Optimal frame completions.

P. G. Massey, M. A. Ruiz and D. Stojanoff
Dpto. de Matemática, FCE-UNLP, La Plata, Bs. As., Argentina and IAM-CONICET *

Abstract

Given a finite sequence of vectors $\mathcal{F}_0$ in $\mathbb{C}^d$ we describe the spectral and geometrical structure of optimal frame completions of $\mathcal{F}_0$ obtained by appending a finite sequence of vectors with prescribed norms, where optimality is measured with respect to a general convex potential. In particular, our analysis includes the so-called Mean Square Error (MSE) and the Benedetto-Fickus' frame potential. On a first step, we reduce the problem of finding the optimal completions to the computation of the minimum of a convex function in a convex compact polytope in $\mathbb{R}^d$. As a second step, we show that there exists a finite set (that can be explicitly computed in terms of a finite step algorithm that depends on $\mathcal{F}_0$ and the sequence of prescribed norms) such that the optimal frame completions with respect to a given convex potential can be described in terms of a distinguished element of this set. As a byproduct we characterize the cases of equality in Lidskii's inequality from matrix theory.

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1 Introduction

A finite sequence of vectors $\mathcal{F} = \{f_i\}_{i=1}^n$ in $\mathbb{C}^d$ is a frame for $\mathbb{C}^d$ if the sequence spans $\mathbb{C}^d$. It is well known that finite frames provide redundant linear encoding-decoding schemes, that have proved useful in real life applications. Conversely, several research problems in this field have arisen in the attempt to apply this theory in different contexts.

Recently, the following frame completion problem was posed in [19]: given an initial sequence $\mathcal{F}_0 = \{f_i\}_{i=1}^{n_o}$ in $\mathbb{C}^d$ and a sequence of positive numbers $\mathbf{a} = (\alpha_i)_{i=1}^k$ then compute the sequences $\mathcal{G} = \{g_i\}_{i=1}^k$ in $\mathbb{C}^d$ whose elements have norms given by the sequence $\mathbf{a}$ and such that the completed sequence $\mathcal{F} = (\mathcal{F}_0\,, \mathcal{G})$ is a frame that minimizes the functional $\mathrm{MSE}(\mathcal{F}) = \mathrm{tr}(S_{\mathcal{F}}^{-1})$, where $S_{\mathcal{F}}$ denotes the frame operator of $\mathcal{F}$. Notice there are other possible functionals - known as *convex potentials* (see [8, 18, 28]) - that we could choose to minimize such as, for example, the frame potential introduced in [2] by Benedetto and Fickus.

A first step toward the solution of this general version of the completion problem was made in [32]. There we showed that under certain hypothesis (feasible cases, see Section 2), optimal frame completions with prescribed norms do not depend on the particular choice of convex functional.On the other hand, it is easy to show examples in which the previous result does not apply (non-feasible cases); in these cases the optimal frame completions with prescribed norms are not known even for the MSE nor the frame potential.

In this paper we consider the frame completion problem of an initial sequence $\mathcal{F}_0$ in $\mathbb{C}^d$, for general sequences $\mathbf{a}$ of prescribed norms and for a fixed convex potential P_f - where f is a strictly convex function - in the non-feasible cases (see Section 2 for motivations and a detailed description of our main problem). In order to deal with the general problem we introduce and develop a class of pairs of positive matrices that are optimal in the sense that they achieve equality in Lidskii's

*e-mail addresses: massey@mate.unlp.edu.ar , mruiz@mate.unlp.edu.ar , demetrio@mate.unlp.edu.ar

inequality (called Lidskii matching matrices, see Appendix II (Section 8)) that allows to reduce the problem to the computation of minimizers of a scalar convex function F (associated to f) in a compact convex domain in $\mathbb{R}^d$ (the same set for every map f). This constitutes a reduction of the optimization problem, that in turn can be tackled with several numerical tools in concrete examples. In fact, the convex domain has a natural and explicit description in terms of majorization, which is an algorithmic notion.

We also study the spectral and geometrical structure of (local) minimizers of P_f in the set of frame completions with prescribed norms, in terms of a geometrical approach to a perturbation problem. We show that optimal completions $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G})$ are frames and they have the property that the vectors of the completing sequence $\mathcal{G}$ are eigenvectors of the frame operator of the complete sequence $\mathcal{F}$. This last result allows for a second reduction of the problem: there is a finite set $E(\mathcal{F}_0\,,\,\mathbf{a})$ in $\mathbb{R}^d$ - that depends only on the initial family $\mathcal{F}_0$ and the finite sequence $\mathbf{a}$ of positive numbers - such that for any fixed convex potential P_f there exists a unique vector $\mu = \mu_f \in E(\mathcal{F}_0\,,\,\mathbf{a})$ (computable by a minimization on the finite set $E(\mathcal{F}_0\,,\,\mathbf{a})$ in terms of F) such that all optimal frame completions for P_f with prescribed norms can be computed in terms of μ.

In both methods, we describe the optimal vector of eigenvalues for the frame operator of the completing sequences. With this data, the optimal frame completions (which satisfy the norm restrictions) can be effectively computed by using a well known algorithm developed in [16] that implements the Schur-Horn theorem.

In all examples that we have computed numerically, we have found that the optimal spectrum of the completing sequences does not depend on the particular choice of convex potential P_f considered. Although at the present we have not been able to prove this fact, we state it as a conjecture. We have also observed two other common features of optimal solutions, that allow to implement an efficient (and considerably faster) algorithm that computes a smaller set than $E(\mathcal{F}_0\,,\,\mathbf{a})$ that also enables to compute the optimal frame completions with prescribed norms with respect to a general convex potential P_f in all the examples considered.

The paper is organized as follows. In Section 2 we state several notions and facts about frame theory in finite dimension and majorization, which is a notion from matrix theory; in this section we describe in detail the main problem of the present paper and some previous related results. In Section 3 we state the main results (about frames) of the paper, and we describe briefly some of their consequences. This section includes several links explaining the role of all other sections and their statements, so that it can be used as a guide for reading the paper. In Section 4 we reduce the problem of computing optimal frame completions with prescribed norms to a set of completions whose frame operators are optimal in the sense that they achieve equality in Lidskii's inequality; we study the case of equality in Lidskii's inequality in Section 8 (Appendix II). We also show that the spectral structure of optimal completions is unique and has some other features. Based on the results in Section 4 it is possible to obtain a first reduction of the problem by showing that the optimal frame completions with prescribed norms for the convex potential P_f can be described in terms of the minimizers of an associated function F in a compact convex polytope in $\mathbb{R}^d$ (as described in Section 3). In Section 5 we introduce a natural metric in the set of completions and study some properties of local minimizers for the completion problem in terms of irreducible sequences. These properties are useful for the following section, and they depend on the geometrical structure of irreducible local minimizers; this study, which involves tools from differential geometry, is postponed to Section 7 (Appendix I). Using these results we show in Section 6 that optimal completions $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G})$ are frames and they have the property that the vectors of the completing sequence $\mathcal{G}$ are eigenvectors of the frame operator $S_{\mathcal{F}}$ of the complete sequence $\mathcal{F}$. Based on this last fact we develop an algorithm (that can be effectively implemented) to compute optimal completions numerically. We include a discussion of other commons features of the numerical solutions from the computed examples. In Section 7 (Appendix I) we apply tools form differential geometry to study some properties of local minimizers which were stated in section 5. Finally, in Section 8 (Appendix II) we introduce pairs of positive matrices, that we call Lidskii

matchings, and describe the structure of these pairs; this corresponds to the study of the case of equality in Lidskii's inequality from matrix theory.

2 Frames and optimal completions with prescribed parameters

In what follows we shall consider the set $\mathbf{F} = \mathbf{F}(n\,,\,d)$ of (n,d)-frames, that is, generating sequences $\mathcal{F} = \{f_i\}_{i=1}^n$ of a d-dimensional complex Hilbert space $\mathcal{H}$. We will denote by $T_{\mathcal{F}}$, $T_{\mathcal{F}}^*$ and $S_{\mathcal{F}}$ the synthesis, analysis and frame operator for $\mathcal{F}$ respectively. For a detailed account of results on frame theory, we refer the reader to [2, 9, 15, 21, 29] and the references therein.

In several applied situations it is desired to construct a sequence $\mathcal{F}$ in such a way that the frame operator of $\mathcal{F}$ is given by some positive definite operator S and the squared norms of the frame elements are prescribed by a sequence of positive numbers $\mathbf{a} = (\alpha_i)_{i=1}^n$. That is, given a positive definite operator S of $\mathcal{H}$ and $\mathbf{a} \in \mathbb{R}^n_{>0}$, to analyze the existence (and construction) of a sequence $\mathcal{F} = \{f_i\}_{i=1}^n$ such that $S_{\mathcal{F}} = S$ and $\|f_i\|^2 = \alpha_i$, for $1 \leq i \leq n$. This is known as the classical frame design problem. It has been treated by several research groups (see for example [1, 7, 10, 12, 16, 17, 18, 25]). In what follows we recall a solution of the classical frame design problem in the finite dimensional setting, in the way that it is convenient for our analysis.

Proposition 2.1 ([1, 28])**.** Let B be a positive operator on $\mathcal{H}$ with (ordered) eigenvalues $\lambda_1 \geq \cdots \lambda_d \geq 0$. Consider $\alpha_1 \geq \cdots \geq \alpha_k > 0$. Then there exists a sequence $\mathcal{G} = \{g_i\}_{i=1}^k$ in $\mathcal{H}$ with frame operator $S_{\mathcal{G}} = B$ such that $\|g_i\|^2 = \alpha_i$ for every $1 \leq i \leq k$ if and only if

$$\sum_{i=1}^{j} \alpha_i \leq \sum_{i=1}^{j} \lambda_i \quad , \qquad \text{for} \quad 1 \leq j \leq \min\{k\,,\,d\} \qquad \text{and} \qquad \sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{d} \lambda_i \ . \tag{1}$$

□

The family of inequalities described in (1) imply that the vector of eigenvalues of B must *majorize* $\mathbf{a}$. Majorization between vectors is a notion from matrix analysis theory that plays a key role in our work and will be used throughout the paper. Given x, $y \in \mathbb{R}^d$ we say that x is **submajorized** by y, and write $x \prec_w y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \text{for every} \quad k \in \mathbb{I}_d \ ,$$

where $x^{\downarrow} \in \mathbb{R}^d$ denotes the vector obtained from x by rearrangement of its entries in non-increasing order. If $x \prec_w y$ and $\sum_{i=1}^d x_i = \sum_{i=1}^d y_i$, then we say that x is **majorized** by y, and write $x \prec y$. If the two vectors x and y have different size, we write $x \prec y$ if the extended vectors (completing with zeros to have the same size) satisfy the previous relationship (as in Eq. (1)).

We shall also use the notation $\prec$ (resp. $\prec_w$) when we (spectrally) compare a pair of self-adjoint operators in a finite dimensional Hilbert space $\mathcal{H}$: $S_1 \prec S_2$ if $\lambda(S_1) \prec \lambda(S_2)$, where $\lambda(S) \in (\mathbb{R}^d)^{\downarrow}$ is the ordered vector of eigenvalues of S, counting multiplicities.

Recently, researchers have made a step forward in the classical frame design problem and have asked about the structure of **optimal** frames with prescribed parameters. For example, consider the following problem posed in [19]: let $\mathcal{F}_0 = \{f_i\}_{i=1}^{n_\circ}$ be a sequence in a d-dimensional Hilbert space $\mathcal{H}$. Consider a sequence $\mathbf{a} = (\alpha_i)_{i=1}^k$ of positive numbers such that $\operatorname{rk} S_{\mathcal{F}_0} \geq d-k$ and denote by $n = n_\circ + k$. Then the problem is to construct a sequence

$$\mathcal{G} = \{f_i\}_{i=n_\circ+1}^{n} \text{ in } \mathcal{H} \quad \text{with} \quad \|f_{n_\circ+i}\|^2 = \alpha_i \quad \text{for} \quad 1 \leq i \leq k \ ,$$

such that the resulting completed sequence is a frame $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) = \{f_i\}_{i=1}^n \in \mathbf{F}(n\,,\,d)$ that minimizes the so called mean square error i.e., the functional $\mathrm{MSE}(\mathcal{F}) = \mathrm{tr}\ S_{\mathcal{F}}^{-1}$, among all possible such completions. It is worth pointing out that the MSE terminology comes from the theory of approximations of a vector x from $(\langle x\,,f_i\rangle + \epsilon_i)_{i=1}^n$ where each ϵ_i is an additive error term: when ϵ_i are independently distributed with each having mean zero and variance σ^2, it can be seen that the MSE of the reconstruction of x using the canonical dual can be simplified in terms of the trace of the inverse of the frame operator of $\mathcal{F}$ ([19]).

Note that there are other possible ways to measure robustness (optimality) of the completed frame $\mathcal{F}$ as above. For example, we can consider optimal (minimizing) completions, with prescribed norms, for the Benedetto-Fickus' potential. In this case we search for a frame $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) = \{f_i\}_{i=1}^n \in \mathbf{F}(n\,,\,d)$, with $\|f_{n_\circ+i}\|^2 = \alpha_i$ for $1 \leq i \leq k$, and such that its frame potential $\mathrm{FP}\,(\mathcal{F}) = \mathrm{tr}\ S_{\mathcal{F}}^2$ is minimal among all possible such completions. Indeed, this problem has been considered before in the particular case in which $\mathcal{F}_0 = \emptyset$ in [2, 13, 20, 23, 29].

In this paper we shall consider the problem of optimal completion with prescribed norms, where optimality is measured with respect to general convex potentials, i.e. we consider minimizers for the (generalized) convex potential associated to a convex function $f : [0\,,\,\infty) \to [0\,,\,\infty)$, denoted P_f, given by

$$P_f(\mathcal{F}) = \mathrm{tr}\ f(S_{\mathcal{F}}) = \sum_{j=1}^{d} f(\lambda_j(S_{\mathcal{F}})\,) \quad \text{for} \quad \mathcal{F} = \{f_i\}_{i=1}^n \in \mathcal{H}^n\ . \tag{2}$$

It is clear that these potentials generalize the frame potential and MSE ($f(t) = t^2$ and $f(t) = t^{-1}$ respectively).

Remark 2.2. A well known result concerning the majorization preorder between vectors is the following: Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subseteq \mathbb{R}$. Given $x,\, y \in I^d$ then $x \prec y \implies \sum_{i=1}^{d} f(x_i) \leq \sum_{i=1}^{d} f(y_i)$ (see for example [3]).

Moreover, if $x \prec_w y$ and f is a strictly convex function such that $\mathrm{tr}\ f(x) = \mathrm{tr}\ f(y)$ then there exists a permutation σ of $\{1, \ldots, d\}$ such that $y_i = x_{\sigma(i)}$ for $1 \leq i \leq d$. $\triangle$

This suggest the study of minimizers for majorization in order to find the frame operators of the optimal completions with respect to a generalized potential P_f. In order to describe our main problem we first fix the notation that we shall use throughout the paper.

Definition 2.3. Let $\mathcal{F}_0 = \{f_i\}_{i=1}^{n_\circ} \in \mathcal{H}^{n_\circ}$ and $\mathbf{a} = (\alpha_i)_{i=1}^k$ be a sequence of positive numbers such that $d - \mathrm{rk}\, S_{\mathcal{F}_0} \leq k$. Define $n = n_\circ + k$. Then

1. In what follows we say that $(\mathcal{F}_0\,,\,\mathbf{a})$ are initial data for the completion problem (CP).
2. Let $f : [0, \infty) \to \mathbb{R}$ be a strictly convex function. In those statements which use this map we shall say that $(\mathcal{F}_0\,,\,\mathbf{a}\,,\,f)$ are initial data for the completion problem (CP).
3. For these data we consider the set of completions

$$\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) = \Big\{\, \{f_i\}_{i=1}^n \in \mathcal{H}^n : \{f_i\}_{i=1}^{n_\circ} = \mathcal{F}_0 \quad \text{and} \quad \|f_{n_\circ+i}\|^2 = \alpha_i \ \text{ for } \ 1 \leq i \leq k\Big\}$$

and the set of frame operators of these completions

$$\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0) = \{S_{\mathcal{F}} :\ \mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\}\ .$$

When the initial data $(\mathcal{F}_0\,,\,\mathbf{a})$ are fixed, we shall use throughout the paper the notations $S_0 = S_{\mathcal{F}_0}$, $\lambda = \lambda(S_0) = (\lambda_i)_{i=1}^d$ are the eigenvalues of S_0 counting multiplicities and arranged in a non-increasing order, and $n = n_\circ + k$.

