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MULTIPARAMETER QUANTUM GROUPS, BOSONIZATIONS AND COCYCLE DEFORMATIONS

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ABSTRACT. The multiparameter quantized enveloping algebras $U_{\mathbf{q}}(\mathfrak{g}_A)$ constructed by Pei, Hu and Rosso [Quantum affine algebras, extended affine Lie algebras, and their applications, 145–171, Amer. Math. Soc., Providence, 2010] are presented as the pointed Hopf algebras $\widetilde{\mathcal{U}}(\mathcal{D}_{\text{red}}, \ell)$ defined by Andruskiewitsch and Schneider [Ann. of Math. (2) 171 (2010), 375–417]. The result is applied to show that under a certain assumption $U_{\mathbf{q}}(\mathfrak{g}_A)$ depends, up to cocycle deformation, on only one parameter in each connected component of the associated Dynkin diagram. In the special case that $\mathfrak{g}_A$ is simple, this was already shown by Pei, Hu and Rosso in an alternative way.

1. Introduction

Let A, H be two complex Hopf algebras with bijective antipode and let $\pi : A \to H$ be a Hopf algebra projection that admits a Hopf algebra section $\iota : H \to A$. Then $A \simeq R\#H$, the Radford–Majid product or *bosonization* of R over H, where $R = A^{\operatorname{co}\pi}$ is a braided Hopf algebra in the category of Yetter–Drinfeld modules ${}^H_H\mathcal{YD}$ over H. Conversely, given a braided Hopf algebra R in ${}^H_H\mathcal{YD}$, its bosonization $R\#H$ is an ordinary Hopf algebra with a projection to H, see [28]. This fact plays a crucial role in the classification of finite-dimensional pointed Hopf algebras [9] and in the description of quantum groups arising from deformation of enveloping algebras of semisimple Lie algebras, in particular their quantum Borel subalgebras.

Let A be a Hopf algebra with bijective antipode such that its coradical A_0 is a Hopf subalgebra. Then the graded object $\operatorname{gr} A$ associated to the coradical filtration is again a Hopf algebra and $\operatorname{gr} A \simeq R\#A_0$. A key point in the lifting method to classify pointed Hopf algebras, where A_0 is a group algebra, relies on the description of R as a Nichols algebra. If $\mathcal{A}$ is a pointed Hopf algebra such that $\operatorname{gr} \mathcal{A} \simeq R\#A_0$, it is said that $\mathcal{A}$ is a lifting of R over A_0. This procedure can be generalized for other types of filtrations, such as the standard filtration, see [2].

Let H be a Hopf algebra with bijective antipode and $R \in {}^H_H\mathcal{YD}$ a braided Hopf algebra. An open question concerning this problem is whether all possible

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liftings of a bosonization $R\#H$ can be obtained by a (left) 2-cocycle deformation on the multiplication. Positive answers were obtained for $H = \mathbb{C}\Gamma$, a group algebra over a finite group Γ, by different authors using different methods. Among them, [19, 23, 24] solved the case when Γ is abelian and R is a quantum linear space, [17] when Γ is a symmetric group $\mathbb{S}_n$ with $n \geq 3$, [16] when Γ is a dihedral group $\mathbb{D}_m$ with $m = 4t \geq 12$, and [18] for pointed Hopf algebras associated to affine racks. Moreover, in [1] a systematic procedure to construct liftings as cocycle deformations via Hopf–Galois objects is described. Variations of this problem were also studied by other authors; see for example [10, 11] and references therein.

Hence, a natural question is to describe the set $\mathcal{Z}^2(A, \mathbb{C})$ of Hopf 2-cocycles on a bosonization $A = R\#H$. Since H is a Hopf subalgebra of A, the restriction of any Hopf 2-cocycle on A gives a Hopf 2-cocycle on H. This restriction admits a section that gives an injective map $\mathcal{Z}^2(H, \mathbb{C}) \hookrightarrow \mathcal{Z}^2(A, \mathbb{C})$. In particular, any Hopf 2-cocycle σ on H defines a Hopf 2-cocycle $\tilde{\sigma}$ on A.

Let θ be a positive integer. Let $U_{\mathbf{q}}(\mathfrak{g}_A)$ be the multiparameter quantum group associated to a generalized Cartan matrix A defined in [27], with $\mathbf{q} = (q_{ij})_{1 \leq i,j \leq \theta}$ and $q_{ij} \in \mathbb{C}^\times, q_{ii} \neq 1$ for all $1 \leq i, j \leq \theta$. In order to treat these quantum groups in a unified way, we describe them explicitly in Theorem 3.8 as a family of *reductive* pointed Hopf algebras $\widetilde{\mathcal{U}}(\mathcal{D}_{\text{red}}, \ell)$ given in [5]. These are also associated to the generalized Cartan matrix A. A similar description is given in [20] using bicharacters. This allows us to study $U_{\mathbf{q}}(\mathfrak{g}_A)$ as a quotient of a bosonization of a pre-Nichols algebra.

Using results on cocyles on bosonizations, we show in Theorem 3.6 that if q_{ii} is a positive real number for all $1 \leq i \leq \theta$, the algebras $\widetilde{\mathcal{U}}(\mathcal{D}_{\text{red}}, \ell)$ depend, up to cocycle deformation, on only one parameter on each connected component of the Dynkin diagram associated to A. This relation was previously described in [8] and [29] as a twist-equivalence between the matrices associated to the braiding. Here we re-interpret them as 2-cocycle deformations.

As a consequence, we prove that the algebras $U_{\mathbf{q}}(\mathfrak{g}_A)$ depend, up to cocycle deformation, on only one parameter on each connected component of the Dynking diagram. In case $\mathfrak{g}_A$ is simple, we obtain a result of Pei, Hu and Rosso [27, Theorem 28] — see Theorem 3.8 and Corollaries 3.9, 3.11.

The paper is organized as follows. In Section 2 we fix notation and recall some known facts on Hopf 2-cocycles, Yetter–Drinfeld modules, Nichols algebras and bosonizations. In particular, in Subsection 2.4 we treat cocycle deformations of Hopf algebras given by bosonizations $A = R\#H$ and describe the relation between $\mathcal{Z}^2(A, \mathbb{C})$ and $\mathcal{Z}^2(H, \mathbb{C})$. At the end of this section we provide an example on finite-dimensional pointed Hopf algebras over symmetric groups. In Section 3 we first introduce the family of pointed Hopf algebras $\widetilde{\mathcal{U}}(\mathcal{D}_{\text{red}}, \ell)$ given in [5] and prove Theorem 3.6. Then we explicitly describe the multiparameter quantum groups $U_{\mathbf{q}}(\mathfrak{g}_A)$ as a family of these pointed Hopf algebras and apply to them the results on cocycle deformations.

2. Preliminaries

In this section we fix notation and recall some definitions and known results that are used along the paper.

2.1. **Conventions.** We work over the field $\mathbb{C}$ of complex numbers, although most of our study could be carried out over a general algebraically closed field of characteristic zero. We denote by $\mathbb{C}^\times$ the group of units of $\mathbb{C}$. If Γ is a group, we denote by $\widehat{\Gamma}$ the character group. By convention, $\mathbb{N} = \{0, 1, \dots\}$. If A is an algebra and $g \in A$ is invertible, then $g \triangleright a = gag^{-1}$, $a \in A$, denotes the inner automorphism defined by g.

Our references for the theory of Hopf algebras are [26, 30, 28]. We use standard notation for Hopf algebras; the comultiplication is denoted by Δ and the antipode by $\mathcal{S}$. The left adjoint representation of H on itself is the algebra map $\operatorname{ad} : H \to \operatorname{End}(H)$, $\operatorname{ad}_l x(y) = x_{(1)} y \mathcal{S}(x_{(2)})$, $x, y \in H$; we shall write ad for ad_l, omitting the subscript l unless strictly needed. There is also a right adjoint action given by $\operatorname{ad}_r x(y) = \mathcal{S}(x_{(1)}) y x_{(2)}$. Note that both ad_l and ad_r are multiplicative. The set of group-like elements of a coalgebra C is denoted by $G(C)$. We also denote by $C^+ = \operatorname{Ker} \varepsilon$ the augmentation ideal of C, where $\varepsilon : C \to \mathbb{C}$ is the counit of C. Let $g, h \in G(H)$; the set of (g, h)-primitive elements is given by $P_{g,h}(H) = \{x \in H : \Delta(x) = x \otimes g + h \otimes x\}$. We call $P_{1,1}(H) = P(H)$ the set of primitive elements.

Let $A \xrightarrow{\pi} H$ be a Hopf algebra map; then

$$A^{\operatorname{co} H} = A^{\operatorname{co} \pi} = \{a \in A \mid (\operatorname{id} \otimes \pi)\Delta(a) = a \otimes 1\}$$

denotes the subalgebra of right coinvariants and ${}^{\operatorname{co} H}A = {}^{\operatorname{co} \pi}A$ denotes the subalgebra of left coinvariants.

For $n > 0$ and $q \in \mathbb{C}^\times$, $q \neq 1$, define

$$(n)_q = \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + q + 1,$$
$$(n)_q! = (n)_q (n-1)_q \cdots (2)_q (1)_q \text{ and } (0)_q = 1,$$
$$\binom{n}{k}_q = \frac{(n)_q}{(k)_q (n-k)_q}.$$

It is well-known that

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q. \tag{1}$$

A *braided vector space* is a pair (V, c) with V a vector space and $c \in \operatorname{Aut}(V \otimes V)$ that satisfies the braid equation, that is $(c \otimes \operatorname{id})(\operatorname{id} \otimes c)(c \otimes \operatorname{id}) = (\operatorname{id} \otimes c)(c \otimes \operatorname{id})(\operatorname{id} \otimes c) \in \operatorname{End}(V \otimes V \otimes V)$.

Let $(\mathcal{C}, \otimes, a, l, r, \mathbf{1})$ be a monoidal category and denote by $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ the flip functor given by $\tau(X, Y) = (Y, X)$ for all $X, Y \in \mathcal{C}$. A *braiding* on $\mathcal{C}$ is a natural isomorphism $c : \otimes \to \otimes\tau$ that satisfies the hexagon axiom for any $U, V, W \in \mathcal{C}$:

$$a_{V,W,U}\, c_{U,V\otimes W}\, a_{U,V,W} = (\operatorname{id}_V \otimes c_{U,W})\, a_{V,U,W}\, (c_{U,V} \otimes \operatorname{id}_W),$$
$$a^{-1}_{W,U,V}\, c_{U\otimes V,W}\, a^{-1}_{U,V,W} = (c_{U,W} \otimes \operatorname{id}_V)\, a^{-1}_{U,W,V}\, (\operatorname{id}_U \otimes c_{V,W}).$$

A *braided monoidal category* is a pair $(\mathcal{C}, c)$ where $\mathcal{C}$ is a monoidal category and c is a braiding on $\mathcal{C}$; see [21, Ch. XIII] for details. If $\mathcal{C}$ is strict, the equalities above are equivalent to

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})\,(c_{U,V} \otimes \mathrm{id}_W), \qquad c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V)\,(\mathrm{id}_U \otimes c_{V,W}).$$

In particular, if $V \in \mathcal{C}$ then $(V, c_{V,V})$ is a braided vector space.

2.2. **Deforming cocycles.** Let A be a Hopf algebra. Recall that a convolution invertible linear map σ in $\mathrm{Hom}_{\mathbb{C}}(A \otimes A, \mathbb{C})$ is a *normalized Hopf 2-cocycle* if

$$\sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)}) = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c)$$

and $\sigma(a, 1) = \varepsilon(a) = \sigma(1, a)$ for all $a, b, c \in A$; see [26, Sec. 7.1].

