Brane-world cosmology and varying G

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October 30, 2018

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We consider a brane-world cosmological model coupled to a bulk scalar field. Since the brane tension turns out to be proportional to Newton coupling G, in such a model a time variation of G naturally occurs. By resorting to available bounds on the variation of G, the parameters of the model are constrained. The constraints coming from nucleosynthesis and CMB result to be the severest ones.

PACS numbers: 04.50.Kd, 11.25.-w, 04.80.Cc.

1 Introduction

The first ideas of a varying Newton's coupling G come from 1937, when P. A. M. Dirac introduced his famous Large Numbers Hypothesis [1, 2]. In the 60s, in an attempt to reconciliate Mach Principle with General Relativity (GR), Brans and Dicke developed their well known scalar-tensor theory of gravity [3]. Following Jordan's ideas, Brans and Dicke generalized GR including a varying G, whose dynamics was governed by a scalar field. See [4] for a detailed review on varying fundamental constants.

In addition to the effects of introducing a dynamical coupling constant, we know that gravitational interaction is also sensitive to the existence of extra dimensions, which could manifest themselves at short distances. In this paper, we will be concerned with models that incorporate both a varying G and a higher dimensional set up.

Although the idea of extra dimensions is not new either [5, 6], the advent of modern (string) theories has brought to the fore the higher dimensional scenarios more recently. One of the string inspired models that have attracted much attention in the last decade is the Randall-Sundrum model (RS), which consists of an effective brane-world embedded in an orbifold of AdS_5 space [7]. The property of the RS-like models that is interesting for our purpose is the relation between the tension of the brane, and the Newton constant of the 4-dimensional effective theory. Models with non-constant brane tension thus lead to a time-varying G, as we will discuss below.

The idea of this paper is to confront particular brane-world models with constraints coming form observational cosmological data. The particular model we will consider here is that of [8], which is motivated by supergravity in singular spaces. We will consider this model as a working example to show how observational data could be used to constrain parameters of this type of scenarios.

The observational data we will use to constrain the model are of rather different types. For instance, we have data coming from planetary/geological scale: Observations due to space missions to Mercury, Mars, Venus and the Moon in the 70s, determined that if G varies in time, its variation is less than 10^{-11} per year. On the other hand, in the late 70s, many works appeared relating the relative variation of G with planetary radius [9, 10]. McElhinny *et al.* studied the evolution of the Earth's radius and extended their study to the Moon, Mars and Mercury, and constrained $\Delta G/G$ in specific moments close to Solar System formation.

At cosmological scale, a variation of G leads to modifications in the Friedmann equation. The direct consequences of these variations are changes in the cooling rate of the universe and in the computed primordial abundances of He and Li. In [11], Accetta *et al.* used this relation between varying G and light elements abundances to give a bound to the relative variation of G. This variation (its absolute value, in fact) turns to be less than 40% since Big-Bang nucleosynthesis (BBN). CMB anisotropies are also sensitive to a varying G. Chan *et al.* concluded in [12] that the relative variation of G since recombination is less than 10%.

In this work, we explore a five-dimensional gravitational model, *alla* RS, with a scalar field in the bulk that modificates the brane tension, which induces a variation in G. The variations of G predicted by the model, depending on two parameters, will be then compared with the observational data mentioned above. That is, the aim is to constrain the possible values of these parameters, using experimental bounds.

This work is organized as follows: In Section 2 we discuss the Randall-Sundrum-like model coupled to a scalar field, which induces variation of effective Newton constant in four dimensions. In Section 3 we survey bounds on the variations of G and the observations. In Section 4, we combine observational data of Section 3 with the predictions of the model, and use this to constrain the parameters. In particular, we give bounds on the 5D-Planck mass, supersymmetry breaking scale, and the cosmological constant in the bulk. In Section 5, we draw some conclusions.

2 A brane-world scenario

2.1 Field equations

The RS-like scenarios propose that our universe is a 3-brane embedded in a curved asymptotically hyperbolic 5-dimensional bulk, or an orbifold of it. One can also include matter in the brane as well as in the bulk [8]; here we consider a scalar field ϕ. The brane is located at the origin of the fifth dimension, which we denote x_5. This dimension has Z_2 symmetry in our case.

Fields of the Standard Model live on the brane, while gravitational interaction (and the scalar field) is free to propagate in the bulk. Bulk action is given by

$$S_{\text{bulk}} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g_{(5)}}\Big(R - \frac{3}{4}\Big((\partial\phi)^2 + U(\phi)\Big)\Big), \qquad (1)$$

where R is the curvature scalar associated to the 5-dimensional metric $g^{(5)}_{AB}$; $U(\phi)$ is the bulk potential, which depends on the scalar field ϕ, and $\kappa_5^2 = 1/M_5^3$, being M_5 the Planck mass in 5D.

The action of the brane depends on its tension $U_B(\phi)$ (brane potential). In our case, it is a function of the scalar ϕ and of the confined matter; namely

$$S_{\text{brane}} = \int d^4x\sqrt{-g_{(4)}}\Big(- \frac{3}{2\kappa_5^2}U_{\text{B}}(\phi(x_5=0)) + L_{\text{matter}}\Big), \qquad (2)$$

with $g^{\mu\nu}_{(4)} = \delta^\mu_M\delta^\nu_N g^{MN}_{(5)}|_{x_5=0}$. In this paper, Latin indices in capital letters go from 0 to 5 (excluding 4), Greek indices go from 0 to 3, and Latin indices in regular letters, from 1 to 3.

The matter content of the 5D space is characterized by the energy-momentum tensor, which can be derived from the total action and has the bulk and brane contributions; namely

$$T_{AB} = T^{\text{bulk}}_{AB} + T^{\text{brane}}_{AB}, \qquad (3)$$

with

$$T^{\text{bulk}\ A}{}_{B} = \frac{3}{4}\left(\partial^A\phi\partial_B\phi - \frac{1}{2}g^{(5)\ A}{}_{B}(\partial\phi)^2\right) - \frac{3}{8}g^{(5)\ A}{}_{B}U(\phi), \qquad (4)$$

$$T^{\text{brane}\ A}{}_{B} = \left(-\frac{3}{2} g^{(5)\ A}{}_{B} U_{\text{B}}(\phi) + \tau^{\text{matter}\ A}{}_{B}\right)\delta(x_5), \tag{5}$$

and

$$\tau^{\text{matter}\ A}{}_{B} = diag(-\rho_{\text{m}}, p_{\text{m}}, p_{\text{m}}, p_{\text{m}}, 0). \tag{6}$$

Tensor τ^{matter} is related to the ordinary matter on the brane.

The energy density ρ_{m} and the pressure p_{m} are independent of the position in the brane, so one recovers an homogeneous cosmology in four dimensions. The equation of state that relates these quantities is taken to be $p_{\text{m}} = \omega_{\text{m}}\rho_{\text{m}}$.