Problem: (Optimal completions with prescribed norms with respect to P_f) Let $(\mathcal{F}_0\,,\,\mathbf{a}\,,\,f)$ be initial data for the CP as in 2.3 . Construct all possible $\mathcal{F}\in\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ that are the minimizers of P_f in $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$. $\triangle$

Our analysis of the completed frame $\mathcal{F}=(\mathcal{F}_0\,,\,\mathcal{G})$ will depend on $\mathcal{F}$ through $S_{\mathcal{F}}$. Hence, the following description of $\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)$ plays a central role in our approach.

Proposition 2.4. *An operator $S\in\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)$ if and only if $S-S_{\mathcal{F}_0}$ is a positive semi-definite operator on $\mathcal{H}$ and $\mathbf{a}\prec\lambda(S-S_{\mathcal{F}_0})$, where $\lambda(S-S_{\mathcal{F}_0})$ is the vector of eigenvalues of $S-S_{\mathcal{F}_0}$ counting multiplicities.*

Proof. Observe that if $\mathcal{F}=(\mathcal{F}_0\,,\,\mathcal{G})\in\mathcal{H}^n$ then $S_{\mathcal{F}}=S_{\mathcal{F}_0}+S_{\mathcal{G}}$. The result follows applying Proposition 2.1 to $B=S_{\mathcal{F}}-S_{\mathcal{F}_0}$ (which must be nonnegative since $S\in\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)$). $\square$

In view of the Remark 2.2 and a spectral characterization of a specific set of matrices, in [32] is described a special case, known as *feasible case* of optimal completions.

Remark 2.5 (Optimal completion problem with prescribed norms: the feasible case)**.** Consider the following set of positive perturbations of a positive semidefinite operator S_0: given $t>\operatorname{tr}S_0$ and $k\in\mathbb{N}$, $k\leq d$,

$$U_t(S_0,k)=\{S_0+B:\ B\text{ positive semidefinite }\ ,\ \operatorname{rk}B\leq k\ ,\ \operatorname{tr}(S_0+B)\ =\ t\ \}\ ,$$

In [32, Theorem 3.12] it is shown that there exist $\prec$-minimizers in $U_t(S_0,k)$. Indeed, there exists $\nu=\nu_{\lambda\,,k}(t)$ - that can be effectively computed by simple algorithms - such that $S\in U_t(S_0,k)$ is a $\prec$-minimizer if and only if $\lambda(S)=\nu$.

Now, let $\mathcal{F}_0=\{f_i\}_{i=1}^{n_{\mathrm{o}}}\in\mathcal{H}^{n_{\mathrm{o}}}$ and $\mathbf{a}=(\alpha_i)_{i=1}^k$ be a sequence of positive numbers. Denote by $S_0=S_{\mathcal{F}_0}$ and let $t=\operatorname{tr}S_0+\operatorname{tr}\mathbf{a}$. We say that the completion problem for $(\mathcal{F}_0\,,\,\mathbf{a})$ is *feasible* if $\mu\stackrel{\text{def}}{=}\nu-\lambda$ satisfies that $\mathbf{a}\prec\mu$, where $\nu=\nu_{\lambda\,,k}(t)$ is as above. In this case, in [32] it is shown that $\lambda(S-S_0)=\mu^{\downarrow}$ for any S which is a $\prec$-minimizer in $U_t(S_0,k)$. Hence we conclude that $S\in\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)$. Moreover, Proposition 2.4 also shows that $\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)\subseteq U_t(S_0,k)$ and therefore S is a $\prec$-minimizer in $\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)$. In this case, as a consequence of Remark 2.2 any completion $\mathcal{F}\in\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ such that $S_{\mathcal{F}}=S$ is a minimizer of P_f for any convex function $f:[0,\infty)\to\mathbb{R}$. That is, in the feasible case we have structural solutions of the completion problem, in the sense that these solutions do not depend on the particular choice of convex potential considered.

Nevertheless, it is easy to construct examples in which the completion problem for $(\mathcal{F}_0\,,\,\mathbf{a})$ is not feasible (see [32] or Example 6.7 below) for wich the structure of the optimal completions with these norms is not known, even for the MSE. $\triangle$

3 Main results

Here we describe the main results of the present paper, for the convenience of the reader. The proofs of these results, as well as detalied descriptions of some of their applications, will be presented in the following sections.

Let $(\mathcal{F}_0\,,\,\mathbf{a})$ be initial data for the CP as in 2.3 . As we shall see in Section 4, the minimizers for the CP lie on the set of frame completions which achieve equality in Lidskii's inequality, namely:

$$\left\{\,\mathcal{F}=(\mathcal{F}_0\,,\,\mathcal{G})\in\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0):\ \lambda(S_{\mathcal{F}})=\left(\lambda(S_{\mathcal{F}_0})+\lambda^{\uparrow}(S_{\mathcal{G}})\right)^{\downarrow}\right\}\ . \tag{3}$$

Recall that the notation $\lambda(S)$ is used to describe the vector of eigenvalues of S, counting multiplicities and such that its entries are arranged in non-increasing order, while the arrows $\downarrow$ and $\uparrow$ are used to indicate that the vectors are rearranged so that the entries are listed in non-increasing or

non-decreasing order. Consider the set of ordered vectors $\{\mu \in (\mathbb{R}^d_{\geq 0})^{\uparrow} : \ \mathbf{a} \prec \mu\}$. It is easy to see that this set is compact and convex (for example, it is bounded since the condition $\mathbf{a} \prec \mu$ requires $\|\mu\|_1 = \operatorname{tr} \mu = \operatorname{tr} \mathbf{a} = \|\mathbf{a}\|_1$). Therefore the shifted set $\Lambda^{\mathrm{op}}_{\mathbf{a}}(\lambda) = \{\lambda^{\downarrow} + \mu^{\uparrow} : \ \mu \in \mathbb{R}^d_{\geq 0} \quad \text{and} \quad \mathbf{a} \prec \mu\}$ is also compact and convex. We shall see in Theorem 4.4 that this set characterizes the spectrum $\lambda(S_{\mathcal{F}})$ for every $\mathcal{F} = (\mathcal{F}_0\,, \mathcal{G})$ liying in the set described in (3), and that the frame operators $S_{\mathcal{F}_0}$ and $S_{\mathcal{G}}$ commute.

Theorem 3.1. Let $\mathcal{F}_0 = \{f_i\}_{i=1}^{n_{\mathrm{o}}}$ and $\mathbf{a} = (\alpha_i)_{i=1}^{k}$ be a sequence of positive numbers such that $d - \operatorname{rk} S_{\mathcal{F}_0} \leq k$. Let λ be the vector of eigenvalues of $S_{\mathcal{F}_0}$. Let $f : [0, \infty) \to \mathbb{R}$ be a strictly convex function. Then there exists a vector $\mu(\lambda\,, \mathbf{a}\,, f) = \mu$ such that $\mu = \mu^{\uparrow}$, $\mathbf{a} \prec \mu$ and

1. $\mathcal{F} = (\mathcal{F}_0\,, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ is a global minimizer of $P_f \iff \lambda(S_{\mathcal{F}}) = \big(\lambda(S_{\mathcal{F}_0}) + \lambda^{\uparrow}(S_{\mathcal{G}})\big)^{\downarrow}$ and $\lambda^{\uparrow}(S_{\mathcal{G}}) = \mu$.

2. The vector μ is uniquely determined by the conditions $\mu = \mu^{\uparrow}$, $\mathbf{a} \prec \mu$ and

$$\sum_{i=1}^{d} f(\lambda_i + \mu_i) = \min_{\nu \in \Lambda^{\mathrm{op}}_{\mathbf{a}}(\lambda)} \sum_{i=1}^{d} f(\nu_i) = \min \ \Big\{ \sum_{i=1}^{d} f(\lambda_i + \gamma_i) : \ \gamma \in (\mathbb{R}^d_{\geq 0})^{\uparrow} \quad \text{and} \quad \mathbf{a} \prec \gamma \Big\} \ . \tag{4}$$

Proof. See Theorem 4.6. □

Remark 3.2 (First reduction of the optimal CP problem)**.** Let $(\mathcal{F}_0\,, \mathbf{a}\,, f)$ be initial data for the CP as in 2.3 . Consider the compact convex set $\{\lambda^{\downarrow} + \mu^{\uparrow} : \ \mu \in \mathbb{R}^d_{\geq 0} \quad \text{and} \quad \mathbf{a} \prec \mu\} \subseteq \mathbb{R}^d_{\geq 0}$. Since the strictly convex function $\gamma \in \mathbb{R}^d \mapsto \sum_{i=1}^d f(\gamma_i)$ is also lower semi-continuous, there exists a unique minimizer $\nu = \nu(\lambda\,, \mathbf{a}\,, f)$ such that

$$\sum_{i=1}^{d} f(\nu_i) \leq \sum_{i=1}^{d} f(\lambda_i + \gamma_i) \quad \text{for every} \quad \gamma \in (\mathbb{R}^d_{\geq 0})^{\uparrow} \quad , \quad \mathbf{a} \prec \gamma \ . \tag{5}$$

We remark that ν could have some zero entries, so that the minimizers of the CP would not be frames. We shall show that this is not the case in Proposition 5.5.

Theorem 3.1 states that $\mu(\lambda\,, \mathbf{a}\,, f) = \nu(\lambda\,, \mathbf{a}\,, f) - \lambda$. Thus, a completion $\mathcal{F} = (\mathcal{F}_0\,, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ such that $\lambda(S_{\mathcal{F}}) = \big(\lambda(S_{\mathcal{F}_0}) + \lambda^{\uparrow}(S_{\mathcal{G}})\big)^{\downarrow}$ is an optimal completion with respect to P_f if and only if $\lambda(S_{\mathcal{G}}) = (\nu(\lambda\,, \mathbf{a}\,, f) - \lambda)^{\downarrow}$. Thus, the minimization problem in Eq. (5) constitutes a reduction of the CP to a optimization problem in $\mathbb{R}^d$ that in turn can be tackled with several numerical tools in concrete examples. Notice that the set of $\lambda^{\downarrow} + \mu^{\uparrow}$ such that $\mu \in \mathbb{R}^d_{\geq 0} \quad \text{and} \quad \mathbf{a} \prec \mu$ has a natural and explicit description in terms of majorization, which is an algorithmic notion. △

Using the results about equality in Lidskii's inequality of Section 8 (or Theorem 4.4), and a detailed study of the geometry of local minimizers for the CP in terms of decompositions into irreducible subfamilies of the completing sequences (see Section 5 and Section 7 - Appendix I), we can show the following result (for a more detailed formulation - and its proof - see Theorem 6.1):

Theorem 3.3. Assume that $\mathcal{F} = (\mathcal{F}_0\,, \mathcal{G})$ is a global minimizer of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$. Then

1. The frame operator $S_{\mathcal{F}} = S_{\mathcal{F}_0} + S_{\mathcal{G}}$ is invertible, so that $\mathcal{F}$ is a frame.

2. Every vector of the sequence $\mathcal{G}$ is an eigenvector of the frame operator $S_{\mathcal{F}}$.

Proof. It is a consequence of Theorem 6.1. □

Remark 3.4. Let $(\mathcal{F}_0\,, \mathbf{a}\,, f)$ be initial data for the CP as in 2.3 and let $\lambda = \lambda(S_{\mathcal{F}_0})$. Using item 2 of Theorem 3.3, it follows that there exists a finite set $E(\mathcal{F}_0\,, \mathbf{a})$, described in Remark 6.2, which can be algorithmically computed and allows to reduce the optimization problem for finding minimizers for the CP of Remark 3.2 to a finite process in the following sense (see Theorem 6.3):

1. The vector $\mu = \mu(\lambda\,,\,\mathbf{a}\,,\,f) \in (\mathbb{R}^d_{\geq 0})^{\uparrow}$ of Theorem 3.1 satisfies that $\mu \in E(\mathcal{F}_0\,,\,\mathbf{a})$.

2. Moreover, this vector μ is uniquely determined by the equation

$$\sum_{i=1}^d f(\lambda_i + \mu_i) \;\; = \;\; \min\,\{\sum_{i=1}^d f(\lambda_i + \gamma_i) : \gamma \in E(\mathcal{F}_0\,,\,\mathbf{a})\,\}\;, \tag{6}$$

That is, $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ is a P_f global minimizer if and only if $\lambda(S_{\mathcal{F}}) = \big(\,\lambda(S_{\mathcal{F}_0}) + \lambda^{\uparrow}(S_{\mathcal{G}})\,\big)^{\downarrow}$, $\mu = \lambda^{\uparrow}(S_{\mathcal{G}}) \in E(\mathcal{F}_0\,,\,\mathbf{a})$ and it satisfies Eq. (6). In the second part of section 6 we describe in detail the corresponding algorithm, we discuss its complexity, and we show several examples. We remark that the algorithmic construction of the finite set $E(\mathcal{F}_0\,,\,\mathbf{a})$ is based on the fact that any vector of the completing sequence of a minimizer must be a eigenvector of the frame operator. Hence $E(\mathcal{F}_0\,,\,\mathbf{a})$ arises from an intrinsic structure of this problem, and it is not merely a reduction to extremal points of the convex set $\{\lambda^{\downarrow} + \mu^{\uparrow} :\; \mu \in \mathbb{R}^d_{\geq 0} \quad \text{and} \quad \mathbf{a} \prec \mu\}$. Moreover, as we point out in Remark 6.9, based on this structure the set $E(\mathcal{F}_0\,,\,\mathbf{a})$ could be reduced, getting significant simplifications of the complexity of the optimization problem. $\triangle$

4 The spectrum of the minimizers of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$

In this section we reduce the problem of computing optimal frame completions with prescribed norms to a set of completions whose frame operators achieve equality in Lidskii's inequality. We also show that the spectral structure of optimal completions is unique and has some other properties that will be considered in the following sections.

Let $(\mathcal{F}_0\,,\,\mathbf{a})$ be initial data for the CP as in 2.3 . Let $\mu \in \mathbb{R}^d_{\geq 0}$ be such that $\mathbf{a} \prec \mu$. We consider the set

$$\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0\,,\,\mu) \;\stackrel{\text{def}}{=}\; \{\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) :\; \lambda(S_{\mathcal{G}}) = \mu^{\downarrow}\} \subseteq \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\;.$$

Recall that if $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G})$ then $S_{\mathcal{F}} = S_{\mathcal{F}_0} + S_{\mathcal{G}}$. By Proposition 2.4 we get the following partition:

$$\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) = \bigsqcup \{\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0\,,\,\mu) : \mu \in \mathbb{R}^d_{\geq 0}\,,\,\mathbf{a} \prec \mu\}\;. \tag{7}$$

As a consequence of Proposition 2.4, we shall deal with the spectrum of $S_{\mathcal{F}}$, $S_{\mathcal{F}_0}$, and $B = S_{\mathcal{F}} - S_{\mathcal{F}_0}$. These eigenvalues are related with a family of inequalities provided by a known result of Lidskii:

Theorem 4.1 (Lidskii's inequality)**.** *Let A, B be a pair of $d \times d$ Hermitian matrices, with eigenvalues $\lambda(A)$, $\lambda(B) \in (\mathbb{R}^d)^{\downarrow}$. Then $\lambda^{\downarrow}(A) + \lambda^{\uparrow}(B) \prec \lambda(A+B)$,* $\square$

Lidskii's inequality plays an important role in our study of optimal frame completion problems. Moreover, the case of equality, i.e. when $(\lambda^{\downarrow}(A) + \lambda^{\uparrow}(B)\,)^{\downarrow} = \lambda(A+B)$ plays a central role in this paper. We completely characterize such pair of matrices - that we call Lidskii matching matrices - in Appendix II (see Section 8). Next we consider the spectral structure of each of the slices in the partition above.

