An exact formula for the radiation of a moving quark in $\mathcal{N} = 4$ super Yang Mills

Diego Correa,[a] **Johannes Henn,**[b] **Juan Maldacena**[b] **and Amit Sever**[b,c]

[a]*Instituto de Física La Plata, Universidad Nacional de La Plata,*
C.C. 67, 1900 La Plata, Argentina

[b]*School of Natural Sciences, Institute for Advanced Study,*
Princeton, NJ 08540, U.S.A.

[c]*Perimeter Institute for Theoretical Physics,*
Waterloo, Ontario N2J 2W9, Canada

E-mail: correa@fisica.unlp.edu.ar, jmhenn@ias.edu, malda@ias.edu,
asever@ias.edu

ABSTRACT: We derive an exact formula for the cusp anomalous dimension at small angles. This is done by relating the latter to the computation of certain 1/8 BPS Wilson loops which was performed by supersymmetric localization. This function of the coupling also determines the power emitted by a moving quark in $\mathcal{N} = 4$ super Yang Mills, as well as the coefficient of the two point function of the displacement operator on the Wilson loop. By a similar method we compute the near BPS expansion of the generalized cusp anomalous dimension.

Contents

1 Introduction

In this article we study some aspects of the cusp anomalous dimension of locally BPS Wilson loops in $\mathcal{N}=4$ super Yang Mills. We consider the leading order expansion of Γ_{cusp} at small angles,

$$\Gamma_{\text{cusp}}(\phi) = -B(\lambda, N)\phi^2 + o(\phi^4) \tag{1.1}$$

where we defined a function which we call the "Bremsstrahlung function" B. B is a function of the coupling. We compute it exactly at all values of the coupling and for all N, by relating it to a computation that uses supersymmetric localization [1–12]. We derive an exact expression for B as

$$B = \frac{1}{2\pi^2}\lambda\partial_\lambda \log\langle W_\circledcirc\rangle \tag{1.2}$$

$$\langle W_\circledcirc\rangle = \frac{1}{N}L_{N-1}^1\left(-\frac{\lambda}{4N}\right)e^{\frac{\lambda}{8N}}\,, \qquad \lambda = g_{\text{YM}}^2 N \tag{1.3}$$

$$B = \frac{1}{4\pi^2}\frac{\sqrt{\lambda}I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} + o(1/N^2) \tag{1.4}$$

where L is the modified Laguerre polynomial and $W_\circledcirc$ is the 1/2 BPS circular Wilson loop computed in [1, 11, 12]. The last line gives the planar expression. The result is for the U(N) theory. For SU(N) we simply subtract the U(1) contribution, $B_{\text{SU}(N)} = B_{\text{U}(N)} - \frac{\lambda}{16\pi^2 N^2}$. This quantity B also determines the energy emitted by a moving quark

$$\Delta E = 2\pi B \int dt(\dot{v})^2 \tag{1.5}$$
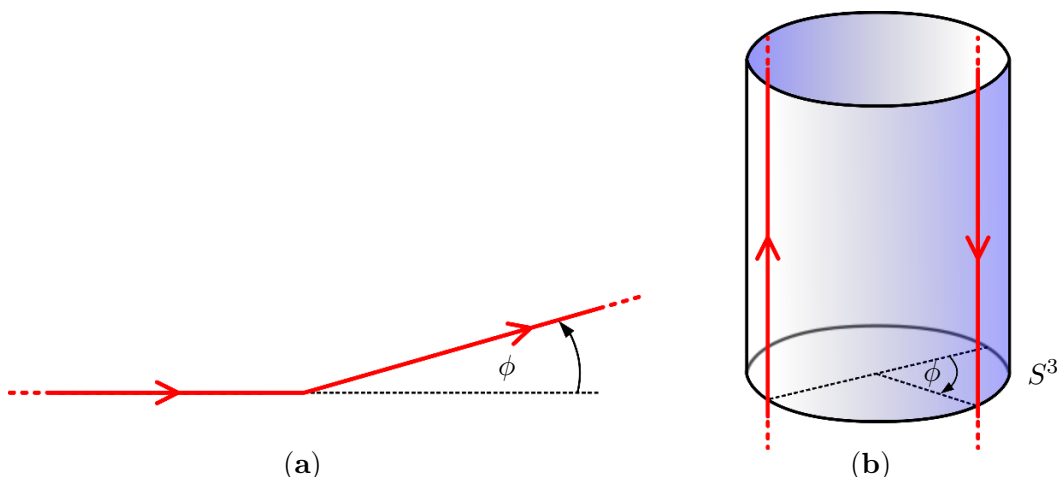
Figure 1. (a) A Wilson line that makes a turn by an angle ϕ. (b) Under the plane to cylinder map, the same line is mapped to a quark anti-quark configuration. The quark and antiquark are sitting at two points on S^3 at a relative angle of $\pi - \phi$. Of course, they are extended along the time direction.

in the small velocity limit. The result for any velocity can be obtained by performing a boost and it is the same old formula that one has in electrodynamics, up to the replacement $\frac{2e^2}{3} \rightarrow 2\pi B$, see [13] for a discussion at strong coupling. Its appearance in (1.5) is what prompted us to call it the Bremsstrahlung function.

The cusp anomalous dimension is an interesting quantity that is related to a variety of physical observables as particular cases.

Originally it was defined in [14] as the logarithmic divergence that arises for a Wilson loop operator when there is a cusp in the contour. A cusp is a region where a straight line makes a sudden turn by an angle ϕ, see figure 1(a). In that case the Wilson loop develops a logarithmic divergence of the form

$$\langle W \rangle \sim e^{-\Gamma_{\text{cusp}}(\phi,\lambda) \log \frac{L}{\tilde{\epsilon}}} \tag{1.6}$$

where L is an IR cutoff and $\tilde{\epsilon}$ a UV cutoff. One can also consider the continuation $\phi = i\varphi$ so that now φ is a boost angle in Lorentzian signature.

Γ_{cusp} is related to a variety of physical observables:

- It characterizes the IR divergences that arise when we scatter massive colored particles in the planar limit. Here φ is the boost angle between two external massive particle lines. And for each consecutive pair of lines in the color ordered diagram we get a factor of (1.6), where L is the IR cutoff and $\tilde{\epsilon}$ is set by the sum of the momenta of the two consecutive particles. The angle is given by $\cosh\varphi = -\frac{p_1 \cdot p_2}{\sqrt{p_1^2 p_2^2}}$. See e.g. [15, 16] and references therein.

- In $\mathcal{N} = 4$ super Yang Mills, the Regge limit of the four point massive scattering amplitudes on the Coulomb branch of $\mathcal{N} = 4$ SYM is governed by the cusp anomalous

dimension. As $t \gg s, m^2$, we have that $\log \mathcal{A} \sim \log t \, \Gamma_{\text{cusp}}(\varphi)$ [17], where $\mathcal{A}$ is the planar amplitude, divided by its value at tree level.

