MACI, 2 (2009), 225-228 E.M.Mancinelli, E.A.Santillan Marcus, D.A.Tarzia (Eds.)

ANALYSIS OF 2D TIME-DOMAIN SEISMOELECTRIC MODELING

Juan E. Santos[♭, †] and Fabio I. Zyserman[♭]

[♭] *CONICET, Fac. de Cs. Astronómicas y Geofísicas, Universidad Nacional de La Plata, Paseo del Bosque s/n, B1900FWA - La Plata, Argentina*
[†] *Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, Indiana, 47907-2067, USA; santos@math.purdue.edu*

Abstract: This work analyzes the equations for 2D seismoelectric modeling in poroelastic fluid-saturated media. The model involves the simultaneous solution of Biot's equations of motion and Maxwell's equations, coupled via an electrokinetc coefficient and employs absorbing boundary conditions at the artificial boundaries. Results on existence and uniqueness of the solution of the differential problem are presented.

Keywords: *Seismoelectric Modeling, Poroelasticity, Electromagnetics, Finite element methods.*
2000 AMS Subject Classification: 65M60

1 PROBLEM STATEMENT

Let $\Omega = \Omega_a \cup \Omega_p$ be a 2D-rectangular domain, where Ω_a and Ω_p are associated with the air and subsurface (disjoint) parts of Ω, respectively. For 2D PSVTM seismoelectric modeling the electric and magnetic fields E and H_2, the solid displacement vector u^s and the relative fluid displacement vector u^f satisfy the equations [1, 2]

$$\varepsilon \frac{\partial E}{\partial t} + \sigma E - \text{curl } H_2 + L_0 \frac{\eta}{\kappa_0} \frac{\partial u^f}{\partial t} = 0, \quad \Omega, \tag{1}$$

$$\text{curl } E + \frac{\partial H_2}{\partial t} = 0, \quad \Omega, \tag{2}$$

$$\rho_b \frac{\partial^2 u^s}{\partial t^2} + \rho_f \frac{\partial^2 u^f}{\partial t^2} - \nabla \cdot \tau(u) = F^{(s)}, \quad \Omega_p, \tag{3}$$

$$\rho_f \frac{\partial^2 u^s}{\partial t^2} + m \frac{\partial^2 u^f}{\partial t^2} + \frac{\eta}{\kappa_0} \frac{\partial u^f}{\partial t} u^f + \nabla p_f = F^{(f)}, \quad \Omega_p, \tag{4}$$

$$\tau_{lm}(u) = 2G\, \varepsilon_{lm}(u^s) + \delta_{lm} \left(\lambda_c \nabla \cdot u^s + \alpha K_{av} \nabla \cdot u^f \right), \quad \Omega_p, \tag{5}$$

$$p_f(u) = -\alpha K_{av} \nabla \cdot u^s - K_{av} \nabla \cdot u^f, \quad \Omega_p. \tag{6}$$

This problem arises when modeling the propagation of man generated waves in a poroelastic fluid saturated medium which give rise, through the electrokinetic coupling coefficient L_0 to detectable electromagnetic waves and seismic waves. All coefficients appearing in (1)-(6) are the usual ones in the respective theories[1, 2, 3, 4]; they are assumed to be real. Notice that if $L_0 = 0$ the above problem decouples into two separate ones, namely one associated with Biot's equations, and another one associated with Maxwell's equations.