Using a 2-cocycle σ it is possible to define a new algebra structure on A by deforming the multiplication, which we denote by A_σ. Moreover, A_σ is indeed a Hopf algebra with $A = A_\sigma$ as coalgebras, deformed multiplication $m_\sigma = \sigma * m * \sigma^{-1} : A \otimes A \to A$ given by

$$m_\sigma(a, b) = a \cdot_\sigma b = \sigma(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}\sigma^{-1}(a_{(3)}, b_{(3)}) \qquad \text{for all } a, b \in A,$$

and antipode $\mathcal{S}_\sigma = \sigma * \mathcal{S} * \sigma^{-1} : A \to A$ given by (see [14] for details)

$$\mathcal{S}_\sigma(a) = \sigma(a_{(1)}, \mathcal{S}(a_{(2)}))\mathcal{S}(a_{(3)})\sigma^{-1}(\mathcal{S}(a_{(4)}), a_{(5)}) \qquad \text{for all } a \in A.$$

We denote by $\mathcal{Z}^2(A, \mathbb{C})$ the set of normalized Hopf 2-cocycles on A. Let $\tau, \sigma : A \otimes A \to \mathbb{C}$ be two linear maps. We denote by $\tau * \sigma : A \otimes A \to \mathbb{C}$ the linear map given by the convolution, that is $(\tau * \sigma)(a, b) = \tau(a_{(1)}, b_{(1)})\sigma(a_{(2)}, b_{(2)})$ for all $a, b \in A$.

Remark 2.1. Assume $A = \mathbb{C}\Gamma$, with Γ a group. Then a normalized *Hopf* 2-cocycle on A is equivalent to a 2-cocycle $\varphi \in \mathcal{Z}^2(\Gamma, \mathbb{C})$, that is a map $\varphi : \Gamma \times \Gamma \to \mathbb{C}^\times$ such that, for all $g, h, t \in \Gamma$,

$$\varphi(g, h)\varphi(gh, t) = \varphi(h, t)\varphi(g, ht) \quad \text{and} \quad \varphi(g, e) = \varphi(e, g) = 1.$$

2.3. **Yetter–Drinfeld modules, Nichols algebras and bosonization.** Let H be a Hopf algebra with bijective antipode. A Yetter–Drinfeld module over H is a left H-module and a left H-comodule with comodule structure denoted by $\delta : V \to H \otimes V$, $v \mapsto v_{(-1)} \otimes v_{(0)}$, such that

$$\delta(h \cdot v) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)} \qquad \text{for all } v \in V,\ h \in H.$$

Let ${}^H_H\mathcal{YD}$ be the category of Yetter–Drinfeld modules over H with H-linear and H-colinear maps as morphisms. The category ${}^H_H\mathcal{YD}$ is monoidal and braided. Indeed, if $V, W \in {}^H_H\mathcal{YD}$, then $V \otimes W$ is the tensor product over $\mathbb{C}$ with the diagonal action and coaction of H and braiding $c_{V,W} : V \otimes W \to W \otimes V$, $v \otimes w \mapsto v_{(-1)} \cdot w \otimes v_{(0)}$ for all $v \in V, w \in W$.

If $H = \mathbb{C}\Gamma$ is a group algebra of a group Γ, we denote this category simply by ${}^\Gamma_\Gamma\mathcal{YD}$. In this case, $V \in {}^\Gamma_\Gamma\mathcal{YD}$ corresponds to a Γ-graded vector space $V = \bigoplus_{g\in\Gamma} V_g$ which is a left Γ-module such that each homogeneous component V_g, $g \in \Gamma$, is stable under the action of Γ. Here, the Γ-grading yields the left $\mathbb{C}\Gamma$-comodule structure by $\delta : V \to \mathbb{C}\Gamma \otimes V$, $\delta(v) = g \otimes v$ if v is homogeneous of degree $g \in \Gamma$.

For $V, W \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$, the braiding is given by $c_{V,W}(v \otimes w) = g \cdot w \otimes v$, for all $v \in V_g$, $w \in W$ and $g \in \Gamma$. If Γ is finite, then ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is a semisimple category.

Let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ and $g \in \Gamma$, $\chi \in \widehat{\Gamma}$. We denote by

$$V_g^\chi = \{v \in V : \ \delta(v) = g \otimes v, \ h \cdot v = \chi(h)v \ \ \forall h \in \Gamma\},$$

the Yetter–Drinfeld submodule given by the g-homogeneous elements with diagonal action of Γ given by χ. In case Γ is finite abelian, the pairs (g, χ) with $g \in \Gamma$ and $\chi \in \widehat{\Gamma}$ parametrize the simple modules and for all $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ we have that $V = \bigoplus_{g\in\Gamma,\chi\in\widehat{\Gamma}} V_g^\chi$.

Since ${}^{H}_{H}\mathcal{YD}$ is a braided monoidal category, we may consider Hopf algebras in ${}^{H}_{H}\mathcal{YD}$. For $V \in {}^{H}_{H}\mathcal{YD}$, the tensor algebra $T(V) = \oplus_{n\geq 0} T^n(V)$ is an $\mathbb{N}$-graded algebra and coalgebra in the braided category ${}^{H}_{H}\mathcal{YD}$ where the elements of $V = T(V)(1)$ are primitive.

Let (V, c) be a finite-dimensional braided vector space. We say that the braiding $c : V \otimes V \to V \otimes V$ is *diagonal* [8, Def. 1.1] if there exists a basis $x_1, \dots, x_\theta$ of V and non-zero scalars q_{ij} such that $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ for all $1 \leq i, j \leq \theta$. The braiding is called *generic* if it is diagonal and q_{ii} is not a root of unity for all $1 \leq i \leq \theta$, and it is called *positive* if it is generic and q_{ii} is a positive real number for all $1 \leq i \leq \theta$. We say that two finite-dimensional braided vector spaces of diagonal type (V, c) and (W, d) with matrices (q_{ij}) and $(\hat{q}_{ij})$ are *twist-equivalent* [7, Def. 3.8] if $\dim V = \dim W$, $q_{ii} = \hat{q}_{ii}$, and

$$q_{ij}q_{ji} = \hat{q}_{ij}\hat{q}_{ji} \qquad \text{for all } 1 \leq i, j \leq \theta.$$

We are particularly interested in one class of braided Hopf algebras in these categories, which turn out to be crucial in the theory: the (pre-) Nichols algebras.

Definition 2.2. Let $I(V) \subseteq T(V)$ be the largest $\mathbb{N}$-graded ideal and coideal such that $I(V) \cap V = 0$. We call $\mathfrak{B}(V) = T(V)/I(V)$ the *Nichols algebra* of V. In particular, $\mathfrak{B}(V) = \bigoplus_{n\geq 0} \mathfrak{B}^n(V)$ is an $\mathbb{N}$-graded Hopf algebra in ${}^{H}_{H}\mathcal{YD}$.

Given a braided vector space (V, c), one may construct the Nichols algebra $\mathfrak{B}(V, c) = \mathfrak{B}(V)$ in a way similar to the construction above, by taking a quotient of the tensor algebra $T(V)$ by the homogeneous two-sided ideal given by the kernel of a homogeneous symmetrizer: Let $\mathbb{B}_n$ be the braid group of n letters. Since c satisfies the braid equation, it induces a representation of $\mathbb{B}_n$, $\rho_n : \mathbb{B}_n \to \mathbf{GL}(V^{\otimes n})$ for each $n \geq 2$. Consider the morphisms

$$Q_n = \sum_{\sigma\in\mathbb{S}_n} \rho_n(M(\sigma)) \in \operatorname{End}(V^{\otimes n}),$$

where $M : \mathbb{S}_n \to \mathbb{B}_n$ is the Matsumoto section corresponding to the canonical projection $\mathbb{B}_n \twoheadrightarrow \mathbb{S}_n$. Then the Nichols algebra $\mathfrak{B}(V)$ is the quotient of the tensor algebra $T(V)$ by the two-sided ideal $\mathcal{J} = \bigoplus_{n\geq 2} \operatorname{Ker} Q_n$.

A *pre-Nichols algebra* is an intermediate graded braided Hopf algebra between $T(V)$ and $\mathfrak{B}(V)$; see [23, 24].

Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$ with multiplication m_R. For $x, y \in P(R)$, we define the braided adjoint action of x on y by

$$\mathrm{ad}_c(x)(y) = m_R(x \otimes y) - m_R \circ c_{R\otimes R}(x \otimes y) = xy - (x_{(-1)} \cdot y)x_{(0)}.$$

This element is also called the *braided commutator* of x and y.

2.3.1. *Bosonization and Hopf algebras with a projection.* Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$. The procedure to obtain a usual Hopf algebra from the braided Hopf algebra R and H is called bosonization or Radford–Majid product, and it is usually denoted by $R\#H$. As a vector space, $R\#H = R\otimes H$ and the multiplication and comultiplication are given by the smash-product and smash-coproduct, respectively. That is, for all $r, s \in R$ and $g, h \in H$, we have

$$\begin{aligned}(r\#g)(s\#h) &= r(g_{(1)} \cdot s)\#g_{(2)}h,\\ \Delta(r\#g) &= r^{(1)}\#(r^{(2)})_{(-1)}g_{(1)} \otimes (r^{(2)})_{(0)}\#g_{(2)},\\ \mathcal{S}(r\#g) &= (1\#\mathcal{S}_H(r_{(-1)}g))(\mathcal{S}_R(r_{(0)})\#1),\end{aligned}$$

where $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$ denotes the comultiplication in $R \in {}^H_H\mathcal{YD}$ and $\mathcal{S}_R$ the antipode. Clearly, the map $\iota : H \to R\#H$ given by $\iota(h) = 1\#h$ for all $h \in H$ is an injective Hopf algebra map, and the map $\pi : R\#H \to H$ given by $\pi(r\#h) = \varepsilon_R(r)h$ for all $r \in R$, $h \in H$ is a surjective Hopf algebra such that $\pi \circ \iota = \mathrm{id}_H$. Moreover, it holds that $R = (R\#H)^{\mathrm{co}\,\pi}$.

Conversely, let A be a Hopf algebra with bijective antipode and $\pi : A \to H$ a Hopf algebra epimorphism admiting a Hopf algebra section $\iota : H \to A$ such that $\pi \circ \iota = \mathrm{id}_H$. Then $R = A^{\mathrm{co}\,\pi}$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ called the *diagram* of A and $A \simeq R\#H$ as Hopf algebras. See [28, 11.6] for more details.

2.4. **On cocycle deformations and bosonizations.** In this subsection we collect some results on the construction of 2-cocycles on bosonizations of Hopf algebras.

Let H be a Hopf algebra with bijective antipode, R a braided Hopf algebra in ${}^H_H\mathcal{YD}$ and $A = R\#H$. To avoid any confusion, in this section we denote by $\rightharpoonup: H \otimes R \to R$ the action of H on R.