- The IR divergences of massless particles are characterized by $\Gamma_{\text{cusp}}^{\infty}$ which is the coefficient of the large φ behavior of the cusp anomalous dimension [15, 18], $\Gamma_{\text{cusp}} \sim \varphi \, \Gamma_{\text{cusp}}^{\infty}$. For $\mathcal{N} = 4$ super Yang Mills, $\Gamma_{\text{cusp}}^{\infty}$ was computed in the seminal paper [19]. Note that $\Gamma_{\text{cusp}}^{\infty}$ is also sometimes called the "cusp anomalous dimension" though it is a particular limit of the general, angle dependent, "cusp anomalous dimension" defined in (1.6).

- By the plane to cylinder map this quantity is identical with the energy of a static quark and anti-quark sitting on a spatial three sphere at an angle $\pi - \phi$. See figure 1(b). In other words, it is the quark anti-quark potential on an S^3.

- As we already remarked, in the small angle limit it behaves as in (1.1), and it is related to the amount of power radiated by a moving quark. This function B also appears as the coefficient of the two point function of the displacement operator on a straight Wilson loop. In other words, $\langle\!\langle \mathbb{D} \, \mathbb{D} \rangle\!\rangle = 12B$, where the double brackets denote the expectation values on the Wilson line. This relation is completely general and is proven in section 4.2. This two point function of the displacement operator has appeared in discussions of wavy lines and the loop equation [20, 21].

In $\mathcal{N} = 4$ super Yang Mills we can also introduce a second angle at a cusp [22]. This second angle is related to the fact that the locally supersymmetric Wilson loop observable contains a coupling to a scalar. This coupling selects a direction $\vec{n}$, where $\vec{n}$ is a point on S^5. The Wilson loop operator is given by

$$W \sim Tr[Pe^{i \oint A \cdot dx + \oint |dx| \vec{n} \cdot \vec{\Phi}}] \tag{1.7}$$

where we wrote it in Euclidean signature. One can consider a loop with a constant direction $\vec{n}$, with $\partial_\tau \vec{n} = 0$. When we have a cusp we could consider the possibility of changing the direction $\vec{n}$ by an angle θ, $\cos\theta = \vec{n} \cdot \vec{n}'$, where $\vec{n}$ and $\vec{n}'$ are the directions before and after the cusp. Thus, we have a cusp anomalous dimension, $\Gamma_{\text{cusp}}(\phi, \theta)$, which is a function of two angles ϕ and θ. The former is the obvious geometric angle and the latter is an internal angle. The same generalized cusp anomalous dimension $\Gamma_{\text{cusp}}(\phi, \theta)$ also characterizes the planar IR divergences that arise when scattering massive w-bosons on the Coulomb branch of $\mathcal{N} = 4$ SYM. There, $\cos\theta = \vec{n}_1 \cdot \vec{n}_2$ is the angle between the Coulomb branch expectation values $\langle \vec{\Phi} \rangle = \text{diag}(m_1 \vec{n}_1, m_2 \vec{n}_2, \dots)$ associated to a pair of color adjacent external w-bosons $W_{1,i}$ and $W_{i,2}$. This generalized cusp anomalous dimension was computed to leading and subleading order in weak and strong coupling in refs. [22] and [23, 24], respectively.

When $\theta = \phi$ (and also when $\theta = -\phi$) the configuration is supersymmetric and therefore the cusp anomalous dimension vanishes. In this case the Wilson loop is BPS. It is a particular case of the 1/4 BPS loops considered in [25]. We will show that the leading order term in the expansion of Γ_{cusp} around the supersymmetric value is

$$\Gamma_{\text{cusp}}(\phi, \theta) = -(\phi^2 - \theta^2) \frac{1}{1 - \frac{\phi^2}{\pi^2}} B(\tilde{\lambda}) + o((\phi^2 - \theta^2)^2), \qquad \tilde{\lambda} = \lambda\left(1 - \frac{\phi^2}{\pi^2}\right) \tag{1.8}$$

where B is the same function appearing in (1.2). For $\theta = 0$ and small ϕ it reduces to (1.1).

Finally, we should mention that in [26] a TBA system of integral equations is derived for $\Gamma_{\text{cusp}}(\phi, \theta)$. This is done via the standard approach of integrability in the gauge/string duality, see [27] for a review. In particular, a limit of those integral equations computes the function B, see [26] for the details. Thus the cusp anomalous dimension at small angles allows us to connect results computed using integrability with results computed using localization. In [26] Γ_{cusp}, and the function B, is computed to three loops, matching the expansion of the function B we have in this paper.

2 Relating Γ_{cusp} at small angles to the circular Wilson loop

The derivation consists of the following steps. First we consider a 1/4 BPS Wilson loop considered in [2] (see also [3–5, 8–10]). It has a parameter θ_0.[1] Taking the derivative with respect to θ_0 we can compute the two point function of the scalar field on the Wilson loop. This two point function of the scalar field also gives the expansion of the cusp anomalous dimension around $\phi = \theta = 0$.

In this section we will set $\phi = 0$ and we will expand in θ. This is equivalent to expanding in ϕ, since Γ_{cusp} vanishes for BPS Wilson loops with $|\theta| = |\phi|$. Then for small values of ϕ and θ we expect

$$\Gamma_{\text{cusp}} = (\theta^2 - \phi^2)B, \qquad \text{for} \quad \phi, \, \theta \ll 1 \tag{2.1}$$

2.1 Relating derivatives of the Wilson loop to the scalar two point function

In [2], a 1/4 BPS circular Wilson loop with the following value of the scalar profile was considered:

$$n_1 + in_2 = \sin\theta_0 \, e^{i\tau}, \qquad n_3 = \cos\theta_0, \tag{2.2}$$

where τ is the coordinate along the circle (which we take to have radius one). Here $\vec{n} = (n_1, n_2, n_3, 0, 0, 0)$ is the profile for the scalar field in the Wilson loop, $W = \text{Tr}Pe^{i\oint A + \oint d\tau \, \vec{n} \cdot \vec{\Phi}}$. The spatial part is the same as before (a unit circle). For $\theta_0 = 0$ we have the 1/2 BPS circular Wilson loop, whose expectation value is a non-trivial known function of λ and N [1, 11]. For $\theta_0 = \pi/2$ we have the Zarembo 1/4 BPS loop [25] that has zero expectation value.

In [2] it was conjectured that the Wilson loop expectation value would be given by the same expression as the circular one, but with a redefined coupling constant. This was further supported in [8] using a localization argument[2] (see also [3–5, 9, 10, 28] for further discussion and checks).

In other words, we have

$$\langle W_{\theta_0} \rangle(\lambda) = \langle W_{\circledcirc} \rangle(\lambda'), \qquad \lambda' = \lambda \cos^2\theta_0 \tag{2.3}$$

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[1]Not to be confused with θ introduced above.

[2]To complete the proof the determinant for the fluctuations in the normal directions to the localization locus would be needed.

