal{N}^{[\perp]}$, see [2, Prop. 1.4.19].

Given $C \in L(\mathcal{H}, \mathcal{K})$, its restriction $C|_{N(C)^\perp} : N(C)^\perp \to R(C)$ admits a linear inverse $(C|_{N(C)^\perp})^{-1} : R(C) \to N(C)^\perp$. Then, the Moore–Penrose inverse of C is the linear operator $C^\dagger : R(C) + R(C)^\perp \to \mathcal{H}$ defined by

$$C^\dagger y = \begin{cases} (C|_{N(C)^\perp})^{-1} y & \text{if } y \in R(C); \\ 0 & \text{if } y \in R(C)^\perp. \end{cases}$$

Note that $C^\dagger$ is densely-defined on $\mathcal{K}$, and it is well-known that $C^\dagger \in L(\mathcal{K}, \mathcal{H})$ if and only if $R(C)$ is closed.

Hereafter, given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let $CR(\mathcal{H}, \mathcal{K})$ denotes the set of bounded closed-range operators from $\mathcal{H}$ into $\mathcal{K}$. The following are some properties of the Moore–Penrose inverse of a closed-range operator:

Proposition 2.3. *Given $C \in CR(\mathcal{H}, \mathcal{K})$,*

1. *$CC^\dagger = P_{R(C)}$ and $C^\dagger C = P_{N(C)^\perp}$, the orthogonal projections onto $R(C)$ and $N(C)^\perp$, respectively. In particular, $CC^\dagger C = C$ and $C^\dagger CC^\dagger = C^\dagger$.*
2. *$C^* \in CR(\mathcal{K}, \mathcal{H})$ and $(C^*)^\dagger = (C^\dagger)^*$.*
3. *If $U \in L(\mathcal{K}), V \in L(\mathcal{H})$ are unitary operators, then $(UCV)^\dagger = V^* C^\dagger U^*$.*

The Moore–Penrose inverse has been thoroughly studied along the years, see e.g. [5] for a complete exposition on this subject.

As a consequence of Proposition 2.3, if $\mathcal{H}$ and $\mathcal{K}$ are two Krein spaces and $C \in CR(\mathcal{H}, \mathcal{K})$ then $C^\# \in CR(\mathcal{K}, \mathcal{H})$ and $(C^\#)^\dagger = (C^\dagger)^\#$.

3. Indefinite least-squares problems

Along this work, the following indefinite least-squares problem is considered: Let $\mathcal{H}$ and $\mathcal{K}$ be two Krein spaces with fundamental symmetries $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$, respectively. Given an operator $C \in CR(\mathcal{H}, \mathcal{K})$ and a vector $y \in \mathcal{K}$, find $u \in \mathcal{H}$ such that

$$[\,y - Cu, y - Cu\,]_{\mathcal{K}} = \min_{x \in \mathcal{H}} [\,y - Cx, y - Cx\,]_{\mathcal{K}}. \tag{3.1}$$

The next lemma shows necessary and sufficient conditions for the existence of indefinite least-squares solutions (ILSS) of the equation $Cx = y$. A proof can be found in [6, Theorem 8.4] or in [12, Lemma 3.1].

Lemma 3.1. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ and $y \in \mathcal{K}$. Then, $u \in \mathcal{H}$ is an ILSS of the equation $Cx = y$ if and only if $R(C)$ is $J_{\mathcal{K}}$-nonnegative and $y - Cu \in R(C)^{[\perp]}$.*

Hence, in order to have a well-posed indefinite least-squares problem it is necessary that $y \in R(C) + R(C)^{[\perp]}$. Note that the set of admissible points $R(C) + R(C)^{[\perp]}$ is always dense in $(R(C)^{\circ})^{[\perp]}$.

Proposition 3.2. *Let $C \in CR(\mathcal{H}, \mathcal{K})$. Then, $Cx = y$ admits an ILSS for every $y \in (R(C)^{\circ})^{[\perp]}$ if and only if $R(C)$ is a $J_{\mathcal{K}}$-nonnegative pseudo-regular subspace of $\mathcal{K}$.*

Proof. Note that $Cx = y$ admits an ILSS for every $y \in (R(C)^{\circ})^{[\perp]}$ if and only if $(R(C)^{\circ})^{[\perp]} \subseteq R(C) + R(C)^{[\perp]}$ and $R(C)$ is $J_{\mathcal{K}}$-nonnegative. But

$$(R(C)^{\circ})^{[\perp]} = \overline{R(C) + R(C)^{[\perp]}},$$

and the equivalence follows. □

In particular, $Cx = y$ admits an ILSS for every $y \in \mathcal{K}$ if and only if $R(C)$ is a uniformly J-positive subspace of $\mathcal{K}$, see also [12, Proposition 3.2].

