CONTINUOUS COHESION OVER SETS

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ABSTRACT. A pre-cohesive geometric morphism $p : \mathcal{E} \to \mathcal{S}$ satisfies *Continuity* if the canonical $p_!(X^{p^*S}) \to (p_!X)^S$ is an iso for every X in $\mathcal{E}$ and S in $\mathcal{S}$. We show that if $\mathcal{S} = \mathbf{Set}$ and $\mathcal{E}$ is a presheaf topos then, p satisfies Continuity if and only if it is a quality type. Our proof of this characterization rests on a related result showing that Continuity and Sufficient Cohesion are incompatible for presheaf toposes. This incompatibility raises the question whether Continuity and Sufficient Cohesion are ever compatible for Grothendieck toposes. We show that the answer is positive by building some examples.

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1. Outline

In several papers Lawvere has emphasised the fact that "even within geometry (that is, even apart from their algebraic/logical role) toposes come in (at least) two varieties: as spaces (possibly generalized, treated via the category of sheaves of discrete sets), or as categories of spaces" (analytical, algebraic, topological, combinatorial, etc.). See [11] and references therein. Moreover, he proposed loc. cit. a set of axioms which help to distinguish the toposes in the second variety from those in the first. We will work with the version of this set of axioms presented in the more recent *Axiomatic Cohesion* [12].

During the International Category Theory Conference 2011 (Vancouver) Lawvere explained that the 'Continuity' condition that appears in Definition 2(b) in [12] was intended to capture 'continuous' models of Axiomatic Cohesion, in opposition to 'combinatorial'

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ones such as simplicial sets. He later urged for an explicit theory of 'combinatorial' toposes and conjectured a result saying that such combinatorial toposes cannot satisfy Continuity.

In the present first attempt to materialize this idea of 'continuous vs combinatorial' I restrict to a familiar base topos **Set**, propose a reasonable class of 'combinatorial' toposes over **Set** and prove a result of the sort suggested above.

In Sections 2 and 3 we recall the necessary material from [12, 7, 15]. In particular, we recall the definitions of: pre-cohesive geometric morphism, quality type, Sufficient Cohesion, and Continuity.

In Section 4 we discuss presheaf toposes and prove the following dichotomy: a pre-cohesive presheaf topos is either sufficiently cohesive or a quality type. (See Corollary 4.6.)

Section 5 recalls the construction of the colimit functor $\widehat{\mathcal{C}} \to \mathbf{Set}$ emphasizing certain aspects that are needed later in the paper.

Sections 6 and 7 are devoted to the proof of one of our main results. Namely, Theorem 7.4, which states that a pre-cohesive presheaf topos satisfies Continuity if and only if it is a quality type. This is a nice positive characterization but, unfortunately, our proof uses an argument by contradiction. A key ingredient is the fact (Proposition 7.3) that sufficiently cohesive presheaf toposes cannot satisfy Continuity. This naturally raises the question of whether Sufficient Cohesion and Continuity are ever compatible for Grothendieck toposes. The remaining sections of the paper show that the answer is positive.

Section 8 introduces the concept of *connector* which is implicitly used in Section 9 to prove a sufficient condition on a site to induce a cohesive and sufficiently cohesive Grothendieck topos (Proposition 9.6), and explicitly in Section 10 to discuss three concrete examples. Let us quickly mention two of them: the monoids of linear and of polynomial endos of the unit interval may be extended to sites satisfying our sufficient condition. These sites are not subcanonical though, so we also describe concrete subcanonical alternatives. In particular, the monoid of piecewise linear endos on the unit interval may be equipped with a subcanonical coverage such that the induced topos is cohesive and sufficiently cohesive.

2. Categories of cohesion

Let $\mathcal{E}$ and $\mathcal{S}$ be cartesian closed extensive categories.

2.1. Definition. The category $\mathcal{E}$ is called *pre-cohesive (relative to $\mathcal{S}$)* if it is equipped with a string of adjoint functors

$$\begin{tikzcd} \mathcal{E} \arrow[d, shift right=6, "p_!"'] \arrow[d, shift left=2, "p_*"] \\ \mathcal{S} \arrow[u, shift right=2, "p^*"] \arrow[u, shift left=6, "p^!"'] \end{tikzcd}$$

with $p_! \dashv p^* \dashv p_* \dashv p^!$ and such that:

1. $p^* : \mathcal{S} \to \mathcal{E}$ is full and faithful.

2. $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products.

3. (Nullstellensatz) The canonical natural transformation $\theta : p_* \to p_!$ is (pointwise) epi.

For brevity we may also say that $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive. Let us fix one such p. As explained immediately after Definition 2 in [12], the two downward functors p_* and $p_!$ express the opposition between 'points' and 'pieces'. For this reason, one pictures the transformation $\theta : p_* X \to p_! X$ as assigning, to each point in X, the piece where it lies. Also, an object X in $\mathcal{E}$ will be called *connected* if $p_! X = 1$. An object Y in $\mathcal{E}$ will be called *contractible* if Y^A is connected for all A.

2.2. Definition. We say that *Sufficient Cohesion* holds if for every X in $\mathcal{E}$ there exists a monic map $X \to Y$ with Y contractible.

We may also say that $p : \mathcal{E} \to \mathcal{S}$ is *sufficiently cohesive*. 'Sufficient' here meaning sufficient cohesion to provide a contractible envelope for any space or, as we will recall in Proposition 3.1, to provide a connection of any truth value with True.

Definition 1 in [12] introduces a contrasting class of categories of cohesion.

2.3. Definition. The pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ is called a *quality type* if the canonical natural transformation $\theta : p_* \to p_!$ is an iso.

In other words, the pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ is a quality type if the full reflective subcategory $p_! \dashv p^* : \mathcal{S} \to \mathcal{E}$ is a quintessential localization in the sense of [5]. In terms of the intuition suggested above, p is a quality type if, for every X in $\mathcal{E}$, every piece of X has a unique point. Concerning terminology, in [12], Lawvere defines *intensive/extensive qualities* as special functors (between [pre-]cohesive categories) whose codomains have the special feature that θ is an iso. This seems enough to justify transplanting from functorial higher order logic the idea of a 'type' as a possible codomain. So, roughly speaking, quality types are the codomains of qualities. The next result explains in what sense sufficiently cohesive categories and quality types are contrasting.

2.4. Proposition. [p.47 in [12].] *If $p : \mathcal{E} \to \mathcal{S}$ is both sufficiently cohesive and a quality type then $\mathcal{S}$ is inconsistent.*

After Proposition 2.4 it is fair to say that Sufficient Cohesion is a positive way of ensuring that 'points' and 'pieces' are different concepts.

The definition of category of cohesion in [12] requires an extra condition that, for reasons that will become evident, we keep separate. The pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ determines a natural transformation $p_!(X^{(p^* S)}) \to (p_! X)^S$, with X in $\mathcal{E}$ and S in $\mathcal{S}$.

2.5. Definition. We say that (the axiom of) *Continuity* holds if the canonical transformation $p_!(X^{(p^* S)}) \to (p_! X)^S$ is an iso.

Definition 2 in [12] can now be formulated as follows.

2.6. DEFINITION. The category $\mathcal{E}$ is a *category of cohesion* (relative to $\mathcal{S}$) if it is equipped with a pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ that satisfies Continuity.

We may also say that $p : \mathcal{E} \to \mathcal{S}$ is *cohesive*.

Notice that over $\mathcal{S} = \mathbf{Set}$, Continuity simply says that for every object X in $\mathcal{E}$ and set S the canonical $p_!(\prod_{s\in S} X) \to \prod_{s\in S}(p_! X)$ is an iso. In other words, that $p_!$ preserves powers.

The axiom of Continuity is the key ingredient that makes the Hurewicz category determined by $p : \mathcal{E} \to \mathcal{S}$ into a quality type over $\mathcal{S}$ (see Theorem 1 in [12]). Although we are not going to consider the Hurewicz category explicitly, the reader should always bear in mind its relation with Continuity since this relation is an important motivation for the present paper.

We end this section with some remarks on the choice of terminology. The general principle was to be consistent with the terminology in [12]. So, in particular, we do not introduce new names for concepts already existing loc. cit.; but, of course, in order to study Continuity we needed a name for the concept loosely described as 'cohesive minus the Continuity condition'. This is how pre-cohesive categories arise. I believe it is a sensible choice of terminology. Apart from being consistent with [12], notice that the stronger notion (cohesive) receives an undecorated name (just as in the case of presheaves and sheaves). Unfortunately, the choice of terminology is not perfect and, since there is a risk of some confusion, we emphasize the following.

2.7. REMARK. A sufficiently cohesive pre-cohesive category need not be cohesive. Recall that a pre-cohesive category is sufficiently cohesive if it has 'enough contractible objects' and it is cohesive if it satisfies Continuity. They are two different things. For example, we will show that every sufficiently cohesive pre-cohesive presheaf topos fails to satisfy Continuity and so, by Definition 2.6, is not cohesive.

3. Cohesive toposes

Let $\mathcal{E}$ and $\mathcal{S}$ be toposes and $p : \mathcal{E} \to \mathcal{S}$ be a geometric morphism. We say that p is *pre-cohesive* if $p^* \dashv p_* : \mathcal{E} \to \mathcal{S}$ extends (necessarily in an essentially unique way) to a string of adjoints $p_! \dashv p^* \dashv p_* \dashv p^!$ making $\mathcal{E}$ into a pre-cohesive category over $\mathcal{S}$.

In the standard terminology for geometric morphisms, $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive if it is connected (i.e. p^* is full and faithful), essential (i.e. p^* has a left adjoint $p_!$), local (i.e. p_* has a full and faithful right adjoint) and, moreover, the Nullstellensatz holds and $p_!$ preserves finite products.