Let $\sigma \in \mathcal{Z}^2(H, \mathbb{C})$. Then the map $\tilde{\sigma} : A \otimes A \to \mathbb{C}$ given by

$$\tilde{\sigma}(r\#h, s\#k) = \sigma(h, k)\varepsilon_R(r)\varepsilon_R(s) \qquad \text{for all } r, s \in R,\ h, k \in H,$$

is a normalized Hopf 2-cocycle such that $\tilde{\sigma}|_{H\otimes H} = \sigma$. Moreover, H_σ is a Hopf subalgebra of $A_{\tilde{\sigma}}$ and the map $\mathcal{Z}^2(H, \mathbb{C}) \to \mathcal{Z}^2(A, \mathbb{C})$ given by $\sigma \mapsto \tilde{\sigma}$ gives a section of the map $\mathcal{Z}^2(A, \mathbb{C}) \to \mathcal{Z}^2(H, \mathbb{C})$ induced by the restriction; in particular, it is injective. See e.g. [24, Section 5], [13, Prop. 4.2].

Proposition 2.3. [24, Prop. 5.2] *Let σ and $\tilde{\sigma}$ be as above. Then $A_{\tilde{\sigma}} = R_\sigma\#H_\sigma$, where $R_\sigma = R$ as coalgebras, and the product is given by*

$$a \cdot_\sigma b = \sigma(a_{(-1)}, b_{(-1)})a_{(0)}b_{(0)} \quad \textit{for all } a, b \in R. \tag{2}$$

Moreover, $R_\sigma \in {}^{H_\sigma}_{H_\sigma}\mathcal{YD}$ *with the action of* H_σ *given by*

$$h \rightharpoonup_\sigma a = \sigma(h_{(1)}, a_{(-1)})(h_{(2)} \rightharpoonup a_{(0)})_{(0)}\sigma^{-1}((h_{(2)} \rightharpoonup a_{(0)})_{(-1)}, h_{(3)}) \quad \text{for all } h \in H_\sigma, a \in R_\sigma. \tag{3}$$

□

Remark 2.4. In case R is a (pre-) Nichols algebra, by [1] and [22, Thm. 2.7 and Cor. 3.4] R_σ is also a (pre-) Nichols algebra and the action is described by (3).

Remark 2.5. Assume $H = \mathbb{C}\Gamma$ and $\sigma \in \mathcal{Z}^2(\Gamma, \mathbb{C})$. Let $h \in \Gamma$ and $a \in R$ be a homogeneous element of degree $g \in \Gamma$; in particular, $\delta(a) = g \otimes a$ and $\Delta_A(a) = a \otimes 1 + g \otimes a$. Then (3) yields

$$h \rightharpoonup_\sigma a = h \cdot_\sigma a \cdot_\sigma h^{-1} = \sigma(h, g)\sigma^{-1}(hgh^{-1}, h)h \rightharpoonup a. \tag{4}$$

Remark 2.6. In [12] the authors introduced another type of cocycle deformation on a Hopf algebra, which is closely related to the one given above. We describe it shortly. Let Γ be an abelian group and A a Hopf algebra that is $\Gamma \times \Gamma$-graded. Given any $\varphi \in \mathcal{Z}^2(\Gamma, \mathbb{C})$, define a new product on A by

$$h \underset{\varphi}{*} k := \varphi(\eta, \kappa)\varphi\big(\eta', \kappa'\big)^{-1}\, h \cdot k \tag{5}$$

for all homogeneous $h, k \in H$ with degrees $\big(\eta, \eta'\big), \big(\kappa, \kappa'\big) \in \Gamma \times \Gamma$. With this multiplication, the new algebra $A^{(\varphi)}$ is a Hopf algebra with the same coalgebra structure and unit as A. Assume $A = R\#\mathbb{C}\Gamma$ is given by a bosonization over an abelian group Γ. Then, the coaction of $\mathbb{C}\Gamma$ on the elements of R induces a $\Gamma \times \Gamma$ grading on A with $\deg g = (g, g)$ for all $g \in \Gamma$ and $\deg(x) = (g, 1)$ if $\delta(x) = g \otimes x$, with $x \in R$ a homogeneous element, and we have that $A^{(\varphi)} = A_{\tilde{\varphi}}$, where $\tilde{\varphi}$ is the Hopf 2-cocycle on A induced by φ. In particular, for x, y homogeneous elements of R of degree g and h respectively, we have that

$$x \underset{\varphi}{*} y = \varphi(g, h)\varphi(1, 1)^{-1}xy = \varphi(x_{(-1)}, y_{(-1)})x_{(0)}y_{(0)},$$

which coincides with formula (2).

Remark 2.7. Let A be a Hopf algebra and $\sigma \in \mathcal{Z}^2(A, \mathbb{C})$, $\tau \in \mathcal{Z}^2(A_\sigma, \mathbb{C})$. Then $\tau * \sigma \in \mathcal{Z}^2(A, \mathbb{C})$. As is known, the set $\mathcal{Z}^2(A, \mathbb{C})$ is not a group in general. It is the case if the cocycles are *lazy*; see for example [13]. Nevertheless, there is a groupoid structure on the set of all Hopf 2-cocycles.

Let $\mathcal{Z}$ be the groupoid whose objects are Hopf algebras A and arrows labelled by the set of 2-cocycles $\{\alpha_\sigma : A \to A_\sigma : \sigma \in \mathcal{Z}^2(A, \mathbb{C})\}$. The source and target maps are given by $s(\alpha_\sigma) = A, t(\alpha_\sigma) = A_\sigma$, and the composition by $\alpha_\tau \circ \alpha_\sigma = \alpha_{\tau*\sigma}$ for $\sigma \in \mathcal{Z}^2(A, \mathbb{C})$ and $\tau \in \mathcal{Z}^2(A_\sigma, \mathbb{C})$.

Clearly, the identity arrow is given by $\mathrm{id}_A = \alpha_{\varepsilon_A}$, and since $(A_\sigma)_{\sigma^{-1}} = A_{\sigma*\sigma^{-1}} = A = A_{\sigma^{-1}*\sigma} = (A_{\sigma^{-1}})_\sigma$, each arrow is invertible with inverse $\alpha_{\sigma^{-1}} : A_\sigma \to A$.

2.4.1. *An example on pointed Hopf algebras over* $\mathbb{S}_n$. We describe now an example where two non-isomorphic families of finite-dimensional pointed Hopf algebras over $\mathbb{S}_n$ are cocycle deformation of each other. The 2-cocycle is given by the product of a 2-cocycle associated to a Hochschild 2-cocycle on a Nichols algebra and a group 2-cocycle found by Vendramin [31] which serves as twisting of the 2-cocycle asociated to the rack of transpositions in $\mathbb{S}_n$. For this purpose we need to introduce first some terminology (see [3, Def. 1.1] for more details).

Racks and Nichols algebras. A *rack* is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \to X$ is a function, such that $\phi_i = i \triangleright (\cdot) : X \to X$ is a bijection for all $i \in X$ satisfying that $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$. A group G is a rack with $x \triangleright y = xyx^{-1}$ for all $x, y \in G$. If $G = \mathbb{S}_n$, then we denote by $\mathcal{O}_j^n$ the conjugacy class of all j-cycles in $\mathbb{S}_n$.

Let $(X, \triangleright)$ be a rack. A *rack 2-cocycle* $q : X \times X \to \mathbb{C}^\times$, $(i, j) \mapsto q_{ij}$ is a function such that

$$q_{i,j\triangleright k}\, q_{j,k} = q_{i\triangleright j, i\triangleright k}\, q_{i,k}, \quad \text{for all } i, j, k \in X.$$

It determines a braiding c^q on the vector space $\mathbb{C}X$ with basis $\{x_i\}_{i\in X}$ by $c^q(x_i \otimes x_j) = q_{ij} x_{i\triangleright j} \otimes x_i$ for all $i, j \in X$. We denote this braided vector space $(\mathbb{C}X, c^q)$ by $M(X, q)$ and the Nichols algebra associated with it by $\mathfrak{B}(X, q)$.

Let X be a subrack of a conjugacy class $\mathcal{O}$ in Γ, q a rack 2-cocycle on X and $\varphi \in \mathcal{Z}^2(\Gamma, \mathbb{C})$. Then the map $q^\varphi : X \times X \to \mathbb{C}^\times$ given by

$$q^\varphi_{xy} = \varphi(x, y)\varphi^{-1}(x \triangleright y, x) q_{xy}, \quad \text{for all } x, y \in X, \tag{6}$$

is a rack 2-cocycle.

If X is any rack, q a rack 2-cocycle on X and $\varphi : X \times X \to \mathbb{C}^\times$, then define q^φ by (6). It can be shown that q^φ is a rack 2-cocycle if and only if

$$\begin{aligned}\varphi(x, z)\varphi(x \triangleright y, x \triangleright z)\varphi(x \triangleright (y \triangleright z), x)\varphi(y \triangleright z, y) \\ = \varphi(y, z)\varphi(x, y \triangleright z)\varphi(x \triangleright (y \triangleright z), x \triangleright y)\varphi(x \triangleright z, x)\end{aligned}$$

for any $x, y, z \in X$. If X is a subrack of a group Γ and $\varphi \in Z^2(\Gamma, \mathbb{C})$, then $\varphi|_{X\times X}$ satisfies the equation above.

Definition 2.8. Let $q, q' : X \times X \to \mathbb{C}^\times$ be rack 2-cocycles on X. We say that q and q' are *twist equivalent* if there exists $\varphi : X \times X \to \mathbb{C}^\times$ such that $q' = q^\varphi$ as in (6).

On Nichols algebras over $\mathbb{S}_n$. Let $X = \mathcal{O}_2^n$ be the rack of transpositions with $n \geq 3$ and consider the cocycles:

$$-1 : \mathcal{O}_2^n \times \mathcal{O}_2^n \to \mathbb{C}^\times, \quad (j, i) \mapsto \mathrm{sg}(j) = -1;$$

$$\chi : \mathcal{O}_2^n \times \mathcal{O}_2^n \to \mathbb{C}^\times, \quad (j, i) \mapsto \chi_i(j) = \begin{cases} 1, & \text{if } i = (a, b) \text{ and } j(a) < j(b), \\ -1, & \text{if } i = (a, b) \text{ and } j(a) > j(b), \end{cases}$$

for all $i, j \in \mathcal{O}_2^n$. By [25, Ex. 6.4], [4, Theorem 6.12], the Nichols algebras are given by

(a) $\mathfrak{B}(\mathcal{O}_2^n,-1)$; generated by the elements $\{x_{(\ell m)}\}_{1\leq \ell<m\leq n}$ satisfying for all $1\leq a<b<c\leq n$, $1\leq e<f\leq n$, $\{a,b\}\cap\{e,f\}=\emptyset$ the identities

$$0=x_{(ab)}^2=x_{(ab)}x_{(ef)}+x_{(ef)}x_{(ab)}=x_{(ab)}x_{(bc)}+x_{(bc)}x_{(ac)}+x_{(ac)}x_{(ab)}.$$

(b) $\mathfrak{B}(\mathcal{O}_2^n,\chi)$; generated by the elements $\{x_{(\ell m)}\}_{1\leq \ell<m\leq n}$ satisfying for all $1\leq a<b<c\leq n$, $1\leq e<f\leq n$, $\{a,b\}\cap\{e,f\}=\emptyset$ the identities

$$0=x_{(ab)}^2=x_{(ab)}x_{(ef)}-x_{(ef)}x_{(ab)}=x_{(ab)}x_{(bc)}-x_{(bc)}x_{(ac)}-x_{(ac)}x_{(ab)},$$

$$0=x_{(bc)}x_{(ab)}-x_{(ac)}x_{(bc)}-x_{(ab)}x_{(ac)}.$$

For $3\leq n\leq 5$ these Nichols algebras are finite-dimensional. If $n>5$ it is not known if this is the case. It turns out that the cocycles associated to them are twist equivalent.