Before describing the indefinite least-squares solutions of $Cx = y$, observe that the minimum value of $L(x) = [\,y - Cx, y - Cx\,]$, $x \in \mathcal{H}$, is attained at the projections (by means of normal projectors) of y onto $R(C)$.

Lemma 3.3. *Given $C \in CR(\mathcal{H}, \mathcal{K})$ such that $R(C)$ is a J-nonnegative pseudo-regular subspace of $\mathcal{K}$ and $y \in R(C) + R(C)^{[\perp]}$,*

$$\min_{x \in \mathcal{H}} [\,y - Cx, y - Cx\,] = [\,(I - Q)y, (I - Q)y\,],$$

where $Q \in L(\mathcal{K})$ is any J-normal projection onto $R(C)$.

Proof. Since $R(C)$ is pseudo-regular, by [19, Thm. 4.3] there exists a J-normal projection $Q \in L(\mathcal{K})$ onto $R(C)$. Then, for any $x \in \mathcal{H}$,

$$\begin{aligned} [\,y - Cx, y - Cx\,] &= [\,(y - Qy) + (Qy - Cx), (y - Qy) + (Qy - Cx)\,] \\ &= [\,(I - Q)y, (I - Q)y\,] + 2\,\mathrm{Re}[\,(I - Q)y, Qy - Cx\,] + [\,Qy - Cx, Qy - Cx\,] \\ &\geq [\,(I - Q)y, (I - Q)y\,] + 2\,\mathrm{Re}[\,(I - Q)y, Qy - Cx\,], \end{aligned} \tag{3.2}$$

because $Qy - Cx \in R(C)$ which is a $J_{\mathcal{K}}$-nonnegative subspace. Furthermore, by Remark 2.1, $y \in R(C) + R(C)^{[\perp]}$ implies that $Q^{\#}(I-Q)y = 0$ and

$$[(I-Q)y, Qy - Cx] = [(I-Q)y, Q(y-Cx)] = [Q^{\#}(I-Q)y, y - Cx] = 0.$$

Therefore,

$$[y - Cx, y - Cx] \geq [(I-Q)y, (I-Q)y]. \qquad \square$$

Also, note that the pseudo-regularity of $R(C)$ is equivalent to the boundedness of the Moore–Penrose inverse of $C^{\#}C$:

Lemma 3.4. *Given $C \in CR(\mathcal{H}, \mathcal{K})$, $R(C)$ is pseudo-regular if and only if $R(C^{\#}C)$ is closed.*

Proof. Since $R(C)$ is closed, note that $R(C^{\#}C)$ is closed if and only if $R(C) + N(C^{\#}) = R(C) + R(C)^{[\perp]}$ is closed, see [16, Corollary 2.5]. Thus, $R(C^{\#}C)$ is closed if and only if $R(C)$ is a pseudo-regular subspace of $\mathcal{K}$. $\square$

Given $C \in CR(\mathcal{H}, \mathcal{K})$ and $y \in R(C) + R(C)^{[\perp]}$, observe that $C^{\#}y \in R(C^{\#}C)$. Then,

$$u_y := (C^{\#}C)^{\dagger} C^{\#} y, \tag{3.3}$$

is a solution of the *normal equation*:

$$C^{\#}(Cx - y) = 0. \tag{3.4}$$

In particular, u_y is the unique solution of the normal equation in $N(C^{\#}C)^{\perp}$ and the set of solutions of (3.4) is the affine manifold

$$u_y + N(C^{\#}C).$$

The following is the main result of this section. It shows that the solutions of the ILSP associated to the equation $Cx = y$ are the solutions of the normal equation $C^{\#}(Cx - y) = 0$, but it also characterizes them in terms of the J-normal projections onto $R(C)$.

Theorem 3.5. *Given $C \in CR(\mathcal{H}, \mathcal{K})$, if $R(C)$ is a J-nonnegative pseudo-regular subspace of $\mathcal{K}$ and $y \in R(C) + R(C)^{[\perp]}$, the following conditions are equivalent:*

1. *$u \in \mathcal{H}$ is an ILSS of $Cx = y$;*
2. *$u \in \mathcal{H}$ is a solution of the normal equation $C^{\#}(Cx - y) = 0$;*
3. *$Cu - Qy \in R(C)^{\circ}$ for any J-normal projection Q onto $R(C)$.*

If $y \notin R(C)$ the above conditions are also equivalent to:

4. *there exists a J-normal projection Q onto $R(C)$ such that $Cu = Qy$.*

Moreover, the set of ILSS of $Cx = y$ coincides with the affine manifold

$$u_y + N(C^{\#}C),$$

where $u_y = (C^{\#}C)^{\dagger} C^{\#} y$.