It is relevant to mention that under certain reasonable conditions the Nullstellensatz is stronger than what the above may suggest. If $p : \mathcal{E} \to \mathcal{S}$ is bounded, connected and locally connected and $\mathcal{S}$ has a nno then the Nullstellensatz implies that p is pre-cohesive. That is: $p_!$ preserves finite products and p_* has a right adjoint. In fact, more is true: see Theorem 3.4 and Proposition 3.5 in [7].

(The reader should be warned that the topological intuition that supports the standard terminology for geometric morphisms, e.g. 'locally connected', is not as useful in the present context as it is among localic morphisms. See the Proposition following Axiom 2 in [11] and also Lemma 1.1 in [7]. The original 'molecular' terminology [1] may be a more neutral alternative.)

Assume from now on that $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive.

Naturally, among toposes, Sufficient Cohesion and quality types admit alternative characterizations.

3.1. Proposition. [p.47 in [12].] *A pre-cohesive $p : \mathcal{E} \to \mathcal{S}$ is sufficiently cohesive if and only if the subobject classifier of $\mathcal{E}$ is connected (i.e. $p_!\Omega = 1$).*

In order to state a characterization of quality types let us first make a small remark.

3.2. Lemma. *The map $\theta_U : p_*U \to p_!U$ is an iso for every subterminal $U \to 1$ in $\mathcal{E}$.*

Proof. The diagram

$$\begin{tikzcd} p_*U \arrow[r, "\theta_U"] \arrow[d] & p_!U \arrow[d] \\ p_*1 \arrow[r, "\theta_1"'] & p_!1 \end{tikzcd}$$

shows that θ_U is mono, because the left vertical map is mono and the bottom one is iso. Since θ is epi by the Nullstellensatz, θ_U is an iso. ∎

The characterization of quality types may now be stated as follows.

3.3. Proposition. [3.7 in [7].] *Assume that $p : \mathcal{E} \to \mathcal{S}$ is locally connected. Then p is a quality type if and only if $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite limits. Moreover, in this case, every connected object of $\mathcal{E}$ has a unique point.*

Proof. The last part of the statement above does not appear in the statement of 3.7 in [7], but it does appear in the proof. Let us repeat part of the argument. Consider points $a, a' : 1 \to A$ in $\mathcal{E}$ and let $U \to 1$ be the equalizer of a and a'. If A is connected and $p_!$ preserves equalizers then $p_!U = 1$. By Lemma 3.2 we have a map $1 = p_!U \to p_*U$ and, by adjointness, a map $1 = p^*1 \to U$. Then the mono $U \to 1$ must be an iso and hence $a = a'$. ∎

Propositions 3.1 and 3.3 characterize sufficiently cohesive toposes and quality types respectively. What about Continuity? As explained in Definition 2 of [12], it holds if $\mathcal{S}$ is the category of finite sets. In particular, this implies that the sufficiently cohesive topos of finite reversible graphs (Section V loc.cit.) satisfies Continuity. On the other hand, Continuity does not hold for simplicial sets when considered as a pre-cohesive topos over **Set** (see comment below Theorem 1 loc.cit.). This is no accident, in Theorem 7.4 we prove that a pre-cohesive presheaf topos satisfies Continuity if and only if it is a quality type.

4. Pre-cohesive presheaf toposes

Let $\mathcal{E}$ be a topos. Let us say that a pre-cohesive $\mathcal{E} \to \mathbf{Set}$ is *combinatorial* if $\mathcal{E}$ is a presheaf topos. (Of course, this is not intended to be lasting terminology. We introduce it here for emphasis.) It is well-known that presheaf toposes are locally connected so all the material in [7] applies.

Recall (Section 5 in [1]) that for a locally connected (or molecular) $p : \mathcal{E} \to \mathbf{Set}$, an object A is connected (in the sense that $p_!A = 1$) if and only if A is *indecomposable* (i.e. it has exactly two complemented subobjects). Since representable objects are indecomposable in a presheaf topos we have that $p_!(\mathcal{C}(\_, C)) = 1$ for every C in $\mathcal{C}$.

Denoting the topos of presheaves on a small category $\mathcal{C}$ by $\widehat{\mathcal{C}}$, the following fact is a corollary of the results in [7]. We recall some of the details.

4.1. Proposition. *Let $\mathcal{C}$ be a small category whose idempotents split. The canonical $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive if and only if $\mathcal{C}$ has a terminal object and every object of $\mathcal{C}$ has a point.*

Proof. The canonical $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is locally connected and $p_!C = 1$ for every representable C in $\widehat{\mathcal{C}}$. Example C3.6.3(b) in [6] shows that p is local if and only is $\mathcal{C}$ has a terminal object. In this case, of course, p is connected. So we can assume that $\mathcal{C}$ has a point and then $p_*X = \widehat{\mathcal{C}}(1, X) = X1$ for every X in $\widehat{\mathcal{C}}$. If the Nullstellensatz holds then $\mathcal{C}(1, C) = p_*C \to p_!C = 1$ is epi and so every object of $\mathcal{C}$ has a point. For the converse assume that every object of $\mathcal{C}$ has a point and let P in $\widehat{\mathcal{C}}$. Recall that an element of $p_!P$ may be described as a 'tensor' $x \otimes C$ with $x \in PC$. The natural transformation $\theta : p_*P \to p_!P$ sends each $y \in P1$ to the tensor $y \otimes 1$. Since every C in $\mathcal{C}$ has a point any tensor $x \otimes C$ is equal to one of the form $y \otimes 1$. (See also Lemma 5.5.)

So far we have proved that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is local and the Nullstellensatz holds if and only if $\mathcal{C}$ has a terminal object and every object of $\mathcal{C}$ has a point. Proposition 1.6(iii) in [7] shows that, in this case, $p_! : \widehat{\mathcal{C}} \to \mathbf{Set}$ preserves finite products (because $\mathcal{C}$ is cosifted). ∎

The simplest non-trivial example of a category $\mathcal{C}$ satisfying the conditions in the statement of Proposition 4.1 consists of two objects, one of them terminal and the other non-terminal but having exactly one point. In this case, the resulting $\widehat{\mathcal{C}} \to \mathbf{Set}$ is a quality type (see Section VI in [12]).

The simplest sufficiently cohesive example is the topos $\widehat{\Delta_1}$ of (reflexive) graphs [11]. A similar example is that of reversible graphs used in [12].

To discuss further examples it seems useful to organize them in terms of their classifying role. I believe that this is justified by the following quotation borrowed from [8]: "Since these combinatorial categories are usually toposes, some light is shed on their particularity by determining what kind of structure they classify (in the established categorical sense, e.g., the simplicial topos classifies total orders with distinct endpoints, and a simple cubical example classifies strictly bipointed objects). Concretely, there are many different theories of algebraic structure for which the unit interval is a model, and having chosen one, this structure should be preserved by geometric realization".

4.2. Example. Sufficiently cohesive pre-cohesive presheaf toposes:

1. (Total orders with distinct endpoints.) As quoted above, this is the classical topos $\widehat{\Delta}$ of simplicial sets.

2. (Non trivial Boolean algebras.) This topos may be described as that of presheaves on the category of non-empty finite sets [9].

3. ('Connected' distributive lattices.) A distributive lattice is called *connected* if it has exactly two complemented elements. The classifying topos may be described as presheaves on finite connected posets. This is the *Gaeta topos* associated to the theory of distributive lattices. See Section 2 in [13].

4. (Strictly bipointed objects.) The classifier for the theory presented by two constants 0 and 1 and the sequent '$0 = 1 \vdash \bot$' is a presheaf topos. It may be described as $\mathbf{Set}^{\mathcal{A}}$ where $\mathcal{A}$ is the category non-trivial finitely presented algebras for the algebraic theory of bipointed objects. More concretely, the objects of $\mathcal{A}$ are finite bipointed sets such that the distinguished points are different. The maps in $\mathcal{A}$ are just functions preserving selected points. As far as I know, this topos has not been systematically studied.

5. ('Connected' $\mathbb{C}$-algebras.) A $\mathbb{C}$-algebra is called *connected* if it has exactly two idempotents. The classifying topos may be described as presheaves on the opposite of the category of finitely presented connected $\mathbb{C}$-algebras. This is the Gaeta topos associated to the theory of $\mathbb{C}$-algebras (see the last paragraph of p. 109 in [10]). Notice that, in contrast with the previous examples, the site of the present one is not locally finite. See also [15].

Of course, given a category $\mathcal{C}$ satisfying the conditions in Proposition 4.1, we don't always have immediate access to a simple presentation for the theory classified by $\widehat{\mathcal{C}}$.

4.3. Example. [The topos of *ball complexes* [17].] The category $\mathcal{B}$ of balls has objects B_n with $n \in \mathbb{N}$ (to be thought of as solid n-balls) and, for any n, two maps $\delta_0, \delta_1 : B_n \to B_{n+1}$ (pictured as the inclusion of upper and lower hemispheres) and a common retraction $p : B_{n+1} \to B_n$ for δ_0 and δ_1 "squashing the ball onto its solid equator" (Chapter 1 loc. cit.). Also, for any n, the maps in the diagram below

$$\begin{tikzcd} B_n \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & B_{n+1} \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & B_{n+2} \end{tikzcd}$$

satisfy $\delta_j \delta_i = \delta_k \delta_i$ and no further relations. The presheaf topos $\widehat{\mathcal{B}}$ is pre-cohesive and sufficiently cohesive but I don't know a simple presentation for the theory it classifies. All the information I know about models of this theory is proved in Section 2.8 of [17]: if $F : \mathcal{B} \to \mathbf{Set}$ is a filtering functor then $F(B_n)$ has at most $2n + 1$ elements.