Theorem 2.9. [31, Theorem 3.8] *Let $n\geq 4$. The rack 2-cocycles χ and -1 associated to $\mathcal{O}_2^n$ are twist equivalent.* □

Remark 2.10. The twist given by Theorem 2.9 is defined using a group 2-cocycle $\varphi\in\mathcal{Z}^2(\mathbb{S}_n,\mathbb{C})$. In particular, $-1=\varphi(x,y)\varphi^{-1}(x\triangleright y,x)\chi(x,y)$ for all $x,y\in\mathcal{O}_2^n$.

Cocycles on pointed Hopf algebras over $\mathbb{S}_n$. Assume $n\geq 4$. Let $\varphi\in\mathcal{Z}^2(\mathbb{S}_n,\mathbb{C})$ be the group 2-cocycle given in Remark 2.10. Denote again by φ the associated Hopf 2-cocycle in $\mathcal{Z}^2(\mathbb{C}\mathbb{S}_n,\mathbb{C})$ and by $\tilde{\varphi}\in\mathcal{Z}^2(A,\mathbb{C})$ the Hopf 2-cocycle on the bosonization $A=\mathfrak{B}(\mathcal{O}_2^n,\chi)\#\mathbb{C}\mathbb{S}_n$. Then by Proposition 2.3 we have that

$$\mathfrak{B}(\mathcal{O}_2^n,-1)\#\mathbb{C}\mathbb{S}_n\simeq\mathfrak{B}(\mathcal{O}_2^n,\chi)_\varphi\#\mathbb{C}\mathbb{S}_n\simeq(\mathfrak{B}(\mathcal{O}_2^n,\chi)\#\mathbb{C}\mathbb{S}_n)_{\tilde{\varphi}}.$$

Let $\Lambda,\Gamma\in\mathbb{C}$ and $t=(\Lambda,\Gamma)$. Denote by $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$ the algebra generated by $\{a_i,h_r : i\in\mathcal{O}_2^n, r\in\mathbb{S}_n\}$ satisfying the following relations for $r,s,j\in\mathbb{S}_n$ and $i\in\mathcal{O}_2^n$:

$$h_e=1,\quad h_rh_s=h_{rs},\quad h_ja_i=-a_{j\triangleright i}h_j,\quad a_{(12)}^2=0,$$
$$a_{(12)}a_{(34)}+a_{(34)}a_{(12)}=\Lambda(1-h_{(12)}h_{(34)}),$$
$$a_{(12)}a_{(23)}+a_{(23)}a_{(13)}+a_{(13)}a_{(12)}=\Gamma(1-h_{(12)}h_{(23)}).$$

This algebra is indeed a Hopf algebra with the structure determined by h_σ being a grouplike element and a_σ being a $(1,h_\sigma)$-primitive for all $\sigma\in\mathcal{O}_2^n$. Consequently, it is a pointed Hopf algebra with diagram $\mathfrak{B}(\mathcal{O}_2^n,-1)$; see [15] for details. If $t=(2\lambda,3\lambda)$ with $\lambda\in\mathbb{C}^\times$, we know that $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$ is a cocycle deformation of the bosonization $\mathfrak{B}(\mathcal{O}_2^n,-1)\#\mathbb{C}\mathbb{S}_n$. The explicit cocycle is given in the theorem below; it was also shown in [17] by other methods.

Briefly, let X be a rack, q a rack 2-cocycle and $\{x_\tau\}_{\tau\in X}$ be homogeneous elements in $V=M(X,q)\in{}^{\mathbb{S}_n}_{\mathbb{S}_n}\mathcal{YD}$. Then the linear combination of tensor products of linear functionals δ_τ given by $\delta_\tau(x_\mu)=\delta_{\tau,\mu}$ for all $\mu,\tau\in X$ give rise to a Hochschild 2-cocycle $\eta=\sum_{\tau,\mu\in X}a_{\tau,\mu}d_\tau\otimes\mu$ by defining it via

$$\eta(\mathfrak{B}^m(V)\otimes\mathfrak{B}^n(V))=0\quad\text{ if }(m,n)\neq(1,1).$$

If this cocycle is invariant under the action of $\mathbb{S}_n$, i.e. $\eta^h(x,y)=\eta(h_{(1)}\rightharpoonup x,h_{(2)}\rightharpoonup y)=\eta(x,y)$ for all $x,y\in\mathfrak{B}(V)$ and $h\in\mathbb{S}_n$, then one may define a

Hochschild 2-cocycle on $A=\mathfrak{B}(V)\#\mathbb{C}\mathbb{S}_n$ by

$$\tilde{\eta}(x\#h,y\#k)=\eta(x,h\rightharpoonup y)\varepsilon(k)\qquad \text{for all } x,y\in\mathfrak{B}(V),\ h,k\in\mathbb{S}_n.$$

Moreover, $\sigma=e^{\tilde{\eta}}=\sum_{i=0}^{\infty}\frac{\tilde{\eta}^{*i}}{i!}: A\otimes A\to\mathbb{C}$ is a well-defined convolution invertible map with convolution inverse $e^{-\tilde{\eta}}$. By [16, Corollary 2.6] this map $\sigma=e^{\tilde{\eta}}$ is a Hopf 2-cocycle. See [16, 2.1] for more details.

Theorem 2.11. [16, Theorem 4.10 (i)] *Let $A=\mathfrak{B}(\mathcal{O}_2^n,-1)\#\mathbb{C}\mathbb{S}_n$ and $\sigma_\lambda=e^{\tilde{\eta}_\lambda}$ a Hopf 2-cocycle with $\eta_\lambda=\frac{\lambda}{3}\sum_{\mu,\tau\in\mathcal{O}_2^n}d_\tau\otimes d_\mu$ and $\lambda\in\mathbb{C}$. Then $A_{\sigma_\lambda}\simeq\mathcal{H}(\mathcal{Q}_n^{-1}[(2\lambda,3\lambda)])$ for $n\geq 4$.* □

We end this section with the following result.

Corollary 2.12. *Let $\sigma_\lambda=e^{\tilde{\eta}_\lambda}\in\mathcal{Z}^2(\mathfrak{B}(\mathcal{O}_2^n,-1)\#\mathbb{C}\mathbb{S}_n,\mathbb{C})$ and $\tilde{\varphi}\in\mathcal{Z}^2(\mathfrak{B}(\mathcal{O}_2^n,\chi)\#\mathbb{C}\mathbb{S}_n,\mathbb{C})$ be the Hopf 2-cocycles defined above. Then*

$$\mathcal{H}(\mathcal{Q}_n^{-1}[(2\lambda,3\lambda)])\simeq(\mathfrak{B}(\mathcal{O}_2^n,\chi)\#\mathbb{C}\mathbb{S}_n)_{\sigma_\lambda*\tilde{\varphi}}.$$

Proof. Since $\sigma_\lambda=e^{\tilde{\eta}_\lambda}$ is a Hopf 2-cocycle on $\mathfrak{B}(\mathcal{O}_2^n,-1)\#\mathbb{C}\mathbb{S}_n$ and this algebra is isomorphic to $(\mathfrak{B}(\mathcal{O}_2^n,\chi)\#\mathbb{C}\mathbb{S}_n)_{\tilde{\varphi}}$ for $\varphi\in\mathcal{Z}^2(\mathbb{S}_n,\mathbb{C})$, the claim follows by Remark 2.7. □

3. Multiparameter quantum groups, pre-Nichols algebras and cocycle deformations

In this section we show explicitly that certain classes of multiparameter quantum groups can be described using the theory of pointed Hopf algebras developed by Andruskiewitsch and Schneider [9, 5].

First we introduce these families of pointed Hopf algebras and then show that under some assumptions they are cocycle deformations of certain (one-parameter) families of pointed Hopf algebras.

3.1. On pointed Hopf algebras associated to generalized Cartan matrices.

Let θ be a positive integer and $(a_{ij})_{1\leq i,j\leq\theta}$ a generalized Cartan matrix, that is, a matrix with integer entries such that $a_{ii}=2$ for all $1\leq i\leq\theta$, and for all $1\leq i,j\leq\theta$, $i\neq j$, $a_{ij}\leq 0$, and if $a_{ij}=0$, then $a_{ji}=0$.

Definition 3.1. [5, Def. 3.2] A *reduced YD-datum of Cartan type*

$$\mathcal{D}_{\text{red}}=\mathcal{D}(\Gamma,(L_i)_{1\leq i\leq\theta},(K_i)_{1\leq i\leq\theta},(\chi_i)_{1\leq i\leq\theta},(a_{ij})_{1\leq i,j\leq\theta})$$

consists of an abelian group Γ, $K_i,L_i\in\Gamma$ and characters $\chi_i\in\widehat{\Gamma}=\operatorname{Hom}(\Gamma,\mathbb{C}^\times)$ satisfying for all $1\leq i,j\leq\theta$ that

$$K_iL_i\neq 1,\qquad q_{ij}=\chi_j(K_i)=\chi_i(L_j),$$
$$q_{ij}q_{ji}=q_{ii}^{a_{ij}},\quad q_{ii}\neq 1,\ 0\leq -a_{ij}<\operatorname{ord}(q_{ii})\leq\infty.$$

A reduced YD-datum D_{red} is called *generic* if $\chi_i(K_i)=q_{ii}$ is not a root of unity, for all $1\leq i\leq\theta$. A *linking parameter* ℓ for $\mathcal{D}_{\text{red}}$ is a family $\ell=(\ell_i)_{1\leq i\leq\theta}$ of non-zero elements in $\mathbb{C}$.

Let $\mathbb{I} = \{1, 2, \dots, \theta\}$. We have an equivalence relation on $\mathbb{I}$: for $i \neq j \in \mathbb{I}$ we say that $i \sim j$ if and only if there are $i_1, \dots, i_t \in \mathbb{I}$ with $t \geq 2$, $i_1 = i$, $i_t = j$ and $q_{i_k,i_{k+1}} q_{i_{k+1},i_k} \neq 1$ for all $1 \leq k < t$. We denote by $\mathcal{X}$ the set of equivalence classes. This equivalence can be described as usual in terms of the Cartan matrix. Indeed, for all $1 \leq i, j \leq \theta$, $i \sim j$ if and only if there are $i_1, \dots, i_t \in \mathbb{I}$, $t \geq 2$ with $i_1 = i$, $i_t = j$, and $a_{i_k,i_{k+1}} \neq 0$ for all $1 \leq k < t$.

