SPECIAL ECCENTRIC VERTICES FOR CHORDAL AND DUALLY CHORDAL GRAPHS AND RELATED CLASSES

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Abstract: It is known that any vertex of a chordal graph has an eccentric vertex which is simplicial. Here we prove similar properties in related classes of graphs where the simplicial vertices will be replaced by other special types of vertices.

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1 BASIC DEFINITIONS

For a graph G, $V(G)$ denotes the set of its vertices and $E(G)$ that of its edges. A *complete* is a set of pairwise adjacent vertices. The subgraph *induced* by $A \subseteq V(G)$, $G[A]$, has A as vertex set and two vertices are adjacent in $G[A]$ if they are adjacent in G.

Given two vertices v and w of a graph G, the *distance* between v and w, or $d(v, w)$, is the length of a shortest path connecting v and w in G. When such a path does not exist it may be said that $d(v, w) = \infty$. For a vertex $v \in V(G)$, the *open neighborhood* of v, $N(v)$, is the set of all vertices adjacent to v. The *closed neighborhood* of v, $N[v]$, is defined by $N[v] = N(v) \cup \{v\}$. The *disk* centered at vertex v with radius k is the set of vertices at distance at most k from v and it is indicated by $N^k[v]$. The *eccentricity* of v is $ecc(v) = max\{d(v, w), w \in V(G)\}$. A vertex w in G is called *eccentric* of v if no vertex in $V(G)$ is further away from v than w, that is, if $ecc(v) = d(v, w)$.

The *kth-power*, G^k, of a graph G is a graph which has the same vertices as G, being two of them adjacent in G^k if the distance between them is at most k in G.

A *chord* of a cycle is an edge joining two nonconsecutive vertices of the cycle. *Chordal* graphs are defined as those without chordless cycles of length at least four.

A vertex v is *simplicial* if $N[v]$ is a complete. A linear ordering $v_1v_2...v_n$ of vertices of a graph G is called a *perfect elimination ordering* if, for $1 \leq i \leq n$, v_i is simplicial in $G_i = G[\{v_i, ..., v_n\}]$.

One of the most classical characterizations of chordal graphs states that a graph is chordal if and only if it has a perfect elimination ordering.

A vertex $w \in N[v]$ is a *maximum neighbor* of v if $N^2[v] \subseteq N[w]$. A linear ordering $v_1...v_n$ of vertices of G is a *maximum neighborhood ordering* if, for all $1 \leq i \leq n$, v_i has a maximum neighbor in G_i. *Dually chordal graphs* can be defined as those possessing a maximum neighborhood ordering.

2 ECCENTRIC VERTICES

We can see that vertices with a maximum neighbor are as important for dually chordal graphs as simplicial vertices are for chordal graphs. Then, if any vertex of a chordal graph has a simplicial eccentric vertex the question wether any vertex of a dually chordal graph has an eccentric vertex with a maximum neighbor arises. The answer is yes, and before proving it we need some previous results.

Lemma 1 *[1] If G is a dually chordal graph and A is a subset of $V(G)$ such that any pair of vertices of A is at a distance not greater than 2, then there is a vertex w with $A \subseteq N[w]$.*

Lemma 2 *[1] If G is dually chordal then G^2 is chordal.*

Lemma 3 *Let G be a dually chordal graph and v a simplicial vertex in G^2. Then v has a maximum neighbor in G.*

Proof. As v is simplicial in G^2 the distance in G between any pair of vertices of $N^2[v]$ is at most 2. Applying Lemma 1 gives a vertex w such that $N^2[v] \subseteq N[w]$. Then w is a maximum neighbor of v. □

Now the major result can be proved. From now on it will be assumed that G is always a connected graph. Otherwise the proofs are trivial.