Proposition 4.2. *Consider the previous notations and fix $\mu \in (\mathbb{R}^d_{\geq 0})^{\uparrow}$ such that $\mathbf{a} \prec \mu$. Then the vector $\nu = (\lambda(S_{\mathcal{F}_0}) + \mu^{\uparrow})^{\downarrow}$ is a $\prec$-minimizer in $\{\lambda(S_{\mathcal{F}}) : \mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0\,,\,\mu)\}$.*

Proof. Notice that the set of all frame operators $S_{\mathcal{G}}$ such that $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0\,,\,\mu)$ is closed under unitary equivalence. Indeed, if U is any unitary operator on $\mathcal{H}$, then $U\,S_{\mathcal{G}}\,U^*$ is the frame operator of the sequence $U \cdot \mathcal{G} = \{U f_i\}_{i=n_o+1}^{n}$. Then it is clear that $\nu \in \{\lambda(S_{\mathcal{F}}) : \mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0\,,\,\mu)\}$. On the other hand, given $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0\,,\,\mu)$, then Lidskii's inequality 4.1 states that the vector $\nu \prec \lambda(S_{\mathcal{F}_0} + S_{\mathcal{G}}) = \lambda(S_{\mathcal{F}})$. This establishes that ν is a $\prec$-minimizer in the set $\{\lambda(S_{\mathcal{F}}) : \mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0\,,\,\mu)\}$. $\square$

Remark 4.3. Consider the previous notations and fix $\mu \in (\mathbb{R}^d_{\geq 0})^{\uparrow}$ such that $\mathbf{a} \prec \mu$. Let $f : [0\,,\,\infty) \to [0\,,\,\infty)$ be a strictly convex function and let P_f be the convex potential induced by f. By Remark 2.2 and and Proposition 4.2 we see that, if $\lambda = \lambda(S_{\mathcal{F}_0})$ then

$$\mathcal{F} \in \operatorname{argmin}\{P_f(\mathcal{G}) : \mathcal{G} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0\,,\,\mu)\} \iff \lambda(S_{\mathcal{F}}) = (\lambda + \mu)^{\downarrow} = (\,\lambda^{\downarrow} + \mu^{\uparrow}\,)^{\downarrow}\,. \tag{8}$$

That is, if we consider the partition of $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ described in Eq. (7), then in each slice $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0\,,\,\mu)$ the minimizers of the potential P_f are characterized by the spectral condition (8).

This shows that in order to search for global minimizers of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ we can restrict our attention to the set of frame completions which achieve equality in Lidskii's inequality:

$$\mathcal{C}^{\mathrm{op}}_{\mathbf{a}}(\mathcal{F}_0) \stackrel{\mathrm{def}}{=} \left\{\, \mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) :\ \lambda(S_{\mathcal{F}}) = \left(\,\lambda(S_{\mathcal{F}_0}) + \lambda^{\uparrow}(S_{\mathcal{G}})\,\right)^{\downarrow} \right\}.$$

Indeed, Eqs. (7) and (8) show that if $\mathcal{F}$ is a minimizer of P_f in $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ then $\mathcal{F} \in \mathcal{C}^{\mathrm{op}}_{\mathbf{a}}(\mathcal{F}_0)$, i.e.

$$\operatorname{argmin}\{P_f(\mathcal{F}) : \mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\} \;=\; \operatorname{argmin}\{P_f(\mathcal{F}) : \mathcal{F} \in \mathcal{C}^{\mathrm{op}}_{\mathbf{a}}(\mathcal{F}_0)\}\,. \qquad \triangle$$

The following theorem, based on the study of equality in Lidskii's inequality (cf. Theorem 8.8 in Appendix II) together with a careful analysis of sums of ordered vectors (cf. Proposition 8.6 and Remark 8.7), gives a strong characterization of the sequences in $\mathcal{C}^{\mathrm{op}}_{\mathbf{a}}(\mathcal{F}_0)$ which will be a key result in order to characterize the minimizers for the CP.

Theorem 4.4. *Let $(\mathcal{F}_0\,,\,\mathbf{a})$ be initial data for the CP as in 2.3 . Denote by $\lambda = \lambda(S_{\mathcal{F}_0})$. Then*

1. *The spectral picture $\{\lambda(S_{\mathcal{F}}) :\ \mathcal{F} \in \mathcal{C}^{\mathrm{op}}_{\mathbf{a}}(\mathcal{F}_0)\} = \{\nu^{\downarrow} :\ \nu = \lambda^{\downarrow} + \mu^{\uparrow} :\ \mu \in \mathbb{R}^d_{\geq 0} \quad \text{and} \quad \mathbf{a} \prec \mu\}$.*
2. *If $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}^{\mathrm{op}}_{\mathbf{a}}(\mathcal{F}_0)$, with $\lambda^{\uparrow}(S_{\mathcal{G}}) = \mu$, then there exists an orthonormal basis $\{v_i\}_{i=1}^d$ of $\mathbb{C}^d$ such that $S_{\mathcal{F}_0}\, v_i = \lambda_i\, v_i$ and*

$$S_{\mathcal{G}} = \sum_{i=1}^{d} \mu_i \cdot v_i \otimes v_i \implies S_{\mathcal{F}} = S_{\mathcal{F}_0} + S_{\mathcal{G}} = \sum_{i=1}^{d} (\lambda_i + \mu_i)\, v_i \otimes v_i\,. \tag{9}$$

In particular, the frame operators $S_{\mathcal{F}_0}$ and $S_{\mathcal{G}}$ commute.

Proof. 1. It is an immediate consequence of the definition of $\mathcal{C}^{\mathrm{op}}_{\mathbf{a}}(\mathcal{F}_0)$.

2. Let $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}^{\mathrm{op}}_{\mathbf{a}}(\mathcal{F}_0)$. Then the frame operator $S_{\mathcal{G}}$ is a Lidskii matching matrix for $S_{\mathcal{F}_0}$ in the sense of Eq. (31) (see Section 8). Hence, the existence of an ONB $\{v_i\}_{i=1}^d$ satisfying Eq. (9) follows from Theorem 8.8. □

Remark 4.5. The advantage in considering the set $\{\lambda + \mu^{\uparrow} : \mu \in \mathbb{R}^d_{\geq 0} \text{ and } \mathbf{a} \prec \mu\}$ instead of the the spectral picture $\{\lambda(S_{\mathcal{F}}) :\ \mathcal{F} \in \mathcal{C}^{\mathrm{op}}_{\mathbf{a}}(\mathcal{F}_0)\}$ is that it is easy to check that the former is a convex set (although its elements are not necesarily ordered vectors). This fact will play an important role in the following results. On the other hand, note that the ordered joint diagonalization in Eq. (9) (which follows from Theorem 8.8) is not a direct consequence of the fact that the frame operators $S_{\mathcal{F}_0}$ and $S_{\mathcal{G}}$ commute. △

Theorem 4.6. *There exists a vector $\mu(\lambda\,,\,\mathbf{a}\,,\,f) = \mu \in (\mathbb{R}^d_{\geq 0})^{\uparrow}$ such that $\mathbf{a} \prec \mu$ and*

1. *$\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ is a global minimizer of $P_f \iff \lambda(S_{\mathcal{F}}) = \left(\,\lambda(S_{\mathcal{F}_0}) + \lambda^{\uparrow}(S_{\mathcal{G}})\,\right)^{\downarrow}$ and $\lambda^{\uparrow}(S_{\mathcal{G}}) = \mu$.*

2. The vector μ is uniquely determined by the conditions $\mu = \mu^{\uparrow}$, $\mathbf{a} \prec \mu$ and

$$\sum_{i=1}^{d} f(\lambda_i + \mu_i) = \min \Big\{ \sum_{i=1}^{d} f(\lambda_i + \gamma_i) : \ \gamma \in (\mathbb{R}^d_{\geq 0})^{\uparrow} \quad \text{and} \quad \mathbf{a} \prec \gamma \Big\} \ . \tag{10}$$

Proof. Recall that

$$P_f(\mathcal{F}) = \operatorname{tr} f(S_{\mathcal{F}}) = \sum_{i=1}^{d} f(\lambda_i(S_{\mathcal{F}})) \quad \text{for every} \quad \mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) \ . \tag{11}$$

Since the set $\{\lambda^{\downarrow} + \mu^{\uparrow} : \ \mu \in \mathbb{R}^d_{\geq 0} \quad \text{and} \quad \mathbf{a} \prec \mu\}$ is compact and convex and $F(\gamma) = \sum_{i=1}^{d} f(\gamma_i)$ is strictly convex and invariant under permutations of the entries γ_i , every local minimizer of F in this set coincide with a unique global minimizer denoted by $\nu = \nu(\mathbf{a}\,,\,\lambda\,,\,f)$. Denote by $\mu = \nu - \lambda$, which clearly satisfies that $\mu = \mu^{\uparrow}$ and $\mathbf{a} \prec \mu$.

Recall that given $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ then a necessary condition for $\mathcal{F}$ to be a global minimizer of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ is that $\mathcal{F} \in \mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$ (see Remark 4.3). Hence, by item 1 in Theorem 4.4, the fact that F is permutation invariant and Eq. (11) we conclude that $\mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ is a global minimizer of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ if and only if

$$\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0) \quad \text{and} \quad \lambda(S_{\mathcal{F}}) = \big(\,\lambda + \lambda^{\uparrow}(S_{\mathcal{G}})\,\big)^{\downarrow} = \nu^{\downarrow} \ . \tag{12}$$

Denote by $\rho = \lambda^{\uparrow}(S_{\mathcal{G}})$ for such a minimizer. Then $\mathbf{a} \prec \rho = \rho^{\uparrow}$ and hence $\lambda + \rho$ is a minimizer of F. Then $\lambda + \rho = \nu$ and $\rho = \mu$. The converse is clear. This shows items 1 and 2. □

Remark 4.7. Majorization between vectors in $\mathbb{R}^d$ is intimately related with the class of doubly stochastic $d \times d$ matrices, denoted by $\mathrm{DS}(d)$. Recall that a $d \times d$ matrix $D \in \mathrm{DS}(d)$ if it has non-negative entries and each row sum and column sum equals 1.

It is well known (see [3]) that given $x\,,\,y \in \mathbb{R}^d$ then $x \prec y$ if and only if there exists $D \in \mathrm{DS}(d)$ such that $D\,y = x$. As a consequence of this fact we see that if $x_1\,,\,y_1 \in \mathbb{R}^r$ and $x_2\,,\,y_2 \in \mathbb{R}^s$ are such that $x_i \prec y_i$, $i = 1\,,\,2$, then $x = (x_1\,,\,x_2) \prec y = (y_1\,,\,y_2)$ in $\mathbb{R}^{r+s}$. Indeed, if D_1 and D_2 are the doubly stochastic matrices corresponding the previous majorization relations then $D = D_1 \oplus D_2 \in \mathrm{DS}(r+s)$ is such that $D\,y = x$. △

The following results, which are rather technical consequences of Theorem 4.6, will be used in the proof of Theorem 6.1 (and Theorem 3.3).

Lemma 4.8. *The vector $\mu = \mu(\lambda\,,\,\mathbf{a}\,,\,f) \in (\mathbb{R}^d_{\geq 0})^{\uparrow}$ of Theorem 4.6 also satisfies that*

$$0 < \mu_i = \mu_{i+1} \implies \lambda_i = \lambda_{i+1} \qquad \textit{for every} \quad 1 \leq i \leq d-1 \ . \tag{13}$$

Proof. Assume that $0 < \mu_i = \mu_{i+1}$ but $\lambda_i > \lambda_{i+1}$ for some $1 \leq i \leq d-1$. We denote by ρ the vector obtained from μ by replacing the i-th and $(i+1)$-th entries of μ by

$$\rho_i = \mu_i - \varepsilon \ \text{ and } \ \rho_{i+1} = \mu_{i+1} + \varepsilon \ , \ \text{ where } \ 0 < \varepsilon < \min\{\tfrac{\lambda_i - \lambda_{i+1}}{2}\,,\,\mu_i\} \ .$$

Although it is possible that $\rho \neq \rho^{\uparrow}$, the facts that $(\mu_i\,,\,\mu_{i+1}) \prec (\rho_i\,,\,\rho_{i+1})$ and $\mu_j = \rho_j$ for every $j \neq i\,,\,i+1$ imply, by Remark 4.7, that $\mu \prec \rho$ and hence $\mathbf{a} \prec \mu \prec \rho$. Using Proposition 2.4 and fixing an ONB $\{v_i\}_{i=1}^d$ for $S_{\mathcal{F}_0}$ such that $S_{\mathcal{F}_0}\,v_i = \lambda_i\,v_i$ for every $1 \leq i \leq d$, we deduce that there exists $\mathcal{F}\,' = (\mathcal{F}_0\,,\,\mathcal{G}\,') \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ such that $\lambda(S_{\mathcal{G}'}) = \rho^{\downarrow}$ and $\lambda(S_{\mathcal{F}'}) = (\lambda+\rho)^{\downarrow}$. Recall that $\nu = \lambda + \mu$. Since $\rho_{i+1} - \rho_i = 2\,\varepsilon < \lambda_i - \lambda_{i+1}$, then

$$\nu_i = \lambda_i + \mu_i > \lambda_i + \rho_i > \lambda_{i+1} + \rho_{i+1} > \lambda_{i+1} + \mu_{i+1} = \nu_{i+1} \ ,$$

while $\nu_j = \lambda_j + \mu_j = \lambda_j + \rho_j$ for every $j \neq i\,,\,i+1$. Then, by Remark 4.7, we conclude that $\lambda + \rho \prec \nu$ and $(\lambda+\rho)^{\downarrow} \neq \nu^{\downarrow}$. Hence, if f is strictly convex the previous facts imply that $P_f(\mathcal{F}\,') < \sum_{i=1}^{d} f(\nu_i)$, which contradicts the characterization of minimizers given in Eq. (12). □

Recall that given two (orthogonal) projections P, Q of $\mathcal{H}$, we say that Q is a sub-projection of P if $R(Q) \subseteq R(P)$ or equivalently if $PQ = Q = QP$.

Corollary 4.9. *Let $(\mathcal{F}_0, \mathbf{a}, f)$ be initial data for the CP as in 2.3 . Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ be a global minimizer of P_f . Denote by $S_0 = S_{\mathcal{F}_0}$. If $z \in \sigma(S_{\mathcal{G}}) \setminus \{0\}$ then there exists $w \in \sigma(S_0)$ such that $\ker(S_{\mathcal{G}} - z) \subseteq \ker(S_0 - w)$. In particular, if P denotes a sub-projection of the spectral projection $P(z)$ of $S_{\mathcal{G}}$ onto its eigenspace $\ker(S_{\mathcal{G}} - z)$, then P and S_0 commute.*

Proof. By Remark 4.3, the P_f-minimality of $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$ in $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ implies that $\mathcal{F} \in \mathcal{C}^{\mathrm{op}}_{\mathbf{a}}(\mathcal{F}_0)$. Then, by Theorem 4.4, there exists an ONB $\{v_i\}_{i=1}^d$ such that $S_0 v_i = \lambda_i v_i$ and Eq. (9) holds. Denote by $S_1 = S_{\mathcal{G}}$, $\mu = \lambda^{\uparrow}(S_1)$ and fix $z \in \sigma(S_1) \setminus \{0\}$. Consider the indices

$$m(z) = \min\{1 \leq i \leq d : \mu_i = z\} \quad \text{and} \quad M(z) = \max\{1 \leq i \leq d : \mu_i = z\} \ .$$

By Eq. (13) in Lemma 4.8 we know that there exists $w \in \sigma(S_0)$ such that $\lambda_i = w$ for every $m(z) \leq i \leq M(z)$. Then, we can use Eq. (9) and deduce that

$$\ker(S_{\mathcal{G}} - z) = \mathrm{span}\{v_i : \ m(z) \leq i \leq M(z)\} \subseteq \ker(S_0 - w) \ .$$

Therefore, any sub-projection P of $P(z)$ must satisfy that $P \cdot S_0 = S_0 \cdot P = wP$. □

5 Local minimizers and irreducible sequences

The following notions have a key role in the characterization of local and global minimizers for the completion problem.

Definition 5.1. Given a sequence $\mathcal{G} = \{g_i\}_{i=1}^k$ in $\mathcal{H}^k$ we say that

1. $\mathcal{G}$ is **irreducible** if it can not be partitioned into two mutually orthogonal subsequences.
2. A **partition of $\mathcal{G}$ into irreducible subfamilies** is a family $\{\mathcal{G}_i\}_{i=1}^p$ given by a partition $\Pi = \{J_i\}_{i=1}^p$ of $\{1, \ldots, k\}$ in such a way that each $\mathcal{G}_i = \{f_j\}_{j \in J_i}$ satisfies that:
   - The subspaces $W_i = \mathrm{span}\{\mathcal{G}_i\}$ $(1 \leq i \leq p)$ are mutually orthogonal, so that $S_{\mathcal{G}} = \bigoplus_{i=1}^p S_{\mathcal{G}_i}$.
   - Each subfamily $\mathcal{G}_i$ is irreducible. △

Remark 5.2. It is easy to see that every sequence $\mathcal{G} = \{f_i\}_{i=1}^k \subseteq \mathcal{H}^k$ has a unique partition into irreducible subfamilies. Indeed, consider the subspace $\mathcal{R} = \mathrm{span}\{\mathcal{G}\} \subseteq \mathbb{C}^d$ and the (non-unital) $*$-subalgebra $\mathcal{M}(\mathcal{G}) = \{f_i \otimes f_i : \ 1 \leq i \leq k\}' \cap \{A : A = P_{\mathcal{R}} A P_{\mathcal{R}}\}$. If $\mathcal{G}$ is not irreducible, then $\mathcal{M}(\mathcal{G})$ contains a unique sequence of minimal orthogonal projections $\{Q_i\}_{i=1}^P$ such that $Q_i Q_j = 0$ for $i, j \leq p$ such that $i \neq j$ and $\sum_{i=1}^P Q_i = P_{\mathcal{R}}$ (with $p > 1$). Then

$$Q_i f_j = \varepsilon(i,j) f_j \quad \text{for every} \quad 1 \leq i \leq p \quad \text{and} \quad 1 \leq j \leq k,$$

where $\varepsilon(i,j) \in \sigma(Q_i) = \{0, 1\}$. Let $J_i = \{1 \leq j \leq k : \ \varepsilon(i,j) = 1\}$ for $1 \leq i \leq p$. Then the partition $\Pi = \{J_i\}_{i=1}^p$ has the desired properties. △

In applied situations it is quite useful to understand the structure of local minimizers of objective functions. In our case, the study of local minimizers allows us to give a detailed description of the geometrical structure of global minimizers. We shall consider the punctual metric d_P on the set $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$, given by

$$d_P(\mathcal{F}, \mathcal{F}') = \|T_{\mathcal{F}} - T_{\mathcal{F}'}\| \ ,$$

where $\|\cdot\|$ denotes the spectral norm. Given a strictly convex function $f:[0\,,\,\infty)\to[0\,,\,\infty)$, we study the geometrical and spectral structure of d_P-local minimizers $\mathcal{F}$ of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ or $\mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$.