A reduced YD-datum of Cartan type is said of *DJ-type* (Drinfeld–Jimbo type) if the Cartan matrix is symmetrizable, i.e. there exist d_i relatively prime positive integers such that $d_i a_{ij} = d_j a_{ji}$ for all $1 \leq i, j \leq \theta$, and for all $I \in \mathcal{X}$ there exists $q_I \in \mathbb{C}^\times$ such that

$$q_{ij} = q_I^{d_i a_{ij}} \qquad \text{for all } i \in I,\ 1 \leq j \leq \theta.$$

In particular, $q_{ij} = 1$ if $a_{ij} = 0$. We denote this datum by

$$\mathcal{D}_q = (\Gamma, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (q_I)_{I \in \mathcal{X}}, (a_{ij})_{1 \leq i,j \leq \theta}).$$

Definition 3.2. [9, Def. 2.4] Let

$$\mathcal{D}_{\text{red}} = (\Gamma, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i,j \leq \theta})$$

be a reduced YD-datum of Cartan type and $\ell = (\ell_i)_{1 \leq i \leq \theta}$ a linking parameter. $\widetilde{\mathcal{U}}(\mathcal{D}_{\text{red}}, \ell) = \widetilde{\mathcal{U}}$ is the algebra generated by the elements $g \in \Gamma$, x_i, y_i with $1 \leq i \leq \theta$, satisfying the relations

$$\begin{aligned}
&g^{\pm 1} h^{\pm 1} = h^{\pm 1} g^{\pm 1}, \qquad g^{\pm 1} g^{\mp 1} = 1,\\
&g x_i g^{-1} = \chi_i(g) x_i, \qquad g y_i g^{-1} = \chi_i(g)^{-1} y_i,\\
&x_i y_j - \chi_j^{-1}(K_i) y_j x_i = -\delta_{ij} \ell_i (K_i L_i - 1),\\
&\operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0, \qquad \operatorname{ad}_c(y_i)^{1-a_{ij}}(y_j) = 0,
\end{aligned}$$

for all $g, h \in \Gamma$, $1 \leq i, j \leq \theta$, where

$$\begin{aligned}
\operatorname{ad}_c(x_i)(x_j) &= x_i x_j - \chi_j(K_i) x_j x_i = x_i x_j - q_{ij} x_j x_i, \quad 1 \leq i, j \leq \theta,\\
\operatorname{ad}_c(y_i)(y_j) &= y_i y_j - \chi_j^{-1}(L_i) y_j y_i = y_i y_j - q_{ji}^{-1} y_j y_i, \quad 1 \leq i, j \leq \theta.
\end{aligned}$$

The algebra $\widetilde{\mathcal{U}}$ is a Hopf algebra with its structure determined by $g \in \Gamma$ being grouplike, x_i being $(1, K_i)$-primitive and y_i being $(1, L_i)$-primitive for all $1 \leq i \leq \theta$. In particular, it is a pointed Hopf algebra with $G(\widetilde{\mathcal{U}}) = \Gamma$.

Remark 3.3. The data $(\Gamma, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta})$ is called simply a YD-reduced datum. Let V, W be vector spaces with basis $\{x_i\}_{1 \leq i \leq \theta}$ and $\{y_i\}_{1 \leq i \leq \theta}$, respectively. Then any YD-reduced datum defines on V and W a Yetter–Drinfeld module structure over $\mathbb{C}\Gamma$ given by $x_i \in V_{K_i}^{\chi_i}$ and $y_i \in W_{L_i}^{\chi_i^{-1}}$ for all $1 \leq i \leq \theta$, that is,

$$\begin{aligned}
\delta(x_i) &= K_i \otimes x_i, \qquad & g \cdot x_i &= \chi_i(g) x_i,\\
\delta(y_i) &= L_i \otimes y_i, \qquad & g \cdot y_i &= \chi_i^{-1}(g) y_i,
\end{aligned}$$

for all $g \in \Gamma$. The braidings $c_V = c_{V,V}$ of V and c_W of W are given by

$$c_V(x_i \otimes x_j) = K_i \cdot x_j \otimes x_i = \chi_j(K_i) x_j \otimes x_i = q_{ij} x_j \otimes x_i, \quad 1 \le i, j \le \theta,$$
$$c_W(y_i \otimes y_j) = L_i \cdot y_j \otimes y_i = \chi_j^{-1}(L_i) y_j \otimes y_i = q_{ji}^{-1} y_j \otimes y_i, \quad 1 \le i, j \le \theta;$$

in particular, they are of diagonal type, and the corresponding adjoint actions are given by

$$\mathrm{ad}_c(x_i)(x_j) = x_i x_j - \chi_j(K_i) x_j x_i = x_i x_j - q_{ij} x_j x_i, \quad 1 \le i, j \le \theta,$$
$$\mathrm{ad}_c(y_i)(y_j) = y_i y_j - \chi_j^{-1}(L_i) y_j y_i = y_i y_j - q_{ji}^{-1} y_j y_i, \quad 1 \le i, j \le \theta.$$

The pre-Nichols algebras $R(\mathcal{D})$, $R(\mathcal{D}, V)$ and $R(\mathcal{D}, W)$ associated to the reduced YD-datum described above are given by the quotient (braided) Hopf algebras

$$R(\mathcal{D}) = T(V \oplus W)/(\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j),\ \mathrm{ad}_c(y_i)^{1-a_{ij}}(y_j),\ 1 \le i \ne j \le \theta),$$
$$R(\mathcal{D}, V) = T(V)/(\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j),\ 1 \le i \ne j \le \theta),$$
$$R(\mathcal{D}, W) = T(W)/(\mathrm{ad}_c(y_i)^{1-a_{ij}}(y_j),\ 1 \le i \ne j \le \theta).$$

Since $c_{W,V} c_{V,W} = \mathrm{id}$, we have that $R(\mathcal{D}) \simeq R(\mathcal{D}, V) \otimes R(\mathcal{D}, W)$ (see [24]). By abuse of notation, we denote the images of the elements x_i, y_j in $R(\mathcal{D})$ again by x_i, y_j. It is well-known that the elements $\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j)$, $1 \le i \ne j \le \theta$, are primitive in the free algebra $T(V)$ (see, for example, [6, A.1]), hence they generate a Hopf ideal.

It follows that $\widetilde{\mathcal{U}}(\mathcal{D}_{\mathrm{red}}, \ell)$ is the Hopf algebra given by the quotient of the bosonization $R(\mathcal{D}) \# \mathbb{C}\Gamma$ modulo the ideal generated by the elements

$$x_i y_j - \chi_j^{-1}(K_i) y_j x_i - \delta_{ij} \ell_i (K_i L_i - 1) \qquad \text{for all } 1 \le i, j \le \theta,$$

where we identify $x_i = x_i \# 1$, $y_i = y_i \# 1$ and $K_i = 1 \# K_i$, $L_i = 1 \# L_i$ for all $1 \le i \le \theta$.

Remark 3.4. In case the braiding is positive and generic, the pre-Nichols algebras $R(\mathcal{D}, V)$ and $R(\mathcal{D}, W)$ coincide with the Nichols algebras $\mathfrak{B}(V)$ and $\mathfrak{B}(W)$ respectively, that is, the ideals $I(V)$ and $I(W)$ are generated by the *quantum Serre relations* $\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j)$, $\mathrm{ad}_c(y_i)^{1-a_{ij}}(y_j)$ associated to the braided commutators (see [8, Theorem 4.3] and references therein). These Serre relations are not enough to define the ideal $I(V \oplus W)$; see Remark 3.5.

Remark 3.5. In case the braiding is positive and generic, the ideal $I(V \oplus W) \subseteq T(V \oplus W)$ is generated by $I(V)$, $I(W)$ and $x_i y_j - \chi_j^{-1}(K_i) y_j x_i$ for all $1 \le i, j \le \theta$; see [5, Remark 1.10]. In particular, we have that $\widetilde{\mathcal{U}}(\mathcal{D}_{\mathrm{red}}, 0) \simeq \mathfrak{B}(V \oplus W) \# \mathbb{Z}^{2\theta}$.

We present now a result that translates the notion of twist-equivalence of matrices of diagonal braidings ([8, Prop. 2.2], [29, Theorem 2.1]) to cocycle deformations. It states that, under some assumptions, these families of pointed Hopf algebras depend only on one parameter for each connected component, up to cocycle deformations.

Theorem 3.6. *Let Γ be a free abelian group of rank 2θ with generators $(L_i)_{1\le i\le\theta}$, $(K_i)_{1\le i\le\theta}$. Let $\mathcal{D}_{\rm red}$ be a reduced YD-datum of Cartan type and $\ell=(\ell_i)$ a linking parameter. If $1\neq q_{ii}$ is a positive real number for all $1\le i\le\theta$, then $\widetilde{\mathcal{U}}(\mathcal{D}_{\rm red},\ell)$ is a cocycle deformation of a Hopf algebra $\widetilde{\mathcal{U}}(\mathcal{D}_q,\ell)$ associated with a reduced YD-datum $\mathcal{D}_q$ of DJ-type.*

Proof. By [29, Theorem 2.1] we have that the Cartan matrix $(a_{ij})_{1\le i,j\le\theta}$ is symmetrizable, with symmetrizing diagonal matrix $(d_i)_{1\le i\le\theta}$, and there is a collection of positive numbers $(q_I)_{I\in\mathcal{X}}$ such that (q_{ij}) is twist-equivalent to $(\hat{q}_{ij})$, where $\hat{q}_{ij}=q_I^{d_ia_{ij}}$ for all $i,j\in I$.

If we order the group generators by $L_1,\dots,L_\theta,K_1,\dots,K_\theta$ and take the corresponding characters $\chi_1^{-1},\dots,\chi_\theta^{-1},\chi_1,\dots,\chi_\theta$, the matrix of the braiding in $V\oplus W$ is given by

$$p_{ij}=\begin{cases} q_{ji}^{-1} & \text{if } 1\le i,j\le\theta,\\ q_{ij}^{-1} & \text{if } 1\le i\le\theta,\ \theta+1\le j\le 2\theta,\\ q_{ji} & \text{if } \theta+1\le i\le 2\theta,\ 1\le j\le\theta,\\ q_{ij} & \text{if } \theta+1\le i,j\le 2\theta.\end{cases}$$

Let $\mathcal{D}_q=\mathcal{D}(\Gamma,(L_i)_{1\le i\le\theta},(K_i)_{1\le i\le\theta},(q_I)_{I\in\mathcal{X}},(a_{ij})_{1\le i,j\le\theta})$ be the reduced YD-datum of DJ-type associated with $(\hat{q}_{ij})$. Denote by $\widehat{V},\widehat{W}$ the braided vector spaces associated with this datum and by $(\hat{p}_{ij})_{i,j}$ the matrix of the braiding in $\widehat{V}\oplus\widehat{W}$. Let $\widetilde{\mathcal{U}}(\mathcal{D}_q,\ell)$ be the corresponding pointed Hopf algebra.