To solve equations (1)-(6) in our 2D domain Ω we need a collection of boundary conditions. Let Γ denote the boundary of Ω and let $\Gamma_{a,p} = \overline{\Omega}_a \cap \overline{\Omega}_p$ denote the free surface (the surface of the Earth) Also let $\Gamma_a = \partial\Omega_a \setminus \Gamma_{a,p}$, $\Gamma_p = \partial\Omega_p \setminus \Gamma_{a,p}$ denote the artificial boundaries of Ω_a and Ω_p, respectively. Also, if Γ_s is either an inner interface in Ω or a part of the boundaries Γ, Γ_p or $\Gamma_{a,p}$, set

$$\mathcal{G}_{\Gamma_s}(u) \quad = \quad \Big(\tau(u)\nu \cdot \nu, \tau(u)\nu \cdot \chi, p_f(u) \Big)^t, \tag{7a}$$

$$S_{\Gamma_s}(u) \quad = \quad \Big(u^s \cdot \nu, u^s \cdot \chi, u^f \cdot \nu \Big)^t, \tag{7b}$$

where , where ^{t} denotes the transpose, ν is the unit outer normal on Γ_s and χ is a unit tangent on Γ_s oriented counterclockwise.

Then, for $\omega > 0$ consider the solution of (1)-(6) with the absorbing boundary conditions [5], [6]

$$-\varepsilon^{1/2} E \cdot \chi + H_2 = 0, \quad \text{on} \quad \Gamma, \tag{8}$$

$$-\mathcal{G}_{\Gamma_p}(u) = \mathcal{D} S_{\Gamma_p}(\frac{\partial u}{\partial t}), \quad \text{on} \quad \Gamma_p, \tag{9}$$

the free surface condition

$$-\mathcal{G}_{\Gamma_p}(u) = 0, \quad \text{on} \quad \Gamma_{a,p}, \tag{10}$$

and the initial conditions

$$\begin{aligned} &E(t=0) = E_0, \quad H(t=0) = H_0 \\ &u^s(t=0) = u_0^s, \quad u^f(t=0) = u_0^f, \\ &\frac{\partial u^s}{\partial t}(t=0) = u_1^s, \quad \frac{\partial u^f}{\partial t}(t=0) = u_1^f. \end{aligned} \tag{11}$$

Remark: The matrix $\mathcal{D}$ in (9) is positive definite.

2 A Weak formulation

A variational formulation of our problem can be stated as follows: find $(E, H_2, u^s, u^f) \in H(\text{curl}, \Omega) \times L^2(\Omega) \times [H^1(\Omega_p)]^2 \times H(\text{div}, \Omega_p)$ satisfying

$$\begin{aligned} (\varepsilon \frac{\partial E}{\partial t}, \psi) + (\sigma E, \psi) - (H_2, \text{curl}\, \psi) + \left(L_0 \frac{\eta}{\kappa_0} \frac{\partial u^f}{\partial t}, \psi \right)_{\Omega_p} \\ + \left\langle \left(\frac{\varepsilon}{\mu} \right)^{1/2} E \cdot \chi, \psi \cdot \chi \right\rangle = 0, \quad \psi \in H(\text{curl}, \Omega), \end{aligned} \tag{12}$$

$$(\text{curl}\, E, \varphi) + (\mu \frac{\partial H_2}{\partial t}, \varphi) = 0, \quad \varphi \in L^2(\Omega).$$

$$\begin{aligned} (\mathcal{P} \frac{\partial^2 u}{\partial t^2}, v)_{\Omega_p} + (\frac{\eta}{\kappa_0} \frac{\partial u^f}{\partial t}, v^f)_{\Omega_p} + \mathcal{A}(u, v) + \left\langle \mathcal{D} S_{\Gamma_p}(\frac{\partial u}{\partial t}), S_{\Gamma_p}(v) \right\rangle_{\Gamma_p} \\ = (F, v)_{\Omega_p}, \quad v = (v^s, v^f) \in [H^1(\Omega_p)]^2 \times H(\text{div}, \Omega_p). \end{aligned} \tag{13}$$

In (13) $F = (F^s, F^f)$ and $A(u, v)$ is the bilinear form defined as

$$\begin{aligned} \mathcal{A}(u, v) = \sum_{l,m} (\tau_{lm}(u), \varepsilon_{lm}(v^s))_{\Omega_p} - \left(p_f(u), \nabla \cdot v^f \right)_{\Omega_p} = (\mathbf{M}\, \widetilde{\epsilon}(u), \widetilde{\epsilon}(v))_{\Omega_p}, \\ u, v \in [H^1(\Omega_p)]^2 \times H(\text{div}, \Omega_p), \end{aligned} \tag{14}$$

Remark: the matrices $\mathcal{P}$ and $\mathbf{M}$ are symmetrix and positive definite.