Theorem 1 *Let G be a dually chordal graph and v a vertex of G. There exists an eccentric vertex of v with maximum neighbor.*

Proof. First suppose that $ecc_G(v)$ is odd. As G^2 is chordal we can choose a vertex w simplicial in G^2 which is eccentric of v in G^2. Hence, by Lemma 3, w has a maximum neighbor in G. Note first that, because of the definition of G^2, if two vertices are at distance k in G their distance in G^2 is $\frac{k}{2}$ if k is even or $\frac{k+1}{2}$ if k is odd. Furthermore any eccentric vertex of v in G will be also eccentric in G^2, implying that the eccentricity of v in G^2 equals $\frac{ecc_G(v)+1}{2}$ because $ecc_G(v)$ is odd. By using the definition of G^2 again and that $d_{G^2}(v,w) = \frac{ecc_G(v)+1}{2}$, we have two possible values for $d_G(v,w)$, namely, $ecc_G(v)$ or $ecc_G(v)+1$. The definition of eccentricity implies that $d(v,w) = ecc_G(v)$ and thus w is the required vertex.

If $ecc_G(v)$ is even, let G' be a graph obtained from G by adding a new vertex v' and making it adjacent to v. Then G' is dually chordal. In fact, if $v_1...v_n$ is a maximum neighborhood ordering for G then $v'v_1...v_n$ is a maximum neighborhood ordering of G'. It is valid that $ecc_{G'}(v')$ is odd and by proceeding like in the previous paragraph there is a vertex u with a maximum neighbor in G' (and so in G) such that $d(v',u) = ecc_{G'}(v')$. It can be easily verified that u is the desired vertex. □

Corollary 1 *If G is a nontrivial, i.e., not composed of just one vertex, dually chordal graph then there are two vertices v_1 and v_2 with maximum neighbors and such that $d(v_1,v_2) = diam(G)$.*

Proof. Let $k = diam(G)$ and x,y two vertices with $d(x,y) = k$. Then there exists a vertex v_1 with maximum neighbor and eccentric of x, so $d(x,v_1) = k$. And likewise there is a vertex v_2 with a maximum neighbor and eccentric of v_1 and consequently $d(v_1,v_2) = k$. □

At this moment it is interesting to determine if similar properties are valid for more specific types of graphs. The answer is affirmative and we will prove it for power chordal and doubly chordal graphs.

A graph G is said to be *power chordal* if all of its powers are chordal. It is true that a graph is power chordal if and only if G and G^2 are chordal [1]. A graph is *doubly chordal* if it is chordal and dually chordal. Any vertex of it which is simplicial and has a maximum neighbor is called *doubly simplicial*.

It is known that a power chordal graph is complete or there are two nonadjacent vertices which are simplicial in both G and G^2. The demonstration can be seen in [1]. A similar technique enables to prove the following result:

Theorem 2 *Let G be a power chordal graph. If $v \in V(G)$ then there exists a vertex w eccentric of v in G^2 which is simplicial in both G and G^2.*

Proof. The proof is direct if G^2 is complete. Assume that G^2 is not complete. Since G^2 is chordal we can take a vertex u which is simplicial in G^2 and eccentric of v in G^2. If u is also simplicial in G there is nothing else to do and we can set $w = u$. On the contrary, let x and y be two nonadjacent neighbors of u and S a minimal xy-separator in G. Then S is a complete because G is chordal [2] and $u \in S$.

Let $G[A]$ and $G[B]$ be the connected components of $G-S$ containing x and y respectively. Without loss of generality we can assume that $v \notin A$. It holds that $G[A \cup S]$ is chordal and since S is a complete minimal separator $(G[A \cup S])^2 = G^2[A \cup S]$ and thus $(G[A \cup S])^2$ is also chordal. Then we have two possibilities: either $G[A \cup S]$ is complete or contains two nonadjacent vertices which are both simplicial in $G[A \cup S]$ and $G^2[A \cup S] = (G[A \cup S])^2$ [1]. Whichever the case we conclude that the set A contains a vertex w which is simplicial in $G[A \cup S]$ and $G^2[A \cup S]$. It is evident that w is simplicial in G. Now it will be demonstrated that w is also simplicial in G^2. If $N^2[w] \subseteq A \cup S$ it is obvious. Otherwise, w must be adjacent to a vertex w' in S. If $z \in N^2[w] \cap (A \cup S)$, then $z \in N^2[u]$ because $u \in N^2[w]$ (note that $w \in N[w']$ and $w' \in N[u]$) and w is simplicial in $G^2[A \cup S]$. If $z \in N^2[w] - (A \cup S)$ then again $z \in N^2[u]$ because any path of length two joining w and z (vertices which are in different connected components of $G - S$) must have its intermediate vertex in S, which could be u or adjacent to it because S is a complete. This makes a path between z and u of length at most two possible. Therefore $N^2[w] \subseteq N^2[u]$ and as u is simplicial in G^2 so is w.