If we set $g_i=L_i$ and $g_{i+\theta}=K_i$ for all $1\le i\le\theta$, then by [8, Prop. 2.2] the map $\sigma:\Gamma\times\Gamma\to\mathbb{C}^\times$ given by

$$\sigma(g_i,g_j)=\begin{cases}\hat{p}_{ij}p_{ij}^{-1} & \text{if } i\le j,\\ 1 & \text{otherwise},\end{cases}$$

is a group 2-cocycle. Denote by $\tilde{\sigma}\in\mathcal{Z}^2(T(V\oplus W)\#\mathbb{C}\Gamma,\mathbb{C})$ the Hopf 2-cocycle induced by σ. Then $\tilde{\sigma}$ induces a Hopf 2-cocycle on $\widetilde{\mathcal{U}}(\mathcal{D}_{\rm red},\ell)$ and we have that $\widetilde{\mathcal{U}}(\mathcal{D}_{\rm red},\ell)_{\tilde{\sigma}}\simeq\widetilde{\mathcal{U}}(\mathcal{D}_q,\ell)$. Indeed, by Proposition 2.3 and the proof of [7, Prop. 3.9] we have that $(T(V\oplus W)\#\mathbb{C}\Gamma)_{\tilde{\sigma}}=T(V\oplus W)_\sigma\#\mathbb{C}\Gamma=T(\hat{V}\oplus\hat{W})\#\mathbb{C}\Gamma$. For example, for $i\le j\in I$

$$K_i\cdot_\sigma x_j=\sigma(K_i,K_j)\sigma^{-1}(K_j,K_i)K_i\cdot x_j=\hat{q}_{ij}q_{ij}^{-1}q_{ij}x_j=\hat{q}_{ij}x_j=q_I^{d_ia_{ij}}x_j.$$

Let J be the ideal of $T(V\oplus W)$ generated by the elements $x_iy_j-q_{ij}^{-1}y_jx_i-\delta_{ij}\ell_i(K_iL_i-1)$ for all $1\le i,j\le\theta$. To prove the claim it suffices to show that the corresponding ideal in $T(V\oplus W)_\sigma$ coincides with the ideal $\hat{J}$ generated by the elements $x_iy_j-\hat{q}_{ij}^{-1}y_jx_i-\delta_{ij}\ell_i(K_iL_i-1)$. But by Proposition 2.3 and the definition of σ we have for all $1 \leq i, j \leq \theta$ that

$$\begin{aligned} x_i \cdot_\sigma y_j - q_{ij}^{-1} y_j \cdot_\sigma x_i - \delta_{ij}\ell_i(K_i \cdot_\sigma L_i - 1) \\ &= \sigma(K_i, L_j)x_iy_j - q_{ij}^{-1}\sigma(L_j, K_i)y_jx_i - \delta_{ij}\ell_i(K_iL_i - 1) \\ &= x_iy_j - q_{ij}^{-1}(\hat{q}_{ij}q_{ij}^{-1})^{-1}y_jx_i - \delta_{ij}\ell_i(K_iL_i - 1) \\ &= x_iy_j - \hat{q}_{ij}^{-1}y_jx_i - \delta_{ij}\ell_i(K_iL_i - 1). \qquad \square \end{aligned}$$

3.2. **Multiparameter quantum groups as quotients of bosonizations of pre-Nichols algebras.** In this subsection we show how the multiparameter quantum groups $U_{\mathbf{q}}(\mathfrak{g}_A)$, associated with a symmetrizable generalized Cartan matrix, introduced by Pei, Hu and Rosso [27], can be described using reduced data. These multiparameter quantum groups contain in a unified way families of quantum groups introduced by other authors (see [27] and references therein). Note that in [8] Andruskiewitsch and Schneider characterized all pointed Hopf algebras that can be constructed using a generic datum of finite Cartan type for a free group of finite rank.

Let $\mathfrak{g}_A$ be a symmetrizable Kac–Moody algebra with $A = (a_{ij})_{i,j\in I}$ the associated generalized Cartan matrix, with I a finite set. Let d_i be relatively prime positive integers such that $d_ia_{ij} = d_ja_{ji}$ for all $i, j \in I$. Let Φ be a finite root system with $\Pi = \{\alpha_i : i \in I\}$ a set of simple roots, $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ the root lattice, Φ^+ the set of positive roots with respect to Π, and $Q^+ = \bigoplus_{i\in I} \mathbb{Z}_+\alpha_i$ the positive root lattice. Let $\mathbf{q} = (q_{ij})_{i,j\in I}$ with $q_{ij} \in \mathbb{C}^\times$ and $q_{ii} \neq 1$ for all $i, j \in I$ satisfying

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}} \quad \text{for all } i, j \in I. \tag{7}$$

Definition 3.7. [27, Def. 7] Let $U_{\mathbf{q}}(\mathfrak{g}_A)$ be the unital associative algebra over $\mathbb{C}$ generated by elements e_i, f_i, $\omega_i^{\pm1}$ and ${\omega'}_i^{\pm1}$ with $i \in I$ satisfying the following relations:

(R1) $\omega_i^{\pm1}{\omega'}_j^{\pm1} = {\omega'}_j^{\pm1}\omega_i^{\pm1}, \qquad \omega_i^{\pm1}\omega_i^{\mp1} = {\omega'}_i^{\pm1}{\omega'}_i^{\mp1} = 1,$

(R2) $\omega_i^{\pm1}\omega_j^{\pm1} = \omega_j^{\pm1}\omega_i^{\pm1}, \qquad {\omega'}_i^{\pm1}{\omega'}_j^{\pm1} = {\omega'}_j^{\pm1}{\omega'}_i^{\pm1},$

(R3) $\omega_ie_j\omega_i^{-1} = q_{ij}e_j, \qquad \omega'_ie_j{\omega'}_i^{-1} = q_{ji}^{-1}e_j,$

(R4) $\omega_if_j\omega_i^{-1} = q_{ij}^{-1}f_j, \qquad \omega'_if_j{\omega'}_i^{-1} = q_{ji}f_j,$

(R5) $[e_i, f_j] = \delta_{i,j}\dfrac{q_{ii}}{q_{ii} - 1}(\omega_i - \omega'_i),$

(R6) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1 - a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k e_i^{1-a_{ij}-k} e_j e_i^k = 0 \qquad (i \neq j),$

(R7) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1 - a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k f_i^k f_j f_i^{1-a_{ij}-k} = 0 \qquad (i \neq j).$

$U_{\mathbf{q}}(\mathfrak{g}_A)$ is a Hopf algebra with its coproduct, counit and antipode determined for all $i,j\in I$ by:

$$\begin{aligned}
\Delta(e_i) &= e_i\otimes 1+\omega_i\otimes e_i, & \varepsilon(e_i)&=0, & \mathcal{S}(e_i)&=-\omega_i^{-1}e_i,\\
\Delta(f_i) &= f_i\otimes \omega_i'+1\otimes f_i, & \varepsilon(f_i)&=0, & \mathcal{S}(f_i)&=-f_i{\omega'}_i^{-1},\\
\Delta(\omega_i^{\pm1}) &= \omega_i^{\pm1}\otimes\omega_i^{\pm1}, & \varepsilon(\omega_i^{\pm1})&=1, & \mathcal{S}(\omega_i^{\pm1})&=\omega_i^{\mp1},\\
\Delta({\omega'}_i^{\pm1}) &= {\omega'}_i^{\pm1}\otimes{\omega'}_i^{\pm1}, & \varepsilon({\omega'}_i^{\pm1})&=1, & \mathcal{S}({\omega'}_i^{\pm1})&={\omega'}_i^{\mp1}.
\end{aligned}$$

Next we prove that this quantum group can be described using reduced data.

Definition of $\widetilde{\mathcal{U}}(\mathcal{D}_{\mathrm{red}},\ell)$. Let

- $\theta=|I|$,
- $\Gamma=\mathbb{Z}^{2|I|}$ and denote K_i, L_i with $i\in I$ two (commuting) generators,
- $\chi_i\in\widehat{\Gamma}$ given by $\chi_i(K_j)=q_{ji}$ and $\chi_i(L_j)=q_{ij}$ for all $i,j\in I$.

In particular, we have that $\chi_i(L_j)=\chi_j(K_i)$ for all $i,j\in I$. Since by assumption $K_iL_i\neq 1$ and by (7), $q_{ij}q_{ji}=q_{ii}^{a_{ij}}$ with $q_{ii}\neq 1$, we have that $\mathcal{D}_{\mathrm{red}}=\mathcal{D}(\Gamma,(K_i),(L_i),(\chi_i),(a_{ij}))$ is a reduced YD-datum of Cartan type.

Let V, W be the vector spaces linearly generated by the elements x_i and y_i for all $1\le i\le\theta$. Following the definition of reduced data, both have a Yetter–Drinfeld module structure. In this case, it is given for all $i,j\in I$ by

$$\begin{aligned}
\delta(x_j)&=K_j\otimes x_j, & K_i\cdot x_j&=\chi_j(K_i)x_j=q_{ij}x_j, & L_i\cdot x_j&=q_{ji}x_j,\\
\delta(y_j)&=L_j\otimes y_j, & K_i\cdot y_j&=\chi_j^{-1}(K_i)y_j=q_{ij}^{-1}y_j, & L_i\cdot y_j&=q_{ji}^{-1}y_j.
\end{aligned}$$

Recall that for $\ell=(\ell_i)_{1\le i\le\theta}$ with $\ell_i\in\mathbb{C}^\times$, the pointed Hopf algebra $\widetilde{\mathcal{U}}(\mathcal{D}_{\mathrm{red}},\ell)$ associated with these data is given by the quotient Hopf algebra of the bosonization $R(\mathcal{D})\#\mathbb{C}\mathbb{Z}^{2\theta}$ modulo the ideal generated by

$$x_iy_j-q_{ij}^{-1}y_jx_i-\delta_{ij}\ell_i(K_iL_i-1)\quad\text{for all } i,j\in I.$$

In particular, in $\widetilde{\mathcal{U}}(\mathcal{D}_{\mathrm{red}},\ell)$, x_i is a $(1,K_i)$-primitive and y_i is a $(1,L_i)$-primitive. Indeed, for $x_i\in R(\mathcal{D})\#\mathbb{C}\mathbb{Z}^{2\theta}$ we have $\Delta_R(x_i)=x_i^{(1)}\otimes x_i^{(2)}=x_i\otimes 1+1\otimes x_i$ and

$$\begin{aligned}
\Delta(x_i)&=x_i^{(1)}\#(x_i^{(2)})_{(-1)}\otimes(x_i^{(2)})_{(0)}\#1=(x_i\#1)\otimes(1\#1)+(1\#K_i)\otimes(x_i\#1)\\
&=x_i\otimes 1+K_i\otimes x_i.
\end{aligned}$$

Theorem 3.8. *$U_{\mathbf{q}}(\mathfrak{g}_A)\simeq\widetilde{\mathcal{U}}(\mathcal{D}_{\mathrm{red}},\ell)$ with $\ell_i=\frac{q_{ii}}{q_{ii}-1}$ for all $1\le i\le\theta$.*

Proof. Let $\varphi: U_{\mathbf{q}}(\mathfrak{g}_A)\to\widetilde{\mathcal{U}}(\mathcal{D}_{\mathrm{red}},\ell)$ be the algebra map defined by

$$\varphi(\omega_i)=K_i,\quad \varphi(\omega_i')=L_i^{-1},\quad \varphi(e_i)=x_i,\quad \varphi(f_i)=y_iL_i^{-1}\qquad\text{for all } 1\le i\le\theta.$$

The map φ is an epimorphism, if it is well-defined. To prove that it is well-defined, we show that the relations in $U_{\mathbf{q}}(\mathfrak{g}_A)$ are mapped to 0 by φ. First, notice that the action of K_i and L_i on x_j and y_j yields a commutation relation in $T(V\oplus W)\#\mathbb{C}\mathbb{Z}^{2\theta}$; for example, $(1\#K_i)(x_j\#1)(1\#K_i^{-1}) = [(K_i\cdot x_j\#K_i)(1\#K_i^{-1})] = K_i\cdot x_j\#1$. Clearly, we need to verify only relations (R3)–(R7). For (R3) we have

$$\begin{aligned}\varphi(\omega_i e_j \omega_i^{-1} - q_{ij}e_j) &= K_i x_j K_i^{-1} - q_{ij}x_j = K_i\cdot x_j - q_{ij}x_j = 0,\\ \varphi(\omega_i' e_j {\omega'}_i^{-1} - q_{ji}^{-1}e_j) &= L_i^{-1}x_jL_i - q_{ji}^{-1}x_j = L_i^{-1}\cdot x_j - q_{ji}^{-1}x_j = 0.\end{aligned}$$