Einstein equations reads

$$G_{AB} \equiv R_{AB} - \frac{1}{2} R\ g^{(5)}_{AB} = \kappa_5^2 \left(T^{\text{bulk}}_{AB} + T^{\text{brane}}_{AB}\right) = \kappa_5^2 T_{AB}. \tag{7}$$

Let us propose the following ansatz for the metric:

$$ds^2 = -A^2(t, x_5)dt^2 + B^2(t, x_5)dx_i dx^i + C^2(t, x_5)dx_5^2. \tag{8}$$

We are interested in cosmological scale solutions, so we assume an isotropic and homogeneous metric in the three spatial coordinates. That is

$$ds^2 = a^2(t, x_5)b^2(x_5)(-dt^2 + dx_5^2) + a^2(t, x_5)\Omega_{ij} dx^i dx^j, \tag{9}$$

where Ω_{ij} is the metric of a 3-dimensional space with constant curvature:

$$\Omega_{ij} = \delta_{ij}\left(1 + \frac{K}{4} x^l x^m \delta_{lm}\right)^{-2}, \tag{10}$$

where the values $K = 0, 1, -1$ correspond to a (spatially) flat, closed or open universe, respectively. Since observational evidences are consistent with a spatially flat Universe [14], then we assume $K = 0$.

It is important to note that in (9) we chose a conformal gauge for the $(0-5)$ part of the metric. In this gauge, the brane is placed in a fixed position, $x_5 = 0$, *i.e.* the fixed point of the Z_2 symmetry in the fifth dimension. Function b only depends on the spatial coordinate x_5.

To derive the brane dynamics, one must verify that, although the equations of motion must be restricted to it, these equations have to be satisfied in the bulk as well. The brane "proper time" is

$$d\tau = ab|_{x_5=0} dt, \tag{11}$$

and the differential of the normal vector to its surface is given by

$$dy = ab|_{x_5=0} dx_5. \tag{12}$$

From now on, we write $\dot{f} = \frac{df}{d\tau}$, $f' = \frac{df}{dy}$.

The Israel-Darmôise junction conditions describe how a brane with a given energy-momentum tensor can be embedded in a higher dimensional space-time. These equations yield

$$\frac{a'}{a}|_{y=0} = -\frac{1}{6}\kappa_5^2\rho, \tag{13}$$

$$\frac{b'}{b}|_{y=0} = \frac{1}{2}\kappa_5^2(\rho + p), \tag{14}$$

where equations ρ is the energy density, and p is the pressure on the brane.

On the other hand, the boundary condition for the scalar ϕ is [15]

$$\phi'|_{y=0} = \frac{\partial U_{\mathrm{B}}}{\partial \phi}|_{y=0}. \tag{15}$$

The total energy density and pressure on the brane can be written as a sum of two contributions: a term related to confined matter, and a second one, related to the tension, which in this case depends on the scalar field. Thus, we have

$$\rho = \rho_{\mathrm{m}} + \frac{3}{2\kappa_5^2}U_{\mathrm{B}},\ p = p_{\mathrm{m}} - \frac{3}{2\kappa_5^2}U_{\mathrm{B}}. \tag{16}$$

In what follows, all quantities will be evaluated on the brane, *i.e.* at $y = 0$. Restricting the $(0-5)$ component of the Einstein equations to the brane, and using boundary conditions (13) and (14) we obtain the energy conservation equation

$$\dot{\rho} = -3H(\rho + p) - 2T^0{}_5, \tag{17}$$

where $H \equiv \frac{\dot{a}}{a}|_{y=0}$ is the Hubble parameter on the brane.

Using the explicit form for ρ and p from (16), total energy density conservation law transforms into a conservation law for the energy of the brane; that is,

$$\dot{\rho_{\mathrm{m}}} = -3H(\rho_{\mathrm{m}} + p_{\mathrm{m}}). \tag{18}$$

Time variation of the scalar field energy density $3U_{\mathrm{B}}/2$ cancels the term involving $T^0{}_5$, since the latter can be written as $T^0{}_5 = -\frac{3}{4}\phi'\dot{\phi} = -\frac{3}{4}\dot{U}_{\mathrm{B}}$.

The solution for the brane energy conservation is

$$\rho_{\mathrm{m}} = \rho_0 a^{-3(1+\omega_{\mathrm{m}})}, \tag{19}$$

as in standard cosmology.

On the other hand the Friedmann equation on the brane is a consequence of the $(5-5)$ component of Einstein equations. For a brane containing matter coupled to a scalar field ϕ, Friedmann equation reads

$$H^2 = \frac{\kappa_5^4}{36}\rho_{\rm m}^2 + \frac{\kappa_5^2}{12}U_{\rm B}\rho_{\rm m} - \frac{1}{16a^4}\int d\tau \frac{da^4}{d\tau}(\dot{\phi}^2 - 2V) - \frac{\kappa_5^2}{12a^4}\int d\tau a^4 \rho_{\rm m}\frac{dU_{\rm B}}{d\tau} + \frac{A}{a^4}, \tag{20}$$

with

$$V = \frac{1}{2}\Big(U_{\rm B}^2 - (\frac{\partial U_{\rm B}}{\partial \phi})^2 + U\Big). \tag{21}$$

The set of equations is completed by the Klein-Gordon equation for the scalar field [16, 17]; namely

$$\ddot{\phi} + 4H\dot{\phi} + \frac{1}{2}\big(\frac{1}{3} - \omega_{\rm m}\big)\rho_{\rm m}\frac{\partial U_{\rm B}}{\partial \phi}\kappa_5^2 = -\frac{\partial V}{\partial \phi} + \Delta\Phi, \tag{22}$$

where

$$\Delta\Phi = \frac{\partial^2 \phi}{\partial y^2}|_{y=0} - \frac{\partial U_{\rm B}}{\partial \phi}|_{y=0}\frac{\partial^2 U_{\rm B}}{\partial \phi^2}|_{y=0}. \tag{23}$$

Following [16] and [17], we consider

$$\Delta\Phi = 0. \tag{24}$$

Einstein equations have been used to write (22) in this form (see [16]).

2.2 Physical considerations

Friedmann equation (20) is not conventional. In contrast to the standard one, (20) presents terms that depend on the field ϕ, a quadratic term in the energy density on the brane (present also in absence of the scalar), and an additional term that goes like a^{-4}.

By the time of primordial nucleosynthesis, corrections coming from brane models, including the term proportional to the square of the energy density in Friedmann equation, must be negligible. Otherwise, the rate of expansion would be modified and the computation of light elements abundances would be inconsistent with observations. In this non-conventional scenario, the freezing temperature of proton to neutron ratio T_C would be of the order of $(2-3)$ MeV, while in standard cosmology it is $T_C \sim (0.7-0.8)$ MeV, consistent with He abundance. The difference between both temperatures is a direct consequence of the fact that Hubble parameter is linear with T^4, and not with T^2, generating a slower cooling of the Universe [18]. However, corrections might be important during the inflationary period.

Let us be reminded of the fact that in standard cosmology Friedmann equation is

$$H_{\rm stand}^2 = \frac{8\pi G}{3}\rho_{\rm m} + \frac{\Lambda_4}{3}, \tag{25}$$

where Λ_4 is the cosmological constant in four dimensions. Then, the quadratic term in $\rho_{\rm m}$ in (20) can be identified with the first term in (25); that is

$$\frac{U_{\rm B}(\phi)}{12}\kappa_5^2 = \frac{8\pi G}{3}. \tag{26}$$

It is clear that in our model, Newton constant in 4D varies as it depends on ϕ; *i.e.* it is possible to find time variation of G.