Proof. By Lemma 3.1, assuming the J-nonnegativity of $R(C)$, u is an ILSS of $Cx = y$ if and only if $y - Cu \in R(C)^{[\perp]} = N(C^{\#})$. Then, the equivalence *1.* $\leftrightarrow$ *2.* follows.

2. $\leftrightarrow$ *3.*: By Remark 2.1, $(I - Q)y \in R(C)^{[\perp]} = N(C^{\#})$ for any J-normal projection $Q \in L(\mathcal{K})$ onto $R(C)$. Hence, $u \in \mathcal{H}$ is a solution of $C^{\#}(Cx - y) = 0$ if and only if $C^{\#}(Cu - Qy) = 0$, or equivalently, $Cu - Qy \in R(C)^{\circ}$.

2. $\leftrightarrow$ *4.*: Assume that $y \notin R(C)$ and u is a solution of $C^{\#}(Cx - y) = 0$. Then, $y = Cu + z$ with $z \in R(C)^{[\perp]} \setminus R(C)$. So, there exists a regular subspace $\mathcal{T}$ of $R(C)^{[\perp]}$ such that $z \in \mathcal{T}$ and $R(C)^{[\perp]} = \mathcal{T}[\dot{+}]R(C)^{\circ}$. Also, consider a regular subspace $\mathcal{M}$ of $R(C)$ such that $R(C) = \mathcal{M}[\dot{+}]R(C)^{\circ}$. Then, note that $R(C)^{\circ}$ is a J-neutral subspace of the Krein space $\mathcal{K}' = (\mathcal{M} + \mathcal{T})^{[\perp]}$. So, it is well-known that there exists a neutral dual companion $\mathcal{N}$ of $R(C)^{\circ}$ in $\mathcal{K}'$, see [11]. Furthermore, $R(C)^{\circ}$ is a hypermaximal neutral subspace of $\mathcal{K}'$ [2, Prop. 1.4.19] because

$$\begin{aligned}(R(C)^{\circ})^{[\perp]_{\mathcal{K}'}} &= (R(C)^{\circ})^{[\perp]} \cap \mathcal{K}' \\ &= (R(C) + R(C)^{[\perp]}) \cap (\mathcal{M} + \mathcal{T})^{[\perp]} = \\ &= (\mathcal{M} \dot{+} \mathcal{T} \dot{+} R(C)^{\circ}) \cap (\mathcal{M} + \mathcal{T})^{[\perp]} = R(C)^{\circ}.\end{aligned}$$

Thus, $(\mathcal{M} + \mathcal{T})^{[\perp]} = \mathcal{K}' = \mathcal{N} \dot{+} (R(C)^{\circ})^{[\perp]_{\mathcal{K}'}} = \mathcal{N} \dot{+} R(C)^{\circ}$ and the following decomposition of $\mathcal{K}$ holds:

$$\mathcal{K} = \mathcal{M}[\dot{+}](R(C)^{\circ} \dot{+} \mathcal{N})[\dot{+}]\mathcal{T}.$$

Given the projection $Q = P_{R(C)//\mathcal{T}+\mathcal{N}} \in L(\mathcal{K})$, it is easy to see that $Q^{\#} = P_{\mathcal{M}+\mathcal{N}//R(C)^{[\perp]}}$. Therefore, Q is J-normal and it satisfies $Qy = Q(Cu + z) = Cu$.

Conversely, if $Cu = Qy$ for some J-normal projection $Q \in L(\mathcal{K})$ onto $R(C)$ then, by Remark 2.1, $y - Cu = (I - Q)y \in R(C)^{[\perp]} = N(C^{\#})$. Therefore, $C^{\#}(Cu - y) = 0$.

Finally, recall that the set of solutions of the normal equation (which in this case coincides with the ILSS of $Cx = y$) is the affine manifold $u_y + N(C^{\#}C)$, where $u_y = (C^{\#}C)^{\dagger}C^{\#}y$. $\square$

Remark 3.6. Given $C \in CR(\mathcal{H}, \mathcal{K})$ with pseudo-regular range $R(C)$, the equivalences *2.* $\leftrightarrow$ *3.* $\leftrightarrow$ *4.* in Theorem 3.5 holds independently of the (semi)definiteness of the range. Hence, Theorem 3.5 also characterizes the solutions of the normal equation $C^{\#}(Cx - y) = 0$ for $C \in CR(\mathcal{H}, \mathcal{K})$ with an arbitrary pseudo-regular range $R(C)$.