Now we expand this equation around $\theta_0 = 0$. When θ_0 is small $\lambda' \sim \lambda(1-\theta_0^2)$ and we have

$$\frac{\langle W_{\theta_0}\rangle - \langle W_{\theta_0=0}\rangle}{\langle W_{\theta_0=0}\rangle} \sim -\theta_0^2\,\lambda\partial_\lambda\log\langle W_\odot\rangle \qquad (2.4)$$

Now the left hand side of (2.4) also has an alternative expression in terms of the two point function for a scalar field. Namely, we can write

$$\frac{\langle W_{\theta_0}\rangle - \langle W_{\theta_0=0}\rangle}{\langle W_{\theta_0=0}\rangle} = \theta_0^2 I \qquad (2.5)$$

$$= \frac{1}{2}\int\limits_0^{2\pi}d\tau\int\limits_0^{2\pi}d\tau'\,\hat{n}^i(\tau)\hat{n}^j(\tau')\,\langle\!\langle\Phi^i(\tau)\Phi^j(\tau')\rangle\!\rangle$$

where Φ^i are the scalars in the 1 and 2 directions and $\hat{n}^i$ are unit vectors in the [12] plane. The double brackets denote expectation values along the contour

$$\langle\!\langle\mathcal{O}(t_1)\mathcal{O}(t_2)\rangle\!\rangle = \frac{\langle Tr[P\,\mathcal{O}(t_1)e^{\int_{t_2}^{t_1}iA\cdot dx+|dx|\vec{n}\cdot\vec{\Phi}}\mathcal{O}(t_2)e^{\int_{t_1}^{t_2}iA\cdot dx+|dx|\vec{n}\cdot\vec{\Phi}}]\rangle}{\langle Tr[Pe^{i\oint A\cdot dx+\oint|dx|\vec{n}\cdot\vec{\Phi}}]\rangle}$$

where $\mathcal{O}$ are in the adjoint representation and inserted along the loop.

Any second order term from n_3 vanishes because it is a one point function, and the conformal symmetry of the 1/2 BPS line implies that one point functions are zero. We now use that

$$\langle\!\langle\Phi^i(\tau)\Phi^j(\tau')\rangle\!\rangle = \frac{\gamma\,\delta_{ij}}{2[1-\cos(\tau-\tau')]} \qquad (2.6)$$

This just follows from a conformal transformation of the expectation value on the straight line[3]

$$\langle\!\langle\Phi(t)\Phi(0)\rangle\!\rangle = \frac{\gamma}{t^2} \quad \to \quad \langle\!\langle\Phi(\tau)\Phi(0)\rangle\!\rangle = \frac{\gamma}{2[1-\cos(\tau)]} \qquad (2.10)$$

We can now do the integral (2.6) as

$$I = \gamma\frac{\pi}{2}\int\limits_{\tilde{\epsilon}}^{2\pi-\tilde{\epsilon}}dt\,\frac{\cos t}{[1-\cos(t)]} = \frac{2\gamma\pi}{\tilde{\epsilon}} - \pi^2\gamma = -\pi^2\gamma \qquad (2.11)$$

where we discarded a power law UV divergence. Inserting this in (2.5) and equating it to (2.4) we derive an expression for the coefficient γ

$$\gamma = \frac{1}{\pi^2}\lambda\partial_\lambda\log\langle W_\odot\rangle \qquad (2.12)$$

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[3]The simplest way to derive this is to write the correlator,

$$\langle\Phi\Phi\rangle = -\frac{1}{2X\cdot X'} \qquad (2.7)$$

in projective coordinates X, with $X^2 = X^+X^- + \sum_{i=1}^4 X_i^2 = 0$. Then write these in two gauges

$$(X^+, X^-, X^1, X^2, X^3, X^4)_{\text{straight}} = (1, -t^2, t, 0, 0, 0) \qquad (2.8)$$

$$(X^+, X^-, X^1, X^2, X^3, X^4)_{\text{circle}} = (1, -1, \cos\tau, \sin\tau, 0, 0) \qquad (2.9)$$

2.2 Relating the two point function of the scalar to $\frac{1}{2}\partial_\theta^2 \Gamma_{\text{cusp}}$

We now need to relate γ to the second derivative of the cusp anomalous dimension. For that purpose we view the cusp as coming from the energy of two static quarks that are sitting at opposite points on S^3. When we vary the relative internal angle of these two quarks we get

$$\Gamma_{\text{cusp}} = \theta^2 I_c\,, \qquad I_c = -\frac{1}{2}\int\limits_{-\infty}^{\infty} d\tau\, \langle\!\langle \Phi(\tau)\Phi(0) \rangle\!\rangle \tag{2.13}$$

where now τ is the time in the $R \times S^3$ coordinates. We have set one of the operators at zero and extracted an overall length of time factor. The minus sign comes from the relation between the partition function and the energy, $Z = e^{-\Gamma_{\text{cusp}}(\text{Length of Time})}$. In (2.13) we have the correlator for two points on a Wilson loop along the same line in global coordinates. We transform from the plane to global coordinates.[4] Then the integral in (2.13) is

$$I_c = -\frac{\gamma}{4}\left[\int\limits_{-\infty}^{-\tilde\epsilon} + \int\limits_{\tilde\epsilon}^{\infty}\right]\frac{d\tau}{(\cosh\tau - 1)} = -\frac{\gamma}{\tilde\epsilon} + \frac{\gamma}{2} = \frac{\gamma}{2} \tag{2.15}$$

where we again discarded a power law UV divergent term. In conclusion, we find that

$$\Gamma_{\text{cusp}}(\phi = 0, \theta) = \theta^2\frac{\gamma}{2} + o(\theta^4) \tag{2.16}$$

Of course, since Γ_{cusp} vanishes for $\theta = \phi$, this also determines the expansion of the cusp for small ϕ, $\Gamma_{\text{cusp}} = -(\phi^2 - \theta^2)\gamma/2$. Inserting the value of γ computed in (2.12) we obtain

$$\Gamma_{\text{cusp}} = -B(\phi^2 - \theta^2)\,, \qquad \text{for} \quad \phi,\ \theta \ll 1\,, \qquad \text{with} \quad B = \frac{1}{2\pi^2}\lambda\partial_\lambda\langle W_\odot\rangle \tag{2.17}$$

where $\langle W_\odot\rangle$ is the 1/2 BPS circular Wilson loop (1.3).

We can expand this function at weak and strong coupling and, in the planar limit, we find

$$B = \frac{\lambda}{16\pi^2} - \frac{\lambda^2}{384\pi^2} + \frac{\lambda^3}{6144\pi^2} - \frac{\lambda^4}{92160\pi^2} + O\left(\lambda^5\right) \tag{2.18}$$

$$B = \frac{\sqrt{\lambda}}{4\pi^2} - \frac{3}{8\pi^2} + \frac{3}{32\pi^2\sqrt{\lambda}} + \frac{3}{32\pi^2\lambda} + O\left(\lambda^{-3/2}\right) \tag{2.19}$$

The first two terms in each expansion agree with the results in [24]. The three loop term in the weak coupling expansion agrees with the explicit three loop computation in [29]. It also agrees with the three loop expansion of the TBA equations in [26]. Figure 2 shows a plot of B for $\lambda \in [0, 30]$.