2.2.1 Bulk and brane potentials

We consider a functional form for the potential $U(\phi)$ coming from the supergravity models in singular spaces studied in [15]. Following these results, one finds

$$U = \left(\frac{\partial W}{\partial \phi}\right)^2 - W^2, \tag{27}$$

where $W(\phi)$ is the so called superpotential.

We study the case in which the superpotential in an exponential function of the field

$$W(\phi) = 4ke^{\alpha\phi}, \tag{28}$$

where $[k^{-1}] = L$ and α is a real number.

The brane potential is defined through the superpotential by

$$U_{\rm B} = TW, \tag{29}$$

where T is a real number related to the scale of supersymmetry breaking [19].

Having the functional relation between U_B and the scalar field, given by (28) and (29), one can find $G(\phi)$ using (26); thus,

$$G(\phi) = \frac{k}{8\pi}\kappa_5^2 T e^{\alpha\phi}. \tag{30}$$

The expectation value of ϕ today is assumed to be zero by convention as a boundary condition. Then, Newton "constant" would be given by

$$G_{\rm today}(\phi) = \frac{k}{8\pi}\kappa_5^2 T. \tag{31}$$

2.2.2 Working hypothesis

The form of the Friedmann equation with all the new contributions is quite abstruse. Then, in order to solve the model, some approximations and assumptions have to be taken into account. We discuss these below.

First, the term proportional to a^{-4}, can be considered as a correction to the radiation density. This term is usually referred as dark radiation. Here, we assume $A = 0$ in (20). We also consider a low energy regime, *i.e.* we neglect the term proportional to ρ_{m}^2 in (20). It is possible to do this under the condition $\rho_{\mathrm{m}} \ll \rho_{crit}$, with

$$\rho_{\mathrm{crit}} = \frac{3U_{\mathrm{B}}}{\kappa_5^2} = \frac{12}{\kappa_5^2} kT \sim 4.6 \times 10^{33} \frac{g}{cm^3}, \tag{32}$$

where (28), (29), and bounds on k, κ_5^2 and T consistently found *a posteriori* in Sections 3 and 4, have been used. In the studied period (between BBN and today) the density remains below this critical density. We also assume that the time evolution of the scalar field ϕ in the brane proper time τ is much slower than the one of the scale factor a. It is possible to extract $\dot{\phi}^2$ from the first integral in (20) in this adiabatic regime.

A non-dissipative approximation of the potential will be also considered. The brane potential U_{B} is basically Newton constant on the brane, up to multiplicative constants. Following the adiabatic approximation, it is reasonable to suppose that the contribution correspondent to this term might be negligible. The term $dU_{\mathrm{B}}/d\tau$, as well as the other terms of order $\dot{\phi}$ and $\dot{a}$, contribute to higher order estimation of $G(\phi)$ than the one we study here.

Finally, and consequently with the assumptions above, the square of the time derivative of ϕ, *i.e.* the kinetic energy of the field, is lower than other terms in Friedmann equation. It is possible to make a simply calculation to constrain the current value of the time derivative of the scalar field: Following the approximations, Friedmann equation today, divided by H_0^2 is

$$1 = \Omega_{\mathrm{M}} + \Omega_{\mathrm{R}} + \Omega_{\Lambda} + \Omega_{\partial\phi}, \tag{33}$$

where $\Omega_{\partial\phi} = \dot{\phi}^2(\tau_0)/16$.

Recent data [14] implies that the sum of the first three contributions is close to 0.996, which fixes a limit to the absolute current value of the time derivative of ϕ; namely

$$|\frac{\dot{\phi}}{H_0}(\tau_0)| < 0.22. \tag{34}$$

This heuristic argument supports our approximation hypothesis.

In addition, the exponential dependences appearing in Friedmann and Klein-Gordon equations will be approximated to 1 since ϕ is small.

With all the approximations described above, Friedmann equation takes the form

$$H^2 = T\frac{k}{3}\kappa_5^2\rho_{\rm m} + \frac{\Lambda_{\rm eff}}{3}, \tag{35}$$

with

$$\frac{\Lambda_{\rm eff}}{3} := k^2(T^2-1)(1-\alpha^2). \tag{36}$$

On the other hand, eq.(19) says that, for radiation,

$$\rho_{\rm R} = \rho_{\rm 0R}H_0^4 a^{-4}, \tag{37}$$

and for non-relativistic matter,

$$\rho_{\rm M} = \rho_{\rm 0M}H_0^4 a^{-3}, \tag{38}$$

with $\rho_{\rm 0M,R}$ dimensionless constants.

Thus, Friedmann equation can be rewritten as follows:

$$H^2 = T\frac{k}{3}\kappa_5^2\left(\frac{\rho_{\rm 0M}}{a^3} + \frac{\rho_{\rm 0R}}{a^4}\right)H_0^4 + \frac{\Lambda_{\rm eff}}{3}. \tag{39}$$

After making the identifications

$$T\frac{k}{3}\kappa_5^2\rho_{\rm 0M,R}H_0^2 = \Omega_{\rm M,R}, \tag{40}$$

$$\frac{\Lambda_{\rm eff}}{3H_0^2} = \Omega_\Lambda, \tag{41}$$

we obtain the familiar form for the equation

$$\frac{H^2}{H_0^2} = \frac{\Omega_{\rm M}}{a^3} + \frac{\Omega_{\rm R}}{a^4} + \Omega_\Lambda. \tag{42}$$

Then, the equation for ϕ is

$$\ddot{\phi} + 4H\dot{\phi} + 2\big(\frac{1}{3} - \omega_{\rm m}\big)\rho_{\rm m}\kappa_5^2\alpha kT = -16\alpha\frac{\Lambda_{\rm eff}}{3}. \tag{43}$$

The system of equations above can now be solved in two different epochs: one dominated by radiation and matter, and the other governed by matter and cosmological constant.

2.3 Solution to the Field equations

2.3.1 Radiation and matter dominated epoch

In this epoch the term proportional to $\Lambda_{\rm eff}$ can be neglected. Then, the Fridmann equation takes the form[1]

$$\frac{H_1^2}{H_0^2} = \frac{\Omega_{\rm M}}{a_1^3} + \frac{\Omega_{\rm R}}{a_1^4}, \tag{44}$$

whose solution is

$$a_1(\eta) = \frac{\Omega_{\rm M}}{4}(\eta - \eta_0)^2 - \frac{\Omega_{\rm R}}{\Omega_{\rm M}}, \tag{45}$$

where the integration has been performed in the conformal time η, being $d\tau = \frac{a_1(\eta)}{H_0} d\eta$.

The equation of state for radiation is $p_{\rm rad} = \rho_{\rm rad}/3$, while $p_{\rm mat} = 0$ is the equation for non-relativistic dust. Then, KG equation reads:

$$\ddot{\phi}_1 + 4H_1\dot{\phi}_1 + 2\alpha H_0^2 \frac{\Omega_{\rm M}}{a_1^3} = -16\alpha H_0^2 \Omega_\Lambda, \tag{46}$$

where we used eqs. (40) y (41). The solution for the differential equation is (see Appendix)

$$\phi_1(a_1) = B - \frac{A}{2\sqrt{\Omega_{\rm R}}a_1^2} - \frac{2\Omega_{\rm M}\alpha}{3\Omega_{\rm R}} a_1, \tag{47}$$

where it has been taken into account that scale factor remains small during this regime.