Let us also mention a non-example together with an explicit instance of how to use the classifying role of a pre-cohesive topos.

4.4. Example. [The topological topos [4].] Let $\mathcal{J}$ be the topos of sheaves for the canonical topology on the monoid of continuous endos of the one-point compactification of the discrete space of natural numbers. It is, by definition, a Grothendieck topos but it is not a presheaf topos and it is not pre-cohesive; in fact, the canonical geometric morphism $\mathcal{J} \to \mathbf{Set}$ is not essential (the inverse image $\mathbf{Set} \to \mathcal{J}$ does not preserve products). One of the main features of $\mathcal{J}$ is that it embeds the category of sequential spaces. Moreover, the interval $I = [0, 1]$ is totally ordered in $\mathcal{J}$ and, of course, has two distinct points; so it determines a geometric morphism $r : \mathcal{J} \to \widehat{\Delta}$ such that $r^*[1] = I$ where [1] denotes the object representable by the total order with exactly two elements. Further, as shown in Theorem 8.1 loc. cit., the inverse image $r^* : \widehat{\Delta} \to \mathcal{J}$ coincides with the usual geometric realization.

See also Proposition 10.6 where we present an analogous example of a geometric morphism $\mathcal{F} \to \widehat{\Delta}$ but where $\mathcal{F}$ is cohesive over $\mathbf{Set}$.

Pre-cohesive presheaf toposes should be contrasted with (pre-cohesive) Grothendieck toposes in general, whose colimits may not be calculated 'as in $\mathbf{Set}$'.

4.5. Proposition. *Let $\mathcal{C}$ be a small category with terminal object and such that every object of $\mathcal{C}$ has a point so that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive. Then the following hold:*

1. *p is sufficiently cohesive if and only if there is an object of $\mathcal{C}$ with two distinct points.*

2. *p is a quality type if and only if $\mathcal{C}$ has a zero object (i.e. its terminal object is also initial).*

Proof. The characterization of Sufficient Cohesion is proved in [15]. To prove the second item observe that if $\mathcal{C}$ has initial object then $p_! : \widehat{\mathcal{C}} \to \mathbf{Set}$ preserves finite limits by Example C3.6.17(b) in [6] and so p is a quality type by Proposition 3.3. Conversely, assume that $\theta : p_* \to p_!$ is an iso and let C in $\mathcal{C}$. Since $\mathcal{C}(\_, C)$ is connected, we can calculate:

$$1 = p_!(\mathcal{C}(\_, C)) \cong p_*(\mathcal{C}(\_, C)) = \widehat{\mathcal{C}}(1, \mathcal{C}(\_, C)) = \mathcal{C}(1, C)$$

to conclude that every object of $\mathcal{C}$ has exactly one point. ∎

It follows that, for combinatorial pre-cohesive toposes, the contrast expressed in Proposition 2.4 may be strengthened to the following dichotomy.

4.6. Corollary. *Let $\mathcal{E}$ be a presheaf topos and assume that the canonical $p : \mathcal{E} \to \mathbf{Set}$ is pre-cohesive. Then exactly one of the following holds:*

1. *p is a quality type.*

2. *p is sufficiently cohesive.*

Proof. Let $\mathcal{E} = \widehat{\mathcal{C}}$ for a small category $\mathcal{C}$. Without loss of generality we can assume that idempotents split in $\mathcal{C}$. Proposition 4.1 implies that $\mathcal{C}$ has a terminal object and that every object in $\mathcal{C}$ has a point. Clearly, for such a $\mathcal{C}$, exactly one of the following holds:

1. The terminal object is not initial. (Equivalently, there is an object of $\mathcal{C}$ with two distinct points.)

2. The terminal object is also initial. (Equivalently, every object of $\mathcal{C}$ has exactly one point.)

so the result follows from Proposition 4.5. ■

If the reader accepts that, for pre-cohesive toposes over **Set**, 'presheaf' is a reasonable formalization of 'combinatorial' then Lawvere's suggestion that combinatorial toposes should not satisfy Continuity admits the following confirmation: if a combinatorial pre-cohesive topos satisfies Continuity then it cannot be sufficiently cohesive. Actually, we will prove a stronger fact in Theorem 7.4.

5. Connected components

Fix a small category $\mathcal{C}$ and let $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ be the canonical geometric morphism. We recall well-known material on the functor $p_!$ and introduce some specific notation that aids in the proof of the main results of the paper.

5.1. Definition. For $P \in \widehat{\mathcal{C}}$, we say that a cospan

$$C \xrightarrow[\sigma_l]{} U \xleftarrow[\sigma_r]{} C'$$

connects the elements $x \in PC$ and $x' \in PC'$ if there is a $y \in PU$ such that $x = y \cdot \sigma_l$ and $x' = y \cdot \sigma_r$.

If the cospan $\sigma = (\sigma_l, \sigma_r)$ as above connects the elements x and x' then we may denote the situation by the following diagram

$$x \longleftarrow\!\!\shortmid\; y \longmapsto x'$$

$$C \xrightarrow[\sigma_l]{} U \xleftarrow[\sigma_r]{} C'$$

or simply write $x\sigma x'$.

5.2. Definition. A *path* from C to C' is a sequence of cospans $\sigma_1, \sigma_2, \ldots, \sigma_n$ as below

$$C_0 \xrightarrow[\sigma_{1,l}]{} U_1 \xleftarrow[\sigma_{1,r}]{} C_1 \xrightarrow[\sigma_{2,l}]{} U_2 \xleftarrow[\sigma_{2,r}]{} C_2 \cdots\cdots C_{n-1} \xrightarrow[\sigma_{n,l}]{} U_n \xleftarrow[\sigma_{n,r}]{} C_n$$

with $C_0 = C$ and $C_n = C'$.

The terminology may be non-standard but it only plays a role in auxiliary results so we keep it for brevity.

5.3. Definition. A path $\sigma_1, \sigma_2, \ldots, \sigma_n$ from C to C' as above *connects* elements $x \in PC$ and $x' \in PC'$ if there exists a sequence of elements $x_0, \ldots, x_n$ with $x_0 = x$, $x_n = x'$ and for every $1 \leq i \leq n$, $x_i \in PC_i$ and $x_{i-1}\sigma_i x_i$.

We say that $x \in PC$ and $x' \in PC'$ are *connectable* if there is a path from C to C' that connects x and x'. The next fact follows from Theorem VII.2.2 in [14].

5.4. Corollary. *The set $p_!P$ may be defined as the set $\{(x, C) \mid x \in PC\}$ quotiented by the relation that identifies (x, C) and (x', C') if and only if x and x' are connectable.*

For this reason, elements in $p_!P$ will be denoted by 'tensors' $x \otimes C$ with $x \in PC$. It follows that $x \otimes C = x' \otimes C'$ if and only if x and x' are connectable.

Assume now that the small $\mathcal{C}$ has a terminal object and is such that every object has a point, so the canonical $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive (Proposition 4.1). In this case the Nullstellensatz says that for every P in $\widehat{\mathcal{C}}$, $\theta : P1 \to p_!P$ is surjective. For illustration let us give a direct proof of this fact.

5.5. Lemma. *Let $P \in \widehat{\mathcal{C}}$ and C in $\mathcal{C}$. For every $x \in PC$ there is a $y \in p_*P = P1$ such that $x \otimes C = y \otimes 1$ in $p_!P$.*

Proof. By hypothesis, C has a point $c : 1 \to C$. Define $y = x \cdot c$ and observe that the cospan

$$C \xrightarrow{id} C \xleftarrow{c} 1$$

connects x and y. ■

We will find it convenient to give another concrete description of the kernel pair of the quotient $p_*P = P1 \to p_!P$ 'improving' that given in Corollary 5.4.

5.6. Definition. A path as in Definition 5.2 is called *restricted* if all intermediate objects are terminal. That is, if $C_1 = C_2 = \ldots = C_{n-1} = 1$.

In other words, a path is restricted if it looks as follows:

$$\begin{tikzcd}
 & U_1 & & U_2 & & & U_n & \\
C_0 \arrow[ur, "\sigma_{1,l}"] & & 1 \arrow[ul, "\sigma_{1,r}"'] \arrow[ur, "\sigma_{2,l}"] & & 1 \arrow[ul, "\sigma_{2,r}"'] \arrow[r, dotted, no head] & 1 \arrow[ur, "\sigma_{n,l}"] & & C_n \arrow[ul, "\sigma_{n,r}"']
\end{tikzcd}$$

Naturally, we say that $x \in PC$ and $x' \in PC'$ are *connectable via a restricted path* if there exists a restricted path that connects them.