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[4]This is done by writing the projective coordinates in global coordinates

$$(X^+, X^-, X^1, X^2, X^3, X^4)_{\text{global}} = (e^\tau, -e^{-\tau}, 1, 0, 0, 0) \tag{2.14}$$
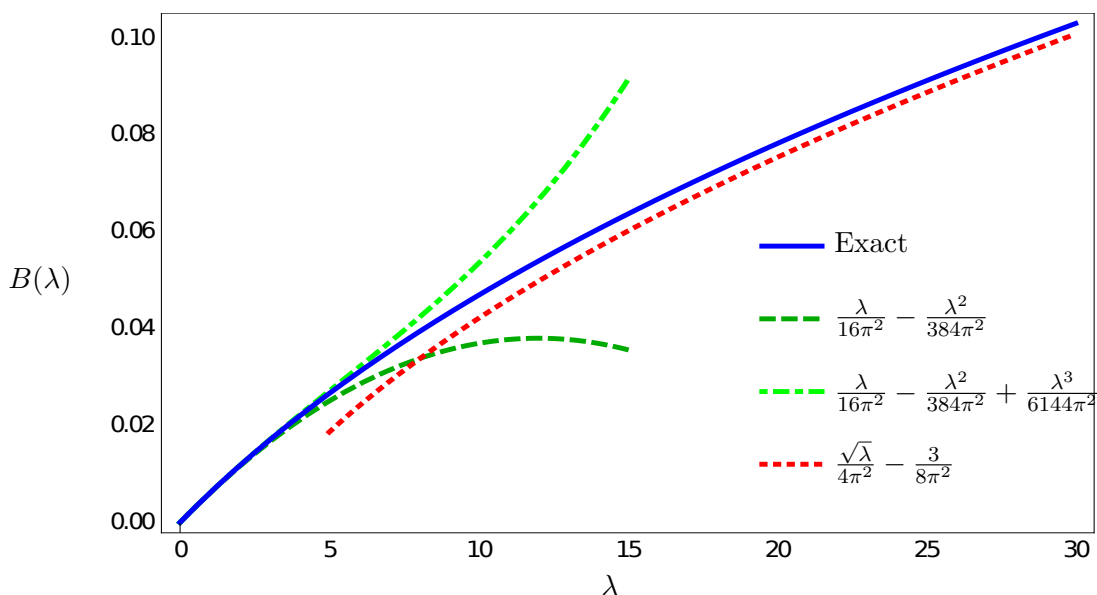
Figure 2. Plot of the Bremsstrahlung function B in the planar limit (solid blue curve). At weak coupling, the lower and upper dashed green curves denote the two- and three-loop approximation, respectively. It is interesting to note that the radius of convergence of the weak coupling expansion is given by the first zero of I_1 in (1.4), which is at $\lambda \sim -14.7$. As one can see in the plot, the perturbative formulas become unreliable in that region. At the same time, we see that the first two orders of the strong coupling result (red dotted curve) give a qualitatively good approximation starting from that region.

3 Near BPS expansion of the generalized $\Gamma_{\text{cusp}}(\phi, \theta)$

Now we turn our attention to the generalized cusp $\Gamma_{\text{cusp}}(\phi, \theta)$. When $\phi = \pm\theta$, it is zero since the configuration is BPS. Here we derive a simple expression for the first order deviation away from the BPS value. Namely, we define a function H as the first term in the expansion away from the BPS limit at $\phi = \theta$,

$$\Gamma_{\text{cusp}} = -(\phi - \theta)H(\phi, \lambda), \qquad \theta - \phi \ll 1 \tag{3.1}$$

We will show below that

$$H(\phi, \lambda) = \frac{2\phi}{1 - \frac{\phi^2}{\pi^2}}B(\tilde{\lambda}), \quad \tilde{\lambda} = \lambda\left(1 - \frac{\phi^2}{\pi^2}\right) \tag{3.2}$$

where B is the same Bremsstrahlung function we had before in (1.2).

In order to derive this formula we need to consider a class of 1/8 BPS Wilson loops discussed in [3–5, 8–10, 28]. These are Wilson loops where the contour lives in an S^2 subspace of R^4 or S^4. These are BPS if the coupling to the scalars is chosen as follows. We consider a six dimensional vector of the form $\vec{n} = (\vec{m}, 0, 0, 0)$ where $\vec{m}$ is a three dimensional unit vector. If we call $\vec{x}$ the three dimensional unit vector parametrizing the S^2, then we
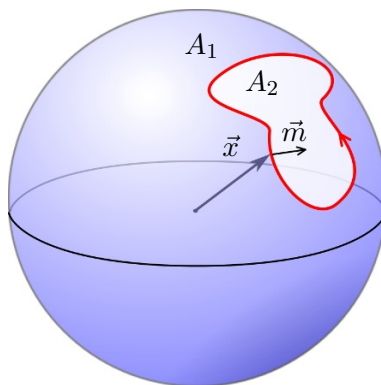
Figure 3. General class of 1/8 BPS Wilson loops. The loop lies on a two sphere S^2. The vector that specifies the scalar couplings can be viewed as the unit vector on R^3 that is orthogonal to the contour and lies on the two sphere, denoted here at a point by $\vec{m}$. The two sphere is divided in two regions, one with area A_1 and the other with area A_2.
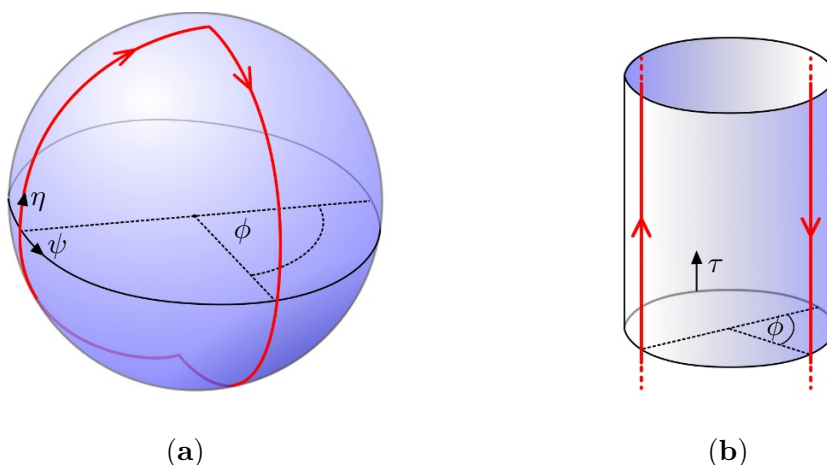
Figure 4. (**a**) Contour with two longitude lines separated by an angle $\pi - \phi$. (**b**) The same contour on the cylinder.

choose

$$\vec{m} = \vec{x} \times \dot{\vec{x}}, \qquad (\vec{x})^2 = (\dot{\vec{x}})^2 = 1, \qquad \dot{\vec{x}} = \frac{d\vec{x}}{dt} \tag{3.3}$$

As conjectured in [3–5], and further argued and discussed in [8–10, 28], the result for a non-intersecting Wilson loop of this kind is given by the answer for the ordinary circular Wilson loop (1.3), but with λ replaced by

$$\lambda \to \tilde{\lambda} = \lambda \frac{4A_1 A_2}{A^2} = \lambda 4 \frac{A_1}{A} \frac{A_2}{A} \tag{3.4}$$

where A_1 and A_2 are the areas of the two sides of the contour and $A = A_1 + A_2$ is the total area of the two sphere.