2.3.2 Matter and cosmological constant dominated epoch

During this regime, the term proportional to the inverse of the fourth power of the scale factor a is negligible when compared to the other two terms. Then, the Friedmann equation reads[2]

$$\frac{H_2^2}{H_0^2} = \frac{\Omega_{\rm M}}{a_2^3} + \Omega_\Lambda. \tag{48}$$

Integrating, one obtains the scale factor during this epoch, namely

$$a_2(\tau) = a_0 \sinh^{\frac{2}{3}}\left(\frac{\sqrt{\Omega_\Lambda}}{2} H_0(\tau - C_0)\right). \tag{49}$$

[1]Subindex 1 refers to the radiation-matter regime.
[2]Subindex 2 refers to the matter-Λ regime.

In this regime, the Klein-Gordon equation is

$$\ddot{\phi} + 4H_2\dot{\phi} + 2\alpha H_0^2 \frac{\Omega_{\mathrm{M}}}{a_2^3} = -16\alpha H_0^2 \Omega_\Lambda. \tag{50}$$

In order to solve this equation it is convenient to consider two cases: $a_2 \ll 1$, that occurs for times close to union time ($\tau \sim \tau_{\mathrm{u}}$); and $a_2 \sim 1$, it is, times near today ($\tau \sim \tau_0$):

- $a_2 \ll 1$:

$$\phi_2(a_2) = D - \frac{2C}{5\sqrt{\Omega_{\mathrm{M}}} a_2^{\frac{5}{2}}} - \frac{4}{5}\alpha \ln(a_2). \tag{51}$$

- $a_2 \sim 1$:

$$\phi_3(a_2) = G - \frac{F}{3\sqrt{\Omega_\Lambda + \Omega_{\mathrm{M}}} a_2^3} - \frac{\alpha}{5(\Omega_\Lambda + \Omega_{\mathrm{M}})}(8\Omega_\Lambda + \Omega_{\mathrm{M}}) a_2^2, \tag{52}$$

where G and F are integration constants. Details of this calculation can be found in the Appendix.

2.3.3 Boundary conditions for the scale factor

To find the integration constants in (45) and (49), one must define the boundary conditions. We take the convention

$$a_2(\tau_0) = 1, \tag{53}$$

where τ_0 is the current value of τ. We also have the matching conditions

$$H_2(\tau_0) = H_0, \tag{54}$$

and

$$H_1(\tau_{\mathrm{u}}) = H_2(\tau_{\mathrm{u}}), \tag{55}$$

where τ_{u} is the junction time between both regimes. In addition, we have

$$a_1(\tau_{\mathrm{u}}) = a_2(\tau_{\mathrm{u}}) = a_{\mathrm{u}}. \tag{56}$$

Constants a_0 and C_0 can be obtained from (53) and (54). Consequently, we have

$$a_2(\tau) = \left(\frac{1}{\Omega_\Lambda} - 1\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}} \left(\frac{3\sqrt{\Omega_\Lambda}}{2} H_0 \left(\tau - \tau_0 + \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda}{1-\Omega_\Lambda}}\right)\right)\right). \tag{57}$$

The constant of integration must be chosen to satisfy the condition $a_1(\tau = 0) = 0$. Then

$$\tau(a_1) = \frac{2}{3H_0\Omega_{\mathrm{M}}^2}(\Omega_{\mathrm{M}}a_1 - 2\Omega_{\mathrm{R}})\sqrt{\Omega_{\mathrm{M}}a_1 + \Omega_{\mathrm{R}}} + \frac{4\Omega_{\mathrm{R}}^{\frac{3}{2}}}{3H_0\Omega_{\mathrm{M}}^2}. \tag{58}$$

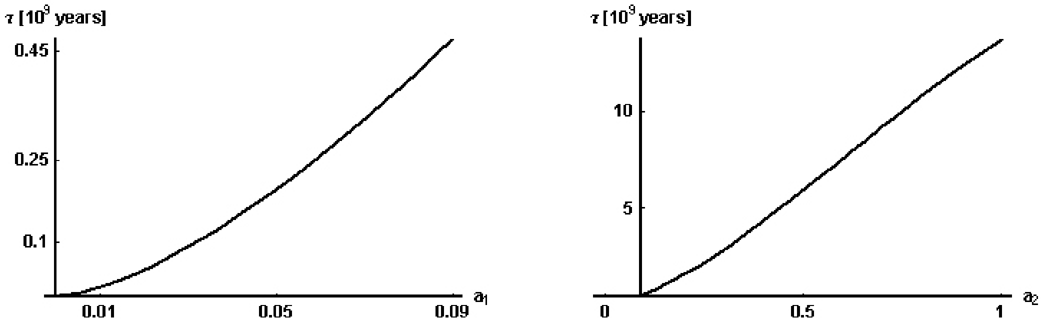

Figure 1: Graphics for τ vs a_1 and τ vs a_2

2.3.4 Boundary conditions for the scalar field

Since the scalar field must be smooth, ϕ_1 (and its derivative) must be equal to ϕ_2 (resp. to its derivative) in the junction time τ_{u}, when $a = a_{\mathrm{u}} = \Omega_{\mathrm{R}}/\Omega_{\Lambda}$. That is,

$$\phi_1(a_{\mathrm{u}}) = \phi_2(a_{\mathrm{u}}), \tag{59}$$

$$\frac{d\phi_1}{da_1}(a_{\mathrm{u}}) = \frac{d\phi_2}{da_2}(a_{\mathrm{u}}). \tag{60}$$

The field ϕ_2 must be equal to ϕ_3 (as well as their derivatives) in an intermediate time τ_{I} between τ_{u} and τ_0. We take τ_{I} as the time equidistant to τ_{u} and τ_0. Thus,

$$\phi_2(a_{\mathrm{I}}) = \phi_3(a_{\mathrm{I}}), \tag{61}$$

$$\frac{d\phi_2}{da_2}(a_{\mathrm{I}}) = \frac{d\phi_3}{da_2}(a_{\mathrm{I}}). \tag{62}$$

The last boundary conditions we need are the values of the field and its derivative today, *i.e.* at τ_0 ($a(\tau_0) = a_0 = 1$):

$$\phi_3(a_0) = 0, \tag{63}$$

$$\frac{d\phi_3}{da_2}(a_0) = \frac{\dot{\phi}_0}{H_0} := p_0. \tag{64}$$

Using (59)-(64) and the values for H_0, $\Omega_{\rm M}$, $\Omega_{\rm R}$ and Ω_Λ listed in Subsection 3 of the Appendix, one obtains the constants of integration A, B, C, D, F and G. Table 1 shows the values of the constants as linear combinations of α and p_0.

Constant	α	p_0
A	0.09469	0.0309
B	682.718	57.114
C	1.745	0.719
D	2.212	0.752
F	2.408	0.998
G	2.0104	0.333

Table 1: Values of the integration constants of the fields which are linear combinations of the parameters α and p_0.