5.7. Lemma. *For every P in $\widehat{\mathcal{C}}$ and $x, x' \in p_*P = P1$, $x \otimes 1 = x' \otimes 1 \in p_!P$ if and only if x and x' are connectable via a restricted path.*

Proof. By Corollary 5.4 we need only prove that if x and x' are connectable then they are connectable via a restricted path. Consider first a path (Definition 5.3) of length 2 connecting x and x' as in the diagram below:

$$x \longleftarrow\!\!\shortmid\; y_1 \longmapsto x_1 \longleftarrow\!\!\shortmid\; y_2 \longmapsto x'$$

$$1 \xrightarrow[\sigma_{1,l}]{} U_1 \xleftarrow[\sigma_{1,r}]{} C_1 \xrightarrow[\sigma_{2,l}]{} U_2 \xleftarrow[\sigma_{2,r}]{} 1$$

By hypothesis there is a point $c : 1 \to C_1$ and it is clear that $x = y_1 \cdot \sigma_{1,l}$ is connectable with $y_1 \cdot \sigma_{1,r} \cdot c = x_1 \cdot c \in P1$. It is also clear that the element $x_1 \cdot c = y_2 \cdot \sigma_{2,l} \cdot c \in P1$ is connectable with $x' = y_2 \cdot \sigma_{2,r}$. This shows that x and x' are connectable via a restricted path. It is easy to apply the same idea to a path of arbitrary length. ■

6. The spear of a topos with nno

Let $\mathcal{E}$ be a topos with nno. Let $\bot, \top : 1 \to I$ be a bipointed object in $\mathcal{E}$. Define R as the codomain of the following coequalizer diagram

$$\begin{tikzcd} \mathbb{N} \times 1 \arrow[r, shift left, "\mathbb{N}\times\top"] \arrow[r, shift right, "suc\times\bot"'] & \mathbb{N} \times I \arrow[r, two heads] & R \end{tikzcd}$$

where $1 \xrightarrow{0} \mathbb{N} \xleftarrow{suc} \mathbb{N}$ is the nno in $\mathcal{E}$. We may picture R as

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \dots \qquad \longrightarrow n \longrightarrow \dots$$

which, in general, is only an intuitive aid; but in the case when $\mathcal{E}$ is the topos of reflexive graphs and I is a single edge the picture above is a precise description.

There is a canonical map $\mathbb{N} \to R$ given by the composition

$$\mathbb{N} \xrightarrow{\langle id,!\rangle} \mathbb{N} \times 1 \xrightarrow{id\times\bot} \mathbb{N} \times I \twoheadrightarrow R$$

in $\mathcal{E}$. The diagonal in the commutative square below

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, "\langle 0,\bot\rangle"'] & \mathbb{N} \arrow[d] \\ \mathbb{N} \times I \arrow[r] & R \end{tikzcd}$$

will be denoted by $0 : 1 \to R$.

The canonical inclusion $\mathbb{N} \to R$ determines, by transposition, a point that we denote by $\vec{\infty} : 1 \to R^{\mathbb{N}}$. On the other hand, the transposition of

$$\mathbb{N} \longrightarrow 1 \xrightarrow{0} R$$

will be denoted by $\vec{0} : 1 \to R^{\mathbb{N}}$. The point $\vec{0} : 1 \to R^{\mathbb{N}}$ may be pictured as the constant sequence $(0, 0, \dots)$ and $\vec{\infty} : 1 \to R^{\mathbb{N}}$ as $(0, 1, 2, \dots)$.

6.1. Lemma. *(With $\mathcal{E}$, I and R as above.) If $\mathcal{D}$ is a topos and $F : \mathcal{E} \to \mathcal{D}$ preserves finite colimits and terminal object then F preserves the nno. If moreover, F preserves finite products then the diagram*

$$\begin{tikzcd} \mathbb{N} \times 1 \arrow[r, shift left, "\mathbb{N}\times F\top"] \arrow[r, shift right, "suc\times F\bot"'] & \mathbb{N} \times FI \arrow[r, two heads] & FR \end{tikzcd}$$

is a coequalizer in $\mathcal{D}$.

Proof. The first part is a corollary of a more general result (see Lemma A2.5.6 in [6]). The second part follows immediately. ■

Lemma 6.1 applies if F is a product preserving left adjoint. For example, if F is the 'pieces' functor of a pre-cohesive topos, or if it is the inverse image of a geometric morphism.

6.2. Lemma. *Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism. If I is connected then so is R. If, moreover, Continuity holds then $R^{\mathbb{N}}$ is also connected.*

Proof. By Lemma 6.1 the following diagram

$$\begin{tikzcd} \mathbb{N} \times 1 \arrow[r, shift left, "id\times p_!\top"] \arrow[r, shift right, "suc\times p_!\bot"'] & \mathbb{N} \times p_! I \arrow[r, two heads] & p_! R \end{tikzcd}$$

is a coequalizer in $\mathcal{S}$. As I is connected, $p_! R$ is the coequalizer of the maps $id, suc : \mathbb{N} \to \mathbb{N}$ and hence $p_! R = 1$ because $\mathbb{N}$ is a nno. To prove the second part of the result calculate

$$p_!(R^{\mathbb{N}}) = p_!(R^{p^*\mathbb{N}}) \cong (p_! R)^{\mathbb{N}} \cong 1^{\mathbb{N}} \cong 1$$

using Continuity. ■

Let us introduce some terminology.

6.3. Definition. The *spear* of a topos with nno is the object R in the coequalizer

$$\begin{tikzcd} \mathbb{N} \times 1 \arrow[r, shift left, "\mathbb{N}\times\top"] \arrow[r, shift right, "suc\times\bot"'] & \mathbb{N} \times \Omega \arrow[r, two heads] & R \end{tikzcd}$$

where $\top, \bot : 1 \to \Omega$ are the standard points of the suboject classifier.

Proposition 3.1 and Lemma 6.2 imply the next fact.

6.4. Corollary. *If the topos $\mathcal{E}$ has a nno and $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive and sufficiently cohesive then the spear R of $\mathcal{E}$ is connected. If, moreover, Continuity holds then $R^{\mathbb{N}}$ is also connected.*

We will use this result to prove that if $p : \mathcal{E} \to \mathcal{S}$ is a pre-cohesive and sufficiently cohesive presheaf topos then p cannot satisfy Continuity.

7. The presheaf toposes that satisfy Continuity

In this section we characterize the pre-cohesive presheaf toposes that satisfy continuity. Let $\mathcal{C}$ be a small category and $p : \mathcal{E} = \widehat{\mathcal{C}} \to \mathbf{Set}$ be the resulting molecular topos over $\mathbf{Set}$. Denote the spear of $\mathcal{E}$ by R.

For any X in $\widehat{\mathcal{C}}$ and C in $\mathcal{C}$,

$$(X^{\mathbb{N}})C = \left(\prod_{i\in\mathbb{N}} X\right) C = \prod_{i\in\mathbb{N}} XC = (XC)^{\mathbb{N}}$$

so, in particular, $R^{\mathbb{N}}C = (RC)^{\mathbb{N}}$.

Recall that the subobject classifier Ω in $\mathcal{E}$ sends each C in $\mathcal{C}$ to the set ΩC of sieves on C. Moreover, for every $g : B \to C$ in $\mathcal{C}$ and $S \in \Omega C$, $(\Omega g)S = S \cdot g = \{h \mid gh \in S\}$.

7.1. Lemma. *The spear of $\mathcal{E}$ may be described as follows.*

1. *For any C in $\mathcal{C}$, RC is the set of pairs (n, S) with $n \in \mathbb{N}$ and S a non-maximal sieve on C.*
2. *For any $g : B \to C$ in $\mathcal{C}$ and $(n, S) \in RC$, $(Rg)(n, S) = (n, S) \cdot g$ is defined by cases:*
   (a) if $S \cdot g \in \Omega B$ is not maximal then $(n, S) \cdot g = (n, S \cdot g)$ and,
   (b) if $S \cdot g \in \Omega B$ is maximal then $(n, S) \cdot g = (n + 1, \emptyset)$.

Also, the canonical map $\mathbb{N} \to R$ sends each $n \in \mathbb{N}C = \mathbb{N}$ to $(n, \emptyset) \in RC$.

Proof. We first sketch the proof that R defined as in the statement is a presheaf. It is clear that identities act as such. So let $f : A \to B$ and $g : B \to C$ be maps in $\mathcal{C}$. For any $(n, S) \in RC$, it is easily seen that $((n, S) \cdot g) \cdot f$ equals either $(n + 1, \emptyset)$ or $(n, (S \cdot g) \cdot f) = (n, S \cdot (gf))$ depending on whether $(S \cdot g) \cdot f = S \cdot (gf)$ is maximal or not. So $((n, S) \cdot g) \cdot f = (n, S) \cdot (gf)$.

There is an obvious function $q_C : (\mathbb{N} \times \Omega)C = \mathbb{N} \times (\Omega C) \to RC$ that sends (n, S) in the domain to $(n + 1, 0)$ or (n, S) depending on whether S is maximal or not. It is easy to check that these functions induce a natural transformation $q : \mathbb{N} \times \Omega \to R$.

It remains to show that the diagram on the left below

$$\mathbb{N}\times 1 \underset{suc\times\bot}{\overset{\mathbb{N}\times\top}{\rightrightarrows}} \mathbb{N}\times\Omega \overset{q}{\twoheadrightarrow} R \qquad\qquad \mathbb{N}\times 1 \underset{suc\times\bot}{\overset{\mathbb{N}\times\top}{\rightrightarrows}} \mathbb{N}\times(\Omega C) \overset{q}{\twoheadrightarrow} RC$$

is a coequalizer in $\mathcal{E}$. Since $\mathcal{E}$ is a presheaf topos it is enough to prove that the fork on the right above is a coequalizer in $\mathbf{Set}$ for each C in $\mathcal{C}$; but this is easy. ∎

Lemma 7.1 implies that for each C in $\mathcal{C}$ there is a 'projection' $\Xi_C = \Xi : RC \to \mathbb{N}$. If $r = (n, S) \in RC$ then $\Xi r = \Xi(n, S) = n$ will be called the *signature* of r.

The map $\Xi^{\mathbb{N}} : (R^{\mathbb{N}})C = (RC)^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ sends each $\rho \in (RC)^{\mathbb{N}}$ to the function $\Xi^{\mathbb{N}}\rho \in \mathbb{N}^{\mathbb{N}}$ defined by $(\Xi^{\mathbb{N}}\rho)i = \Xi(\rho i)$. The function $\Xi^{\mathbb{N}}\rho$ will be called the *signature* of ρ.