The notion of irreducible sequence allows to develop a geometrical study which give strong properties for irreducible d_P-local minimizers. This study is rather technical and it needs several notions and notations, so that we will state and prove these results in Section 7 (Appendix I).

The following result deals with some features of completions $\mathcal{F}\in\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ that are d_P-local minimizers of P_f, under some rather technical assumptions. Nevertheless, this result will apply in case $\mathcal{F}$ is a global minimizer of P_f in $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ (see the proof of Theorem 6.1 below).

Proposition 5.3. *Let $\mathcal{F}=(\mathcal{F}_0\,,\,\mathcal{G})\in\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ be a d_P-local minimizer of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$. Let $\{\mathcal{G}_i\}_{i=1}^p$ be a partition of $\mathcal{G}$ into irreducible subfamilies, where $\mathcal{G}_i=\{f_j\}_{j\in J_i}$ for a partition $\{J_i\}_{i=1}^p$ of the set of indices $\{i:\ n_\circ+1\le i\le n\}$. Assume that $S_{\mathcal{G}_i}$ and S_0 commute, for every $1\le i\le p$. Then there exist positive numbers*

$$c_1\,,\,\dots\,,\,c_p\in\mathbb{R}_{>0}\quad\text{ such that }\quad S_{\mathcal{F}}f_j=c_i\,f_j\ ,\ \ j\in J_i\ ,\ 1\le i\le p\ .$$

Proof. Notice that, by construction, the ranges of the frame operators $S_{\mathcal{G}_i}$ and $S_{\mathcal{G}_j}$ are orthogonal whenever $i\neq j$. Fix $1\le i\le p$. The hypothesis allows us to apply Lemma 7.5 of Appendix I to the sequence $(\mathcal{F}_0\,,\,\mathcal{G}_i)\in\mathcal{C}_{\mathbf{a}_i}(\mathcal{F}_0)$, where $\mathbf{a}_i=(\|f_j\|^2)_{j\in J_i}$. In this case we conclude that there exists $c_i\in\mathbb{R}_{>0}$ such that $(S_{\mathcal{F}_0}+S_{\mathcal{G}_i})\,f_j=c_i\,f_j$, for every $j\in J_i$. Hence,

$$S_{\mathcal{F}}\,f_j=(S_{\mathcal{F}_0}+S_{\mathcal{G}})\,f_j=(S_{\mathcal{F}_0}+\bigoplus_{l=1}^p S_{\mathcal{G}_l})\,f_j=(S_{\mathcal{F}_0}+S_{\mathcal{G}_i})\,f_j=c_i\,f_j\ ,$$

for every $j\in J_i$. □

Let $(\mathcal{F}_0\,,\,\mathbf{a})$ be initial data for the CP as in 2.3 . The key argument in order to characterize the minimizers for the CP is to compute the minimum of a convex map on the compact convex set $\{\mu\in(\mathbb{R}^d_{\ge0})^\uparrow:\ \mathbf{a}\prec\mu\}$. Notice that the set $\{\lambda^\downarrow+\mu^\uparrow:\ \mu\in\mathbb{R}^d_{\ge0}\quad\text{and}\quad\mathbf{a}\prec\mu\}$ contains vectors with zero entries that correspond to completions that are not frames. Fortunately, this is not the case for global minimizers (or even d_P-local minimizers) as we show in Proposition 5.5 below.

Lemma 5.4. *Let $f:[0\,,\,\infty)\to[0\,,\,\infty)$ be a strictly convex function and let $\{a_i\}_{i=1}^n\in\mathbb{R}^n_{>0}$ for some $n\ge d$. If $\mathcal{F}=\{f_i\}_{i=1}^n$ is a d_P-local minimizer of P_f in the set $\{\mathcal{G}=\{g_i\}_{i=1}^n\in\mathcal{H}^n:\ \|g_i\|^2=a_i\,,\ 1\le i\le n\}$, then $\mathcal{F}$ is a frame for $\mathcal{H}$.*

Proof. Let $\Pi=\{J_i\}_{i=1}^p$ be a partition of $\{1,\dots,n\}$ such that, if $\mathcal{F}_i=\{f_j\}_{j\in J_i}$ for $1\le i\le p$, then $\{\mathcal{F}_i\}_{i=1}^p$ is a partition of $\mathcal{F}$ into irreducible subsequences, as in Definition 5.1. Recall that in this case the subspaces $W_i\stackrel{\rm def}{=}\operatorname{span}\{\mathcal{F}_i\}$ $(1\le i\le p)$ are mutually orthogonal. Hence, it is easy to see that each subfamily $\mathcal{F}_i$ is a d_P-local minimizer of P_f in the set

$$\{\{g_j\}_{j\in J_i}:\ g_j\in W_i\,,\ \|g_i\|=\|f_i\|\,,\ j\in J_i\}\,.$$

By [29, Corollary 3] and the properties of Π, each $\mathcal{F}_i$ is a c_i-tight frame for W_i, for some $c_i>0$, $1\le i\le p$. Therefore

$$S_{\mathcal{F}}=\sum_{i=1}^p S_{\mathcal{F}_i}=\sum_{i=1}^p c_i\,P_{W_i}\ .$$

Notice that, in particular, $S_{\mathcal{F}}\,f_j=c_i\,f_j$ for every $j\in J_i$.

Suppose that $\mathcal{F}$ is not a frame for $\mathcal{H}$. Then, there exists $i\in\mathbb{I}_p$ and q, $s\in J_i$ such that $\langle f_q,f_s\rangle\neq0$, because otherwise $\mathcal{F}$ would be a sequence of mutually orthogonal vectors, then $n=d$ and we would have span $\mathcal{F}=\mathcal{H}$. In particular, for this choice of indices we have that $a_s=\|f_s\|^2<c_i$, since

$$c_i\,\|f_s\|^2=\langle S_{\mathcal{F}}\,f_s\,,\,f_s\rangle\ge|\langle f_s\,,\,f_s\rangle|^2+|\langle f_s\,,\,f_q\rangle|^2=\left(\|f_s\|^2+\frac{|\langle f_s\,,\,f_q\rangle|^2}{\|f_s\|^2}\right)\,\|f_s\|^2\ .$$

We are assuming that $\ker S_{\mathcal{F}} \neq \{0\}$. Hence there exists $g \in \ker S_{\mathcal{F}}$ with $\|g\| = \|f_s\|$. Let

$$f_s(t) = \cos\big(\frac{\pi}{2}\, t\big) \cdot f_s + \sin\big(\frac{\pi}{2}\, t\big) \cdot g \quad \text{for every} \quad t \in [0,1]\ ,$$

so that $f_s(0) = f_s$ and $f_s(1) = g$. Notice that $\|f_s(t)\| = \|f_s\|$ for every $t \in [0,1]$. Let $\mathcal{F}(t)$ be the sequence obtained from $\mathcal{F}$ by replacing f_s by $f_s(t)$ and let $\mathfrak{s}(t)$ denote the frame operator of $\mathcal{F}(t)$, for each $t \in [0,1]$. Then

$$\mathfrak{s}(t) = [S_{\mathcal{F}} - (f_s \otimes f_s)] + f_s(t) \otimes f_s(t) \quad \text{for every} \quad t \in [0,1]\ .$$

The inequality $a_s = \|f_s\|^2 < c_i$ implies that $S_{\mathcal{F}} - (f_s \otimes f_s)$ is a positive operator and also that $R(S_{\mathcal{F}} - (f_s \otimes f_s)) = R(S_{\mathcal{F}})$. Indeed, $S_{\mathcal{F}} - (f_s \otimes f_s) = [a_s^{-1}\,(c_i - a_s)] \cdot f_s \otimes f_s + S'$ with S' a positive operator on $\mathcal{H}$; in this case $\lambda(S')$ is obtained from $\lambda(S_{\mathcal{F}})$ by setting one of the occurrences of c_i in $\lambda(S)$ equal to 0, and $f_s \in \ker S'$. Thus,

$$\mathfrak{s}(t) = S' + [a_s^{-1}\,(c_i - a_s) \cdot f_s \otimes f_s + f_s(t) \otimes f_s(t)] \quad \text{with} \quad f_s\,,\, f_s(t) \in \ker S'\ , \tag{14}$$

for every $t \in [0,1]$. Using again the inequality $a_s = \|f_s\|^2 < c_i$, let us define

$$\lambda(t) = \lambda([a_s^{-1}\,(c_i - a_s)] \cdot f_s \otimes f_s + f_s(t) \otimes f_s(t)) = (\lambda_1(t)\,,\, \lambda_2(t)\,,\, 0\,,\, \dots\,,\, 0) \in (\mathbb{R}^d_{\geq 0})^{\downarrow}\ .$$

Then $\lambda(0) = (c_i\,,\, 0\,,\, \dots\,,\, 0)$, $\lambda(1) = (c_i - a_s\,,\, a_s\,,\, 0\,,\, \dots\,,\, 0)^{\downarrow}$ and $\lambda_2(t) > 0$ for $t > 0$. Then there exists $t_0 \in (0,1)$ such that for $0 < t < t_0$, $\lambda_2(t) < \varepsilon$ for $\varepsilon > 0$ such that $\varepsilon < \min_{1 \leq j \leq p} c_j$ and $\varepsilon < \lambda_1(t) = (c_i - \lambda_2(t))$. By the previous remarks, it follows that $\lambda(\mathfrak{s}(t))$ is obtained from $\lambda(S_{\mathcal{F}})$ by replacing one occurrence of c_i by $\lambda_1(t)$ and one occurrence of 0 by $\lambda_2(t)$. Therefore, if $r = \operatorname{rk} S_{\mathcal{F}}$ then $\lambda_j(\mathfrak{s}\,(t)) \leq \lambda_j(S_{\mathcal{F}})$ for $1 \leq j \leq r$ and $\operatorname{tr} S_{\mathcal{F}} = \sum_{j=1}^{r+1} \lambda_j(\mathfrak{s}\,(t)) = \operatorname{tr} \mathfrak{s}\,(t)$ imply that $\lambda(\mathfrak{s}\,(t)) \prec \lambda(S_{\mathcal{F}})$ for $0 < t < t_0$.

These facts show that $\mathcal{F}(t)$ converges to $\mathcal{F}$ with respect to the d_P-metric as $t \to 0^+$, while $P_f(\mathcal{F}(t)) < P_f(\mathcal{F})$ for $t \in (0, t_0)$. This contradicts the assumption that $\mathcal{F}$ is a d_P-local minimum of P_f and thus we should have that $R(S_{\mathcal{F}}) = \mathcal{H}$, i.e. $\mathcal{F}$ is a frame. □

If we fix a strictly convex function $f : [0\,,\, \infty) \to [0\,,\, \infty)$ then (see Remark 4.3) global minimizers of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ actually lie in $\mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$. Hence, we shall focus our interest in the properties of d_P-local minimizers $\mathcal{F} \in \mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$ of P_f.

Proposition 5.5. *Let $(\mathcal{F}_0\,,\, \mathbf{a}\,,\, f)$ be initial data for the CP as in 2.3 . Let $\mathcal{F} = (\mathcal{F}_0\,,\, \mathcal{G}) \in \mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$ be a d_P-local minimizer of P_f on $\mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$. Then $\mathcal{F}$ is a frame, i.e. $S = S_{\mathcal{F}}$ is an invertible operator on $\mathcal{H}$.*

Proof. Denote by $S_0 = S_{\mathcal{F}_0}$, $\lambda(S_0) = \lambda = \lambda^{\downarrow}$, $S_1 = S_{\mathcal{G}}$ and $\lambda(S_1) = \mu^{\downarrow}$ for some $\mathbf{a} \prec \mu = \mu^{\uparrow}$. Since $\mathcal{F} = (\mathcal{F}_0\,,\, \mathcal{G}) \in \mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$, by Theorem 4.4 there exists an ONB $\{v_i\}_{i=1}^d$ such that $S_0 = \sum_{i=1}^d \lambda_i\, v_i \otimes v_i$ and $S = S_0 + S_1 = \sum_{i=1}^d (\lambda_i + \mu_i)\, v_i \otimes v_i$. If S is not invertible, let

$$r = \max\{1 \leq i \leq d : \lambda_i \neq 0\} < \min\{1 \leq j \leq d : \mu_j \neq 0\} - 1\ . \tag{15}$$

Then $\mathcal{H}_r = \operatorname{span}\{v_i : i > r\} = \ker S_0$, and S_1 acts on $\mathcal{H}_r$. The minimality of $\mathcal{F}$ in $\mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$ implies that $\mathcal{G}$ is a d_P-local minimizer of P_f in the set ($n = k + n_{\rm o}$)

$$\mathcal{B}_k(\mathcal{H}_r) \stackrel{\rm def}{=} \{\mathcal{G} = \{g_i\}_{i=1}^k \in \mathcal{H}_r^k :\ \|g_i\|^2 = \alpha_i\,,\ 1 \leq i \leq k\}\ ,$$

because $\lambda(S_0 + S_{\mathcal{G}}) = (\lambda\,,\, \lambda(S_{\mathcal{G}}))^{\downarrow} \implies P_f(\mathcal{F}_0\,,\, \mathcal{G}) = P_f(\mathcal{F}_0) + P_f(\mathcal{G})$ for every $\mathcal{G} \in \mathcal{B}_k(\mathcal{H}_r)$. By Lemma 5.4, we deduce that S_1 is invertible in $\mathcal{H}_r$, contradicting Eq. (15). □

Remark 5.6. From an applied point of view, it would be desirable to verify that local d_P-minimizers are global minimizers of the convex potential P_f (notice that this is a non-trivial fact for the Benedetto-Fickus' frame potential in [2, 12]). Although our techniques allow us to describe the geometrical and some of the spectral structure of local d_P-minimizers, at the present time we are not able to show that local d_P-minimizers are global minimizers. Nevertheless, we conjecture that this is always the case for an arbitrary strictly convex function $f : [0, \infty) \to [0, \infty)$.

6 Structure and computation of global minimizers of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$

In this section we obtain a description of the geometrical structure of global minimizers of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$. This geometrical structure of global minimizers allows us to obtain a finite step algorithm that produces a finite set (that does not depend on f) which completely describes the optimal frame completions $\mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ for P_f .

6.1 On the structure of global minimizers of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$

The goal of this section is the following theorem. We remark that our approach is based on the decomposition into irreducible subfamilies of the completing sequence. It turns out that the geometrical tools and results of Sections 4, 5 and 7 (Appendix I) are essential in the study of the structure of each irreducible subfamily (e.g. see Proposition 5.3).

Theorem 6.1. *Let $(\mathcal{F}_0\,,\,\mathbf{a}\,,\,f)$ be initial data for the CP as in 2.3 . Denote by $\lambda = \lambda(S_{\mathcal{F}_0})$. Then*

1. *There exists a vector $\mu = \mu(\lambda\,,\,\mathbf{a}\,,\,f) \in (\mathbb{R}^d_{\geq 0})^{\uparrow}$ such that $\mathbf{a} \prec \mu$ and*

$$\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) \text{ is a global minimizer of } P_f \iff \mathcal{F} \in \mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0) \text{ and } \lambda^{\uparrow}(S_{\mathcal{G}}) = \mu \ .$$

Assume now that $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G})$ is a global minimizer of P_f on $\mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$. Then

2. *The frame operator $S_{\mathcal{F}} = S_{\mathcal{F}_0} + S_{\mathcal{G}}$ is invertible, so that $\mathcal{F}$ is a frame.*

Let $\{\mathcal{G}_i\}_{i=1}^p$ be a partition of $\mathcal{G}$ into irreducible subfamilies, where $\mathcal{G}_i = \{f_j\}_{j\in J_i}$ for a partition $\{J_i\}_{i=1}^p$ of the set of indices $\{i:\ n_\circ + 1 \leq i \leq n\}$. Then for each $1 \leq i \leq p$

3. *The frame operators $S_{\mathcal{G}_i}$ and $S_{\mathcal{F}_0}$ commute.*

4. *There exists $c_i \in \mathbb{R}_{>0}$ such that $S_{\mathcal{F}}\, f_j = c_i\, f_j \quad$ for every $\quad j \in J_i$.*

Proof. Item 1 was shown in Theorem 4.6.