3 Uniqueness

Let us analyze the uniqueness of the solution of our continous problem. Set $F = 0$ in (13) and set to zero the initial conditions in (11). Then choose $\varphi = H_2$ in (13) to get

$$(\text{curl}\, E, H_2) + (\mu \frac{\partial H_2}{\partial t}, H_2) = 0. \tag{15}$$

Also, choose $\psi = E$ in (12) and use (15) to obtain

$$\begin{aligned} (\varepsilon \frac{\partial E}{\partial t}, E) + (\sigma E, E) + (\mu \frac{\partial H_2}{\partial t}, H_2) + \left(L_0 \frac{\eta}{\kappa_0} u^f, E \right)_{\Omega_p} \\ + \left\langle \left(\frac{\varepsilon}{\mu} \right)^{1/2} E \cdot \chi, E \cdot \chi \right\rangle = 0. \end{aligned} \tag{16}$$

Next, choose $v^s = \frac{\partial u^s}{\partial t}, v^f = \frac{\partial u^f}{\partial t}$ in (13) and add the resulting equation to (16) to get

$$\frac{1}{2}\frac{d}{dt}\left[\left(\mathcal{P}\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right)_{\Omega_p} + (\mathbf{M}\,\widetilde{\epsilon}(u), \widetilde{\epsilon}(u))_{\Omega_p} + (\varepsilon E, E) + (\mu H_2, H_2)\right] \tag{17}$$
$$+\Phi(E, \frac{\partial u^f}{\partial t}) + (\sigma E, E)_{\Omega_a} + \left\langle \left(\frac{\varepsilon}{\mu}\right)^{1/2} E\cdot\chi, E\cdot\chi\right\rangle + \left\langle \mathcal{D}S_{\Gamma_p}(\frac{\partial u}{\partial t}), S_{\Gamma_p}(\frac{\partial u}{\partial t})\right\rangle_{\Gamma_p} = 0,$$

where

$$\Phi(E, \frac{\partial u^f}{\partial t}) = (\sigma E, E)_{\Omega_p} + \left(L_0\frac{\eta}{\kappa_0}u^f, E\right)_{\Omega_p} + (\frac{\eta}{\kappa_0}\frac{\partial u^f}{\partial t}, \frac{\partial u^f}{\partial t})_{\Omega_p}. \tag{18}$$

Set

$$A_{\min,p} = \inf\{A(x_1,x_3), (x_1,x_3)\in\Omega_p\},\ \ A = \sigma, \kappa_0,\ \ \kappa_{0,\max} = \sup\{\kappa_0(x_1,x_3), (x_1,x_3)\in\Omega_p\}.$$

Assume that

$$\sigma_{\min,p} > 0, \quad \kappa_{0,\min,p} > 0, \quad \kappa_{0,\max,p} < \infty, \tag{19}$$
$$C_1 = \min\left(\sigma_{\min,p} - \frac{L_0\eta}{2\kappa_{0,\max}, p}, \frac{\eta}{\kappa_{0,\max}, p} - \frac{L_0\eta}{2\kappa_{0,\max}, p}\right) > 0. \tag{20}$$

Thus,

$$\Phi(E, \frac{\partial u^f}{\partial t}) \ge C_1\left(\|E\|^2_{0,\Omega_p} + \|u^f\|^2_{0,\Omega_p}\right). \tag{21}$$