If $C \in CR(\mathcal{H}, \mathcal{K})$ and $R(C)$ is pseudo-regular, the set $\mathcal{Q}_{R(C)}$ of J-normal projections onto $R(C)$ is related to a family of inner inverses of C, where $X \in L(\mathcal{K}, \mathcal{H})$ is an inner inverse of C if $CXC = C$. Let $\mathcal{I}$ denote the set of solutions $D \in L(\mathcal{K}, \mathcal{H})$ of the equations

$$CXC = C, \quad (CX)^{\#}CX = CX(CX)^{\#}. \tag{3.5}$$

Then, $D \in \mathcal{I}$ if and only if there exist $Q \in \mathcal{Q}_{R(C)}$ and $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$ such that

$$D = C^{\dagger}Q + T.$$

Indeed, if $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of (3.5) then $Q := CD \in \mathcal{Q}_{R(C)}$ and $C^{\dagger}Q = C^{\dagger}CD = P_{N(C)^{\perp}}D$. So, $T := P_{N(C)}D \in L(\mathcal{K}, \mathcal{H})$ satisfies $R(T) \subseteq N(C)$ and $D = C^{\dagger}Q + T$.

Conversely, given $Q \in \mathcal{Q}_{R(C)}$ and $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$, consider $D := C^{\dagger}Q + T$. Then, $CD = CC^{\dagger}Q = P_{R(C)}Q = Q$ implies that D is a solution of (3.5).

The following result describes the solutions of the ILSP associated to $Cx = y$ in terms of these generalized inverses.

Proposition 3.7. *Given $C \in CR(\mathcal{H}, \mathcal{K})$, if $R(C)$ is a J-nonnegative pseudo-regular subspace of $\mathcal{K}$ and $y \in R(C) + R(C)^{[\perp]}$, the following conditions are equivalent:*

1. *$u \in \mathcal{H}$ is an ILSS of $Cx = y$;*
2. *$Dy - u \in N(C^{\#}C)$ for any solution $D \in L(\mathcal{K}, \mathcal{H})$ of (3.5).*

If $y \notin R(C)$ the above conditions are also equivalent to:

3. *there exists a solution of (3.5) such that $Dy = u$.*

Proof. *1.* $\leftrightarrow$ *2.*: Given a solution $D \in L(\mathcal{K}, \mathcal{H})$ of (3.5), consider $Q \in \mathcal{Q}_{R(C)}$ and $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$ such that $D = C^{\dagger}Q + T$. For $u \in \mathcal{H}$, follows that $Dy - u \in N(C^{\#}C)$ if and only if $C^{\#}(Qy - Cu) = 0$, or equivalently, $Qy - Cu \in R(C)^{\circ}$. Thus the equivalence follows from Theorem 3.5.

1. $\leftrightarrow$ *3.*: Given $y \in (R(C)+R(C)^{[\perp]})\backslash R(C)$, suppose that $u = Dy$ where $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of (3.5). It is easy to see that $Q = CD$ is a J-normal projection with $R(Q) = R(C)$. Furthermore, $Cu = CDy = Qy$. By Theorem 3.5, this implies that u is an ILSS of $Cx = y$.

Conversely, if $u \in \mathcal{H}$ is an ILSS of $Cx = y$, Theorem 3.5 states that $Cu = Qy$ for some J-normal projection $Q \in L(\mathcal{K})$. Then, $u = C^{\dagger}Qy + w$, where $w \in N(C)$. Consider $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$ such that $Ty = w$ and define $D = C^{\dagger}Q + T$. Thus, D is a solution of (3.5) and $Dy = C^{\dagger}Qy + Ty = C^{\dagger}Qy + w = u$. $\square$

In the following it is shown that the ILSP associated to the equation $Cx = y$ can be rewritten as an ILSP associated to another equation $C'x = y$, where $C' \in CR(\mathcal{H}, \mathcal{K})$ and $R(C')$ is a uniformly J-positive subspace of $\mathcal{K}$. But this is only true if the vector $y \in \mathcal{K}$ is admissible for the ILSP associated to the equation $Cx = y$ (recall that the ILSP associated to the equation $C'x = y$ is always well-posed).

If $R(C)$ is a $J_{\mathcal{K}}$-nonnegative pseudo-regular subspace of $\mathcal{K}$ and $y \in R(C) + R(C)^{[\perp]}$, then

$$u \in \mathcal{H} \text{ is an ILSS of } Cx = y \quad \Leftrightarrow \quad u \in \mathcal{H} \text{ is an ILSS of } (EC)x = y,$$

where $E = QQ^{\#}$ and Q is any J-normal projection onto $R(C)$.