Let us say that a function $u : \mathbb{N} \to \mathbb{N}$ is *bounded by* $k \in \mathbb{N}$ if $ui \leq k$ for every $i \in \mathbb{N}$. Also, we say that u is *bounded* if it is bounded by some $k \in \mathbb{N}$. We can conveniently extend the terminology to figures of $R^{\mathbb{N}}$. We say that an element of $(R^{\mathbb{N}})C = (RC)^{\mathbb{N}}$ is *bounded* if its signature is.

For example, if we let $\mathcal{C}$ have a final object then the signature of $\vec{0} \in (R^{\mathbb{N}})1 = p_*(R^{\mathbb{N}})$ is the constant function $\mathbb{N} \to \mathbb{N}$ that sends everything to 0. On the other hand, the signature of $\vec{\infty} \in R^{\mathbb{N}}1$ is the identity on $\mathbb{N}$. Hence, $\vec{0}$ is bounded and $\vec{\infty}$ is not.

7.2. LEMMA. *Let $h : C' \to C$ in $\mathcal{C}$ and $\rho \in (RC)^{\mathbb{N}}$. Then ρ is bounded if and only if $\rho \cdot h$ is.*

PROOF. Lemma 7.1 implies that for every $r \in RC$, $\Xi_C r \leq \Xi_{C'}(r \cdot h) \leq (\Xi_C r) + 1$. Now, if ρ in $(RC)^{\mathbb{N}}$ is bounded then there is a $k \in \mathbb{N}$ such that, for every $i \in \mathbb{N}$, the inequality $(\Xi^{\mathbb{N}}\rho)i = \Xi(\rho i) \leq k$ holds. Then

$$(\Xi^{\mathbb{N}}_{C'}(\rho \cdot h))i = \Xi_{C'}((\rho \cdot h)i) = \Xi_{C'}((\rho i) \cdot h) \leq (\Xi_C(\rho i)) + 1 \leq k + 1$$

for every $i \in \mathbb{N}$, so $\Xi^{\mathbb{N}}(\rho \cdot h)$ is bounded by $k + 1$. On the other hand, if $\Xi^{\mathbb{N}}(\rho \cdot h)$ is bounded by k then so is $\Xi^{\mathbb{N}}\rho$. ∎

We can now prove that Sufficient Cohesion and Continuity are incompatible for presheaf toposes.

7.3. PROPOSITION. *Let $\mathcal{E}$ be a presheaf topos such that the canonical $p : \mathcal{E} \to$ **Set** is pre-cohesive. It p is sufficiently cohesive then it does not satisfy Continuity.*

PROOF. Let R be the spear of $\mathcal{E}$. Since p is sufficiently cohesive, R is connected by Corollary 6.4. Consider now the distinguished points $\vec{0}, \vec{\infty} : 1 \to R^{\mathbb{N}}$ and assume, for the sake of contradiction, that p satisfies Continuity. Corollary 6.4 again implies that $R^{\mathbb{N}}$ is connected so $p_!\vec{0} = p_!\vec{\infty} : 1 \to p_!(R^{\mathbb{N}})$. By Corollary 5.4, $\vec{0}$ and $\vec{\infty}$ are connectable and so, by Lemma 7.2, $\vec{0}$ is bounded if and only if $\vec{\infty}$ is. But this is absurd because $\vec{0}$ is bounded and $\vec{\infty}$ is not. ∎

We can summarize the discussion so far as a characterization.

7.4. THEOREM. *Let $\mathcal{E}$ be a presheaf topos and assume that the canonical $p : \mathcal{E} \to$ **Set** is pre-cohesive. Then p satisfies Continuity if and only if p is a quality type.*

PROOF. One implication is trivial: if p is a quality type then $p_! = p_*$ preserves all small products so Continuity holds. For the converse assume that Continuity holds. By Corollary 4.6, p is either sufficiently cohesive or a quality type; but Proposition 7.3 implies that it cannot be sufficiently cohesive. ∎

8. Bipointed objects and connectors

After Definition 2 of [12] Lawvere comments that: "This 'continuity' property [...] also holds if the contrast with $\mathcal{S}$ is determined [...] by an infinitely divisible interval in $\mathcal{E}$". This comment inspired our construction of cohesive and sufficiently cohesive Grothendieck toposes. We formalize here the most basic ingredients.

Let $\mathcal{D}$ be a category with terminal object. (In practice this may be a small category underlying the site of definition of a topos, or it may be the domain of a pre-cohesive geometric morphism.) Let I be an object in $\mathcal{D}$ equipped with two points $0, 1 : 1 \to I$.

We stress that for the moment we are not requiring the points 0 and 1 to be different, nor that I is 'connected' in any sense; but of course, these conditions are in the back of our minds and guide our definitions and results. For example, if $p : \mathcal{D} \to \mathcal{S}$ is a pre-cohesive geometric morphism we would like express that there is some close relationship between the bipointed I and the functor $p_! : \mathcal{D} \to \mathcal{S}$. We propose the following concept.

8.1. Definition. Assume that $p : \mathcal{D} \to \mathcal{S}$ is a pre-cohesive geometric morphism. The bipointed object $0, 1 : 1 \to I$ is called a *connector* for p if the following diagram

$$\begin{tikzcd} p_*(X^I) \arrow[r, shift left, "p_* ev_0"] \arrow[r, shift right, "p_* ev_1"'] & p_* X \arrow[r, "\theta"] & p_! X \end{tikzcd}$$

is a coequalizer in $\mathcal{S}$ for each X in $\mathcal{D}$.

We will show that many pre-cohesive Grothendieck toposes have a connector. The proof consists on identifying special objects in the sites of definition. The next definition does not assume anything on $\mathcal{D}$ but, in practice, it will be a small category underlying a site of definition.

8.2. Definition. An object C in $\mathcal{D}$ is called *arcwise connected* if for every cospan of points $a : 1 \to C \leftarrow 1 : b$ in $\mathcal{D}$ there exists a map $f : I \to C$ making the following diagram

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "a"'] & I \arrow[d, dashed, "f"] & 1 \arrow[l, "1"'] \arrow[dl, "b"] \\ & C & \end{tikzcd}$$

commute.

Let us look at some examples among the sites discussed in Example 4.2.

8.3. Example. [In the site for the classifier of non-trivial Boolean algebras.] Let $\mathcal{C}$ be the category of non-empty finite sets. It may be equipped with the bipointed object $in_0, in_1 : 1 \to 1 + 1$. It is easy to check that every object in this category is arcwise connected. The idea that non-empty finite sets are arcwise connected may sound strange at first; but recall that representables are connected in $\widehat{\mathcal{C}}$. In fact, in this case, the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$ factors through the codiscrete inclusion $\mathbf{Set} \to \widehat{\mathcal{C}}$. See [9].

In contrast, consider the following.

8.4. Example. [In the site for the classifier of connected distributive lattices and Δ.] Let $\mathcal{C}$ be the essentially small category of finite connected posets. Let $0, 1 : 1 \to I$ in $\mathcal{C}$ be the total order with two elements with $0 \leq 1$. In this case, I itself is not arcwise connected because if we let $a = 1 : 1 \to I$ and $b = 0 : 1 \to I$ in Definition 8.2 then no f exists as required there. This argument also shows that the same $0, 1 : 1 \to I$, as an object in Δ, is not arcwise connected.

Assume now that $\mathcal{C}$ is a small category with terminal object and such that every object has a point, so the canonical $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive (Proposition 4.1). Assume further that $\mathcal{C}$ is equipped with a bipointed object $0, 1 : 1 \to I$.

8.5. Definition. A *combinatorial arc* is a path as in Definition 5.2 such that for every $1 \leq i \leq n$, $\sigma_{i,l} = 0 : 1 \to I$ and $\sigma_{i,r} = 1 : 1 \to I$.

In other words, a combinatorial arc is a path of the form

$$1 \xrightarrow[0]{} I \xleftarrow[1]{} 1 \xrightarrow[0]{} I \xleftarrow[1]{} 1 \cdots\cdots 1 \xrightarrow[0]{} I \xleftarrow[1]{} 1$$

and it is clear that every combinatorial arc is a restricted path in the sense of Definition 5.6.

8.6. Lemma. *If every object in $\mathcal{C}$ is arcwise connected (w.r.t. $0, 1 : 1 \to I$) then, for every X in $\widehat{\mathcal{C}}$ and $x, x' \in p_*X = X1$, $x \otimes 1 = x' \otimes 1 \in p_!X$ if and only if x and x' are connectable via a combinatorial arc.*

Proof. Consider a diagram as in Definition 8.2. If the cospan $a : 1 \to C \leftarrow 1 : b$ connects $x, x' \in X1$ via an element $y \in XC$ such that $x = y \cdot a$ and $x' = y \cdot b$ then the cospan $0 : 1 \to I \leftarrow 1 : 1$ also connects x and x', via $y \cdot f \in XI$.

More generally, every restricted path

$$1 \xrightarrow[\sigma_{1,l}]{} U_1 \xleftarrow[\sigma_{1,r}]{} 1 \xrightarrow[\sigma_{2,l}]{} U_2 \xleftarrow[\sigma_{2,r}]{} 1 \cdots\cdots 1 \xrightarrow[\sigma_{n,l}]{} U_n \xleftarrow[\sigma_{n,r}]{} 1$$

connecting $x, x' \in X1$ can be transformed, using the same idea 'cospanwise', into a combinatorial arc also connecting x and x'. ∎

The bipointed object $0, 1 : 1 \to I$ in $\mathcal{C}$ determines a (representable) bipointed object in the topos $\widehat{\mathcal{C}}$.