Next, since the matrix $\mathbf{M}$ is positive definite, Korn' second inequality [7, 8] implies that

$$(\mathbf{M}\,\widetilde{\epsilon}(u), \widetilde{\epsilon}(u))_{\Omega_p} \ge C_2\left(\|u^s\|^2_{1,\Omega_p} + \|u^f\|^2_{H(\mathrm{div},\Omega_p)}\right) \tag{22}$$
$$-C_3\left(\|u^s\|^2_{0,\Omega_p} + \|u^f\|^2_{0,\Omega_p}\right).$$

Set

$$\mathcal{Z} = [H^1(\Omega_p)]^2 \times H(\mathrm{div}, \Omega_p),$$

provided with the natural norm

$$\|u\|_{\mathcal{Z}} = \left(\|u^s\|^2_{1,\Omega_p} + \|u^f\|^2_{H(\mathrm{div},\Omega_p)}\right).$$

Then choose a constant ζ such that $\zeta > C_3$ and define the bilinear form

$$\mathcal{A}_\zeta(u,v) = \mathcal{A}(u,v) + \zeta(u,v),$$

so that $\mathcal{A}_\zeta$ is $\mathcal{Z}$-coercive, *i.e.*,

$$\mathcal{A}_\zeta(u,u) \ge C_2\|u\|^2_{\mathcal{Z}}, \tag{23}$$

Next, add to (17) the inequality

$$\frac{\zeta}{2}\frac{d}{dt}\|u\|^2_{0,\Omega_p} \le \frac{\zeta}{2}\left(\|u\|^2_{0,\Omega_p} + \|\frac{\partial u}{\partial t}\|^2_{0,\Omega_p}\right),$$

integrate in time the resulting inequality and use (21) and (23) to obtain

$$\frac{1}{2}\left[\left(\mathcal{P}\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)_{\Omega_p}(t)+C_2\|u(t)\|_{\mathcal{Z}}^2+(\varepsilon E,E)(t)+(\mu H_2,H_2)(t)\right] \tag{24}$$
$$+C_1\int_0^t\left(\|E(s)\|_{0,\Omega_p}^2+\|u^f(s)\|_{0,\Omega_p}^2\right)ds+\int_0^t(\sigma E,E)_{\Omega_a}(s)ds$$
$$+\int_0^t\left\langle\left(\frac{\varepsilon}{\mu}\right)^{1/2}E\cdot\chi,E\cdot\chi\right\rangle(s)ds+\int_0^t\left\langle\mathcal{D}S_{\Gamma_p}(\frac{\partial u}{\partial t}),S_{\Gamma_p}(\frac{\partial u}{\partial t})\right\rangle_{\Gamma_p}(s)ds$$
$$\leq\int_0^t\left(\|u(s)\|_{0,\Omega_p}^2+\|\frac{\partial u}{\partial t}(s)\|_{0,\Omega_p}^2\right)ds.$$

Assume that ε and μ are bounded below and above by positive constants, so that

$$0<\varepsilon_*\leq\varepsilon\leq\varepsilon^*<\infty,\quad 0<\mu_*\leq\mu\leq\mu^*<\infty. \tag{25}$$

Thus apply Gronwall's lemma in (24), note that all terms in the left-hand side of (24) are nonnegative and use (25) to conclude that

$$\|u(t)\|_{\mathcal{Z}}=0,\quad \|E(t)\|_0=0,\quad \|H_2(t)\|_0=0,\quad \forall t,$$

so that uniqueness holds for the solution of our differential problem. Assuming sufficient regularity on the initial data and the external sources, existence can be derived using the compacteness argument of Lions [9] with an argument similar to that given in [10]. For brevity the argument is ommited. The result is summarized in the following theorem.

Theorem 1 *Assume the validity of* (20) *and* (25) *and that the matrices* $\mathbf{M}$ *and* $\mathcal{D}$ *are positive definite. Then there exists a unique solution of problem* (2)-(4) *with the boundary conditions* (8)-(9) *and the initial conditions* (11).

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