First, observe that $R(EC) = E(R(C) + N(E)) = R(E)$ since $R(E) \subset R(C)$. Hence, $R(EC)$ is uniformly $J_{\mathcal{K}}$-positive and the indefinite least-squares problem associated to the equation $ECx = y$ is well-posed. Then, by Theorem 3.5, $u \in \mathcal{H}$ is an ILSS of $Cx = y$ if and only if $Cu - Qy \in R(C)^{\circ}$. But, $R(C)^{\circ} \subset N(E)$ implies that

$$ECu = E(Cu - Qy) + EQy = Ey,$$

and E is the J-selfadjoint projection onto $R(EC)$. Then, u is an ILSS of $ECx = y$, see e.g. [12, Prop. 3.2].

Proposition 3.8. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is a pseudo-regular subspace of $\mathcal{K}$. Then, $C' = CP_{N(C^{\#}C)^{\perp}} \in CR(\mathcal{H}, \mathcal{K})$ has regular range and, if $y \in R(C) + R(C)^{[\perp]}$,*

$$u \in \mathcal{H} \text{ is an ILSS of } Cx = y \quad \Leftrightarrow \quad u \in \mathcal{H} \text{ is an ILSS of } C'x = y.$$

Proof. Given $C \in CR(\mathcal{H}, \mathcal{K})$, consider the operator $E_0 := C(C^{\#}C)^{\dagger}C^{\#}$. By Lemma 3.4, $E_0 \in L(\mathcal{K})$ and it is easy to check that $E_0^2 = E_0$. As a consequence of Proposition 2.3 the projection E_0 is J-selfadjoint, and $R(E_0)$ is obviously contained in $R(C)$. Then, $R(E_0C) = E_0(R(C) + N(E_0)) = R(E_0)$ and note that

$$E_0C = C(C^{\#}C)^{\dagger}C^{\#}C = CP_{N(C^{\#}C)^{\perp}} = C'.$$

Therefore, $R(C') = R(E_0C) = R(E_0)$ is regular.

Also, $R(E_0) \cap R(C)^\circ = \{0\}$ because $R(C)^\circ \subseteq R(C)^{[\perp]} = N(C^\#) \subseteq N(E_0)$. Since $C = C' + CP_{N(C^\# C)}$ and the range of $CP_{N(C^\# C)}$ coincides with $R(C)^\circ$, it follows that

$$R(C) = R(CP_{N(C^\# C)^\perp}) + R(C)^\circ = R(E_0 C) + R(C)^\circ = R(E_0) \dotplus R(C)^\circ.$$

Therefore, E_0 is a J-selfadjoint projection onto a regular complement of $R(C)^\circ$ in $R(C)$ and, by [19, Thm. 6.9] there exist (at least) a J-normal projection $Q \in L(\mathcal{K})$ such that $E_0 = QQ^\#$. Finally, if $y \in R(C) + R(C)^{[\perp]}$ the discussion above shows that the ILSS of $Cx = y$ and $C'x = y$ coincide. □

4. Minimizers among indefinite least-squares solutions

The following paragraphs are devoted to consider a minimization problem among the indefinite least-squares solutions of $Cx = y$, where $C \in CR(\mathcal{H}, \mathcal{K})$ and $y \in R(C) + R(C)^{[\perp]}$.

Definition 1. A vector $w \in \mathcal{H}$ is a minimal least-squares solution (hereafter MILSS) of $Cx = y$ if w is an ILSS of $Cx = y$ and

$$[w, w]_{\mathcal{H}} \leq [u, u]_{\mathcal{H}}, \quad \text{for every ILSS } u \text{ of } Cx = y.$$

It follows from Theorem 3.5 that, if $R(C)$ is a pseudo-regular $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$ and $y \in R(C) + R(C)^{[\perp]}$, the set of ILSS of $Cx = y$ coincides with

$$u_y + N(C^\# C),$$

where $u_y = (C^\# C)^\dagger C^\# y$. So, $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if

$$[w, w] = \min_{z \in N(C^\# C)} [u_y + z, u_y + z]. \tag{4.1}$$

Thus, if $P_{N(C^\# C)}$ is the orthogonal projection onto $N(C^\# C)$ and $w = u_y + z_w$ is the orthogonal decomposition of w according to $\mathcal{H} = N(C^\# C)^\perp \oplus N(C^\# C)$, note that (4.1) can be rewritten as

$$\begin{aligned}[u_y + z_w, u_y + z_w] &= \min_{z \in N(C^\# C)} [u_y + z, u_y + z] \\ &= \min_{x \in \mathcal{H}} [u_y + P_{N(C^\# C)} x, u_y + P_{N(C^\# C)} x].\end{aligned}$$