8.7. Lemma. *If every object in $\mathcal{C}$ is arcwise connected then the induced bipointed object in $\widehat{\mathcal{C}}$ is a connector for p.*

Proof. We need to check that the diagram on the left below

$$p_*(X^I) \underset{p_* ev_1}{\overset{p_* ev_0}{\rightrightarrows}} p_*X \xrightarrow{\theta} p_!X \qquad\qquad XI \underset{X1}{\overset{X0}{\rightrightarrows}} X1 \xrightarrow{\theta} p_!X$$

is a coequalizer in $\mathbf{Set}$ for each X in $\widehat{\mathcal{C}}$. In this case, $p_*(X^I) = \widehat{\mathcal{C}}(1, X^I) \cong \widehat{\mathcal{C}}(I, X) \cong XI$. So the problem is reduced to proving that the diagram on the right above is a coequalizer; but this follows from Lemma 8.6. ∎

We will use this result to prove that our cohesive examples have connectors, but consider also the following application.

8.8. Example. [The classifier of non trivial Boolean algebras has a connector.] Combine Example 8.3 and Lemma 8.7.

We will also need the next somewhat uglier but more general fact.

8.9. Lemma. *If the bipointed object $0, 1 : 1 \to I$ in $\mathcal{C}$ satisfies that for every cospan of points $a : 1 \to C \leftarrow 1 : b$ there is a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r, "\sigma_{1,l}"] \arrow[drrr, bend right, "a"'] & I \arrow[drr, bend right, "f_1"] & 1 \arrow[l, "\sigma_{1,r}"'] \arrow[r, "\sigma_{2,l}"] \arrow[dr, bend right, "a_1"] & I \arrow[d, "f_2"] & 1 \arrow[l, "\sigma_{2,r}"'] \arrow[r, dotted, no head] \arrow[dl, "a_2"] & 1 \arrow[r, "\sigma_{n,l}"] \arrow[dll, bend left, "a_{n-1}"'] & I \arrow[dlll, bend left, "f_n"] & 1 \arrow[l, "\sigma_{n,r}"'] \arrow[dllll, bend left, "b"] \\
 & & & C & & & &
\end{tikzcd}$$

where the top row is a restricted path all of whose maps equal $0 : 1 \to I$ or $1 : 1 \to I$, then the bipointed object $0, 1 : 1 \to \mathcal{C}(\_, I)$ is a connector in $\widehat{\mathcal{C}}$.

Proof. Essentially the same proof as Lemma 8.7 suitably generalizing Lemma 8.6 to deal with the more general restricted paths of the present statement instead of the simpler combinatorial arcs. ■

We mention three relevant cases where Lemma 8.9 is applicable but Lemma 8.7 is not.

8.10. Example. [The topos of simplicial sets has a connector.] Let $0, 1 : 1 \to I$ in Δ be the total order with two elements $0 \leq 1$. We have already observed that I is not arcwise connected. Loosely speaking, Lemma 8.7 is not applicable because the bipointed object $0, 1 : 1 \to I$ is not 'symmetric'. On the other hand, Lemma 8.9 is applicable because for every cospan $a : 1 \to C \leftarrow 1 : b$ in Δ, there exists an $f : I \to C$ such that one of the diagrams below

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "a"'] & I \arrow[d, "f"] & 1 \arrow[l, "1"'] \arrow[dl, "b"] \\
 & C &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1 \arrow[r, "1"] \arrow[dr, "a"'] & I \arrow[d, "f"] & 1 \arrow[l, "0"'] \arrow[dl, "b"] \\
 & C &
\end{tikzcd}$$

commutes. It follows that $\Delta(\_, I)$ is a connector for the pre-cohesive $\widehat{\Delta} \to \mathbf{Set}$.

8.11. Example. [The classifier of connected distributive lattices has a connector.] Let $\mathcal{C}$ be the essentially small category of finite connected posets. Let $0, 1 : 1 \to I$ in $\mathcal{C}$ be again the total order with two elements with $0 \leq 1$. It is not difficult to check that Lemma 8.9 is applicable and so $\mathcal{C}(\_, I)$ is a connector in $\widehat{\mathcal{C}} \to \mathbf{Set}$.

8.12. Example. [The classifier of strictly bipointed objects.] The reader is invited to prove that the site for this topos has a strictly bipointed object which induces a connector in the topos.

Since we are assuming that $\mathcal{C}$ is equipped with a bipointed object $0, 1 : 1 \to I$ then, in particular, $\mathcal{C}$ has a terminal object so every locally connected coverage on $\mathcal{C}$ is connected.

8.13. Lemma. *Let J be a Grothendieck topology making $(\mathcal{C}, J)$ into a locally connected site such that I is a J-sheaf. If every object in $\mathcal{C}$ is arcwise connected then the bipointed $0, 1 : 1 \to I$ in $\mathbf{Shv}(\mathcal{C}, J)$ is a connector for the pre-cohesive $\mathbf{Shv}(\mathcal{C}, J) \to \mathbf{Set}$.*

Proof. Recall that in the proof of Proposition 1.3 in [7] it is observed that local connectedness of the site $(\mathcal{C}, J)$ implies that the left adjoint $p_! : \mathbf{Shv}(\mathcal{C}, J) \to \mathbf{Set}$ is simply the restriction of the colimit functor $\widehat{\mathcal{C}} \to \mathbf{Set}$ to $\mathbf{Shv}(\mathcal{C}, J)$. In other words, the connected components of a sheaf are those of it considered as a presheaf. The present result then follows from Lemma 8.7. ∎

9. Continuity and Sufficient Cohesion

In this section we prove a sufficient condition for a site to induce a cohesive and sufficiently cohesive topos. Recall that a site $(\mathcal{C}, J)$ is called *locally connected* if each covering sieve (on an object C) is connected as a full subcategory of $\mathcal{C}/C$. A *connected and locally connected* site is a locally connected one with a terminal object. The next result is a rewording of some of the results in [7].

9.1. Proposition. *A bounded geometric morphism $p : \mathcal{E} \to \mathbf{Set}$ is locally connected and pre-cohesive if and only if $\mathcal{E}$ has a connected and locally connected site of definition $(\mathcal{C}, J)$ such that every object of $\mathcal{C}$ has a point.*

Let $\mathcal{C}$ be a small category with terminal object.

9.2. Definition. An *abstract interval* in $\mathcal{C}$ is a bipointed object $0, 1 : 1 \to I$ equipped with monic endos $l, r : I \to I$ and satisfying the following conditions:

1. (Sufficient cohesion) The cospan $0 : 1 \to I \leftarrow 1 : 1$ is disjoint (in the sense that it cannot be completed to a commutative square).

2. The maps $l : I \to I$ and $r : I \to I$ make the following diagrams

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "0"'] & I \arrow[d, "l"] \\ & I \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "1"] \arrow[dr, "1"'] & I \arrow[d, "r"] \\ & I \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, "1"'] & I \arrow[d, "r"] \\ I \arrow[r, "l"'] & I \end{tikzcd}$$

   commute and the square on the right is also a pullback.

These axioms are extracted from very basic intuition about the interval $[0, 1] \subseteq \mathbb{R}$. In particular, the maps l and r reflect the idea that the interval may be partitioned in two. For example, the obvious linear inclusions $[0, \frac{1}{2}] \to [0, 1]$ and $[\frac{1}{2}, 1] \to [0, 1]$ intersect in the point $\frac{1}{2} : 1 \to [0, 1]$.

9.3. Definition. A coverage on $\mathcal{C}$ is called *compatible* with a given abstract interval $(I, 0, 1, l, r)$ if the family $\{l, r\}$ covers I.

It seems natural at this point to discuss a non-example. Let **Arc** $\to$ **Top** be the full subcategory determined by the arcwise connected spaces. It has an obvious interval object with underlying space $[0, 1]$ and it is clear that every object in **Arc** is arcwise connected in the sense of Definition 8.2. One may naturally wonder if the canonical topology on **Arc** is compatible with the interval object (in the sense of Definition 9.3). The answer is negative. This is the result mentioned at the end of p. 239 in [4] (and attributed to Isbell [3]): the sieve generated by $l, r : I \to I$ is not universally effective-epimorphic. See also the first paragraph in p. 502 of [13].

Assume now that $(\mathcal{C}, J)$ is a (connected and) locally connected site.

9.4. Lemma. *If J is compatible with the interval object I then the canonical functor $\mathcal{C} \to \mathbf{Shv}(\mathcal{C}, J)$ sends the square*

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[d, "1"'] & I \arrow[d, "r"] \\ I \arrow[r, "l"'] & I \end{tikzcd}$$

in $\mathcal{C}$ to a pushout in $\mathbf{Shv}(\mathcal{C}, J)$.

Proof. First observe that since toposes have 'effective unions' (see Proposition A1.4.3 in [6]) then the following general fact holds. If, in a topos, the square below

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "b"'] & C \arrow[d, "r"] \\ B \arrow[r, "l"'] & D \end{tikzcd}$$

is a pullback with all maps mono and such that the maps in the cospan $l : B \to D \leftarrow C : r$ are jointly epic then the square is also a pushout. Now assume that the square of monos is a pullback in the site $\mathcal{C}$ and that the cospan $l : B \to D \leftarrow C : r$ J-covers D. The composite $\mathcal{C} \to \widehat{\mathcal{C}} \to \mathbf{Shv}(\mathcal{C}, J)$ of the Yoneda embedding followed by sheafification preserves the pullback and sends the cospan to a jointly epic family, so we can apply the previous observation. ∎

Loosely speaking, every J-sheaf 'perceives' the square in the statement of Lemma 9.4 as a pushout in $\widehat{\mathcal{C}}$.