An interesting contour, considered in [3–5, 9], consists of two longitude lines separated by an angle $\pi - \phi$ and going all the way to the poles, see figure 4(a). In this case, (3.4),

becomes

$$\tilde{\lambda} = \lambda \, 4 \frac{(\pi - \phi)}{2\pi} \frac{(\pi + \phi)}{2\pi} = \lambda(1 - \frac{\phi^2}{\pi^2}) \tag{3.5}$$

The result of this Wilson loop is then

$$\langle W_\phi \rangle = \langle W_\odot \rangle (\tilde{\lambda}) \tag{3.6}$$

where $\langle W_\odot \rangle$ is the 1/2 BPS circular Wilson loop expectation value given in (1.3) and $\tilde{\lambda}$ is in (3.5). Here $\langle W_\phi \rangle$ denotes the BPS contour which is made out of two longitude lines as depicted in figure 4(a).

Notice that this Wilson loop has a BPS cusp with angles $\theta = \phi$ at the two poles. This loop is mapped to two straight lines separated by an angle $\pi - \phi$ in the cylinder, as depicted in figure 4(b). The fact that there is no logarithmic divergence is simply the statement that $\Gamma_{\mathrm{cusp}}(\theta = \phi) = 0$.[5]

It is useful to consider the conformal map between S^4 and $R \times S^3$ in global coordinates, which maps the north and south pole of an $S^2 \subset S^4$ into $\tau = \pm\infty$, where τ is the R coordinate. Under this map the two longitude lines we mentioned above are mapped into two straight lines, extended along R_τ and sitting at a relative angle $\pi - \phi$ on the S^3. See figure 4.

$$ds^2 = d\eta^2 + \cos^2 \eta \, d\Omega_3 = \cos^2 \eta \left(d\tau^2 + d\Omega_3^2 \right) \tag{3.7}$$

$$\sin \eta = \tanh \tau \,, \qquad \cos \eta = \frac{1}{\cosh \tau} \,, \qquad \frac{d\eta}{d\tau} = \frac{1}{\cosh \tau} \tag{3.8}$$

Here the $S^2 \subset S^4$ is described by picking a great circle on the S^3 parametrized by ψ, together with the direction η of (3.7). Then the vector $\vec{x}$ we discussed above is parametrized as

$$\vec{x} = (\sin \eta, \cos \eta \cos \psi, \cos \eta \sin \psi) \tag{3.9}$$

For longitude lines with $\dot{\psi} = 0$ the internal vector $\vec{m}$ reads

$$\vec{m} = \vec{x} \times \dot{\vec{x}} = \dot{\eta} \, (0, \sin \psi, -\cos \psi) \,, \qquad \text{for} \quad \dot{\psi} = 0, \ |\dot{\eta}| = 1 \tag{3.10}$$

We start with a semicircle, or longitude line, at $\psi = 0$ and another at $\psi = \pi - \phi$. The internal vectors are

$$\vec{m}_{\psi=0} = (0, 0, -1) \,, \qquad \vec{m}_{\psi=\pi-\phi} = (0, -\sin \phi, -\cos \phi) \tag{3.11}$$

This configuration gives the supersymmetric cusp $\theta = \phi$. We now consider a small deformation of this contour, which will help us in our arguments. We deform the line that sits at $\psi = 0$ in the following way

$$\psi = \epsilon \, \sigma(\eta) \tag{3.12}$$

where $\sigma(\eta)$ goes to zero at the two poles and it is equal to one as soon as we get to a small distance from the poles. See figure 5. Under the map into the straight lines (3.8), this gives

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[5]The expectation values for the same contour in the sphere and in the cylinder are different because the corresponding conformal map has a quantum anomaly.
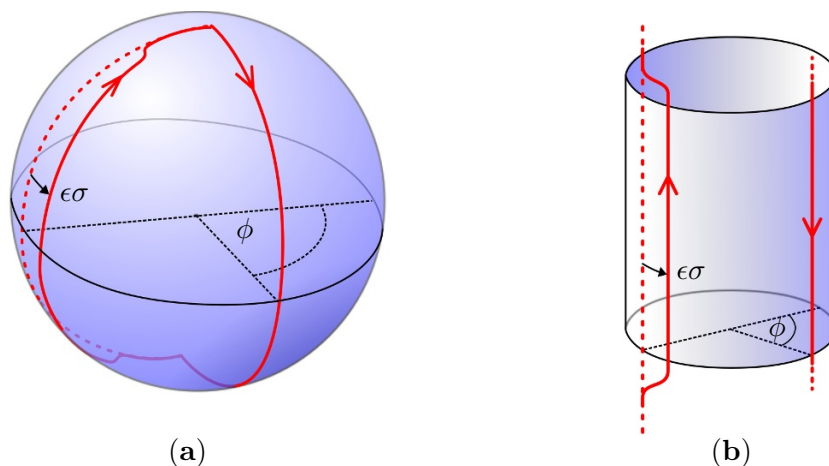
Figure 5. Contour we consider for the argument. We start from two longitude segments and we deformed one of them. The deformed contour goes back to the old one near the poles. The deformation is constant along most of the longitude line. In (**a**) we see the diagram on the sphere. In (**b**) we see the diagram on the cylinder.

a BPS deformation of the two straight lines on $R \times S^3$. One of the lines is untouched. The line at $\psi = 0$ is deformed as in (3.12). This deformation changes the line so that it goes to its original value at $\tau = \pm\infty$. For a large region of values of τ the line and its internal angle are moved to a new location: $\psi = \epsilon$. Then there is a transition region that occurs at a large value of τ. We will compute the change in the vacuum expectation value (VEV) of this Wilson loop in two ways and this will tell us the VEVs of a scalar insertion and the displacement insertion, Φ and $\mathbb{D}$, on the straight lines.

The displacement operator displaces the contour in an orthogonal direction. Starting from a given contour we make an infinitesimal displacement $\delta x^j(t)$, where t is the proper time parameter along the contour. We consider displacements orthogonal to the contour, $\delta\vec{x}(t) \cdot \dot{\vec{x}}(t) = 0$. Then the displacement operator is the functional derivative of the Wilson loop operator with respect to this displacement. In particular, we write the infinitesimal change as

$$\delta W = P \int dt\, \delta x^j(t) \mathbb{D}_j(t) W \tag{3.13}$$

where $\mathbb{D}_j$ is inserted at point t along the contour. In other words $P \mathbb{D}_j(t) W = P e^{i \int_t^\infty A} \mathbb{D}_j(t) e^{i \int_{-\infty}^t A}$.