The proof for (R4) follows the same lines. Since $\mathcal{D}$ is a reduced datum, for (R5) we have

$$\begin{aligned}\varphi([e_i,f_j]) &= x_iy_jL_j^{-1} - y_jL_j^{-1}x_i = x_iy_jL_j^{-1} - \chi_i(L_j^{-1})y_jx_iL_j^{-1}\\ &= x_iy_jL_j^{-1} - q_{ij}^{-1}y_jx_iL_j^{-1} = (x_iy_j - q_{ij}^{-1}y_jx_i)L_j^{-1} = \delta_{ij}\ell_i(K_iL_i-1)L_j^{-1}\\ &= \delta_{ij}\frac{q_{ii}}{q_{ii}-1}(K_i - L_i^{-1}) = \varphi\left(\delta_{ij}\frac{q_{ii}}{q_{ii}-1}(\omega_i - \omega_i')\right).\end{aligned}$$

To verify (R6) and (R7) one notes that their images under φ are the quantum Serre relations in $T(V\oplus W)$, e.g. $\operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j)$ is the image of the left hand side of (R6). Indeed, since

$$\operatorname{ad}_c(x_i)(x_j) = x_ix_j - [(x_i)_{-1}\cdot x_j](x_i)_0 = x_ix_j - (K_i\cdot x_j)x_i = x_ix_j - q_{ij}x_jx_i,$$

one proves by induction that

$$\operatorname{ad}_c(x_i)^n(x_j) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k x_i^{n-k}x_jx_i^k \quad \text{for all } n\in\mathbb{N}. \tag{8}$$

Assuming that the equality holds for $n \in \mathbb{N}$ and using (1) we have

$$\begin{aligned}
\operatorname{ad}_c(x_i)^{n+1}(x_j) &= \operatorname{ad}_c(x_i)(\operatorname{ad}_c(x_i)^n(x_j)) \\
&= \operatorname{ad}_c(x_i)\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k x_i^{n-k}x_jx_i^k\right) \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k [x_i^{n+1-k}x_jx_i^k - K_i\cdot(x_i^{n-k}x_jx_i^k)x_i] \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k [x_i^{n+1-k}x_jx_i^k - q_{ii}^nq_{ij}x_i^{n-k}x_jx_i^{k+1}] \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k x_i^{n+1-k}x_jx_i^k \\
&\qquad + \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^{k+1} q_{ii}^n x_i^{n-k}x_jx_i^{k+1} \\
&= x_i^{n+1}x_j + (-1)^{n+1}q_{ii}^{\frac{n(n+1)}{2}} q_{ij}^{n+1}x_jx_i^{n+1} + \\
&\qquad + \sum_{k=1}^{n}(-1)^k q_{ij}^k\left[\binom{n}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} + \binom{n}{k-1}_{q_{ii}} q_{ii}^{\frac{(k-1)(k-2)}{2}} q_{ii}^n\right] x_i^{n+1-k}x_jx_i^k \\
&= x_i^{n+1}x_j + (-1)^{n+1}q_{ii}^{\frac{n(n+1)}{2}} q_{ij}^{n+1}x_jx_i^{n+1} + \\
&\qquad + \sum_{k=1}^{n}(-1)^k q_{ij}^k q_{ii}^{\frac{k(k-1)}{2}}\left[\binom{n}{k}_{q_{ii}} + \binom{n}{k-1}_{q_{ii}} q_{ii}^{n+1-k}\right] x_i^{n+1-k}x_jx_i^k \\
&= x_i^{n+1}x_j + (-1)^{n+1}q_{ii}^{\frac{n(n+1)}{2}} q_{ij}^{n+1}x_jx_i^{n+1} \\
&\qquad + \sum_{k=1}^{n}(-1)^k q_{ij}^k q_{ii}^{\frac{k(k-1)}{2}}\binom{n+1}{k}_{q_{ii}} x_i^{n+1-k}x_jx_i^k \\
&= \sum_{k=0}^{n+1}(-1)^k q_{ij}^k q_{ii}^{\frac{k(k-1)}{2}}\binom{n+1}{k}_{q_{ii}} x_i^{n+1-k}x_jx_i^k.
\end{aligned}$$

Since $\operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0$ in $R(\mathcal{D})$, and

$$\varphi\left(\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k e_i^{1-a_{ij}-k}e_je_i^k\right) = \operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j),$$

the assertion about (R6) follows. Analogously,

$$\operatorname{ad}_c(y_i)(y_j) = y_iy_j - (L_i\cdot y_j)y_i = y_iy_j - q_{ji}^{-1}y_jy_i = -q_{ji}^{-1}(y_jy_i - q_{ji}y_iy_j),$$

and one may prove by induction that

$$\operatorname{ad}_c(y_i)^n(y_j) = (-1)^n q_{ji}^{-n} q_{ii}^{-\frac{n(n-1)}{2}}\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ji}^k y_i^k y_j y_i^{n-k}\right). \quad (9)$$

Hence,

$$\varphi\left(\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k f_i^k f_j f_i^{1-a_{ij}-k}\right)$$
$$= \sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k (y_iL_i^{-1})^k(y_jL_j^{-1})(y_iL_i^{-1})^{1-a_{ij}-k}$$
$$= \sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k q_{ii}^{\frac{k(k-1)}{2}} q_{ii}^{\frac{(1-a_{ij}-k)(-a_{ij}-k)}{2}}$$
$$\cdot y_i^k L_i^{-k} y_j L_j^{-1} y_i^{1-a_{ij}-k} L_i^{-1+a_{ij}+k}$$
$$= \left(\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{k(k-1)} q_{ii}^{\frac{(1-a_{ij}-k)(-a_{ij}-k)}{2}} q_{ji}^k q_{ii}^{k(1-a_{ij}-k)} q_{ij}^{1-a_{ij}}\right.$$
$$\left.\cdot y_i^k y_j y_i^{1-a_{ij}-k}\right) L_j^{-1} L_i^{-1+a_{ij}}$$

But

$$\frac{k(k-1)}{2} + \frac{(1-a_{ij}-k)(-a_{ij}-k)}{2} + k(1-a_{ij}-k)$$
$$= \frac{k(k-1)}{2} + \frac{-a_{ij}(1-a_{ij}-k) - k(1-a_{ij}-k))}{2} + k(1-a_{ij}-k)$$
$$= \frac{k(k-1)}{2} + \frac{-a_{ij}(1-a_{ij}-k) + k(1-a_{ij}-k)}{2}$$
$$= \frac{1}{2}[k^2 - k - a_{ij} + a_{ij}^2 + a_{ij}k + k - ka_{ij} - k^2] = \frac{1}{2}a_{ij}(a_{ij}-1).$$

Thus, φ of (R7) equals

$$\left(\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ji}^k q_{ii}^{\frac{a_{ij}(a_{ij}-1)}{2}} q_{ij}^{1-a_{ij}} y_i^k y_j y_i^{1-a_{ij}-k}\right) L_j^{-1} L_i^{-1+a_{ij}}$$
$$= q_{ij}^{1-a_{ij}} (-1)^{1-a_{ij}} q_{ji}^{1-a_{ij}} \operatorname{ad}_c(y_i)^{1-a_{ij}}(y_j) L_j^{-1} L_i^{-1+a_{ij}}.$$

Since $\operatorname{ad}_c(y_i)^{1-a_{ij}}(y_j) = 0$ in $R(\mathcal{D})$, the claim about (R7) follows.

Hence, φ is a well-defined algebra map. Moreover, it is a Hopf algebra map, since ω_i, ω_i' and K_i, L_i are grouplike elements, e_i is $(1,\omega_i)$-primitive and f_i is $(\omega_i',1)$-primitive, and the elements x_i and y_i are $(1,K_i)$-primitive and $(1,L_i)$-primitive, respectively, for all $i \in I$.

Now we show φ is an isomorphism. Let $\widetilde{\psi} : R(\mathcal{D})\#\mathbb{C}\mathbb{Z}^{2\theta} \to U_{\mathbf{q}}(\mathfrak{g}_A)$ be the algebra map given by

$$\widetilde{\psi}(1\#K_i) = \omega_i, \quad \widetilde{\psi}(1\#L_i) = {\omega'}_i^{-1}, \quad \widetilde{\psi}(x_i\#1) = e_i, \quad \widetilde{\psi}(y_i\#1) = f_i{\omega'}_i^{-1},$$

for all $i \in \mathbb{I}$. Again, $\widetilde{\psi}$ is clearly a Hopf algebra epimorphism, if it is well-defined. For example, it preserves the algebra structure, if $i \in \mathbb{I}$ we have

$$\widetilde{\psi}((1\#K_i)(x_j\#1)(1\#K_i^{-1})) = \omega_i e_j \omega_i^{-1} = q_{ij} e_j = \chi_j(K_i) e_j = \widetilde{\psi}(K_i \cdot x_j\#1),$$

and the coalgebra structure with $\varepsilon(\omega_i) = 1 = \varepsilon(1\#K_i)$, $\varepsilon(\omega_i') = 1 = \varepsilon(1\#L_i)$, $\varepsilon(e_i) = 0 = \varepsilon(x_i\#1)$, $\varepsilon(f_i {\omega'}_i^{-1}) = 1 = \varepsilon(y_j\#1)$ and

$$\begin{aligned}\Delta(\widetilde{\psi}(y_i\#1)) &= \Delta(f_i {\omega'}_i^{-1}) = f_i {\omega'}_i^{-1} \otimes 1 + {\omega'}_i^{-1} \otimes f_i {\omega'}_i^{-1} \\ &= \widetilde{\psi}(y_i\#1) \otimes 1 + \widetilde{\psi}(1\#L_i) \otimes \widetilde{\psi}(y_i\#1) = (\widetilde{\psi} \otimes \widetilde{\psi})\Delta(y_i\#1).\end{aligned}$$

To see that $\widetilde{\psi}$ is indeed well-defined, we have to check that the quantum Serre relations are mapped to 0. For $i \neq j \in I$ we have by (8) and (R6) that

$$\widetilde{\psi}(\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j)) = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k e_i^{1-a_{ij}-k} e_j e_i^k = 0.$$

Analogously, by (9), (R7) and the same calculation with the exponents as above we have

$$\begin{aligned}
&\widetilde{\psi}(\mathrm{ad}_c(y_i)^{1-a_{ij}}(y_j)) \\
&\quad = (-1)^{1-a_{ij}} q_{ji}^{-1+a_{ij}} q_{ii}^{\frac{a_{ij}(1-a_{ij})}{2}} \\
&\quad\quad \cdot \left(\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ji}^k (f_i {\omega'}_i^{-1})^k (f_j {\omega'}_j^{-1}) (f_i {\omega'}_i^{-1})^{1-a_{ij}-k} \right) \\
&\quad = (-1)^{1-a_{ij}} q_{ji}^{-1+a_{ij}} q_{ii}^{\frac{a_{ij}(1-a_{ij})}{2}} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ji}^k \\
&\quad\quad \cdot q_{ii}^{\frac{k(k-1)}{2} + \frac{(1-a_{ij}-k)(1-a_{ij})}{2}} f_i^k {\omega'}_i^{-k} f_j {\omega'}_j^{-1} f_i^{1-a_{ij}-k} {\omega'}_i^{-1+a_{ij}+k}) \\
&\quad = (-1)^{1-a_{ij}} q_{ji}^{-1+a_{ij}} q_{ii}^{\frac{a_{ij}(1-a_{ij})}{2}} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ji}^k \\
&\quad\quad \cdot q_{ii}^{-\frac{k(k-1)}{2} - \frac{(1-a_{ij}-k)(-a_{ij}-k)}{2}} q_{ji}^{-k} q_{ij}^{-1+a_{ij}+k} q_{ii}^{-k(1-a_{ij}-k)} f_i^k f_j f_i^{1-a_{ij}-k} {\omega'}_j {\omega'}_i^{-1+a_{ij}} \\
&\quad = (-1)^{1-a_{ij}} q_{ji}^{-1+a_{ij}} q_{ii}^{\frac{a_{ij}(1-a_{ij})}{2}} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k \\
&\quad\quad \cdot q_{ii}^{-\frac{k(k-1)}{2} - \frac{(1-a_{ij}-k)(-a_{ij}-k)}{2} - k(1-a_{ij}-k)} q_{ij}^{-1+a_{ij}} f_i^k f_j f_i^{1-a_{ij}-k} {\omega'}_j {\omega'}_i^{-1+a_{ij}} \\
&\quad = (-1)^{1-a_{ij}} q_{ji}^{-1+a_{ij}} q_{ij}^{-1+a_{ij}} \left(\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_{ii}} q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k f_i^k f_j f_i^{1-a_{ij}-k} \right) \\
&\quad\quad \cdot {\omega'}_j {\omega'}_i^{-1+a_{ij}} = 0.
\end{aligned}$$