9.5. Lemma. [Main Lemma] *Assume that J is compatible with the interval object. If X is in $\mathbf{Shv}(\mathcal{C}, J)$ then for every $x, x' \in X1$, x and x' are connectable if and only if they can be connected by a cospan $0 : 1 \to I \leftarrow 1 : 1$.*

Proof. One direction is trivial. So assume that x and x' are connectable. Lemma 8.6 implies that x and x' are connectable by a combinatorial arc. For the moment let us assume that they are connectable by a combinatorial arc of length 2 as in the diagram below.

$$\begin{tikzcd} x & y_1 \arrow[l, mapsto] \arrow[r, mapsto] & x_1 & y_2 \arrow[l, mapsto] \arrow[r, mapsto] & x' \\ 1 \arrow[r, "0"'] & I & 1 \arrow[l, "1"'] \arrow[r, "0"'] & I & 1 \arrow[l, "1"'] \end{tikzcd}$$

Lemma 9.4 implies the existence of a $y \in XI$ such that $y \cdot l = y_1$ and $y \cdot r = y_2$. So the following diagram

$$\begin{tikzcd} x & y_1 \arrow[l, mapsto] & y \arrow[l, mapsto] \arrow[r, mapsto] & x_1 \arrow[r, mapsto] & x' \\ 1 \arrow[r, "0"] \arrow[rr, bend right, "0"'] & I \arrow[r, "l"] & I & I \arrow[l, "r"'] & 1 \arrow[l, "1"'] \arrow[ll, bend left, "1"] \end{tikzcd}$$

shows that x and x' can be connected by a single cospan of the form $0 : 1 \to I \leftarrow 1 : 1$. Iterating the idea we can reduce a combinatorial arc of any length connecting x and x' to a single cospan. ■

We can now formulate our main tool to produce examples of cohesive and sufficiently cohesive Grothendieck toposes.

9.6. Proposition. *Let $(\mathcal{C}, J)$ be a connected and locally connected site such that every object of $\mathcal{C}$ has a point, so that $p : \mathbf{Shv}(\mathcal{C}, J) \to \mathbf{Set}$ is pre-cohesive. Assume that $\mathcal{C}$ is equipped with an abstract interval. If J is compatible with the interval and every object of $\mathcal{C}$ is arcwise connected then p is cohesive and sufficiently cohesive.*

Proof. We must show that $p : \mathbf{Shv}(\mathcal{C}, J) \to \mathbf{Set}$ satisfies Sufficient Cohesion and Continuity. Sufficient Cohesion follows from disjointness of $0, 1 : 1 \to I$ (see [15]). To prove that Continuity holds we must show that the canonical morphism $\gamma : p_!(X^{p^* S}) \to (p_! X)^S$ is an iso.

The fact that γ is epi follows from general considerations about the Nullstellensatz and $\mathbf{Set}$. Indeed, the map $\theta : p_* X \to p_! X$ is epi by the Nullstellensatz, so the product $\theta^S : (p_* X)^S \to (p_! X)^S$ in $\mathbf{Set}$ is also epi by Internal Choice. Moreover, the adjunction $p^* \dashv p_* : \mathcal{E} \to \mathcal{S}$ is strong if we consider $\mathcal{E}$ as an $\mathcal{S}$-category (see p. 129 in [1]) so we have an iso $p_*(X^{p^* S}) \cong (p_* X)^S$. Therefore, the top-right composite in the following commutative diagram

$$\begin{tikzcd} p_*(X^{p^* S}) \arrow[r, "\cong"] \arrow[d, "\theta"'] \arrow[dr] & (p_* X)^S \arrow[d, "\theta^S"] \\ p_!(X^{p^* S}) \arrow[r, "\gamma"'] & (p_! X)^S \end{tikzcd}$$

is epi and hence, the bottom map is also epi.

It remains to show that $\gamma : p_!(X^{p^* S}) \to (p_! X)^S$ is mono. We show that the function $\gamma : p_!(\prod_{s\in S} X) \to \prod_{s\in S} p_! X$ is injective. An element in $p_!(\prod_{s\in S} X)$ may be described as a tensor $(x_s \mid s \in S) \otimes 1$ where $(x_s \mid s \in S)$ is an indexed collection of elements $x_s \in X1$ and γ sends it to the indexed collection $(x_s \otimes 1 \mid s \in S) \in \prod_{s\in S} p_! X$.

Let $(x_s \mid s \in S) \otimes 1$ and $(x'_s \mid s \in S) \otimes 1$ be two elements in the set $p_!(\prod_{s\in S} X)$ such that $(x_s \otimes 1 \mid s \in S) = (x'_s \otimes 1 \mid s \in S)$ in $\prod_{s\in S} p_! X$. This means that for each $s \in S$, $x_s \otimes 1 = x'_s \otimes 1$ in $p_! X$. By Lemma 9.5 there is, for each $s \in S$, a $y_s \in XI$ such that $y_s \cdot 0 = x_s$ and $y_s \cdot 1 = x'_s$. It follows that the collection $(y_s \mid s \in S)$ in $\prod_{s\in S} XI$ witnesses that the cospan $0 : 1 \to I \leftarrow 1 : 1$ connects $(x_s \mid s \in S)$ and $(x'_s \mid s \in S)$ in $\prod_{s\in S} X1$. Hence, $(x_s \mid s \in S) \otimes 1 = (x'_s \mid s \in S) \otimes 1$ in $p_!(\prod_{s\in S} X)$. ∎

10. Three concrete examples

For brevity let us define an *interval* to be a subset $[a, b] \subseteq \mathbb{R}$ with $a \le b \in \mathbb{R}$. Of course, there are two cases: $a = b$ or $a < b$. A function $f : [a, b] \to [c, d]$ is called *linear* if there are $u, v \in \mathbb{R}$ such that $fx = ux + v$ for every $x \in [a, b]$.

Let $\mathbf{L}$ be the category of intervals and linear functions between them. Notice that $\mathbf{L}$ has a terminal object and every object has a point. (Indeed, $\mathbf{L}$ is equivalent to the result of splitting all constant maps in the monoid of linear endos of $[0, 1]$.) Similarly, we can define $\mathbf{A}$ as the category of intervals and polynomial (with coefficients in $\mathbb{R}$) functions between them.

The categories $\mathbf{L}$ and $\mathbf{A}$ have an interval object $([0, 1], 0, 1, l, r)$ with $lx = \frac{x}{2}$ and $rx = \frac{x+1}{2}$. Moreover, all objects are arcwise connected because for every $x, y \in [a, b]$ there is a (unique) linear map $f : [0, 1] \to [a, b]$ such that $f0 = x$ and $f1 = y$.

10.1. Definition. A *partition* of an interval $[a, b] \subseteq \mathbb{R}$ with $a < b$ is a sequence of reals $a < r_1 < \ldots < r_m < b$ for some $m \in \mathbb{N}$. For such a partition we may also denote a by r_0 and b by r_{m+1}.

For any $a < b \in \mathbb{R}$ define $K[a, b]$ to be the set of families

$$([r_i, r_{i+1}] \to [a, b] \mid 0 \le i \le m)$$

of linear inclusions determined by a partition of $[a, b]$ as above. Define also $K1$ to be the set consisting only of the trivial family given by the identity.

10.2. Lemma. *K is a basis for a Grothendieck topology on $\mathbf{L}$ and on $\mathbf{A}$. Moreover, it is connected, locally connected and compatible with the abstract interval described above.*

Proof. Identities cover and it is clear that partitions can be 'composed'. So we need only prove that the 'covering' axiom holds. Let $f : [a, b] \to [c, d]$ be a polynomial map between two intervals in $\mathbb{R}$. Let $c < s < d$ and consider the (finite) set of zeros of the polynomial $fx - s$. This set determines a partition $a < r_1 < \ldots < r_m < b$ and we claim that for every $i \le m$, $f[r_i, r_{i+1}] \subseteq [c, s]$ or $f[r_i, r_{i+1}] \subseteq [s, d]$. For assume otherwise, then there exists a $0 \le j \le m$ such that $f(r_j) < s < f(r_{j+1})$ or $f(r_{j+1}) < s < f(r_j)$. In any case, the intermediate value theorem implies the existence of a $r_j < r < r_{j+1}$ such that $fr = s$. But this is absurd because r would be a 'new' zero of $fx - s$. Using the fact just proved and induction it is easy to show that for every partition $c < s_1 < \ldots < s_n < d$ of $[c, d]$ there is a partition $a < r_1 < \ldots < r_m < b$ such that for every $i \leq m$ there is $j \leq n$ such that $f[r_i, r_{i+1}] \subseteq [s_j, s_{j+1}]$.

The site has a terminal object and the covering families are obviously connected. Finally, the family $\{l, r\}$ is iso to the covering $\{[0, \frac{1}{2}] \to [0, 1], [\frac{1}{2}, 1] \to [0, 1]\}$ in $K[0, 1]$ so K is compatible with the abstract interval. ■

Altogether both sites $(\mathbf{L}, K)$ and $(\mathbf{A}, K)$ satisfy the hypotheses of Proposition 9.6 so they induce cohesive and sufficiently cohesive toposes over $\mathbf{Set}$.

In order to visualize an arbitrary object in the resulting toposes, say $(\mathbf{L}, K)$, we can apply a similar idea to that attributed to Lawvere in p. 239 of [4]. Consider the category $\mathbf{L}$ as consisting of two objects I and 1. For an arbitrary X in $\mathbf{Shv}(\mathbf{L}, K)$, we may think of $X1$ as the set of underlying points and XI as the collection of 'paths' in X. Since each map $1 \to I$ determines a function $XI \to X1$, each 'path' a in XI determines the collection $(a \cdot i \in X1 \mid i : 1 \to I)$ of points of X touched by the path.

The sites $(\mathbf{L}, K)$ and $(\mathbf{A}, K)$ are not subcanonical but it is not difficult to give a subcanonical alternatives. Let us consider the case of $(\mathbf{L}, K)$.