For the locally BPS Wilson loop in $\mathcal{N} = 4$ super Yang Mills there is also an internal angle displacement operator which arises when we change $\vec{n} \to \vec{n} + \delta\vec{n}$. Since $\vec{n}$ has unit norm, $\vec{n} \cdot \delta\vec{n} = 0$. We can write then a small change of the Wilson loop as

$$\delta W = P \left[\int dt\, \delta x^j \mathbb{D}_j\, W + \int dt\, \delta\vec{n} \cdot \vec{\Theta}\, W \right] \tag{3.14}$$

where the angle displacement operator is simply

$$\vec{\Theta} = \vec{\Phi} \tag{3.15}$$

This operator is defined only for directions orthogonal to the original angle $\vec{n}$ of the contour. Again the dimension of this operator is fixed by the symmetries. It has dimension one and it is in the same supermultiplet as the displacement (4.5) under the supergroup preserved by the straight line, $OSp(4^*|4)$. It is a BPS operator. An insertion of a scalar along the same internal direction as the original one $(\vec{n} \cdot \vec{\Phi})$ is not protected [30].

Let us consider the case when ψ varies in η, as in (3.12). We get, to first order in ϵ

$$\vec{m} = \vec{x} \times \dot{\vec{x}} = (0, \epsilon\,\sigma, -1) + \epsilon\,\partial_\eta\sigma \cos\eta\,(\cos\eta, -\sin\eta, 0)$$
$$= (0, \epsilon\,\sigma, -1) + \epsilon\,\partial_\tau\sigma\left(\frac{1}{\cosh\tau}, -\tanh\tau, 0\right) \tag{3.16}$$

where we wrote the result both in the sphere and cylinder coordinates using (3.8). The crucial point here is the term proportional to $\partial_\tau\sigma$ in the second entry. Notice that the term proportional to σ in the second entry is what we expect from a constant change in the internal angle position. The source term for the second scalar is then

$$m_2 = \epsilon[\sigma(\tau) - \tanh\tau\,\partial_\tau\sigma(\tau)] \tag{3.17}$$

We see that $\epsilon\,\sigma$ is the displacement in the angular direction, $\delta\phi = \epsilon\,\sigma$. And, *when σ is constant*, this is the same as the displacement in the θ direction. However, when σ varies, we get an extra source term for the θ variation

$$\delta\phi = \epsilon\,\sigma\,, \qquad\qquad \delta\theta = \epsilon[\sigma - \tanh\tau\,\partial_\tau\sigma] \tag{3.18}$$

Now, for a general contour, if we make a small change in the contour δx^i and a small change in the internal orientation $\delta\vec{n}$, then we find that the change in the expectation value of the Wilson loop is

$$\frac{\delta\langle W\rangle}{\langle W\rangle} = \int dt\left[\langle\!\langle\mathbb{D}_i\rangle\!\rangle\delta x^i(t) + \langle\!\langle\vec{\Phi}\rangle\!\rangle\delta\vec{n}\right] \tag{3.19}$$

This is a general formula, valid for arbitrary displacements and changes in the internal orientation. Note that $\delta\vec{n}$ is orthogonal to $\vec{n}$ and thus the scalar fields in (3.19) are scalars which are orthogonal to the original scalar appearing the loop at that point, which is $\vec{n} \cdot \vec{\Phi}$. Here $\mathbb{D}_i$ is the displacement operator in direction i. See section 4.1 for further discussion of this formula. We can now apply this general formula for the small change of the 1/8 BPS Wilson operator we are considering. We find, to leading order in ϵ,

$$\frac{\delta\langle W\rangle}{\langle W\rangle} = \epsilon\int d\tau\left[(\langle\!\langle\mathbb{D}\rangle\!\rangle + \langle\!\langle\Phi^2\rangle\!\rangle)\sigma(\tau) - \langle\!\langle\Phi^2\rangle\!\rangle\tanh\tau\,\partial_\tau\sigma\right] \tag{3.20}$$

where Φ^2 is the second scalar field. We used that $\langle\!\langle\Phi^1\rangle\!\rangle = 0$, due to a residual $SO(4) \subset SO(6)$ symmetry of the configuration (3.11). The expectation values are the ones on the underformed contour. Since this contour has a time translation symmetry, these expectation values are independent of the Euclidean time τ. They are simply constant and can be pulled out of the integral. On the other hand, the same left hand side can be computed in terms of the general formula (3.4). For the purposes of computing the areas in (3.4) we can approximate $\sigma = 1$, since the change near the poles contributes in a

negligible way to the area. Thus the change in the expectation value is the change we have for the contour along the longitudes when we change the angle by ϵ

$$\frac{\delta\langle W\rangle}{\langle W\rangle} = \epsilon\,\partial_\phi\log\langle W_\phi\rangle = \epsilon\frac{\partial\tilde\lambda}{\partial\phi}\partial_{\tilde\lambda}\log[\langle W_\odot\rangle(\tilde\lambda)] = -4\epsilon\frac{\phi}{(1-\frac{\phi^2}{\pi^2})}B(\tilde\lambda) \tag{3.21}$$

Now this formula, combined with (3.20) will allow us to compute the expectation values of $\langle\mathbb{D}\rangle$ and $\langle\Phi^2\rangle$ as follows. First we note that the BPS condition for $\delta\theta = \delta\phi$ implies that

$$\langle\langle\mathbb{D}\rangle\rangle = -\langle\langle\Phi^2\rangle\rangle \tag{3.22}$$

Then the only contribution to (3.20) comes from the term with a derivative. We can integrate it by parts and find

$$\frac{\delta\langle W\rangle}{\langle W\rangle} = \epsilon\langle\langle\Phi^2\rangle\rangle\int d\tau\frac{1}{\cosh^2\tau}\sigma(\tau) = 2\epsilon\langle\langle\Phi^2\rangle\rangle \tag{3.23}$$

where we used that $\sigma(\tau) = 1$ in a large region around $\tau = 0$ so that we can approximate the integral of $1/\cosh^2(\tau)$ (which is localized around $\tau = 0$) by its integral over $\int_{-\infty}^{\infty} d\tau/\cosh^2(\tau) = 2$.