2. This fact follows from Proposition 5.5.

3. Assume now that $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G})$ is a global minimizer of P_f on $\mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$. Then

$$S_{\mathcal{G}} = \bigoplus_{i=1}^p S_{\mathcal{G}_i} \implies \sigma(S_{\mathcal{G}}) \cup \{0\} = \bigcup_{i=1}^p \sigma(S_{\mathcal{G}_i}) \cup \{0\} \ .$$

Let $P(\alpha)$ (resp. $P_i(\alpha)$) denote the spectral projection of $S_{\mathcal{G}}$ (resp. $S_{\mathcal{G}_i}$) associated with $\alpha \in \sigma(S_{\mathcal{G}})$ (or $P_i(\alpha) = 0$ in case $\alpha \notin \sigma(S_{\mathcal{G}_i})$). Then, for every $1 \leq i \leq p$ we have that

$$S_{\mathcal{G}_i} = \sum_{\alpha\in\sigma(S_{\mathcal{G}})} \alpha\ P_i(\alpha) \quad \text{with} \quad \sum_{i=1}^p P_i(\alpha) = P(\alpha)\ , \quad \alpha \in \sigma(S_{\mathcal{G}})\ .$$

Thus, each $P_i(\alpha)$ is a sub-projection of $P(\alpha)$ for $1 \leq i \leq p$. If we consider $\alpha \in \sigma(S_{\mathcal{G}})$, $\alpha \neq 0$, then Corollary 4.9 shows that $P_i(\alpha)$ commutes with $S_{\mathcal{F}_0}$, for every $i \in \mathbb{I}_p$. This last fact implies that $S_{\mathcal{G}_i}$ commutes with $S_{\mathcal{F}_0}$, for every $i \in \mathbb{I}_p$.

4. It is a consequence of item 3 of this theorem and Proposition 5.3. □

6.2 A finite step algorithm to compute global minimizers

In this section we obtain, as a consequence of Theorem 6.1, an algorithmic solution of the optimal frame completion problem with prescribed norms with respect to a general convex potential P_f. The key idea is the introduction of the following finite set:

Remark 6.2. In order to find the minimizers for the CP with parameters $(\mathcal{F}_0\,,\,\mathbf{a})$ we construct a finite set $E(\mathcal{F}_0\,,\,\mathbf{a})\subseteq(\mathbb{R}^d_{\geq 0})^{\uparrow}$ as follows:

Set $1\leq r\leq d$. Consider a partition $\{K_i\}_{i=1}^p$ of the set $\{d-r+1\,,\,\dots\,,\,d\}$ for some $1\leq p\leq r$, and define the subsequences of $\lambda=\lambda(S_{\mathcal{F}_0})$ given by

$$\Lambda_i=\{\lambda_j\}_{j\in K_i}\in\mathbb{R}^{|K_i|}_{\geq 0}\;,\quad\text{for every}\quad 1\leq i\leq p\;.$$

Consider also a partition $\{J_i\}_{i=1}^p$ of the set $\{1,\dots,k\}$ and define the subsequences of $\mathbf{a}=(\alpha_i)_{i=1}^k\in\mathbb{R}^k$ given by

$$\mathbf{a}_i=\{\alpha_j\}_{j\in J_i}\in\mathbb{R}^{|J_i|}_{>0}\;,\quad\text{for every}\quad 1\leq i\leq p\;.$$

For each $1\leq i\leq p$ define $c_i=|K_i|^{-1}\cdot(\operatorname{tr}\Lambda_i+\operatorname{tr}\mathbf{a}_i)$ and $\Gamma_i=\{c_i-\lambda_j\}_{j\in K_i}$. Let

$$\mu\in\mathbb{R}^d\quad\text{be given by}\quad\mu_j=(\Gamma_i)_j=c_i-\lambda_j\quad\text{if}\quad j\in K_i\;,\tag{16}$$

and $\mu_j=0$ if $j\leq d-r$. We now check whether for every $1\leq i\leq p$ it holds that:

$$\Gamma_i\in\mathbb{R}^{|K_i|}_{\geq 0}\quad,\quad\mathbf{a}_i\prec\Gamma_i\quad\text{and that}\quad\mu=\mu^{\uparrow}\in(\mathbb{R}^d_{\geq 0})^{\uparrow}\;.\tag{17}$$

In this case we declare this μ as a member of $E(\mathcal{F}_0\,,\,\mathbf{a})$. Otherwise we drop this μ. The set $E(\mathcal{F}_0\,,\,\mathbf{a})$ is then obtained by this procedure, as we vary $1\leq r\leq d$ and the partitions previously considered. Therefore, $E(\mathcal{F}_0\,,\,\mathbf{a})$ is a finite set.

A straightforward computation using Proposition 2.1 and Eq. (17) shows that for every $\gamma\in E(\mathcal{F}_0\,,\,\mathbf{a})$ there exists a completion $\mathcal{F}\,'=(\mathcal{F}_0\,,\,\mathcal{G}\,')\in\mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$ such that $\lambda^{\uparrow}(S_{\mathcal{G}'})=\gamma$ and $\lambda(S_{\mathcal{F}'})=(\lambda+\gamma)^{\downarrow}$. We remark that the set $E(\mathcal{F}_0\,,\,\mathbf{a})$ can be explicitly computed in a finite step algorithm, in terms of $\lambda=\lambda(S_{\mathcal{F}_0})$ and $\mathbf{a}$ (see Section 6.3 below for details). $\triangle$

Theorem 6.3. *Let $(\mathcal{F}_0\,,\,\mathbf{a}\,,\,f)$ be initial data for the CP as in 2.3 and let $\lambda=\lambda(S_{\mathcal{F}_0})$. Then*

1. *The vector $\mu=\mu(\lambda\,,\,\mathbf{a}\,,\,f)\in(\mathbb{R}^d_{\geq 0})^{\uparrow}$ of Theorem 6.1 satisfies that $\mu\in E(\mathcal{F}_0\,,\,\mathbf{a})$.*
2. *Moreover, this vector μ is uniquely determined by the equation*

$$\sum_{i=1}^d f(\lambda_i+\mu_i)=\min\,\{\sum_{i=1}^d f(\lambda_i+\gamma_i):\gamma\in E(\mathcal{F}_0\,,\,\mathbf{a})\,\}\;,\tag{18}$$

That is, a completion $\mathcal{F}=(\mathcal{F}_0\,,\,\mathcal{G})\in\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ is a P_f global minimizer if and only if $\mathcal{F}\in\mathcal{C}^{\rm op}_{\mathbf{a}}(\mathcal{F}_0)$, $\mu=\lambda^{\uparrow}(S_{\mathcal{G}})\in E(\mathcal{F}_0\,,\,\mathbf{a})$ and it satisfies Eq. (18).

Proof. Denote by $\mu=\mu(\lambda\,,\,\mathbf{a}\,,\,f)\in(\mathbb{R}^d_{\geq 0})^{\uparrow}$, the vector of Theorem 6.1. Let $\mathcal{F}=(\mathcal{F}_0\,,\,\mathcal{G})$ be a global minimizer of P_f on $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$. In this case, by Proposition 4.2 and Theorem 4.4, $S_{\mathcal{F}_0}$ and $S_{\mathcal{G}}$ commute, $\lambda^{\uparrow}(S_{\mathcal{G}})=\mu$, and $\lambda(S_{\mathcal{F}})=(\lambda+\mu)^{\downarrow}$.

Let $\{\mathcal{G}_i\}_{i=1}^p$ be a partition of $\mathcal{G}$ into irreducible subfamilies, corresponding to the partition $\{J_i\}_{i=1}^p$ of $\{n_\circ+1\,,\,\dots\,,\,n\}$, for some $1\leq p\leq d$. Notice that in this case $S_{\mathcal{G}}=\oplus_{i=1}^p S_{\mathcal{G}_i}$. This last fact shows that there exists a partition $\{K_i\}_{i=1}^p$ of $\{1,\dots,d\}$ such that $\lambda(S_{\mathcal{G}_i})=(\Gamma_i\,,\,0_i)$ where $\Gamma_i=\{\mu_j\}_{j\in K_i}$ and $0_i\in\mathbb{R}^{d-|K_i|}$ for every $1\leq i\leq p$. Then $\mu=(\oplus_{i=1}^p\Gamma_i)^{\uparrow}$.

Fix $i \in \mathbb{I}_p$. Theorem 6.1 implies that there exists $c_i > 0$ such that $S_{\mathcal{F}} f_j = c_i\, f_j$ for every $j \in J_i$ and that $S_{\mathcal{G}_i}$ and $S_{\mathcal{F}_0}$ commute. This fact implies that $S_{\mathcal{F}}|_{R_i} = c_i\, I_{R_i}$, where $R_i = R(S_{\mathcal{G}_i})$ and I_{R_i} denotes the identity operator on R_i. Therefore, we conclude that $c_i = \lambda_j + \mu_j$ for every $j \in K_i$. Hence $\Gamma_i = (c_i - \lambda_j)_{j \in K_i} \in \mathbb{R}_{\geq 0}^{|K_i|}$ and

$$c_i = |K_i|^{-1} \cdot \sum_{j \in K_i} (\lambda_j + \mu_j) = |K_i|^{-1} \cdot (\operatorname{tr} \Lambda_i + \operatorname{tr} \{\alpha_{j-n_\circ}\}_{j \in J_i}) \ ,$$

since $S_{\mathcal{G}_i} = \sum_{j \in J_i} f_j \otimes f_j$. This shows that $\operatorname{tr} S_{\mathcal{G}_i} = \sum_{j \in J_i} \|f_j\|^2 = \sum_{j \in J_i} \alpha_{j-n_\circ}$. Moreover, the previous identity and Proposition 2.1 imply that $\mathbf{a}_i \prec \Gamma_i$, where $\mathbf{a}_i = \{\alpha_{j-n_\circ}\}_{j \in J_i}$. Hence, we conclude that the vector $\mu = \lambda^{\uparrow}(S_{\mathcal{G}}) \in E(\mathcal{F}_0\,, \mathbf{a})$, as defined in Remark 6.2.

As we mentioned before, for every $\gamma \in E(\mathcal{F}_0\,, \mathbf{a})$ there exists a completion $\mathcal{F}\,' = (\mathcal{F}_0\,, \mathcal{G}\,') \in \mathcal{C}_{\mathbf{a}}^{\mathrm{op}}(\mathcal{F}_0)$ such that $\lambda^{\uparrow}(S_{\mathcal{G}\,'}) = \gamma$ and $\lambda(S_{\mathcal{F}\,'}) = (\lambda + \gamma)^{\downarrow}$. Hence the vector μ satisfies Eq. (18). The converse implication now follows from item 1 and Theorem 4.6. □

Remark 6.4. Let $E(\mathcal{F}_0\,, \mathbf{a}) \subseteq (\mathbb{R}_{\geq 0}^d)^{\uparrow}$ be the finite set defined in Remark 6.2 and assume that there exists $\mu \in E(\mathcal{F}_0\,, \mathbf{a})$ such that $\lambda + \mu$ is a $\prec$-minimizer for the set $\lambda + E(\mathcal{F}_0\,, \mathbf{a})$ i.e., such that

$$\lambda + \mu \prec \lambda + \gamma \quad \text{ for every } \quad \gamma \in E(\mathcal{F}_0\,, \mathbf{a}) \ . \tag{19}$$

Then, by Theorem 6.3 and Remark 2.2 we see that μ coincides with $\mu(\lambda\,, \mathbf{a}\,, f)$, the vector of Theorem 6.1, for all strictly convex functions $f : [0\,, \infty) \to [0\,, \infty)$.

That is, given an arbitrary strictly convex function $f : [0\,, \infty) \to [0\,, \infty)$ then a completion $\mathcal{F} = (\mathcal{F}_0\,, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ is a global minimizer of P_f in $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ if and only if $\lambda^{\uparrow}(S_{\mathcal{G}}) = \mu$. Moreover, a similar argument shows that in this case

$$\lambda(S_{\mathcal{F}_0}) + \mu \quad \text{ is a } \prec\text{-minimizer in } \quad \{\lambda(S_{\mathcal{F}_0})^{\downarrow} + \mu^{\uparrow} : \ \mu \in \mathbb{R}_{\geq 0}^d \quad \text{ and } \quad \mathbf{a} \prec \mu\} \ ,$$

Therefore μ (resp. $\lambda(S_{\mathcal{F}_0}) + \mu$) is an structural (spectral) solution to the problem of minimizing P_f, in the sense that the solution does not depend of the particular choice of the strictly convex function f. Such structural solutions exist if we assume that the completion problem is feasible (see Remark 2.5). Numerical examples suggest that such a majorization minimizer always exists (see Section 6.3). These facts induce the following conjecture: △

Conjecture 6.5. Let $(\mathcal{F}_0\,, \mathbf{a})$ be initial data for the CP as in 2.3 . Then there exists $\mu \in E(\mathcal{F}_0\,, \mathbf{a})$ such that $\lambda_{\mathcal{F}_0} + \mu$ satisfies the majorization minimality of Eq. (19). □

6.3 Algorithmic implementation: some examples.

As it was described in the previous section, an algorithm can be developed in order to compute explicitly the set $E(\mathcal{F}_0\,, \mathbf{a})$ and the finite set of possible minimizers $\nu = \lambda + \mu$, $\mu \in E(\mathcal{F}_0\,, \mathbf{a})$ constructed from it. A proposed algorithm scheme is the following:

6.6. Given the initial data $\lambda \in (\mathbb{R}_{\geq 0}^d)^{\downarrow}$ and $\mathbf{a} = (\alpha_i)_{i=1}^k$, we set $n = k + n_\circ$ as before.

Step 1. Let $m = \min\{d,\ k\}$. For each $1 \leq r \leq m$ let $\lambda(r) = (\lambda_j)_{j=d-r+1}^d$. For every $1 \leq p \leq r$,

- We compute all possible partitions of $\lambda(r)$ into p non-empty sets. We do the same with $\mathbf{a}$.
- Fixed a partition for $\lambda(r)$ and one of $\mathbf{a}$, we pair the sets of both partitions and compute for every pair the constant c and check majorization as it was described in Eq. (17).
- In case that the majorization conditions are satisfied for all pairs in these partitions for $\lambda(r)$ and $\mathbf{a}$, the vector μ is constructed as in Eq. (16).

- If $\mu = \mu^{\uparrow}$ then is μ stored in the set $E(\mathcal{F}_0\,,\,\mathbf{a})$.

Step 2. The set $N(\mathcal{F}_0\,,\,\mathbf{a}) = \{\lambda + \mu : \mu \in E(\mathcal{F}_0\,,\,\mathbf{a})\}$ is constructed from that stored data.

Step 3. We search for the vector $\nu \in N(\mathcal{F}_0\,,\,\mathbf{a})$ of minimum Euclidean norm.

Then this ν is a minimizer for the map $F(x) = \sum_{i=1}^{d} x_i^2$ associated to the frame potential on the set $\{\lambda(S_{\mathcal{F}}) : \mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\}$. Moreover $\mu = \nu - \lambda$ is the vector of Theorem 4.6, which allows to construct (via the Schur-Horn algorithm) optimal completions in $\mathcal{C}_{\mathbf{a}}^{\mathrm{op}}(\mathcal{F}_0)$ with respect to the Benedetto-Fickus's frame potential. By Theorem 6.3, the global minimizers corresponding to a different potential in $\mathcal{C}_{\mathbf{a}}^{\mathrm{op}}(\mathcal{F}_0)$ can be computed similarly, i.e. by minimizing the corresponding convex function on the set $N(\mathcal{F}_0\,,\,\mathbf{a})$.

Step 4. Finally, we test if the vector ν obtained in Step 3 is a minimizer for majorization in $N(\mathcal{F}_0\,,\,\mathbf{a})$. In that case, the algorithm succeed in finding the minimizer for every convex potential P_f. $\triangle$

In all examples in which we have applied the previous algorithm, the Step 4 confirmed that the minimizer for the frame potential in $N(\mathcal{F}_0\,,\,\mathbf{a})$ is actually the minimizer for majorization, which suggests a positive answer to the Conjecture 6.5 (see the comments in Remark 6.4).