Hence, if $w = u_y + z_w \in u_y + N(C^\# C)$,

$$w \text{ is a MILSS of } Cx = y \quad \Leftrightarrow \quad z_w \text{ is an ILSS of } P_{N(C^\# C)} x = -u_y. \tag{4.2}$$

By Lemma 3.1, the existence of an ILSS of $P_{N(C^\# C)} x = -u_y$ is equivalent to

$$u_y \in N(C^\# C) + N(C^\# C)^{[\perp]},$$

and the $J_{\mathcal{H}}$-nonnegativity of $N(C^\# C)$. Therefore,

Proposition 4.1. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is a $J_{\mathcal{K}}$-nonnegative pseudo-regular subspace of $\mathcal{K}$ and consider $y \in R(C) + R(C)^{[\perp]}$. Then, there exists a MILSS $w \in \mathcal{H}$ of $Cx = y$ if and only if $N(C^\# C)$ is $J_{\mathcal{H}}$-nonnegative and $u_y \in N(C^\# C) + N(C^\# C)^{[\perp]}$. In this case, the set of MILSS of $Cx = y$ coincides with*

$$(u_y + N(C^\# C)) \cap N(C^\# C)^{[\perp]}.$$

Proof. The equivalence between the existence of a MILSS for $Cx = y$ and the conditions on $N(C^\# C)$ and u_y follows from the discussion above. Also, note that $u_y \in N(C^\# C) + N(C^\# C)^{[\perp]}$ if and only if

$$(u_y + N(C^\# C)) \cap N(C^\# C)^{[\perp]} \neq \varnothing.$$

Now, assume that $w \in \mathcal{H}$ is a MILSS of $Cx = y$. Then, there exists $z_w \in N(C^\# C)$ such that $w = u_y + z_w$ and z_w is an ILSS of $P_{N(C^\# C)}x = -u_y$. By Lemma 3.1, $-u_y - P_{N(C^\# C)}z_w \in N(C^\# C)^{[\perp]}$. So,

$$w = u_y + z_w = u_y + P_{N(C^\# C)}z_w \in (u_y + N(C^\# C)) \cap N(C^\# C)^{[\perp]}.$$

Conversely, suppose that $w \in (u_y + N(C^\# C)) \cap N(C^\# C)^{[\perp]}$. Then, w is an ILSS of $Cx = y$ because $w \in u_y + N(C^\# C)$. Also, there exists $z_w \in N(C^\# C)$ such that $w = u_y + z_w$. Furthermore, since

$$-u_y - P_{N(C^\# C)}z_w = -u_y - z_w = -w \in N(C^\# C)^{[\perp]},$$

$z_w \in N(C^\# C)$ is an ILSS of $P_{N(C^\# C)}x = -u_y$. So, (4.2) implies that $w = u_y + z_w$ is a MILSS of $Cx = y$. □

In the rest of this section it is assumed that $N(C^\# C)$ is a $J_{\mathcal{H}}$-nonnegative pseudo-regular subspace of $\mathcal{H}$, aiming to describe the set of MILSS of $Cx = y$ in terms of J-normal projections.

Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is pseudo-regular and consider $y \in R(C) + R(C)^{[\perp]}$. Then, note that

$$u_y = (C^\# C)^\dagger C^\# y = 0 \quad \text{if and only if} \quad y \in R(C)^{[\perp]}.$$

In this case, $u \in \mathcal{H}$ is an ILSS of $Cx = y$ if and only if $u \in N(C^\# C)$. Moreover, by Proposition 4.1, $u \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if $u \in N(C^\# C)^\circ$.

Lemma 4.2. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is a $J_{\mathcal{K}}$-nonnegative pseudo-regular subspace of $\mathcal{K}$ and consider $y \in (R(C) + R(C)^{[\perp]}) \setminus R(C)^{[\perp]}$. Assume also that $N(C^\# C)$ is a $J_{\mathcal{H}}$-nonnegative pseudo-regular subspace of $\mathcal{H}$. Then, $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if there exists $P \in \mathcal{Q}_{N(C^\# C)}$ such that*

$$w = (I - P)u_y. \tag{4.3}$$

Proof. Given $C \in CR(\mathcal{H}, \mathcal{K})$ with $J_{\mathcal{K}}$-nonnegative pseudo-regular range $R(C)$, let $y \in (R(C) + R(C)^{[\perp]}) \setminus R(C)^{[\perp]}$. By the above remark, $u_y \neq 0$.