Since v and w are in different connected components of $G - S$ any path joining them must include a vertex in S, and so in $N[u]$. We can conclude that $d_G(u, v) \leq d_G(v, w)$ and then $d_{G^2}(v, w)$ is maximum and has the required properties. □

Theorem 3 *Let G be a power chordal graph. If $v \in V(G)$ then there exists an eccentric vertex of v, in G, which is simplicial in G and G^2.*

Proof. The proof is very similar to that of Theorem 1, so we will just give a sketch of it.

We suppose at first that $ecc(v)$ is odd and applying Theorem 2 will give the required vertex.

And if $ecc(v)$ is even the graph G' is again introduced.

□

Corollary 2 *Let G be a doubly chordal graph. If $v \in V(G)$ then there exists an eccentric vertex of v which is doubly simplicial.*

Proof. As G is dually chordal G^2 is chordal, so the previous theorem can be applied to get a vertex w simplicial in G and G^2 and eccentric of v. Because of Lemma 3 w has a maximum neighbor in G, so it is doubly simplicial. □

So far it was possible to prove the existence of eccentric vertices with characteristics distinguishing all the classes related to chordal and dually chordal graphs that have been discussed. One that was not mentioned yet is that of *strongly chordal* graphs and fortunately a similar property can be deduced.

A vertex v of a graph G is *simple* if the set $\{N[u] : u \in N[v]\}$ is totally ordered by inclusion. From this definition we infer that, particularly, simple vertices are simplicial and have a maximum neighbor. A linear ordering $v_1 v_2 ... v_n$ of $V(G)$ is called a *simple elimination ordering* of G if, for $1 \leq i \leq n$, v_i is simple in G_i. Strongly chordal graphs are just those possessing at least one such ordering. One of the main characteristics of strongly chordal graphs is that they are hereditary. In fact, being a strongly chordal graph is equivalent to being a hereditary dually chordal graph.

In connection with eccentric vertices we have the following:

Lemma 4 *Let $v \in V(G)$ and w be a maximum neighbor of v with $ecc(w) > 1$ and u such that $d(u, v) \geq 2$. Then $d(u, v) = d(u, w) + 1$ and any vertex eccentric of w is also eccentric of v and vice versa.*

Proof. The property is true if $d(u, v) = 2$ due to the definition of maximum neighbor, so suppose now that $d(u, v) > 2$. By the triangle inequality $d(u, v) \leq d(u, w) + d(w, v)$, that is, $d(u, v) \leq d(u, w) + 1$. Let $vv1v2...u$ be a shortest path from v to u. Then $wv_2...u$ is a path from w to u of length $d(u, v) - 1$. Therefore $d(u, v) - 1 \geq d(u, w)$ and hence $d(u, v) \geq d(u, w) + 1$. Then the equality $d(u, v) = d(u, w) + 1$ holds. This implies that any vertex eccentric of w is at distance greater than or equal to 3 of v and consequently

$$d(v, u) = ecc(v) \Leftrightarrow d(v, u) = max\{d(v, x) : x \in V(G)\} \Leftrightarrow d(v, u) = max\{d(v, x) : x \in V(G), d(v, x) \geq 3\} \Leftrightarrow$$

$$d(w, u) + 1 = max\{d(w, x) + 1 : x \in V(G), d(w, x) \geq 2\} \Leftrightarrow d(w, u) = max\{d(w, x) : x \in V(G), d(w, x) \geq 2\}$$

$$\Leftrightarrow d(w, u) = max\{d(w, x) : x \in V(G)\} \Leftrightarrow d(w, u) = ecc(w)$$