Example 6.7. Consider the frame $\mathcal{F}_0 \in \mathbf{F}(7\,,\,5)$ whose synthesis operator is

$$T_{\mathcal{F}_0}^{*} = \begin{bmatrix} 0.9202 & -0.7476 & -0.4674 & 0.9164 & 0.1621 & 0.3172 & -0.5815 \\ 0.4556 & 0.0164 & 0.0636 & 1.0372 & -1.6172 & 0.3688 & 0.2559 \\ -0.0885 & -0.3495 & -0.9103 & 0.3672 & -0.6706 & -0.9252 & 0.6281 \\ 0.1380 & -0.4672 & -0.6228 & -0.1660 & 0.9419 & 1.0760 & 1.1687 \\ 0.7082 & 0.2412 & -0.1579 & -1.8922 & -0.4026 & 0.1040 & 1.6648 \end{bmatrix}. \tag{20}$$

In this case $\lambda = \lambda(S_{\mathcal{F}_0}) = (9\,,\,5\,,\,4\,,\,2\,,\,1)$ and $t_0 = \operatorname{tr} S_{\mathcal{F}_0} = 21$. Fix the data $n = 9$ (hence $k = 2$), $\mathbf{a} = (3.5\,,\,2)$ and notice that then $t = t_0 + \operatorname{tr} \mathbf{a} = 26.5$ and $m = d - k = 3$. Then, according to the results in [32] we know that the optimal spectrum for $U_t(S_0\,,\,m)$ is $\nu = \nu_{\lambda\,,m}(26.5) = (9\,,\,5\,,\,4.25\,,\,4.25\,,\,4)$. Therefore, we have that $\nu - \lambda = \mu = (0\,,\,0\,,\,0.25\,,\,2.25\,,\,3)$ so that $\mathbf{a} \not\prec \mu$, that is the completion problem for $(\mathcal{F}_0\,,\,\mathbf{a})$ is not feasible.

Nevertheless, if we apply the algorithm described above, the optimal spectrum μ and ν can be computed, since we can describe the set $N(\mathcal{F}_0\,,\,\mathbf{a})$.

Indeed in this case $N(\mathcal{F}_0\,,\,\mathbf{a}) = \{(9\,,\,5\,,\,4.5\,,\,4\,,\,4)\,,\ (9\,,\,6.5\,,\,5\,,\,4\,,\,2)\}$ so $\nu = (9\,,\,5\,,\,4.5\,,\,4\,,\,4)$ (where $\mu = (0\,,\,0\,,\,0\,,\,2\,,\,3.5)$) and an optimal completion is given by:

$$T_{\mathcal{F}_1}^{*} = \begin{bmatrix} 0.0441 & -1.3541 \\ 0.6901 & 0.5701 \\ -1.2093 & 0.0887 \\ -0.0569 & 0.8836 \\ 0.2371 & -0.7435 \end{bmatrix}. \tag{21}$$

In this case, the vector μ is constructed with the partitions $K_1 = \{2\}$, $K_2 = \{1\}$ of the two smaller eigenvalues in $\lambda = \lambda(S_{\mathcal{F}_0}) = (9\,,\,5\,,\,4\,,\,2\,,\,1)$ which are paired with $J_1 = \{2\}$ and $J_2 = \{3.5\}$ of $\mathbf{a}$, using the notation introduced in Section 6.2.

If we now set $\mathbf{a} = (2\,,\,\frac{1}{4}\,,\,\frac{1}{4}\,,\,\frac{1}{4})$, again the problem is not feasible (see [32]). In this case the algorithm yields a $N(\mathcal{F}_0\,,\,\mathbf{a})$ with 23 elements with a minimizer for majorization given by $\nu = (9\,,\,5\,,\,4\,,\,3\,,\,2.75)$. In this case, the partitions of λ are K_1 and K_2 of previous example, and $J_1 = \{\frac{1}{4}\,,\,\frac{1}{4}\,,\,\frac{1}{4}\}$ and $J_2 = \{2\}$ is the partition of $\mathbf{a}$. Finally, an optimal completion of $\mathcal{F}_0$ with prescribed norms is given by:

$$T_{\mathcal{F}_1}^{*} = \begin{bmatrix} 0.0156 & 0.0156 & 0.0156 & -1.0236 \\ 0.2440 & 0.2440 & 0.2440 & 0.4310 \\ -0.4275 & -0.4275 & -0.4275 & 0.0670 \\ -0.0201 & -0.0201 & -0.0201 & 0.6679 \\ 0.0838 & 0.0838 & 0.0838 & -0.5620 \end{bmatrix}. \tag{22}$$

Example 6.8. If $\mathbf{a} = (5.35\,,\, 4.66\,,\, 3.2\,,\, 2.5\,,\, 1.2\,,\, 1\,,\, 0.65)$ and let $\mathcal{F}_0$ be any family in $\mathbf{F}(n_\circ\,,\, 6)$ such that $\lambda = \lambda(S_{\mathcal{F}_0}) = (5.75\,,\, 5.4\,,\, 4.25\,,\, 4.25\,,\, 3\,,\, 2)$, (this is also a non-feasible example) then $N(\mathcal{F}_0\,,\, \mathbf{a})$ has 744 elements, and a minimizer is $\nu = (7.505\,,\, 7.505\,,\, 7.45\,,\, 6.9167\,,\, 6.9167\,,\, 6.9167)$. In this example, the partitions for λ ($r_0 = 1$) and $\mathbf{a}$ involved in the computation of the optimal μ are $K_1 = \{5.75\,,\, 5.4\,,\, 4.25\}$, $K_2 = \{4.25\}$ and $K_3 = \{3\,,\, 2\}$ and $J_1 = \{2.5\,,\, 1.2\,,\, 1\,,\, 0.65\}$, $J_2 = \{3.2\}$ and $J_3 = \{5.35\,,\, 4.66\}$ respectively. $\triangle$

Remark 6.9. It is worth to say that, despite all possible partitions of the set $\{1, \ldots, m\}$ into k non-empty subsets can be computed using known MATLAB routines, the number of iterations in Step 1 grows rapidly on $m = \min\{d,\, k\}$. Indeed, this number can be computed as

$$\sum_{i=1}^{m}\sum_{j=1}^{i} j\,!\,S(i,j)\,S(k,j),$$

where $m = \min\{k,\, d\}$ and $S(i,j) = \dfrac{1}{j\,!}\displaystyle\sum_{p=0}^{j}(-1)^{j-p}\binom{j}{p}p^i$ is the so-called Stirling number of the second kind, which is the number of ways to partition a set of i objects into j non-empty subsets.

Nevertheless, in the previous examples (and several others considered for this work) it turned out that, besides the fact that Conjecture 6.5 is verified in all examples, the partition of λ and $\mathbf{a}$ in the $\prec$-minimizer consist of sets of **consecutive** elements, both for λ and $\mathbf{a}$. Also, in all examples the partitions are paired in such a way that the partitions with the greater elements of λ corresponds to those of $\mathbf{a}$ with the smaller entries (see the description of Λ_i and J_i in previous examples). Moreover, in all examples considered, the minimizer has the property that the sets of vectors corresponding to the partitions with the greater norms of $\mathbf{a}$ are linearly independent, with the exception of the last partition of $\mathbf{a}$. This structure is consistent with the solution for the classical completion problem with $\mathcal{F}_0 = \emptyset$ (see [2, 13, 29]). Assuming that the partitions of λ and $\mathbf{a}$ corresponding to the optimal spectrum have the properties described above, we can reduce the number of iterations in Step 1 of our algorithm to

$$\sum_{i=1}^{m}\sum_{j=0}^{i}\binom{i}{j} = 2^{m+1} - 2.$$

This allows to develop a faster algorithm (still exponential on m) which tests a smaller set of partitions for λ and $\mathbf{a}$ which reduces considerably the time of computation and data storage. Based on our numerical computations, we conjecture that the previously mentioned properties for the construction of the $\prec$-minimizer always hold. For a detailed formulation of these conjectures (which we omit here) see [33]. $\triangle$

In the following example we compare the algorithm implemented following the scheme in 6.6 and the simplified (and faster) version of this algorithm that assumes some special features of the partitions of λ and $\mathbf{a}$ considered in Step 1 (as described in Remark 6.9 above). In particular, we verify that they produce the same solution to the optimal completion problem with respect to the Benedetto-Fickus' frame potential.

Example 6.10. Given the initial data

$$\lambda = \lambda(S_{\mathcal{F}_0}) = (7\,,\, 6\,,\, 5.5\,,\, 4\,,\, 2.5\,,\, 1\,,\, 0.5\,,\, 0.3) \quad \text{and} \quad \mathbf{a} = (5\,,\, 4.5\,,\, 1.2\,,\, 1\,,\, 0.8\,,\, 0.5)\,,$$

then applying the algorithm described in 6.6 we obtain that the optimal completion with prescribed norms $\mathcal{F} = (\mathcal{F}_0\,,\, \mathcal{G})$ has eigenvalues $\nu = (7\,,\, 6\,,\, 5.5\,,\, 5.3\,,\, 5\,,\, 4\,,\, 3.5\,,\, 3.5)$. If we assume the conjectures of Remark 6.9, then we obtain the same optimal eigenvalues ν, with the partitions $J_1 = \{1.2\,,\, 1\,,\, 0.8\,,\, 0.5\}$, $J_2 = \{4.5\}$, $J_3 = \{5\}$ and $K_1 = \{2.5\,,\, 1\}$, $K_2 = \{0.5\}$, $K_3 = \{0.3\}$ for $\mathbf{a}$ and λ respectively ($r_0 = 5$). But there are only 5 cases constructed from this kind of partitions in a set $N(\mathcal{F}_0\,,\, \mathbf{a})$ with 322 elements. $\triangle$

7 Appendix I: Geometry of irreducible d_P-local minimizers

In what follows we consider a geometrical approach to the study of d_P-local minimizers on $\mathcal{C}_{\mathbf{a}}^{\rm op}(\mathcal{F}_0)$. Our results are based on a perturbation result for finite sequences of vectors from [30]. In what follows we consider the unitary group of a complex and finite dimensional inner product space $\mathcal{R}$, denoted $\mathcal{U}(\mathcal{R})$, together with its natural differential geometric (Lie) structure. Denote also with $L(\mathcal{H})$ (resp. $L(\mathcal{H})_{sa}$) the set of linear (resp. selfadjoint operators) acting on the d-dimensional Hilbert space $\mathcal{H}$.

Let $(\mathcal{F}_0\,,\,\mathbf{a})$ be initial data for the CP as in 2.3 . Fix $\mathcal{F}=(\mathcal{F}_0\,,\,\mathcal{G})=\{f_i\}_{i=1}^n\in\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$, where $n=k+n_{\rm o}$, $\mathcal{R}=R(S_{\mathcal{G}})=\text{span}\{\mathcal{G}\}\subseteq\mathcal{H}$, and $\tau=\text{tr}\;\mathbf{a}=\sum_{i=1}^k\alpha_i>0$. Consider the real vector space

$$L_d(\mathcal{R})_\tau^{sa}\;\stackrel{\rm def}{=}\;\{S\in L(\mathcal{H})_{sa}:\;R(S)\subseteq\mathcal{R}\,,\;\text{tr}\;S=\tau\}\;, \tag{23}$$

the cone of positive operators in $L_d(\mathcal{R})_\tau^{sa}$ is denoted as $L_d(\mathcal{R})_\tau^{+}$, and the affine manifold

$$S_{\mathcal{F}_0}+L_d(\mathcal{R})_\tau^{sa}=\{S_{\mathcal{F}_0}+S:\;S\in L_d(\mathcal{R})_\tau^{sa}\}\subseteq L(\mathcal{H})_{sa}\;.$$

We define the smooth (and d_P-continuous) map

$$\Phi_{\mathcal{F}}:\mathcal{U}(\mathcal{R})^k\to\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\subseteq\mathcal{H}^n\quad\text{ given by }\quad\Phi_{\mathcal{F}}(U_i)_{i=1}^k=\{f_i\}_{i=1}^{n_{\rm o}}\cup\{U_if_{i+n_{\rm o}}\}_{i=1}^k\;. \tag{24}$$

Finally, we consider the smooth map $\Psi_{\mathcal{F}}:\mathcal{U}(\mathcal{R})^k\to S_{\mathcal{F}_0}+L_d(\mathcal{R})_\tau^{sa}$ given by

$$\Psi_{\mathcal{F}}(U_i)_{i=1}^k=S_{\mathcal{F}_0}+\sum_{i=n_{\rm o}+1}^{n}U_if_i\otimes U_if_i=S_{\mathcal{F}'}\quad\text{ where }\quad\mathcal{F}'=\Phi_{\mathcal{F}}(U_i)_{i=1}^k\;. \tag{25}$$

Let us denote by $I^k=(I,\ldots,I)\in\mathcal{U}(\mathcal{R})^k$. It turns out that in several cases (indeed, in a generic case) the map $\Psi_{\mathcal{F}}$ is an open map (in $S_{\mathcal{F}_0}+L_d(\mathcal{R})_\tau^{sa}$) around $\Psi_{\mathcal{F}}(I^k)=S_{\mathcal{F}}$. In order to characterize this situation we consider the notion of irreducible sequence of vectors from Definition 5.1; recall that given a sequence $\mathcal{G}=\{g_i\}_{i=1}^k$ in $\mathcal{H}$ we say that $\mathcal{G}$ is irreducible if it can not be partitioned into two mutually orthogonal subsequences.

Remark 7.1. In [30] we have characterized when the map $\Psi_{\mathcal{F}}$ defined in Eq. (25) is a submersion in terms of certain commutant. Recall that $L_d(\mathcal{R})=\{T\in L(\mathcal{H}):T=P_{\mathcal{R}}\,T\,P_{\mathcal{R}}\}$, which is a (non unital) $*$-subalgebra of $L(\mathcal{H})$.

Then, an immediate application of [30, Theorem 4.2.1.] shows that $\Psi_{\mathcal{F}}$ is a submersion at $I^k\in\mathcal{U}(\mathcal{R})^k$ if and only if the local commutant

$$\mathcal{M}(\mathcal{G})\;\stackrel{\rm def}{=}\;\{f_i\otimes f_i:\;n_{\rm o}+1\leq i\leq n\}'\cap L_d(\mathcal{R})=\mathbb{C}\cdot P_{\mathcal{R}}\;. \tag{26}$$

It is easy to see that the orthogonal projections of $\mathcal{M}(\mathcal{G})$ can be identified with mutually orthogonal subsequences of $\mathcal{G}$. Then $\mathcal{M}(\mathcal{G})=\mathbb{C}\cdot P_{\mathcal{R}}\iff\mathcal{G}$ is irreducible. Thus, we have proved the following statement: $\triangle$

Proposition 7.2. Let $(\mathcal{F}_0\,,\,\mathbf{a})$ be initial data for the CP as in 2.3 . Fix $\mathcal{F}=(\mathcal{F}_0\,,\,\mathcal{G})\in\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$. Denote by $n=k+n_{\rm o}$ and $\mathcal{R}=R(S_{\mathcal{G}})=\text{span}\{\mathcal{G}\}\subseteq\mathcal{H}$. Then the following statements are equivalent:

1. The map $\Psi_{\mathcal{F}}$ of Eq. (25) is a submersion at $I^k\in\mathcal{U}(\mathcal{R})^k$.
2. The sequence $\mathcal{G}$ is irreducible.

In this case, the image of $\Psi_{\mathcal{F}}$ contains an open neighborhood of $\Psi_{\mathcal{F}}(I^k)=S_{\mathcal{F}}$ in $S_{\mathcal{F}_0}+L_d(\mathcal{R})_\tau^{sa}$. Hence, $\Psi_{\mathcal{F}}$ admits a smooth local cross section ψ around $S_{\mathcal{F}}$ such that $\psi(S_{\mathcal{F}})=I^k$. $\square$

Next we state a reformulation of Proposition 7.2, in terms of the distance d_P. This technical fact is necessary in order to prove Theorem 6.1 (through Lemma 7.5 below).

Corollary 7.3. *Consider the smooth map*

$$\mathcal{S} : \left\{ \mathcal{F}' = (\mathcal{F}_0\,,\,\mathcal{G}') \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) \,:\, R(S_{\mathcal{G}'}) \subseteq \mathcal{R} \right\} \to S_{\mathcal{F}_0} + L_d(\mathcal{R})^{sa}_\tau \tag{27}$$

given by $\mathcal{S}(\mathcal{F}') = S_{\mathcal{F}'} = S_{\mathcal{F}_0} + S_{\mathcal{G}'}$. *Then*

1. *The image of* $\mathcal{S}$ *contains an open neighborhood of* $S_{\mathcal{F}}$ *in* $S_{\mathcal{F}_0} + L_d(\mathcal{R})^{sa}_\tau$.
2. *The map* $\mathcal{S}$ *has a* d_P*-continuous local cross section* φ *around* $S_{\mathcal{F}}$ *such that* $\varphi(S_{\mathcal{F}}) = \mathcal{F}$.