If $w \in \mathcal{H}$ is a MILSS of $Cx = y$, consider its orthogonal decomposition $w = u_y + z$, where $z \in N(C^\# C)$. Then, by (4.2), z is an ILSS of the equation $P_{N(C^\# C)}x = -u_y$. Also $u_y \in N(C^\# C)^\perp$ and, by Theorem 3.5, there exists $P \in \mathcal{Q}_{N(C^\# C)}$ such that

$$z = P_{N(C^\# C)}z = P(-u_y) = -Pu_y.$$

Thus, $w = u_y + z = u_y - Pu_y = (I - P)u_y$ for some $P \in \mathcal{Q}_{N(C^\# C)}$.

Conversely, if $w = (I - P)u_y$ for some $P \in \mathcal{Q}_{N(C^\# C)}$ then, since $u_y \in N(C^\# C) + N(C^\# C)^{[\perp]}$,

$$w = (I - P)P^\# u_y + (I - P)(I - P)^\# u_y = (I - P)(I - P)^\# u_y,$$

because, by Proposition 4.1,

$$u_y \in R(P) + R(P)^{[\perp]} = R(P) + N(P^\#) = N((I - P)P^\#).$$

Then, $w \in N(C^\# C)^{[\perp]}$ and, by Proposition 4.1, w is a MILSS of $Cx = y$. □

If $R(C)$ is a pseudo-regular subspace of $\mathcal{K}$, consider $E_0 = C(C^\# C)^\dagger C^\#$. If $y \in R(C) + R(C)^{[\perp]}$ then $Cu_y = E_0 y$ and

$$u_y = C^\dagger C u_y = C^\dagger E_0 y,$$

because $u_y \in N(C^\# C)^\perp \subseteq N(C)^\perp$. Moreover, if $y \in (R(C) + R(C)^{[\perp]}) \setminus R(C)^{[\perp]}$, applying this identity in (4.3) it follows that if $w \in \mathcal{H}$ is a MILSS of $Cx = y$ then there exists $P \in \mathcal{Q}_{N(C^\# C)}$ such that

$$w = (I - P)u_y = (I - P)C^\dagger E_0 y.$$

Furthermore, following the construction made in the proof of Theorem 3.5, it is easy to see that there exists $Q_0 \in \mathcal{Q}_{R(C)}$ such that $E_0 = Q_0^\# Q_0$. Hence, by Remark 2.1, $Q_0^\#(I - Q_0)y = 0$ and

$$w = (I - P)C^\dagger E_0 y = (I - P)C^\dagger Q_0^\# y.$$

Theorem 4.3. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ such that $R(C)$ is a $J_\mathcal{K}$-nonnegative pseudo-regular subspace of $\mathcal{K}$ and consider $y \in (R(C) + R(C)^{[\perp]}) \setminus R(C)^{[\perp]}$. Assume also that $N(C^\# C)$ is a $J_\mathcal{H}$-nonnegative pseudo-regular subspace of $\mathcal{H}$. Then, $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if there exists $P \in \mathcal{Q}_{N(C^\# C)}$ and $Q \in \mathcal{Q}_{R(C)}$ such that*

$$w = (I - P)C^\dagger Q^\# y = (I - P)C^\dagger E y, \tag{4.4}$$

where $E = QQ^\#$.

Proof. Under these assumptions, there exists a MILSS of $Cx = y$. Furthermore, in the discussion above it was shown that, if $w \in \mathcal{H}$ is a MILSS of $Cx = y$ then there exists $P \in \mathcal{Q}_{N(C^\# C)}$ and $Q_0 \in \mathcal{Q}_{R(C)}$ such that

$$w = (I - P)u_y = (I - P)C^\dagger Q_0^\# y = (I - P)C^\dagger E_0 y.$$

Conversely, given $P \in \mathcal{Q}_{N(C^\# C)}$ and $Q \in \mathcal{Q}_{R(C)}$, consider the vector $w = (I - P)C^\dagger Q^\# y$. By Remark 2.1 it follows that $Q^\#(I - Q)y = 0$ and $x := C^\dagger Q^\# y = C^\dagger QQ^\# y$. Then, $Cx = P_{R(C)} QQ^\# y = Q^\# Q y$ and

$$Qy - Cx = Qy - QQ^\# y = Q(I - Q^\#)y \in R(C)^\circ.$$

So, by Theorem 3.5, $x \in u_y + N(C^\# C)$. Also, $w = (I - P)x = (I - P)u_y$ and, following the same arguments as in Lemma 4.2, $w \in N(C^\# C)^{[\perp]}$. Therefore, by Proposition 4.1, w is a MILSS of $Cx = y$. □