In conclusion, by equating (3.23) and (3.21) we have managed to compute the expectation values

$$\langle\langle\mathbb{D}\rangle\rangle = -\langle\langle\Phi^2\rangle\rangle = 2\frac{\phi}{(1-\frac{\phi^2}{\pi^2})}B(\tilde\lambda) \tag{3.24}$$

We now need to relate these expectation values to Γ_{cusp}. All we need is the first order change of Γ_{cusp} as we change the angular position ϕ from its BPS value. This is given by the general formula (3.19) where now we have constant change in $\delta\phi$ and $\delta\theta$. We then get

$$\Gamma_{\text{cusp}} = -\langle\langle\mathbb{D}\rangle\rangle\delta\phi - \langle\langle\Phi^2\rangle\rangle\delta\theta \tag{3.25}$$

This is a general formula. If $\delta\theta = \delta\phi$, this vanishes as expected for a BPS configuration. It also implies, that expanding around $\theta = \phi$ we have

$$\Gamma_{\text{cusp}} = (\phi - \theta)\langle\langle\Phi^2\rangle\rangle = -(\phi - \theta)H, \qquad \phi - \theta \ll 1$$
$$H = -\langle\langle\Phi^2\rangle\rangle = \frac{2\phi}{(1-\frac{\phi^2}{\pi^2})}B(\tilde\lambda) \tag{3.26}$$

which is the formula we wanted to derive (1.8). This is the same as (1.8) because for $\phi - \theta \ll \phi, \theta$ we have $\phi^2 - \theta^2 \sim 2\phi(\phi - \theta)$.

In the weak coupling expansion we can check the first two loops with [24] and the third loop with [29]. One can also check that we get the correct answer at strong coupling from [24], expanding around the $\theta - \phi$ limit. For that purpose, one should take the formulas in appendix B of [24] which are written in terms of two parameters q and p. We then expand around $q = 1$, we find that $\Gamma_{\text{cusp}} = -\frac{\sqrt\lambda}{2\pi p}(\phi - \theta)$, to leading order in $\phi - \theta$. (Here $\phi - \theta$ is proportional to $(q-1)$). We also find that $p = \frac{\pi}{\phi}\sqrt{1 - \frac{\phi^2}{\pi^2}}$, so that we have agreement with (3.26), and the strong coupling form of B in (2.19).

4 Three related observables

In this section we would like to argue that the following three observables are governed by the same function B.

- The derivative of the cusp at zero angle

$$B = \frac{1}{2}\partial_\phi^2 \Gamma_{\text{cusp}}(\phi)\Big|_{\phi=0} \tag{4.1}$$

- The two point function of the displacement operator on a straight Wilson loop

$$\langle\!\langle \mathbb{D}_i(t)\mathbb{D}_j(0)\rangle\!\rangle = \frac{\tilde\gamma\,\delta_{ij}}{t^4}\,, \qquad\qquad \tilde\gamma = 12B \tag{4.2}$$

- The energy emitted by a moving quark, at small velocities,

$$\Delta E = 2\pi B \int dt\,(\dot v)^2 \tag{4.3}$$

The relations between these three observables are general for any conformal gauge theory with a Wilson loop operator. In fact, they are valid for any line defect operator, including 't Hooft line operators, etc.

We first start with general remarks about the displacement operator and then explain the relations.

4.1 The displacement operator for line defect operators

In (3.13) we have written the infinitesimal change in a Wilson loop, due to a displacement orthogonal to the contour. The operator $\mathbb{D}$ is inserted along the contour and it is not a color singlet. It is a particular operator among a large class of operators that we can insert on a Wilson line contour, see e.g. [20, 33, 34] for further examples. In conformal theories these operators are classified by their $\text{SL}(2) \times \text{SU}(2)$ quantum numbers.[6] Since the displacement δx and the time t appearing in (3.13) have canonical dimensions, the dimension of $\mathbb{D}$ is protected and equal to two for all values of the coupling. This is true for any line defect operator for any conformal field theory.

For the ordinary Wilson loop operator, $Tr[Pe^{i\oint A}]$, the displacement operator at weak coupling is given by

$$\mathbb{D}_j = iF_{tj} \tag{4.4}$$

where F_{tj} is the field strength and t is the direction along the loop. (We wrote it in Euclidean signature). On the other hand if we are talking about the locally supersymmetric Wilson loop operator in $\mathcal{N} = 4$ SYM, which is $W = Pe^{i\int A + \int \vec n\cdot\vec\Phi}$, then the displacement operator is

$$\mathbb{D}_j = iF_{tj} + \vec n \cdot \partial_j\vec\Phi \tag{4.5}$$

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[6]$\text{SL}(2) \times \text{SO}(d-1)$ quantum numbers for a d-dimensional conformal theory.

The coefficient of the two point function of the displacement operator on a straight line

$$\langle\!\langle \mathbb{D}_i(t)\mathbb{D}_j(0)\rangle\!\rangle = \frac{\tilde{\gamma}\,\delta_{ij}}{t^4} \tag{4.6}$$

is well defined since the normalization of $\mathbb{D}_i$ is fixed by its physical interpretation in terms of the displacement. $\tilde{\gamma} > 0$ by unitarity (reflection positivity). This number is an important characterization of the line defect operator.[7]

As an aside, the two point function (4.6) determines the expectation value of a Wilson loop with a wavy line profile as [20, 21]

$$\langle W\rangle = 1 + \frac{\tilde{\gamma}}{24}\int dt\int dt'\,\frac{[\dot{\vec{\xi}}(t) - \dot{\vec{\xi}}(t')]^2}{(t - t')^2} + \cdots \tag{4.7}$$

Here $\vec{\xi}(t)$, with $|\vec{\xi}| \ll 1$, $|\dot{\vec{\xi}}| \ll 1$, is a small deviation from a straight line. We simply used that the contour is given by $P e^{\int dt\,\vec{\xi}(t)\cdot\vec{\mathbb{D}}(t)} W$, expanded in ξ, integrated by parts and dropped all power law UV divergent terms. Again this is valid for any line defect operator in any CFT.

4.2 The small angle Γ_{cusp} and the two point function of the displacement operator

Here we relate B, defined as a second derivative of Γ_{cusp}, to $\tilde{\gamma}$, defined as the coefficient of the two point function of the displacement operator (4.6).

We consider the map between the plane (R^4) and the cylinder, $R \times S^3$. This maps the straight line into two lines along the R direction and sitting at the "north" and "south" poles of the S^3. For this configuration Γ_{cusp} vanishes. The first derivative with respect to ϕ also vanishes by the symmetries. The second derivative with respect to ϕ can be computed in terms of the two point function of the displacement operator evaluated in the cylinder coordinates

$$\Gamma_{\mathrm{cusp}} \sim -\phi^2\,\frac{1}{2}\int d\tau\,\langle\!\langle \mathbb{D}(\tau)\mathbb{D}(0)\rangle\!\rangle = -\frac{\phi^2}{2}\int d\tau\,\frac{\tilde{\gamma}}{4(\cosh\tau - 1)^2} \tag{4.8}$$

Here we have used the same conformal change of coordinates that we discussed in (2.14), except that now we have $\langle\!\langle \mathbb{D}\,\mathbb{D}\rangle\!\rangle = 1/(2X\cdot X')^2$ in projective coordinates. In writing (4.8) we have neglected possible terms that involve one point functions, since those would vanish by conformal symmetry. We perform the integral in (4.8) dropping its power law UV divergences to obtain

$$B = \frac{\tilde{\gamma}}{12} \qquad \text{with} \qquad B = -\frac{1}{2}\partial_\phi^2\,\Gamma_{\mathrm{cusp}}(\phi)|_{\phi=0} \tag{4.9}$$

This formula relates B, defined as the second derivative of Γ_{cusp} as in (1.1) and the two point function of the displacement operator (4.6).