□

Theorem 4 *Let G be a strongly chordal graph. If $v \in V(G)$ then there exists an eccentric vertex of v which is simple.*

Proof. It will be by induction on $n = |V(G)|$. The property is obviously valid when $n = 1$. Suppose now that it is always valid when $n = k$, $k \geq 1$, and that G is a strongly chordal graph with $|V(G)| = k + 1$. Given v, the proof will be divided into cases.

Case 1: G has at least one universal vertex.

Let w be a universal vertex of G. If w is simple then G is complete because simple vertices are simplicial and thus the existence of an eccentric simple vertex is evident. Otherwise, it is trivial in case that $v = w$, so assume now that $v \neq w$ and that w is not simple. Then we consider the strongly chordal graph $G - w$. In case that $G - w$ is not connected, any vertex simple in $G - w$ and located in a connected component different from that of v is an eccentric simple vertex for v in G. If $G-w$ is connected, applying the inductive hypothesis yields an eccentric simple vertex u for v in $G - w$. Then again it will be simple and eccentric of v in G.

Case 2: G has not a universal vertex.

Case 2a: v is simple.

Let v' be a maximum neighbor of v. $G - v$ is strongly chordal and applying the inductive hypothesis on this subgraph gives a simple eccentric vertex of v' in $G - v$ which will be named w. Then it is true that $d(v', w) \geq 2$ because otherwise v' would be universal in G. Now, as v' is a maximum neighbor of v in G, and so $N^2[v] \subseteq N[v']$, we conclude that $d(v, w) \geq 3$ and thus the neighborhoods of vertices in $N[w]$ are coincident in G and $G - v$, from what we can deduce that w is simple in G. And because of Lemma 4 w is also eccentric of v.

Case 2.b: v is not simple and there is a simple vertex which is not adjacent to v.

Let w be a simple vertex not adjacent to v. If it is also eccentric we are done. If not, consider the strongly chordal graph $G - w$, which possesses a simple vertex w' eccentric of v. As removing a simplicial vertex does not change the distance between the other vertices (simplicial vertices are never intermediate vertices in shortest paths) w' is also eccentric in G, so it suffices to prove that w' is simple in G. If w' is not simple in G, there is at least one vertex in $N[w']$ whose neighborhood is not the same in G and $G - w$, implying that $w \in N^2[w']$. Let u be a maximum neighbor of w in G. Then u is adjacent to w' and therefore $d(v, w') \leq d(v, u) + 1$, which combined with Lemma 4 implies that $d(v, w') \leq d(v, w)$, contradicting that w was not an eccentric vertex of v. Consequently w' is necessarily simple.

We claim that all these cases are enough to prove the property for every strongly chordal graph. In fact, if v is not simple and is adjacent to all the simple vertices it will be proved that $diam(G) \leq 2$ and thus G has a universal vertex by Lemma 1. Let x and y be vertices such that $d(x, y) = diam(G)$. If $diam(G) \geq 3$ then $\{x, y\} \nsubseteq N[v]$ so we can assume without loss of generality that $x \notin N[v]$. Since all simple vertices are simplicial and adjacent to v we conclude that none of them is adjacent to x. Then, by case **2.b**, x has a simple eccentric vertex x' and thus $d(x, x') = diam(G)$. By case **2.a** we know that x' has a simple eccentric vertex x'' so $d(x', x'') = diam(G)$. But $d(x', x'') \leq 2$, contradicting that $diam(G) \geq 3$. □

Corollary 3

- *If G is a nontrivial power chordal graph there are two vertices v_1 and v_2, simplicial both in G and G^2, such that $d(v_1, v_2) = diam(G)$.*
- *If G is a nontrivial doubly/strongly chordal graph there are two doubly simplicial/simple vertices v_1 and v_2 such that $d(v_1, v_2) = diam(G)$.*

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