Proof. Just define the d_P-continuous local cross section $\varphi = \Phi_{\mathcal{F}} \circ \psi$, where ψ is the smooth local cross section for $\Psi_{\mathcal{F}}$ of Proposition 7.2 and $\Phi_{\mathcal{F}}$ is the map of Eq. (24), which takes values on the domain of $\mathcal{S}$. □

Remark 7.4. Let $(\mathcal{F}_0\,,\,\mathbf{a})$ be initial data for the CP as in 2.3 and let $t = \operatorname{tr}\mathbf{a}$. Denote by $S_0 = S_{\mathcal{F}_0}$ and $\lambda = \lambda(S_0)$. Consider the set

$$U_t(S_0, k) = \{S_0 + B :\ B \in L(\mathcal{H})) \text{ positive }\ ,\ \operatorname{rk} B \leq k\ ,\ \operatorname{tr}(S_0 + B)\ =\ t\ \}\ ,$$

As a consequence of [32, Theorem 3.12], there exist $\prec$-minimizers in $U_t(S_0, k)$. Indeed, there exists a vector $\nu = \nu_{\lambda\,,k}(t) \in (\mathbb{R}^d_{\geq 0})^{\downarrow}$ such that $S \in U_t(S_0, k)$ is a $\prec$-minimizer if and only if $\lambda(S) = \nu$. In this case, there exist $c > 0$ and $\{v_i :\ 1 \leq i \leq d\}$, an ONB such that $S_0\, v_i = \lambda_i v_i$, $\forall i$ such that

1. $S - S_0 = \sum_{i=1}^d \rho_i \cdot v_i \otimes v_i$, where $\rho = \rho(\lambda\,,\,m) = \lambda(S - S_0)^{\uparrow}$;
2. $\nu = (\lambda + \rho^{\uparrow})^{\downarrow}$ and $\lambda_i(S_0) + \rho_i = c$ whenever $\rho_i \neq 0$.

As a consequence of these facts we get $Sf = c\, f$ for every $f \in R(S - S_0)$. Moreover, if $S' \in U_t(S_0, k)$ is another matrix such that $\lambda(S' - S_0)^{\uparrow} = \rho$ and $S' - S_0 = \sum_{i=1}^d \rho_i\, w_i \otimes w_i$, where $\{w_i :\ 1 \leq i \leq d\}$ is some ONB such that $S_0\, w_i = \lambda_i\, w_i$, then $\lambda(S') = \nu$ and S' is a $\prec$-minimizer in $U_t(S_0, k)$.

Assume now that $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ is such that S_0 and $S_{\mathcal{G}}$ commute. Denote by

$$\mathcal{R} = R(S_{\mathcal{G}})\ \ ,\ \ \mu = \lambda^{\uparrow}(S_{\mathcal{G}})\ \ ,\ \ k' = \operatorname{rk} S_{\mathcal{G}}\ \ ,\ \ m' = d - k' = \max\{1 \leq i \leq d : \mu_i = 0\}$$

and $\tau = \operatorname{tr}\mathbf{a}$. Note that $\mathcal{R}$ reduces $S_{\mathcal{F}_0}$. Write $S_{\mathcal{R}} = S_{\mathcal{F}_0}|_{\mathcal{R}} \in L(\mathcal{R})^+$. We get the identity

$$S_{\mathcal{F}_0} + L_d(\mathcal{R})^+_\tau = S_{\mathcal{F}_0}|_{\mathcal{R}^\perp} \oplus \left(S_{\mathcal{R}} + L(\mathcal{R})^+_\tau\right)\ , \tag{28}$$

where $L_d(\mathcal{R})^+_\tau$ is the space defined in Eq. (23). Then,

$$S_{\mathcal{R}} + L(\mathcal{R})^+_\tau = U_s(S_{\mathcal{R}}\,,\,k') \subseteq L(\mathcal{R})\ ,$$

where $s = \tau + \operatorname{tr} S_{\mathcal{R}}$. By the previous comments there exists $S_\tau \in S_{\mathcal{R}} + L(\mathcal{R})^+_\tau$ such that $\lambda(S_\tau) = \nu_{\lambda(S_{\mathcal{R}})\,,k'}(s) \in \mathbb{R}^{k'}_{\geq 0}$, which is a $\prec$-minimizer in $U_s(S_{\mathcal{R}}\,,\,k') = S_{\mathcal{R}} + L(\mathcal{R})^+_\tau$. As a consequence of Eq. (28) and Remark 4.7, we conclude that

$$S_1 \stackrel{\text{def}}{=} S_{\mathcal{F}_0}|_{\mathcal{R}^\perp} \oplus S_\tau \in S_{\mathcal{F}_0} + L_d(\mathcal{R})^+_\tau \text{ is a } \prec\text{-minimizer in } S_{\mathcal{F}_0} + L_d(\mathcal{R})^+_\tau\ .$$

Notice that $\lambda(S_1) = \left(\lambda(S_{\mathcal{F}_0}|_{\mathcal{R}^\perp})\,,\,\lambda(S_\tau)\right)^{\downarrow} \in \mathbb{R}^d$. Moreover, by items 1 and 2 above, we see that in this case there exists an ONB (for $\mathcal{R}$) $\{w_i\}_{i=1}^{k'}$ with $S_{\mathcal{R}}\, w_i = \lambda_i(S_{\mathcal{R}})\, w_i$ such that

$$S_\tau - S_{\mathcal{R}} = \sum_{i=1}^{k'} \rho_i\, w_i \otimes w_i\ ,\quad \text{where} \quad \rho = \lambda(S_\tau - S_{\mathcal{R}})^{\uparrow} \in \mathbb{R}^{k'}\ , \tag{29}$$

and there exists $c \in \mathbb{R}_{>0}$ such that $\lambda_i(S_{\mathcal{R}}) + \rho_i = c$ whenever $\rho_i \neq 0$. Hence, in this case we obtain that

$$S_1 f = c\ f \quad \text{ for every } \quad f \in R(S_\tau - S_{\mathcal{R}}) \subseteq \mathcal{R}\ . \tag{30}$$

△

Lemma 7.5. *Fix a subspace $\mathcal{R} \subseteq \mathbb{C}^d$ which reduces $S_{\mathcal{F}_0}$. Let $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ be a d_P-local minimizer of P_f on the set*

$$\left\{\, \mathcal{F}' = (\mathcal{F}_0\,,\,\mathcal{G}') \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) \,:\, R(S_{\mathcal{G}'}) \subseteq \mathcal{R} \,\right\}\,.$$

Assume further that $S_0 = S_{\mathcal{F}_0}$ and $S_{\mathcal{G}}$ commute and that the sequence $\mathcal{G}$ is irreducible. Then

1. *The frame operator $S_{\mathcal{F}}$ is a $\prec$-minimizer in $S_{\mathcal{F}_0} + L_d(\mathcal{R})^+_\tau$.*
2. *The subspace $\mathcal{R}$ is contained in an eigenspace of $S_{\mathcal{F}}$.*

In particular, there exists $c \in \mathbb{R}_{>0}$ such that $S_{\mathcal{F}}\, f_i = c\, f_i$, for $n_\circ + 1 \leq i \leq n$.

Proof. Let $k' = \operatorname{rk} S_{\mathcal{G}}$, since by hypothesis S_0 and $S_{\mathcal{G}}$ commute, there exists an orthonormal basis $\mathcal{H}$ of eigenvectors of $S_{\mathcal{F}}$ and $S_{\mathcal{G}}$, denoted $\{v_i\}_{i=1}^d$, such that, if $S_{\mathcal{R}} \stackrel{\text{def}}{=} S_0|_{\mathcal{R}} \in L(\mathcal{R})^+_\tau$ then

$$S_{\mathcal{F}} = \sum_{i=1}^{d} \alpha_i \cdot v_i \otimes v_i \quad,\quad S_{\mathcal{R}} = \sum_{i=1}^{k'} \lambda_i(S_{\mathcal{R}}) \cdot v_i \otimes v_i \quad \text{and} \quad S_{\mathcal{G}} = \sum_{i=1}^{k'} \beta_i \cdot v_i \otimes v_i\,,$$

for some $(\alpha_i)_{i=1}^d \in \mathbb{R}^d_{\geq 0}$ and $(\beta_i)_{i=1}^{k'} \in \mathbb{R}^{k'}_{\geq 0}$. Let $\mathfrak{s} : [0,1] \to S_{\mathcal{F}_0} + L_d(\mathcal{R})^+_\tau$ given by

$$\mathfrak{s}(x) = S_{\mathcal{F}_0} + \sum_{i=1}^{k'} [x \cdot \beta_i + (1-x)\cdot \rho_i] \cdot v_i \otimes v_i \quad \text{for} \quad x \in [0,1]\,,$$

where the ρ_i are those of Eq. (29), so that $\mathfrak{s}(0) = S_1 = S_{\mathcal{F}_0}|_{\mathcal{R}^\perp} \oplus S_\tau$ is a $\prec$-minimizer in $S_{\mathcal{F}_0} + L_d(\mathcal{R})^+_\tau$ as in Remark 7.4. Notice that $\mathfrak{s}(x)$ is a segment (so, in particular, a continuous curve) joining $\mathfrak{s}(0) = S_1 = S_{\mathcal{F}_0}|_{\mathcal{R}^\perp} \oplus S_\tau$ and $\mathfrak{s}(1) = S_{\mathcal{F}}$. Consider now the map $h : [0,1] \to \mathbb{R}$ given by

$$\begin{aligned} h(x) \;&= \operatorname{tr} f\big(\mathfrak{s}(x)\big) = \textstyle\sum_{i=1}^d f(\lambda_i(\mathfrak{s}(x))) \\ &= \textstyle\sum_{i=k'+1}^d f(\alpha_i) + \sum_{i=1}^{k'} f\big(\lambda_i(S_{\mathcal{R}}) + x\cdot\beta_i + (1-x)\cdot\rho_i\big) \end{aligned}$$

for every $x \in [0,1]$. Since the sequence $\mathcal{G}$ is irreducible then Corollary 7.3, implies that the map $\mathcal{S} : \left\{\, \mathcal{F}' = (\mathcal{F}_0\,,\,\mathcal{G}') \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) \,:\, R(S_{\mathcal{G}'}) \subseteq \mathcal{R} \,\right\} \to S_0 + L_d(\mathcal{R})^+_\tau$ defined in Eq. (27) has a d_P-continuous local cross section φ around $S_{\mathcal{F}}$ such that $\varphi(S_{\mathcal{F}}) = \mathcal{F}$. Then, the fact that $\mathcal{F}$ is a d_P-local minimizer of P_f implies that h has a local minimizer at $1 \in [0,1]$. But this h is a strictly convex function on $[0,1]$ that has a global minimum at $x = 0$, since $\mathfrak{s}(0)$ is a $\prec$-minimizer in $S_{\mathcal{F}_0} + L_d(\mathcal{R})^+_\tau$.

This implies that h is constant on $[0,1]$ and hence the segment $\lambda(\mathfrak{s}(x))$, $x \in [0,1]$, reduces to a point (since $h(0)$ is the global minimum of a strictly convex map on a convex compact set of vectors). Thus $\beta_i = \rho_i$ for every $1 \leq i \leq k'$. Hence $S_{\mathcal{G}} = S_\tau - S_{\mathcal{R}}$ and $S_{\mathcal{F}} = S_{\mathcal{F}_0}|_{\mathcal{R}^\perp} \oplus S_\tau = S_1$. By Eq. (30) of Remark 7.4, there exists a $c \in \mathbb{R}_{\geq 0}$ such that $S_{\mathcal{F}}\, f_i = S_\tau\, f_i = c\, f_i$ for $n_\circ + 1 \leq i \leq n$ (since $f_i \in \mathcal{R} = R(S_{\mathcal{G}}) = R(S_\tau - S_{\mathcal{R}})$ for these indices). This last fact proves item 2 of the statement. □

8 Appendix II: Equality in Lidskii's inequality

The purpose of this section is to further the study on Lidskii's inequality. Since we shall deal with Hermitian (resp. positive definite and semidefinite) matrices, we fix first the notation used to indicate these sets of matrices. Denote by $\mathcal{M}_d(\mathbb{C})$ the set of $d \times d$ complex matrices. In particular, the results of this section will apply to linear operators on $\mathcal{H}$ by fixing a canonical orthonormal basis in $\mathcal{H}$, which allows a identification $L(\mathcal{H}) \sim \mathcal{M}_d(\mathbb{C})$. By $\mathcal{M}_d(\mathbb{C})_{sa}$ we denote the $\mathbb{R}$-subspace of selfadjoint matrices and $\mathcal{M}_d(\mathbb{C})^+$ is the set of positive semidefinite matrices.

In this section we characterize those matrices

$$S_1 \in \mathcal{M}_d(\mathbb{C})^+ \quad \text{such that} \quad \lambda(S_0+S_1) = \Big(\lambda^{\downarrow}(S_0)+\lambda^{\uparrow}(S_1)\Big)^{\downarrow} . \tag{31}$$

If $S_1 \in \mathcal{M}_d(\mathbb{C})^+$ satisfies Eq. (31) then we say that S_1 is a **Lidskii matching matrix** for S_0. Note that Lidskii matching matrices correspond to the cases of equality in Lidskii's inequality, as stated in Theorem 4.1.

Although we have defined this notion for positive matrices (since we are interested in its application to frame operators) similar definitions and conclusions holds for general hermitian matrices (by translations by convenient multiples of the identity).

8.1 Lidskii matching matrices commute

In this section we study the case of equality in Lidskii's inequality and show that if S_1 is a Lidskii matching for S_0 (i.e. S_1 is as in Eq. (31)) then $S_0\, S_1 = S_1\, S_0$.

We begin by revisiting some classical matrix analysis results. We shall give short proofs of them in order to handle these proofs for the equality cases in which we are interested here.

Lemma 8.1 (Weyl's inequalities)**.** *Let A, B be $d \times d$ Hermitian matrices. Then,*

$$\lambda_j(A+B) \le \lambda_i(A) + \lambda_{j-i+1}(B) \quad \textit{for} \quad i \le j\,, \tag{32}$$

$$\lambda_j(A+B) \ge \lambda_i(A) + \lambda_{j-i+d}(B) \quad \textit{for} \quad i \ge j\,. \tag{33}$$

Moreover, if there exists $i \le j$ (resp. $i \ge j$) such that

$$\lambda_j(A+B) = \lambda_i(A) + \lambda_{j-i+1}(B) \tag{34}$$

(resp. $\lambda_j(A+B) = \lambda_i(A) + \lambda_{j-i+d}\,(B)$) then there exists a unit vector x such that

$$(A+B)\,x = \lambda_j(A+B)\,x\,,\ A\,x = \lambda_i(A)\,x\,,\ B\,x = \lambda_{j-i+1}(B)\,x\ ,$$

(resp. $(A+B)\,x = \lambda_j(A+B)\,x\,,\ A\,x = \lambda_i(A)\,x\,,\ B\,x = \lambda_{j-i+d}\,(B)\,x$).

Proof. We begin by proving (32). Let u_j, v_j and w_j denote the eigenvectors of A, B and $A+B$ respectively, corresponding to their eigenvalues arranged in decreasing order. Let $i \le j$ and consider the three subspaces spanned by the sets $\{w_1,\dots,w_j\}$, $\{u_i,\dots,u_n\}$ and $\{v_{j-i+1},\dots,v_n\}$. Since the dimensions of these subspaces are j, $n-i+1$ and $n-j+i$ respectively, we see that they have a non trivial intersection. If x is a unit vector in the intersection of these subspaces then

$$\lambda_j(A+B) \le \langle\,(A+B)\,x\,,\,x\rangle = \langle A\,x\,,\,x\rangle + \langle B\,x\,,\,x\rangle \le \lambda_i(A) + \lambda_{j-i+1}(B)\ .$$

If we further assume that equality (34) holds for these indices then we deduce that

$$\langle\,(A+B)x\,,\,x\rangle = \lambda_j(A+B) \quad , \quad \langle A\,x\,,\,x\rangle = \lambda_i(A) \quad \text{and} \quad \langle B\,x\,,\,x\rangle = \lambda_{j-i+1}(B)\ .$$

Because x lies in the intersection of the previous subspaces, these last facts imply that $(A+B)\,x = \lambda_j(A+B)\,x$, $A\,x = \lambda_i(A)\,x$ and $\langle B\,x, x\rangle = \lambda_{j-i+1}(B)\,x$. The inequality (33) and the equality (34) for the case $i \ge j$ follow similarly. □

Corollary 8.2 (Weyl's monotonicity principle)**.** *Let A, B be $d\times d$ matrices such that A is Hermitian and B positive. Then*

$$\lambda_j(A+B) \ge \lambda_j(A) \quad \textit{for every} \quad 1 \le j \le d\ . \tag{35}$$

If there exists $J \subseteq \mathbb{I}_d$ such that $\lambda_j(A+B) = \lambda_j(A)$ for every $j \in J$, then there exists an orthonormal system $\{x_j\}_{j\in J}$ such that $A\,x_j = \lambda_j(A)\,x_j$ and $B\,x_j = 0$ for every $j \in J$.