The final expression for the fields can be found in the Appendix. In Figure 2, the behavior of the field for different allowed values[3] of α and p_0 are observed. Figure 2(A) corresponds to $\alpha = 1/\sqrt{20078}$ and $p_0 = -0.02164$, which is the value for p_0 when α has the pointed value. Figure 2(B) shows the graphic for the field, for a null value of α, and $p_0 = 0.22$, which is the upper limit for p_0 coming from Friedmann equation. Finally, Figure 2(C) shows the unique case in which the field does not diverge at the origin, due to the fact that the value of α is such that the linear term in a^{-2} in ϕ_1 (see eq.(102) in the Appendix) is zero for all p_0.

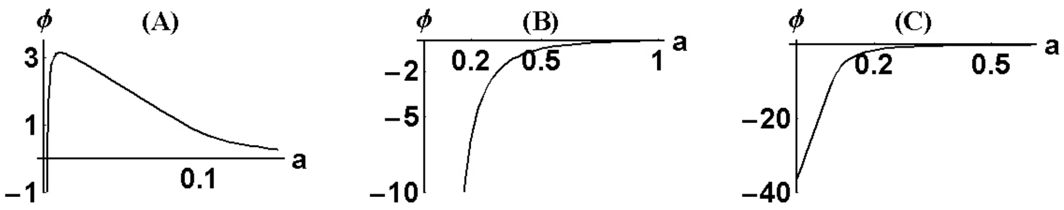

Figure 2: Different behavior of ϕ vs a. (A) $\alpha = 1/\sqrt{20078}$ and $p_0 = -0.02164$; (B) $\alpha = 0$ y $p_0 = 0.22$; (C) $\alpha = -0.326062 p_0$ and $p_0 = 0.22$. In this case the field has no divergence at the origin, and its value there is -36.4087.

[3]Allowed values for α y p_0 are detailed in Section 4.

3 Experimental and observational data on $G(\phi)$

3.1 Corrections to Newtonian potential

Theoretical speculations predict new effects at distances of order less than 1mm. In particular, models with spatial non-compact extra dimensions are of interest because these "internal" dimensions could alter the form of the Newtonian potential.

If the extra dimension is non-compact, as in the case of RS model, there is a continuous of Kaluza-Klein (KK) modes for the gravitational field. The continuous spectra of KK modes leads to a correction to the force between two static masses in the brane. The potential for two point-like masses confined to the brane reads [7]

$$V_{\mathrm{RS}}(r) = G\frac{m_1 m_2}{r}\Big(1 + \frac{1}{r^2 k^2}\Big), \tag{65}$$

where k^{-1} should be of order of the distance of available gravitational tests ($\sim$1mm) or smaller.

Thus, in order to bound the deviation from the Newtonian potential, one must constrain parameter k. Adelberger *et al.* [20] performed experiments with torsion balances to model the correction to Newtonian potential using a power law of the form

$$\Delta V_{12}^{j} = -G\frac{m_1 m_2}{r}\beta_j\Big(\frac{1\mathrm{mm}}{r}\Big)^{j-1}, \tag{66}$$

where the values of j and $|\beta_j|$ are shown in Table 2.

k	$\|\beta_j\|(<)$
2	4.5×10^{-4}
3	1.3×10^{-4}
4	4.9×10^{-5}
5	1.5×10^{-5}

Table 2: Bounds on $|\beta_j|$ for $j = 2, 3, 4, 5$ obtained by Adelberger *et al.* [20].

For $j = 3$ the extra term is

$$\Delta V_{12}^{3} = G\frac{m_1 m_2}{r}|\beta_3|\Big(\frac{1\mathrm{mm}}{r}\Big)^{2}, \tag{67}$$

while the deviation predicted by RS model is

$$\Delta V_{12}^{\mathrm{RS}} = G\frac{m_1 m_2}{r}\Big(\frac{1}{r^2 k^2}\Big). \tag{68}$$

Thus, the value of k is constrained comparing ΔV_{12}^{3} with $\Delta V_{12}^{\mathrm{RS}}$. Then, the characteristic scale at which the effects due to the presence of an extra dimension become important is

$$\frac{1}{k} < 0.01\mathrm{mm}. \tag{69}$$

3.2 Observational bounds on G variation

Bounds on the variation of the Newton constant are obtained from local and cosmological observations. Local observations are related to the solar system, as well as nearby stars. Geological and paleontological data, as well as planetary orbits, stellar densities, and luminosities, are of great importance when studying G, because they are affected by its variation.

3.2.1 Bounds on $\Delta G/G$

Planetary radius variation: In 1961, Egyed proposed that paleomagnetic data could be used for the calculation of Earth paleoradius (past to current planetary radius ratio) in different geological eras. Starting from the hypothesis that the continental material area remained constant during planetary expansion, Egyed found that the ratio between current and past angular separation (paleolatitud) of two given sites is proportional to the paleoradius [9]. A few years later, in 1978, McElhinny *el al.* related Earth radius variation to time evolution of gravitational constant, and extended the analysis to the Moon, Mars and Mercury [10]. According to their work,

$$\frac{\Delta R}{R} = -\gamma\frac{\Delta G}{G}, \tag{70}$$

where ΔR is the variation of the radius R and γ is a constant that depends on the planet structure.

On the other hand, there is another way to write this relation using the paleoradius R_{a}:

$$\frac{\Delta G}{G} = \frac{R_{\mathrm{a}} - 1}{\gamma}. \tag{71}$$

Table 3 summarizes the results for the Earth, the Moon, Mars[4] and Mercury. These results assumes that the surface of each studied planet acquired its current shape by the time indicated in the fourth column.

[4]There are two different analysis for Mars: (A) assumes a 19km expansion during the last 3600 million years; (B) supposes a 1km variation on martian radius in the past 1000 million years. See [10] for details.

Planet	$R_{\mathbf{a}}$	γ	**Time [10^9 years]**	$\lvert\Delta G/G\rvert(<)$
Earth	1.020 ± 0.028	0.085 ± 0.02	0.4	0.62
Moon	1.0000 ± 0.0006	0.0004 ± 0.001	3.9	1.5
Mars (A)	0.9944	0.03 ± 0.01	3.6	0.12
Mars (B)	1.0000 ± 0.0003	0.03 ± 0.01	1.0	0.01
Mercury	1.0000 ± 0.0004	0.02 ± 0.005	3.5	0.02

Table 3: Paleoradius (R_{a}) of the Earth, the Moon, Mars (A and B) and Mercury, and bounds on relative variation of G.

Big-Bang Nucleosynthesis: Bounds of a different sort come from cosmology. In 1990, Accetta *et al.* studied bounds on gravitational constant value during primordial nucleosynthesis, considering neutron mean life measurements [11]. They determined D, ^{3}He and ^{7}Li abundances while varying G, and how this variation affects barion to photon ratio. On the other hand, Copi *et al.* recalculated relative variation of G since BBN, but using only primordial D abundance in quasars [21]. In both works the constraint on relative variation of G is

$$|\frac{\Delta G}{G}|_{\mathrm{BBN}} < 0.4, \tag{72}$$

which means that the relative variation of gravitational constant since BBN is less than 40%, at the 95% confidence level[5].