In the description obtained for the MILSS of $Cx = y$ in the above theorem, the family of operators

$$\{(I - P)C^\dagger E : \ P \in \mathcal{Q}_{N(C^\# C)}\}$$

appears, where E is the J-selfadjoint projection onto an arbitrary complement of $R(C)^\circ$ in $R(C)$. Along the next section, this family is related to some of the generalized inverses of $C' := EC$. Note that, under the assumptions of Theorem 4.3, $R(C') = R(E)$ is regular and $N(C') = N(C^\# C)$ is pseudo-regular.

5. Generalized inverses related to indefinite least-squares problems

The next result describes a family of generalized inverses of a closed-range operator with pseudo-regular range and nullspace.

Proposition 5.1. *Suppose that $C \in CR(\mathcal{H}, \mathcal{K})$ is such that $R(C)$ and $N(C)$ are pseudo-regular subspaces of $\mathcal{K}$ and $\mathcal{H}$, respectively. Then, $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of*

$$\begin{cases} CXC = C, \\ XCX = X, \\ (CX)(CX)^{\#} = (CX)^{\#}(CX), \\ (XC)(XC)^{\#} = (XC)^{\#}(XC), \end{cases} \tag{5.1}$$

if and only if there exist $Q \in \mathcal{Q}_{R(C)}$ and $P \in \mathcal{Q}_{N(C)}$ such that $D = (I - P)C^{\dagger}Q$.

Proof. Given $Q \in \mathcal{Q}_{R(C)}$ and $P \in \mathcal{Q}_{N(C)}$, consider $D = (I - P)C^{\dagger}Q$. Since $CP = 0$,

$$CD = C(I - P)C^{\dagger}Q = CC^{\dagger}Q = P_{R(C)}Q = Q.$$

Also,

$$DC = (I - P)C^{\dagger}QC = (I - P)C^{\dagger}C = (I - P)P_{N(C)^{\perp}} = I - P,$$

because $R(P) = N(C)$. Therefore, CD is a $J_{\mathcal{K}}$-normal projection and DC is a $J_{\mathcal{H}}$-normal projection. Furthermore,

$$CDC = (CD)C = QC = C \quad \text{and} \quad DCD = (DC)D = (I - P)D = D.$$

Conversely, assume that $D \in L(\mathcal{K}, \mathcal{H})$ satisfies the equations in (5.1). Then, note that $Q := CD \in \mathcal{Q}_{R(C)}$, $P := I - DC \in \mathcal{Q}_{N(C)}$ and

$$(I - P)C^{\dagger}Q = (DC)C^{\dagger}(CD) = D(CC^{\dagger}C)D = DCD = D. \qquad \square$$

Let $E \in L(\mathcal{K})$ be a J-selfadjoint projection such that $R(E) \dotplus R(C)^{\circ} = R(C)$. Applying the above proposition to $C' = EC$ it is possible to reinterpret the operators of the form $(I - P)C^{\dagger}E$ (with $P \in \mathcal{Q}_{N(C^{\#}C)}$) as a particular family of generalized inverses of C'.

Corollary 5.2. *Suppose that $C \in CR(\mathcal{H}, \mathcal{K})$ is such that $R(C)$ and $N(C^{\#}C)$ are pseudo-regular subspaces of $\mathcal{K}$ and $\mathcal{H}$, respectively. Consider a J-selfadjoint projection $E \in L(\mathcal{K})$ such that $R(E) \dotplus R(C)^{\circ} = R(C)$. If $C' = EC$ then the operators in the set*

$$\{(I - P)C^{\dagger}E : \ P \in \mathcal{Q}_{N(C^{\#}C)}\},$$

are the solutions in $L(\mathcal{K}, \mathcal{H})$ of

$$\begin{cases} C'XC' = C', \\ XC'X = X, \\ C'X = E, \\ (XC')(XC')^{\#} = (XC')^{\#}(XC'). \end{cases} \tag{5.2}$$

Proof. Consider a J-selfadjoint projection $E \in L(\mathcal{K})$ such that $R(E) \dotplus R(C)^{\circ} = R(C)$. If $C' = EC$ note that

$$R(C') = R(E) \quad \text{and} \quad N(C') = N(C^{\#}C).$$

Then, apply Proposition 5.1 to C'. $\square$

Thus, the statement of Theorem 4.3 can be rephrased as: $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if $u = Dy$ where $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of Eq. (5.2).

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