Proof. Inequality (35) follows easily from Lemma 8.1 (with $i = j$). The second part follows by induction on the set $|J|$: Fix $j_0 \in J$. By Eq. (33) with $i = j = j_0$, there exists a unit vector x_{j_0} such that $A\,x_{j_0} = \lambda_{j_0}(A)\,x_{j_0}$ and $B\,x_{j_0} = \lambda_d(B)\,x_{j_0} = 0$.

This proves the case $|J| = 1$. If $|J| > 1$, consider the space $W = \{x_{j_0}\}^\perp \subseteq \mathbb{C}^d$ which reduces A, B and $A+B$. Let $I = \{j :\ j \in J\,,\ j < j_0\} \cup \{j-1 :\ j \in J\,,\ j > j_0\}$. The operators $A|_W \in L(W)^{\rm sa}$ and $B|_W \in L(W)^+$ satisfy that $\lambda_j(A|_W + B|_W) = \lambda_j(A|_W)$ for every $j \in I$, with $|I| = |J| - 1$. By the inductive hypothesis we can find an orthonormal system $\{x_j\}_{j\in I} \subseteq W$ which satisfies the desired properties. □

Proposition 8.3. *Let A, B be $d \times d$ Hermitian matrices. Then the equality*

$$\big(\,\lambda(A+B) - \lambda(A)\,\big)^\downarrow = \lambda(B) \implies \quad A \text{ and } B \text{ commute}\,.$$

Proof. We can assume that B is not a multiple of the identity. By hypothesis, there exists permutation $\sigma \in \mathbb{S}_d$ such that $\lambda_j(B) = \lambda_{\sigma(j)}(A+B) - \lambda_{\sigma(j)}(A)$ for every $1 \le j \le d$. Therefore, there exists an increasing sequence $\{J_k\}_{k=1}^d$ of subsets of $\{1,\dots,d\}$ such that $|J_k| = k$ and

$$\sum_{j\in J_k} \lambda_j(A+B) - \lambda_j(A) = \sum_{j=1}^k \lambda_j(B) \quad \text{ for every } \quad 1 \le k \le d\,. \tag{36}$$

Let $1 \le k \le d$ be such that $\lambda_{k-1}(B) > \lambda_k(B)$ (recall that $B \neq \alpha\, I$ for $\alpha \in \mathbb{R}$). Let us denote by $B_k = B - \lambda_k(B)\, I$ and notice Eq. (36) also holds if we replace B by B_k.

By construction $\lambda_k(B_k) = 0$ and the orthogonal projection onto the kernel of the positive part $B_k^+ \in \mathcal{M}_d(\mathbb{C})^+$ coincides with the spectral projection of the B associated to the interval $(-\infty, \lambda_k(B)]$. Moreover, $\dim\ker B_k^+ = d - k + 1$.

Since $B_k^+ \in \mathcal{M}_d(\mathbb{C})^+$ and $B_k \le B_k^+$ then Weyl's monotonicity principle implies that

$$\lambda_j(A+B_k) \le \lambda_j(A+B_k^+)\,,\ 1 \le j \le d \implies \sum_{j\in J_{k-1}} \lambda_j(A+B_k) \le \sum_{j\in J_{k-1}} \lambda_j(A+B_k^+)\,.$$

Therefore

$$\begin{aligned}
\sum_{j\in J_{k-1}} \lambda_j(A+B_k) - \lambda_j(A) \ &\le \ \sum_{j\in J_{k-1}} \lambda_j(A+B_k^+) - \lambda_j(A) \\
&\le \ \sum_{j=1}^d \lambda_j(A+B_k^+) - \lambda_j(A) \\
&= \ \operatorname{tr}(A+B_k^+) - \operatorname{tr} A = \sum_{j=1}^{k-1} \lambda_j(B_k)
\end{aligned}$$

since $\lambda_j(A+B_k^+) \ge \lambda_j(A)$ for $1 \le j \le d$ - again by Weyl's monotonicity principle - and since, by hypothesis, $\lambda_k(B_k) = 0$. The inequalities above are the key part of the proof of Lidskii's Theorem 4.1 ($\lambda(A+B) - \lambda(A) \prec \lambda(B)$). But here they are actually equalities, by Eq. (36).

Let $J_{k-1}^c = \{1,\dots,d\} \setminus J_{k-1}$. Then, from the above equalities we get that $\lambda_j(A+B_k^+) = \lambda_j(A)$ for every $j \in J_{k-1}^c$. By Corollary 8.2 there exists an ONS $\{x_j\}_{j\in J_{k-1}^c}$ such that $A\,x_j = \lambda_j(A)\,x_j$ and $B_k^+\,x_j = 0$ for every $j \in J_{k-1}^c$. All these facts together imply that

$$P_k \stackrel{\rm def}{=} \sum_{j\in J_{k-1}^c} x_j \otimes x_j = P_{\ker B_k^+} \quad \text{ and } \quad P_k\,A = A\,P_k\,.$$

Recall that P_k is also the spectral projection of B associated to the interval $(-\infty, \lambda_k(B)]$, for any $1 \leq k \leq d$ such that $\lambda_{k-1}(B) > \lambda_k(B)$. Since the spectral projection of B associated with $(-\infty, \lambda_1(B)]$ equals the identity operator, and B is a linear combination of the projections P_k and I, we conclude that A and B commute. □

Now we are ready to prove that if $S_1 \in \mathcal{M}_d(\mathbb{C})^+$ is as in Eq. (31) then $S_0\, S_1 = S_1\, S_0$.

Theorem 8.4. *Let S_0 , S_1 be Hermitian such that $\lambda(S_0 + S_1) = \big(\lambda(S_0) + \lambda^{\uparrow}(S_1)\big)^{\downarrow}$. Then S_0 and S_1 commute.*

Proof. Take $B = S_0 + S_1$ and $A = -S_1$. Therefore $-\lambda(A) = \lambda^{\uparrow}(-A) = \lambda^{\uparrow}(S_1)$, so that $\lambda(A+B) - \lambda(A) = \lambda(S_0) + \lambda^{\uparrow}(S_1)$. Hence A and B satisfy the assumptions in Proposition 8.3 and they must commute. In this case S_0 and S_1 also commute. □

8.2 Characterization of Lidskii matching matrices

Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and let $S_1 \in \mathcal{M}_d(\mathbb{C})^+$ be a Lidskii matching matrix for S_0. Then, Theorem 8.4 implies that $S_0\, S_1 = S_1\, S_0$ and hence there exists a common ONB of eigenvectors for S_0 and S_1. In order to completely describe S_0 and S_1 we first consider some technical results.

We begin by fixing some notations. Let $\lambda \in \mathbb{R}^d_{>0}$. For every $1 \leq j \leq d$ we define the set

$$L(\lambda\,,\, j) = \{1 \leq i \leq d : \lambda_i = \lambda_j\}\ .$$

If we assume that $\lambda = \lambda^{\downarrow}$ or $\lambda = \lambda^{\uparrow}$ then the sets $L(\lambda\,,\, j)$ are formed by consecutive integers. In the first case we have that $\lambda_i < \lambda_j \implies k > l$ for every $k \in L(\lambda\,,\, i)$ and $l \in L(\lambda\,,\, j)$.

Given a permutation $\sigma \in \mathbb{S}_d$ and $\lambda \in \mathbb{R}^d_{>0}$ we denote by $\lambda_\sigma = (\lambda_{\sigma(1)}\,,\, \ldots\,,\, \lambda_{\sigma(d)})$. Observe that

$$\lambda = \lambda_\sigma \iff \lambda = \lambda_{\sigma^{-1}} \iff \sigma\big(L(\lambda\,,\, j)\big) = L(\lambda\,,\, j) \quad \text{for every} \quad 1 \leq j \leq d\ . \tag{37}$$

The following inequality is well known (see for example [3, II.5.15]):

Proposition 8.5 (Rearrangement inequality for products of sums)**.** Let $\lambda\,,\, \mu \in \mathbb{R}^d_{>0}$ be such that $\lambda = \lambda^{\downarrow}$ and $\mu = \mu^{\uparrow}$. Then $\prod_{i=1}^d (\lambda_i + \mu_i) \geq \prod_{i=1}^d (\lambda_i + \mu_{\sigma(i)})$ for every permutation $\sigma \in \mathbb{S}_d$.

The following result deals with the case of equality in the last inequality.

Proposition 8.6. *Let $\lambda,\, \mu \in \mathbb{R}^d_{>0}$ be such that $\lambda = \lambda^{\downarrow}$ and $\mu = \mu^{\uparrow}$. Let $\sigma \in \mathbb{S}_d$ be such that*

$$(\lambda + \mu)^{\downarrow} = (\lambda + \mu_\sigma)^{\downarrow}\ .$$

Moreover, assume that σ also satisfies that:

$$\text{if}\ \ 1 \leq r\,,\, s \leq d\ \ \text{are such that}\ \ \mu_{\sigma(r)} = \mu_{\sigma(s)}\ \ \text{with}\ \ \sigma(r) < \sigma(s)\ \ \text{then}\ \ r < s\ . \tag{38}$$

Then the permutation σ satisfies that $\lambda = \lambda_\sigma$.

Proof. For every $\tau \in \mathbb{S}_d$ let $F(\tau) = \prod_{i=1}^d (\lambda_i + \mu_{\tau(i)})$. By the hypothesis and Proposition 8.5,

$$F(\sigma) = F(\mathrm{id}) = \max_{\tau \in \mathbb{S}_d} F(\tau)\ .$$

Assume that $\lambda \neq \lambda_{\sigma^{-1}}$. In this case there exists $1 \leq j\,,\, k \leq d$ such that

$$\mu_j < \mu_k \quad \text{and} \quad \lambda_{\sigma^{-1}(j)} < \lambda_{\sigma^{-1}(k)}\ . \tag{39}$$

Indeed, let j_0 be the smallest index such that σ^{-1} does not restrict to a permutation on $L(\lambda\,,\, j_0)$. Then, there exists $j \in L(\lambda\,,\, j_0)$ such that $\sigma^{-1}(j) \notin L(\lambda\,,\, j_0)$. As $\sigma^{-1}(L(\lambda\,,\, j_0) \setminus \{j\}) \neq L(\lambda\,,\, j_0)$ there also exists $k \notin L(\lambda\,,\, j_0)$ such that $\sigma^{-1}(k) \in L(\lambda\,,\, j_0)$. They have the required properties:

- First note that $\lambda_{\sigma^{-1}(j)} < \lambda_{j_0} = \lambda_{\sigma^{-1}(k)}$ (and then also $\sigma^{-1}(j) > \sigma^{-1}(k)$) because $\sigma^{-1}(j)$ can not be in $L(\lambda\,,\,j_0)$ nor in $L(\lambda\,,\,r)$ for any $r < j_0$ (where σ^{-1} acts as a permutation).

- A similar argument shows that $j < k$. We have used in both cases that the sets $L(\lambda\,,\,j)$ are formed by consecutive integers, since the vector λ is decreasingly ordered.

- Observe that $j < k \implies \mu_j \leq \mu_k$. So it suffices to show that $\mu_j \neq \mu_k$. Let us denote by $r = \sigma^{-1}(j)$ and $s = \sigma^{-1}(k)$. The previous items show that $r > s$ and $\sigma(r) < \sigma(s)$. Hence the equality $\mu_j = \mu_{\sigma(r)} = \mu_{\sigma(s)} = \mu_k$ is forbidden by our hypothesis (38).

So Eq. (39) is proved. Consider now the permutation $\tau = \sigma^{-1} \circ (j\,,\,k)$, where $(j\,,\,k)$ stands for the transposition of the indices j and k. Straightforward computations show that

$$(\lambda_{\sigma^{-1}(j)} + \mu_j)\,(\lambda_{\sigma^{-1}(k)} + \mu_k) - (\lambda_{\sigma^{-1}(j)} + \mu_k)\,(\lambda_{\sigma^{-1}(k)} + \mu_j) = (\lambda_{\sigma^{-1}(j)} - \lambda_{\sigma^{-1}(k)})\,(\mu_k - \mu_j) \overset{(39)}{<} 0\ .$$

From the previous inequality we conclude that $F(\mathrm{id}) = F(\sigma) < F(\tau) \leq F(\mathrm{id})$. This contradiction arises from the assumption $\lambda \neq \lambda_{\sigma^{-1}}$. Therefore $\lambda = \lambda_{\sigma^{-1}} \overset{(37)}{=} \lambda_\sigma$ as desired. □

Remark 8.7. Let $\lambda\,,\,\mu \in \mathbb{R}^d_{>0}$ be such that $\lambda = \lambda^\downarrow$ and $\mu = \mu^\uparrow$. Let $\tau \in \mathbb{S}_d$ be such that $(\lambda + \mu)^\downarrow = (\lambda + \mu_\tau)^\downarrow$. Then, by considering convenient permutations of the sets $L(\mu\,,\,j)$ we can always replace τ by σ in such a way that $\mu_\sigma = \mu_\tau$ and such that this σ satisfies the condition (38) of Proposition 8.6. Hence, in this case $(\lambda + \mu)^\downarrow = (\lambda + \mu_\sigma)^\downarrow$ and the previous result applies. △

Theorem 8.8 (Equality in Lidskii's inequality)**.** *Let $S_0\,,\,S_1$ be $d \times d$ positive matrices such that S_1 is a Lidskii matching matrix for S_0. Let $\lambda = \lambda(S_0)$ and $\mu = \lambda^\uparrow(S_1)$. Then there exists an orthonormal basis $\{v_i\}_{i=1}^d$ such that*

$$S_1 = \sum_{i=1}^d \mu_i \cdot v_i \otimes v_i \qquad \textit{and} \qquad S_0 + S_1 = \sum_{i=1}^d (\lambda_i + \mu_i)\, v_i \otimes v_i\ . \tag{40}$$

Proof. Let us assume further that $S_0\,,\ S_1$ are invertible matrices so that $\lambda\,,\,\mu \in \mathbb{R}^d_{>0}$. By Theorem 8.4 we see that S_0 and S_1 commute. Then, there exists an orthonormal basis $\mathcal{B} = \{w_i\}_{i=1}^d$ such that $S_0\, w_i = \lambda_i\, w_i$ and $S_1\, w_i = \mu_{\tau(i)}\, w_i$ for every $1 \leq i \leq d$, and for some permutation $\tau \in \mathbb{S}_d$. Therefore

$$\big(\,\lambda + \mu\,\big)^\downarrow \overset{(31)}{=} \lambda(S_0 + S_1) = \big(\,\lambda + \mu_\tau\,\big)^\downarrow\ .$$

By Remark 8.7 we can replace τ by $\sigma \in \mathbb{S}_d$ in such a way that $\mu_\tau = \mu_\sigma$, $(\lambda+\mu)^\downarrow = (\lambda+\mu_\sigma)^\downarrow$ and σ satisfies the hypothesis (38). Hence, by Proposition 8.6, we deduce that $\lambda_{\sigma^{-1}} = \lambda$. Therefore one easily checks that the ONB formed by the vectors $v_i = w_{\sigma^{-1}(i)}$ for $1 \leq i \leq d$ (i.e. the rearrangement $\mathcal{B}_{\sigma^{-1}}$ of $\mathcal{B}$) is still a ONB for S_0 and λ, but it now satisfies Eq. (40).

In case S_0 or S_1 are not invertible, we can argue as above with the matrices $\tilde{S}_0 = S_0 + I$ and $\tilde{S}_1 = S_1 + I$. These matrices are invertible and such that $\tilde{S}_1$ is a Lidskii matching for $\tilde{S}_0$. Further, $\lambda_i(\tilde{S}_0) = \lambda_i(S_0)+1$ and $\lambda_i(\tilde{S}_1) = \lambda_i(S_1)+1$, $\forall 1 \leq i \leq d$. Hence, if $\{v_i\}_{i=1}^d$ has the desired properties for $\tilde{S}_0$ and $\tilde{S}_1$ then this ONB also has the desired properties for S_0 and S_1. □

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