Cosmic background anisotropies: Power spectra of cosmic microwave background anisotropies (CMBA) can be useful while constraining G variations in cosmological scales. In [12], gravitational constant stabilization (convergence to its current value) and its relation to CMBA are studied in detail. Two possible parametrizations of G are considered: one, corresponding to an instantaneous stabilization, and the other to a stabilization linear with the scale factor a.

If the stabilization is linear, the relative variation of G since recombination ($z \sim 1000$) is

$$|\frac{\Delta G}{G}|_{\mathrm{CMB}} < 0.1. \tag{73}$$

at the 95% confidence level[6]. Thus, the relative variation of G (its absolute

[5]It is important to note that the constraint is for the absolute value of $\frac{\Delta G}{G}|_{\mathrm{BBN}}$.
In [21] the constraints are: $0.85 < \frac{G_{\mathrm{BBN}}}{G_0} < 1.21$, at the 68.3% confidence level, and $0.71 < \frac{G_{\mathrm{BBN}}}{G_0} < 1.43$, at the 95% confidence level. G_0 is the present value of the Newton constant.
In this work we made use of the last constraint.

[6]Again, we aware the reader that the constraint is for the absolute value of $\frac{\Delta G}{G}|_{\mathrm{CMB}}$.

value) since recombination is less than 10%.

3.2.2 Bounds on $\dot{G}/G$

Lunar Laser Ranging (LLR) has been measuring the position of the Moon with respect to the Earth during more than thirty years, with a precision of 1cm. The misions *Appolo* 11, 14 y 15, and russian-french *Lunakhod* 1 y 4 carried retro-reflectors to the Moon, which reflect laser pulses sent from the Earth. LLR data are used to constrain Weak Equivalence Principle, post-newtonian parameters and $\dot{G}/G$.

According to 2004 data in [22], the maximum variation allowed to gravitational constant today is

$$\frac{\dot{G}}{G}(\tau_0) = (4 \pm 9) \times 10^{-13}\mathrm{yr}^{-1}. \tag{74}$$

3.2.3 Bound on $\ddot{G}/G$

Now, let us discuss the constraints coming from observational bounds on $\ddot{G}/G$. For that purpose, we consider a model slightly different to brane cosmology.

Other model which considers scalar fields is the scalar-tensorial theory of Brans and Dicke of 1961. This theory contains a scalar field governing G dynamics. In usual notation we have [3],

$$G(\varphi) = \frac{1}{\varphi}. \tag{75}$$

G time dependence is described by

$$G(\tau) \sim \tau^{-\mathrm{n}}, \tag{76}$$

where $\mathrm{n} = 2/(4+3\omega)$, and ω is a model parameter[7], which measures the deviation from General Relativity (GR). GR results are reobtained when ω goes to infinity. Considering (76), it can be shown that

$$\frac{\ddot{G}}{G}(\tau) = \mathrm{n}(\mathrm{n}+1)\tau^{-2}. \tag{77}$$

In [12] the constraints are: $0.95 < \frac{G_{\mathrm{CMB}}}{G_0} < 1.05$, for a G variation modeled by a step function, and $0.89 < \frac{G_{\mathrm{CMB}}}{G_0} < 1.13$, for a variation modeled by a linear function of the scale factor. Both constraints are at the 95% confidence level.
In this work we made use of the last constraint.

[7]In Brans-Dicke work $\omega = const.$, but there are more complex models where $\omega = \omega(\phi)$.

Benvenuto, Althaus and Torres in [23] gave a bound to the absolute value of ω, coming from white dwarfs evolution and its relation with a varying G. According to their results, the calculated luminosities differ from the observed ones for $|\omega| < 5000$. Then, the allowed values range is $|\omega| > 5000$.

Equation (77) evaluated today (τ_0=13730 million years) together with the constrain on ω, give a bound on the variation of the second time derivative of G:

$$-3.55648 \times 10^{-40}\mathrm{s}^{-2} < \frac{\ddot{G}}{G}(\tau_0) < 7.11082 \times 10^{-40}\mathrm{s}^{-2}. \tag{78}$$

At this point, one could ask about the relation between Brans-Dicke model and the one studied in this work. The explanation is the following: Brans-Dicke theory in the limit $\omega \to \infty$ gives back GR; while GR and brane cosmology should be equivalent in the studied limit. Then, at first order, BD and brane cosmology are equivalent and constraints on ω can be translated into constrains on brane model parameters. However, it is not clear that the differences between BD and brane cosmology lead to a great discrepancy on the limit (78). Then, we study the inclusion and exclusion of this bound in following analysis.

4 Constraining model parameters

4.1 Parameters α and $\dot{\phi}(\tau_0)$

G variations, *i.e.* $\Delta G/G$, $\dot{G}/G$ and $\ddot{G}/G$, can be written in terms of ϕ as follows:

$$\frac{\Delta G}{G}(a) = -\alpha\phi(a), \tag{79}$$

$$\frac{\dot{G}}{G}(a) = \alpha\dot{\phi}(a), \tag{80}$$

$$\frac{\ddot{G}}{G}(a) = \alpha^2\dot{\phi}^2(a) + \alpha\ddot{\phi}(a). \tag{81}$$

Then, observational bounds discussed above restrict the possible values of α and p_0. The strongest restrictions on α and p_0 come from BBN and recombination. The largest value for the modulus of α is fixed by the combination of BBN and CMB restrictions; that is

$$|\alpha| < \frac{1}{\sqrt{20078}} = 0.007, \tag{82}$$

while $|p_0| < 0.22$.

The allowed values are shown in Figure 3. Allowed values are inside the surface which perimeter is given by the green hyperboles.

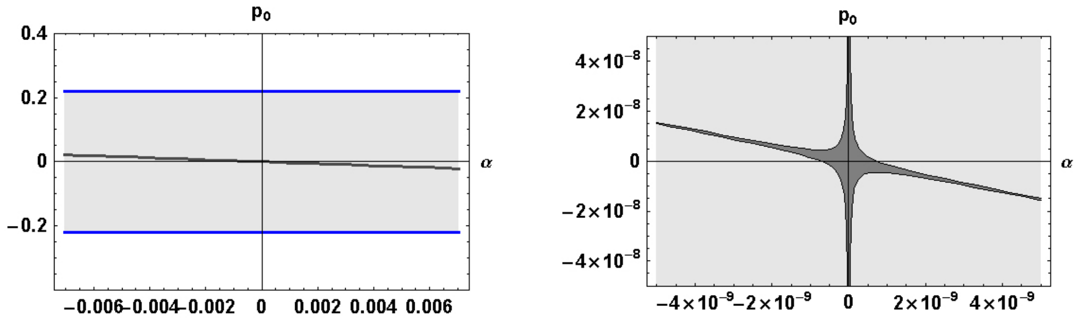

Figure 3: Allowed values for α and p_0. The figure on the right is a zoom-in of the left one. Allowed values are located inside the dark gray contour, and satisfy $|\alpha| < 0.007$ y $|p_0| < 0.22$. Blue lines in the left figure establish the bound for the largest value for $|p_0|$, a condition coming from Friedmann equation today.