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[7]We thank D. Gaiotto for a discussion on this.

4.3   Relation between B and the energy emitted by a moving quark

In this section we relate the energy emitted by a moving quark to the two point function (4.6) of the displacement operator, which via (4.9) is related to the cusp anomalous dimension at small angles.

The energy emitted by a moving quark, for small velocities can be written as

$$\Delta E = A \int dt \, (\dot v)^2 \tag{4.10}$$

We now relate A to $\tilde\gamma$ as follows. We consider a specific small displacement of the form $\delta x = \eta \left(e^{i\omega t} + e^{-i\omega t}\right)$, where η is a small quantity. The probability that the Wilson loop absorbs a quantum of energy ω is given in terms of the two point function of the displacement operator, which couples to δx. The absorption probability is

$$p_{\rm abs} = \parallel \eta \int dt \, e^{-i\omega t} \mathbb{D}(t)|0\rangle \parallel^2 = T \, |\eta|^2 \int dt \, e^{i\omega t} \langle\!\langle \mathbb{D}(t)\mathbb{D}(0)\rangle\!\rangle \tag{4.11}$$

$$= T|\eta|^2\tilde\gamma \int\limits_{-\infty}^{\infty} dt \, e^{i\omega t} \frac{1}{(t - i\epsilon)^4} = \frac{\pi}{3}|\eta|^2 \, \tilde\gamma \, \omega^3 \, T \tag{4.12}$$

where T is a long time cutoff. Here we consider the two point function in the explicit ordering stated, which results in the corresponding $i\epsilon$ prescription. To get the total absorbed energy we need to multiply (4.11) by a factor of ω. Then we write $\omega^4|\eta|^2$ as the integral of the acceleration

$$2\omega^4|\eta|^2 T = \int\limits_{0}^{T} dt \, |\delta\ddot x|^2 \tag{4.13}$$

so that the emitted energy is

$$\Delta E = \frac{\pi}{6}\tilde\gamma \int dt \, (\dot v)^2 = 2\pi B \int dt \, (\dot v)^2 \tag{4.14}$$

where we used (4.9). This is again valid for any line defect operator in any conformal field theory. In the weak coupling limit it reduces to the usual electrodynamics formula. The strong coupling form of A was explicitly computed in [13], which, of course, agrees with (4.14) and the strong coupling form of B (2.19). It was also show in [13] that the energy emission formula is also the same (up to the overall coefficient) as in electrodynamics for any velocity. This follows from boost invariance. See [35] for more discussion on properties of the radiation emitted by a moving quark at strong coupling in theories with a gravity dual.

5   Conclusions

In this paper we have derived an exact expression for the cusp anomalous dimension at small angles. This corresponds to a small deviation from the straight line configuration. The result is written in eqs. (1.2) and (1.3) and it is valid for all couplings and all N, in the U(N) theory. The result was derived by relating the small deviation from the straight

Wilson line to the two point function of the displacement operator on a straight Wilson line. The two point function of this displacement operator can then be obtained by considering the 1/4 (and 1/8) BPS family of Wilson loops discussed in [2–5, 8–10, 28]. This family also contains the circular Wilson loop but it can be computed exactly as a function of the parameters that parametrize the family. The leading order deviation from the circular Wilson loop is also given in terms of the two point function of the displacement operator. This allows us to compute this two point function, using the exact results conjectured in [2–5] and proved in [8, 12].

We have also emphasized that the small angle limit of the cusp is a function which appears also in the two point function of the displacement operator of a Wilson loop and in the formula for the energy radiated by a moving quark. These relations are general and valid for any line defect operator in any CFT.

The answer in the planar limit is given by a ratio of Bessel functions (1.4). The function I_2 also appears in the expectation value of the operator $\text{Tr}[Z^2]$ in the presence of a Wilson loop [12, 36]. This operator is in the same supermultiplet as the lagrangian. So it is likely that this explains why this ratio is equal to the derivative of the loop with respect to the coupling. Since the lagrangian is the same supermultiplet as the stress tensor, and a Ward identity relates the stress tensor to the displacement operator [37], it is likely that this could also explain the relations derived in this paper in an alternative fashion.

A very similar ratio of Bessel functions was recently conjectured in [38] to be related to the first derivative with respect to the spin at zero spin for twist J operators. The formula in [38] says that

$$\lim_{S \to 0} \frac{\Delta(S) - J}{S} = \frac{\sqrt{\lambda}}{J} \frac{I_J'(\sqrt{\lambda})}{I_J(\sqrt{\lambda})} \tag{5.1}$$

Of course, one needs to analytically continue in the spin S to be able to write this formula. This looks similar to (1.4), except that we would have to set $J = 1$. But there are no closed string operators with $J = 1$!. On a Wilson loop there are twist one operators, so maybe there is a connection between the function B and this small spin limit. But we leave this problem to the future.

Basically the same idea can be used to compute the leading deviation from the BPS limit for the generalized cusp anomalous dimension $\Gamma_{\text{cusp}}(\phi, \theta)$. The leading deviation is the coefficient of the term proportional to $\phi - \theta$. This is a non-trivial, but simple, function of the angle (3.2). Expanded in perturbation theory, this function of the angle has the form $\lambda^L \phi(\phi^2 - \pi^2)^{L-1}$, where L is the loop order. Note that the cusp anomalous dimension is intimately related to amplitudes. As we discussed in the introduction, it governs the IR divergences for the scattering of massive particles in the presence of massless gluons. As such, it is a non-trivial function of the angle with a given transcendentality. In $\mathcal{N} = 4$ SYM, we have a total transcendentality $2L$ for an L loop diagram. The $\log \mu_{\text{IR}}$ soaks up one unit of transcendentality. Thus $\Gamma_{\text{cusp}}(\phi)$ has transcendentality $2L - 1$ at L loops, which is precisely what we see in this formula. For counting the transcendentality, we should assign $\phi = \log e^\phi$ transcendentality one. While a scattering amplitude is a function of many variables, Γ_{cusp} is just a function of one variable ϕ. We see that in this limit it is a particularly simple function, which can be computed at all loop orders!.

As will be shown in [26] the same function B can be computed from an integral equation of the TBA type. This then connects the traditional integrability approach with localization. A similar connection would be more exciting for the ABJM theory since it would allows us to compute the undetermined function of λ which is present in the integrability approach [39–42].

Note added. Shortly after this paper appeared, the same result for the power radiation B (1.2) was obtained in [31] by relating it to the correlation function of a Wilson line and the stress tensor [32]. That correlation function is another observable computed by function B.

Acknowledgments

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