Moreover, identifying $x_i = x_i\#1$, $y_i = y_i\#1$ and $K_i = 1\#K_i$, $L_i = 1\#L_i$, by (R5) we have that

$$\begin{aligned}\widetilde{\psi}(x_iy_j - q_{ij}^{-1}y_jx_i) &= e_if_j{\omega'}_j^{-1} - q_{ij}^{-1}f_j{\omega'}_j^{-1}e_i = (e_if_j - q_{ij}^{-1}q_{ij}f_je_i){\omega'}_j^{-1}\\ &= [e_i, f_j]{\omega'}_j^{-1} = \delta_{i,j}\frac{q_{ii}}{q_{ii}-1}(\omega_i - {\omega'}_i){\omega'}_j^{-1}\\ &= \delta_{i,j}\frac{q_{ii}}{q_{ii}-1}(\omega_i{\omega'}_i^{-1} - 1) = \widetilde{\psi}(\delta_{i,j}\ell_i(K_iL_i - 1)),\end{aligned}$$

for all $i, j \in \mathbb{I}$. Thus $\widetilde{\psi}$ induces a Hopf algebra epimorphism $\psi : \widetilde{\mathcal{U}}(\mathcal{D}_{\mathrm{red}}, \ell) \to U_{\mathbf{q}}(\mathfrak{g}_A)$ such that $\varphi \circ \psi = \mathrm{id} = \psi \circ \varphi$, implying that φ is an isomorphism. □

Corollary 3.9. *Assume $\mathfrak{g}_A$ is simple and let $U_q(\mathfrak{g}_A)$ be the one-parameter quantum group of Drinfeld–Jimbo type. Let $\widetilde{\mathcal{U}}(\mathcal{D}_q, \ell)$ be the pointed Hopf algebra associated with the reduced YD-datum of DJ-type $\mathcal{D}_q = (\Gamma, (L_i)_{1\leq i\leq\theta}, (K_i)_{1\leq i\leq\theta}, q, (a_{ij})_{1\leq i,j\leq\theta})$ and ℓ with $\ell_i = \frac{q}{q-1}$ for all $i \in I$, and denote $\widehat{\mathcal{U}}(\mathcal{D}_q, \ell) = \widetilde{\mathcal{U}}(\mathcal{D}_q, \ell)/(K_i - L_i^{-1})$. Then $U_q(\mathfrak{g}_A) \simeq \widehat{\mathcal{U}}(\mathcal{D}_q, \ell)$.*

Proof. By [27, Remark 9] we know that $U_q(\mathfrak{g}_A) \simeq U_{\mathbf{q}}(\mathfrak{g}_A)/(\omega_i' - \omega_i^{-1})$ for $q_{ij} = q^{d_ia_{ij}}$ for all $i, j \in I$. Moreover, by Theorem 3.8 we know that $U_{\mathbf{q}}(\mathfrak{g}_A) \simeq \widetilde{\mathcal{U}}(\mathcal{D}_q, \ell)$ for $\mathcal{D}_q$ the reduced YD-datum of DJ-type described above, since $\mathfrak{g}_A$ is connected and the assumption on the q_{ij}'s. The result follows since the isomorphism factors through the quotients. □

Remark 3.10. With the assumptions above, we have that $U_{\mathbf{q}}(\mathfrak{g}_A) = U_{q,q^{-1}}(\mathfrak{g}_A)$, the special case of the two-parameter quantum group; see [27, Remark 9]. Then $U_{q,q^{-1}}(\mathfrak{g}_A) \simeq \widetilde{\mathcal{U}}(\mathcal{D}_q, \ell)$.

3.3. **Multiparameter quantum groups as cocycle deformations.** In this subsection we apply Theorems 3.6 and 3.8 to multiparameter quantum groups. In particular, we obtain another description of [27, Theorem 28]. *From now on we assume that $1 \neq q_{ii}$ is a positive real number for all $i \in I$ and all $I \in \mathcal{X}$.*

The existence of all positive roots for every q_{ii} ensures that for each $I \in \mathcal{X}$ and for each $i \in I$ there exists $q_i \in \mathbb{C}^\times$ such that $q_{ii} = q_i^{2\,d_i}$. Moreover, we may choose in each connected component $J \in \mathcal{X}$ a unique constant value $q_i =: q_J$ for all $i \in J$.

Let $\widetilde{\mathcal{U}}(\mathcal{D}_q, \ell)$ be the pointed Hopf algebra associated to the reduced YD-datum of DJ-type given by $\mathcal{D}_q = \mathcal{D}(\Gamma, (L_i)_{1\leq i\leq\theta}, (K_i)_{1\leq i\leq\theta}, (q_J)_{J\in\mathcal{X}}, (a_{ij})_{1\leq i,j\leq\theta})$.

Corollary 3.11. *There exists a group 2-cocycle $\sigma \in \mathcal{Z}^2(\Gamma, \mathbb{C})$ such that $\widetilde{\mathcal{U}}(\mathcal{D}_q, \ell) \simeq U_{\mathbf{q}}(\mathfrak{g}_A)_{\tilde{\sigma}}$. In particular, if $\mathfrak{g}_A$ is simple and $q_{ii} = q^{2d_i}$ for all $i, j \in I$, we have that $U_{q,q^{-1}}(\mathfrak{g}_A) \simeq U_{\mathbf{q}}(\mathfrak{g}_A)_{\tilde{\sigma}}$.*

Proof. By Theorem 3.8 we know that $U_{\mathbf{q}}(\mathfrak{g}_A) \simeq \widetilde{\mathcal{U}}(\mathcal{D}_{\mathrm{red}}, \ell)$ with $\ell_i = \frac{q_{ii}}{q_{ii}-1}$ for all $i \in I$. Since the braiding is positive and generic, Theorem 3.6 implies that $U_{\mathbf{q}}(\mathfrak{g}_A)$ is a cocycle deformation of a pointed Hopf algebra associated to a reduced YD-datum of DJ-type. For, the proof of Theorem 3.6 gives a group 2-cocycle σ such that $\widetilde{\mathcal{U}}(\mathcal{D}_q,\ell) \simeq \widetilde{\mathcal{U}}(\mathcal{D}_{\mathrm{red}},\ell)_{\tilde{\sigma}}$. Taking the corresponding 2-cocycle induced by the isomorphism we have the assertion.

If $\mathfrak{g}_A$ is simple and $q_{ij} = q^{d_i a_{ij}}$ for all $i,j \in I$, by Remark 3.10 we have that $\widetilde{\mathcal{U}}(\mathcal{D}_q,\ell) \simeq U_{q,q^{-1}}(\mathfrak{g}_A)$. □

Remark 3.12. Assume $\mathfrak{g}_A$ is simple. The result above was previously obtained in [27], where a Hopf 2-cocycle σ is defined in $U_{q,q^{-1}}(\mathfrak{g}_A)$. We show that this cocycle comes from a group 2-cocycle on Γ.

First we fix the notation $\omega_\lambda := \prod_{i\in I}\omega_i^{\lambda_i}$ and $\omega'_\lambda := \prod_{i\in I}{\omega'_i}^{\lambda_i}$ for every $\lambda = \sum_{i\in I}\lambda_i\alpha_i \in Q$. Similarly, we shall also write

$$q_{\mu\nu} := \prod_{i,j\in I} q_{ij}^{\mu_i\nu_j} \qquad \forall\, \mu = \sum_{i\in I}\mu_i\,\alpha_i,\ \nu = \sum_{j\in I}\nu_j\,\alpha_j \in Q$$

Let $\sigma : U_{q,q^{-1}}(\mathfrak{g}_A)\otimes U_{q,q^{-1}}(\mathfrak{g}_A) \to \mathbb{C}$ be the unique $\mathbb{C}$-linear form on $U_{\mathbf{q}}(\mathfrak{g}_A)$ such that

$$\sigma(x,y) := \begin{cases} q_{\mu\nu}^{\frac{1}{2}} & \text{if } x=\omega_\mu \text{ or } x=\omega'_\mu,\ y=\omega_\nu \text{ or } y=\omega'_\nu,\\ 0 & \text{otherwise.}\end{cases}$$

Then by [27, Prop. 27 and Thm. 28], $\sigma \in \mathcal{Z}^2(U_{q,q^{-1}}(\mathfrak{g}_A),\mathbb{C})$ and $U_{\mathbf{q}}(\mathfrak{g}_A) \simeq U_{q,q^{-1}}(\mathfrak{g}_A)_\sigma$.

On the other hand, we know that $U_{q,q^{-1}}(\mathfrak{g}_A)$ is a quotient of a bosonization $T(V\oplus W)\#\mathbb{C}\Gamma$ with $\Gamma = \mathbb{Z}^{2\theta}$. As in Remark 2.6, we have a $\Gamma\times\Gamma$ grading on $T(V\oplus W)$ induced by the coaction on the Yetter–Drinfeld module; for example, ω_i has degree (ω_i,ω_i), e_i has degree $(\omega_i,1)$ and f_i has degree $(1,{\omega'_i}^{-1})$ for all $i\in I$. Consider now the group 2-cocycle $\varphi\in\mathcal{Z}^2(\Gamma,\mathbb{C})$ given by $\varphi = \sigma|_{\Gamma\times\Gamma}$, that is,

$$\varphi(h,k) := q_{\mu\nu}^{\frac{1}{2}} \qquad \text{if } h=\omega_\mu \text{ or } h=\omega'_\mu,\ k=\omega_\nu \text{ or } k=\omega'_\nu,$$

and let $\tilde{\varphi}$ be the 2-cocycle defined on $T(V\oplus W)\#\mathbb{C}\Gamma$. Since the group is abelian and $e_i\cdot_{\tilde{\varphi}} f_j = e_i f_j$ for all $i,j\in I$, we have that $e_i\cdot_{\tilde{\varphi}} f_j - f_j\cdot_{\tilde{\varphi}} e_i = [e_i,f_j]$ and consequently $\tilde{\varphi}$ defines a Hopf 2-cocycle on $U_{q,q^{-1}}(\mathfrak{g}_A)$. Since $\sigma(x,y) = 0 = \varepsilon(x)\varepsilon(y)$ if $x,y\notin\Gamma$, it follows that $\sigma = \tilde{\varphi}$ and hence $U_{\mathbf{q}}(\mathfrak{g}_A) = U_{q,q^{-1}}(\mathfrak{g}_A)_\sigma$.

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