In [24], Brax and Davis find two theoretical values for α, emerging from supergravity in singular spaces. This values are $\alpha = 1/\sqrt{3}$ and $\alpha = -1/\sqrt{12}$. From the analysis we have just done, based on observational constraints, the absolute value of α is bounded by (82) and then, it excludes these theoretical values.

4.1.1 Statistical analysis

Now, we would like to investigate how robust our constraints are. That is, we will study how the constraints change if one excludes ones or others data.

Excluding G relative variation since Big-Bang nucleosynthesis data, and using the other data, *i.e.* $\Delta G/G|_{\rm CMB}$, $\Delta G/G|_{\rm paleoradii}$, $\dot{G}/G|_{\rm today}$, $\ddot{G}/G|_{\rm today}$ and $|p_0| < 0.22$, it can be deduced that the strongest constraint on α absolute value is given by the combination of $\Delta G/G|_{\rm CMB}$ and $\ddot{G}/G|_{\rm today}$.

In this case, we have

$$|\alpha| < \frac{1}{\sqrt{1787}} = 0.024. \qquad (83)$$

If we exclude CMB data, in addition to BBN data, and make a similar analysis, we have that the constraint on $|\alpha|$ is fixed by the combination of

$\ddot{G}/G|_{\rm today}$ and $\dot{G}/G|_{\rm today}$:

$$|\alpha| < \frac{1}{\sqrt{237}} = 0.065. \tag{84}$$

Omitting observational data of the second derivative of G, we obtain the same results that those at the beginning of this subsection. This is because the severest constraint on $|\alpha|$ is given by the combination of BBN and CMB data.

If we exclude BBN data, in addition to $\ddot{G}/G|_{\rm today}$, α is constrained by the combination of $\Delta G/G|_{\rm CMB}$ and $\dot{G}/G|_{\rm today}$ cotes, being

$$|\alpha| < \frac{1}{\sqrt{243}} = 0.064. \tag{85}$$

Finally, if we only take into account the bounds coming from today´s value of G derivative, and those related to planetary paleoradius (*i.e.* BBN, CMB and $\ddot{G}/G|_{\rm today}$ limits, excluded), $|\alpha|$ has no bound.

Table 4 condenses the results obtained in data analysis.

$\lvert\alpha\rvert$	**Obtained with:**
$1/\sqrt{3} = 0.577$	SUGRA
$1/\sqrt{12} = 0.289$	SUGRA
$1/\sqrt{237} = 0.065$	$\ddot{G}/G\rvert_{\rm today}$; $\dot{G}/G\rvert_{\rm today}$
$1/\sqrt{243} = 0.064$	CMB; $\dot{G}/G\rvert_{\rm today}$
$1/\sqrt{1787} = 0.024$	CMB; $\ddot{G}/G\rvert_{\rm today}$
$1/\sqrt{20078} = 0.007$	BBN; CMB
no limit	$\ddot{G}/G\rvert_{\rm today}$; Mercury paleoradius

Table 4: Constraints on α obtained combining G variations observational data. The first two values are equalities found with supergravity (SUGRA) in singular spaces. The other values are cotes and should be read "$|\alpha| <$ ...". The severest constraint is given by Big-Bang nucleosynthesis (BBN) and CMB. All the constrains, except the last one, exclude the first two values.

In the studied cases, the upper bound on $|p_0|$ is fixed by Friedmann equation evaluated today, *i.e.* $|p_0| < 0.22$. In addition, it should be said that $|\alpha|$ is always below 0.065, except in the last case, where its value has no bound.

Analog analysis can be done, but assuming a different bound on BD parameter: $|\omega| < 500$ (instead of $|\omega| < 5000$). Table 5 shows the results.

The difference with respect to the results obtained with $|\omega| < 5000$ lies in the constraint of $|\alpha|$ found combining CMB data and $\ddot{G}/G|_{\rm today}$ bound. In this case, the constraint on $|\alpha|$ is less restrictive since it is three times bigger.

$\lvert\alpha\rvert$	**Obtained with**
$1/\sqrt{231} = 0.066$	$\ddot{G}/G\rvert_{\text{today}}$; $\dot{G}/G\rvert_{\text{today}}$
$1/\sqrt{243} = 0.064$	CMB; $\dot{G}/G\rvert_{\text{today}}$
$1/\sqrt{178} = 0.075$	CMB; $\ddot{G}/G\rvert_{\text{today}}$
$1/\sqrt{20078} = 0.007$	BBN; CMB
no limit	$\ddot{G}/G\rvert_{\text{today}}$; Mercury paleoradius

Table 5: Constrains on α when combining observational data of G variation, in the case $|\omega| < 500$.

4.2 5D parameters, T and Λ_{eff}

The 5-dimensional parameters of the model can also be constrained, as well as the 4-D cosmological constant Λ_{eff}. Using H_0 and Ω_Λ from Subsection 3 of the Appendix, the constraint on k from (69), and $|\alpha| < 1/\sqrt{20078}$, we have the results of Table 6.

Parameter	Bound
κ_5^2	$< 2.8 \times 10^{-99} s^3$
$M_5 = 1/\kappa_5^{\frac{2}{3}}$	$> 4.7 \times 10^8 GeV$
Λ_5	$< -4.2 \times 10^{27} s^{-2}$
$\lvert T\rvert$	$< 1 + 3.05 \times 10^{-63}$
Λ_{eff}	$= 4.8 \times 10^{-18} s^{-2}$

Table 6: Constrains on 5D parameters, T and Λ_{eff}.

The same results are obtained when the other constraints on α from Tables 4 and 5 are used. Constraints on κ_5^2 and M_5 are in accord with the ones predicted in [25, 26, 27].

5 Conclusions

In this work, we studied a brane-world cosmological model in which variation of the Newton coupling G emerges naturally. This model is inspired in supergravity in singular spaces [24]. By resorting to available observational data, we manage to constrain the parameters of the theory. Light elements abundances, coming from Big Bang nucleosynthesis (He, Li y D)(eq. (72)), CMB (eq. (73)) and, near in time, planetary radii variations, allowed to constrain the relative variation of G, *i.e.* $\Delta G/G$ (see Table 3). Measurements of Lunar position with respect to the Earth with LLR offered us a bound on today's value of the G time derivative to G ratio (eq. (74)). Combining the severest constraints on G variations, *i.e.* those coming from Big Bang nucleosynthesis and CMB, a bound for the absolute value of the parameter α was obtained. In fact, this parameter must be less than 0.007, and the value of $|p_0|$ is bounded while evaluating Friedmann equation today.

Statistical analysis was performed on the results to analyze how robust the bounds are against the exclusion of particular set of data. That is, we studied how the upper bound of $|\alpha|$ gets affected if one excludes different sets of data. Results are shown in Tables 4 and 5. It is worth mentioning that the upper bound for $|\alpha|$ is always less than $1/\sqrt{178} \sim 0.07$ (except in the case where Big Bang nucleosynthesis and CMB data are excluded, which turn out to be the most important ones).

Our analysis presents a method to investigate the phenomenological viability of models that, among other features, predict time variation of the fundamental couplings. It could be interesting to extend our analysis to other brane-world type scenarios.

Acknowledgments

The authors are grateful to G. Giribet for useful discussions. The work of L.A. was supported by CONICET.

A Field equations for the scalar field

A.1 Matter and radiation regime

During matter and radiation epoch. Klein-Gordon equation reads

$$\ddot{\phi}_1 + 4H_1\dot{\phi}_1 + 2\alpha H_0^2 \frac{\Omega_{\mathrm{M}}}{a_1^3} = -16\alpha H_0^2 \Omega_\Lambda, \tag{86}$$

where we used (40) and (41).

Differentiating with respect to a_1, we have

$$\frac{d\psi_1}{da_1} + \frac{4}{a_1}\psi_1 = \frac{-2\alpha H_0}{\sqrt{\Omega_{\mathrm{M}} a_1 + \Omega_{\mathrm{R}}}} \Big(\frac{\Omega_{\mathrm{M}}}{a_1^2} + 8\Omega_\Lambda a_1 \Big), \tag{87}$$

where we used (44) and $\psi_1 = \dot{\phi}_1$.

Being reminded of the fact that during this regime a_1 is small, the function on the right hand side can be approximated by its first term in power expansion for $a_1 \sim 0$, thus

$$\frac{d\psi_1}{da_1} + \frac{4}{a_1}\psi_1 = -2\alpha H_0 \frac{\Omega_{\rm M}}{\sqrt{\Omega_{\rm R}} a_1^2}. \tag{88}$$

Integrating, we find

$$\psi_1(a_1) = \frac{-2H_0\Omega_{\rm M}\alpha}{3\sqrt{\Omega_{\rm R}} a_1} + \frac{AH_0}{a_1^4}, \tag{89}$$

where A is an integration constant.

To find $\phi_1(a_1)$, it is necessary to integrate once again, taking into account that

$$\frac{d\phi_1}{da_1} = \frac{1}{a_1 H_1}\frac{d\phi_1}{d\tau} = \frac{1}{a_1 H_1}\psi_1(a_1). \tag{90}$$

Then, we have

$$\phi_1(a_1) = B + \int da_1 \psi(a_1) \frac{a_1}{H_0\sqrt{\Omega_{\rm M} a_1 + \Omega_{\rm R}}} \simeq B + \int da_1 \psi(a_1) \frac{a_1}{H_0\sqrt{\Omega_{\rm R}}}, \tag{91}$$

Where B is a constant, and where we only considered the first term in the power expansion of $\frac{a_1}{\sqrt{\Omega_{\rm M} a_1 + \Omega_{\rm R}}}$.

The solution for the differential equation is

$$\phi_1(a_1) = B - \frac{A}{2\sqrt{\Omega_{\rm R}} a_1^2} - \frac{2\Omega_{\rm M}\alpha}{3\Omega_{\rm R}} a_1. \tag{92}$$

A.2 Matter and cosmological constant regime

During this regime the Klein-Gordon equation is

$$\ddot{\phi} + 4H_2\dot{\phi} + 2\alpha H_0^2 \frac{\Omega_{\rm M}}{a_2^3} = -16\alpha H_0^2, \Omega_\Lambda \tag{93}$$

which, written as a function of a_2 derivatives, is

$$\frac{d\psi}{da_2} + \frac{4}{a_2}\psi = \frac{-2\alpha H_0}{a_2^{\frac{5}{2}}\sqrt{\Omega_{\rm M} + \Omega_\Lambda a_2^3}} (\Omega_{\rm M} + 8\Omega_\Lambda a_2^2). \tag{94}$$

- First, consider $a_2 \ll 1$:

$$\frac{1}{a_2^{\frac{5}{2}}\sqrt{\Omega_{\rm M} + \Omega_\Lambda a_2^3}} (\Omega_{\rm M} + 8\Omega_\Lambda a_2^2) \simeq \frac{\sqrt{\Omega_{\rm M}}}{a_2^{\frac{5}{2}}}. \tag{95}$$

Replacing this equation in (94), we have

$$\psi_2(a_2) = -\frac{4H_0\sqrt{\Omega_\mathrm{M}}\alpha}{5a_2^{\frac{3}{2}}} + \frac{CH_0}{a_2^4}, \tag{96}$$

where C is an integration constant.

The solution for the field comes form the integral

$$\phi_2(a_2) = D + \int da_2\psi_2(a_2)\frac{1}{a_2H_2} \simeq D + \int da_2\psi_2(a_2)\frac{1}{H_0}\sqrt{\frac{a_2}{\Omega_\mathrm{M}}}, \tag{97}$$

where D is a constant, and (48) and (96) were used. Also, $1/a_2H_2$ was approximated by its first order in the power expansion, for $a_2 \ll 1$. After integrating, we have

$$\phi_2(a_2) = D - \frac{2C}{5\sqrt{\Omega_\mathrm{M}}a_2^{\frac{5}{2}}} - \frac{4}{5}\alpha\ln(a_2). \tag{98}$$

- Now, consider $a_2 \sim 1$:

$$\frac{1}{a_2^{\frac{5}{2}}\sqrt{\Omega_\mathrm{M} + \Omega_\Lambda a_2^3}}(\Omega_\mathrm{M} + 8\Omega_\Lambda a_2^2) \simeq \frac{8\Omega_\Lambda + \Omega_\mathrm{M}}{\sqrt{\Omega_\Lambda + \Omega_\mathrm{M}}}. \tag{99}$$

As above, and making the approximation

$$\frac{1}{a_2H_2} \simeq \frac{1}{H_0\sqrt{\Omega_\Lambda + \Omega_\mathrm{M}}}, \tag{100}$$

we obtain the approximate value for the field

$$\phi_3(a_2) = G - \frac{F}{3\sqrt{\Omega_\Lambda + \Omega_\mathrm{M}}a_2^3} - \frac{\alpha}{5(\Omega_\Lambda + \Omega_\mathrm{M})}(8\Omega_\Lambda + \Omega_\mathrm{M})a_2^2, \tag{101}$$

where G and F are constants.

A.3 Final form for the field

The values we take for the cosmological parameters appearing in the expression of the fields are $H_0 = 22.69 \times 10^{-19}\mathrm{s}^{-1}$; $\Omega_\mathrm{M} = 0.28$; $\Omega_\Lambda = 0.716$; $\Omega_\mathrm{R} = 4.6 \times 10^{-5}$ (see [14] for further details).

The values for the integration constants can be found with the parameters above and with the boundary conditions discussed in Section 2. The value for the scale factor in the intermediate time a_I is 0.542. Then,

$$\phi_1(a_1) = 682.718\alpha + 57.1142p_0 - \frac{6.9539\alpha + 2.2674p_0}{a_1^2} - 4029.68\alpha a_1, \quad (102)$$

$$\phi_2(a_2) = -(543.028\alpha + 224.698p_0) + (15241.3\alpha + 6283.55p_0)(a_2 - a_\mathrm{u}) - \\ - (297480\alpha + 122591p_0)(a_2 - a_\mathrm{u})^2, \quad (103)$$

$$\phi_3(a_2) = p_0(a_2 - a_0) - (6.0313\alpha + 2p_0)(a_2 - a_0)^2, \quad (104)$$

with $a_\mathrm{u} = 0.0897$ and $a_0 = 1$.

Notice that ϕ_2 series is around $a_\mathrm{u} = 0.0897$, while ϕ_3 series is around $a_0 = 1$.

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