10.3. Definition. Let $f : [a, b] \to [c, d]$ be a continuous map between intervals. A *dissection* of f is a family $(g_v : I_v \to [a, b] \mid v \in V)$ in $K[a, b]$ such that every $fg_v : I_v \to [c, d]$ is in $\mathbf{L}$. The map f is called *(continuous) piecewise-linear* if it has a dissection.

Experience with traditional examples suggests the following fact.

10.4. Lemma. *Identities are piecewise-linear and piecewise-linear maps are closed under composition.*

Proof. Identities are piecewise-linear because the trivial covering family in KB is a dissection of the identity on B. To prove closure under composition let f and g as below

$$B_0 \xrightarrow{g} B_1 \xrightarrow{f} B_2$$

be piecewise-linear, and let the following covering families $(f_u : L_u \to B_1 \mid u \in U) \in KB_1$ and $(g_v : L_v \to B_0 \mid v \in V) \in KB_0$ be corresponding dissections. In particular, the composite $gg_v : L_v \to B_1$ is in $\mathbf{L}$ for every $v \in V$. So, as $(\mathbf{L}, K)$ is a site, there is a covering $(h_{v,w} : L_{v,w} \to L_v \mid w \in W_v) \in KL_v$ such that for every $w \in W_v$ there is a $u \in U$ and a commutative diagram

$$\begin{tikzcd}
L_{v,w} \arrow[rr] \arrow[d, "h_{v,w}"'] & & L_u \arrow[d, "f_u"] \\
L_v \arrow[r, "g_v"'] & B_0 \arrow[r, "g"'] & B_1
\end{tikzcd}$$

in $\mathbf{L}$. Now, the composite $L_u \xrightarrow{f_u} B_1 \xrightarrow{f} B_2$ is also in $\mathbf{L}$ by hypothesis so the composite

$$L_{v,w} \xrightarrow{h_{v,w}} L_v \xrightarrow{g_v} B_0 \xrightarrow{g} B_1 \xrightarrow{f} B_2$$

is also in $\mathbf{L}$, for every $v \in V$ and $w \in W_v$. But the indexed collection

$$(g_v h_{v,w} : L_{v,w} \to B_0 \mid v \in V, w \in W_v)$$

is in KB_0 because K is the basis of a Grothendieck topology on $\mathbf{L}$. Hence, this family is a dissection witnessing that $fg : B_0 \to B_2$ is piecewise-linear. ∎

Let $\mathbf{L}_p$ be the category whose objects are those of $\mathbf{L}$ and whose maps are the piecewise-linear ones. There is an obvious inclusion $\mathbf{L} \to \mathbf{L}_p$.

10.5. Lemma. *The function K is a basis for a Grothendieck topology on $\mathbf{L}_p$ and the induced $\mathbf{Shv}(\mathbf{L}, K) \to \mathbf{Shv}(\mathbf{L}_p, K)$ is an equivalence. Moreover, the site $(\mathbf{L}_p, K)$ is subcanonical.*

Proof. We must prove that the covering axiom holds for the pair $(\mathbf{L}_p, K)$. This can be proved using an argument similar to that of Lemma 10.4. The equivalence between the sheaf toposes will follow from an application of the Comparison Lemma (Theorem C2.2.3 in [6]). This result is formulated in terms of Grothendieck coverages so we give some of the details. Let J_p assign to each object A of $\mathbf{L}_p$ the set J_pA of sieves that contain all the maps in some K-cover. By Exercise III.3 in [14] J_p is a Grothendieck coverage on $\mathbf{L}_p$ and $\mathbf{Shv}(\mathbf{L}_p, K) = \mathbf{Shv}(\mathbf{L}_p, J_p)$. We now prove that $\mathbf{L} \to \mathbf{L}_p$ is a dense subcategory in the sense of Definition C2.2.1 in [6]. Since $\mathbf{L} \to \mathbf{L}_p$ is bijective on objects we need only worry about proving that for any map $f : B' \to B$ in $\mathbf{L}_p$ there is a J_p-covering sieve of B' generated by maps g for which the composite fg is in $\mathbf{L}$. For this, just take any dissection of f. Now the Comparison Lemma implies that restricting along $\mathbf{L} \to \mathbf{L}_p$ induces an equivalence $\mathbf{Shv}(\mathbf{L}_p, J_p) \to \mathbf{Shv}(\mathbf{L}, J)$ where J is the Grothendieck coverage defined by declaring JA to be the set of sieves of the form $S \cap \mathbf{L}$ for some $S \in J_pA$. It is clear that a sieve in $\mathbf{L}$ is in JA if and only if it contains all the maps in some K-cover. In other words, J is the Grothendieck coverage induced by the basis K on $\mathcal{A}$. So $\mathbf{Shv}(\mathbf{L}, J) = \mathbf{Shv}(\mathbf{L}, K)$ and, altogether:

$$\mathbf{Shv}(\mathbf{L}_p, K) = \mathbf{Shv}(\mathbf{L}_p, J_p) \xrightarrow{\cong} \mathbf{Shv}(\mathbf{L}, J) = \mathbf{Shv}(\mathbf{L}, K)$$

is an equivalence.

To prove subcanonicity we must show that $\mathbf{L}_p(\_, A)$ is a K-sheaf for the site $(\mathbf{L}_p, K)$ for any object A in $\mathbf{L}_p$. So let $(g_v : B_v \to B \mid v \in V)$ be a cover of B in K and let $(f_v : B_v \to A \mid v \in V)$ be a compatible family of maps in $\mathbf{L}_p$. The K-coverings are finite closed coverings in the classical sense and so they are effective epimorphic in **Top** (Proposition 4 § 3.2 in [2]). Hence, there exists a unique continuous $f : B \to A$ amalgamating the f_v's. Moreover, the fact that each of the f_v's is piecewise-linear implies that f also is. ∎

An analogous argument produces a subcanonical site for $\mathbf{Shv}(\mathbf{A}, K)$ in terms of piecewise-polynomial maps.

An alternative argument might show that the full image of $\mathbf{L} \to \widehat{\mathbf{L}} \to \mathbf{Shv}(\mathbf{L}, K)$ coincides with $\mathbf{L}_p$. The topos $\mathbf{Shv}(\mathbf{L}, K) = \mathbf{Shv}(\mathbf{L}_p, K)$ may be a natural setting to develop piecewise linear topology as treated in [16]. We still do not have enough evidence to support this claim, but the following seems relevant.

Let us denote the pre-cohesive topos of simplicial sets by $p : \widehat{\Delta} \to \mathbf{Set}$ and the 'piecewise linear topos' above by $f : \mathbf{Shv}(\mathbf{L}, K) \to \widehat{\Delta}$.

10.6. Proposition. *There exists a geometric morphism $g : \mathbf{Shv}(\mathbf{L}, K) \to \widehat{\Delta}$ such that the diagram on the left below commutes*

$$\begin{tikzcd} \mathbf{Shv}(\mathbf{L}, K) \arrow[r, "g"] \arrow[dr, "f"'] & \widehat{\Delta} \arrow[d, "p"] \\ & \mathbf{Set} \end{tikzcd} \qquad \begin{tikzcd} p_*(g_* X) \arrow[r, "\theta"] \arrow[d, "="'] & p_!(g_* X) \arrow[d, "\lambda"] \\ f_* X \arrow[r, "\theta"'] & f_! X \end{tikzcd}$$

commutes and, also, g 'preserves pieces' in the sense that there exists a natural iso $\lambda : p_! g_* \to f_!$ such that the square on the right above commutes for every X in $\mathbf{Shv}(\mathbf{L}, K)$.

Proof. Let us denote $\mathbf{Shv}(\mathbf{L}, K) = \mathbf{Shv}(\mathbf{L}_p, K)$ by $\mathcal{F}$. To prove the existence of the morphism $g : \mathcal{F} \to \widehat{\Delta}$ recall that $\widehat{\Delta}$ classifies total orders with distinct endpoints. For brevity let us call these structures *orders*, as in VIII.8 of [14]. (See also Section 7 in [4].)

It is clear from the discussion of $\mathcal{F}$ above that the sheafification of $\mathbf{L}(\_, I)$ in $\widehat{\mathbf{L}}$ is $\mathbf{L}_p(\_, I)$ in $\mathcal{F}$. For comfort we write $\mathtt{I}$ instead of $\mathbf{L}(\_, I)$. Let us first prove that $\mathtt{I}$ is a partial order with distinct endpoints $0, 1 : 1 \to \mathtt{I}$ in $\widehat{\mathbf{L}}$. Let $PI \subseteq \mathtt{I}^2 I = \mathbf{L}(I, I) \times \mathbf{L}(I, I)$ be the subset of those pairs (α, β) of linear maps such that for every $x \in I = [0, 1]$, $\alpha x \leq \beta x$ where '$\leq$' here is the standard linear order of $[0, 1]$. It is easy to check that this defines a presheaf P on $\mathbf{L}$ such that the subobject $P \to \mathtt{I}^2$ is a partial order with the intended (distinct) endpoints. The sheafification functor $\mathbf{a} : \widehat{\mathbf{L}} \to \mathcal{F}$ preserves this structure so $\mathbf{a}P \to (\mathbf{a}\mathtt{I})^2$ is a partially ordered set in $\mathcal{F}$ with distinct endpoints. To prove that it is a linear order we must show that $\mathbf{a}P + (\mathbf{a}P)^{\mathrm{op}} \to (\mathbf{a}\mathtt{I})^2$ is epi in $\mathcal{F}$, where $(\mathbf{a}P)^{\mathrm{op}} \to (\mathbf{a}\mathtt{I})^2$ is the opposite of the partial order $\mathbf{a}P \to (\mathbf{a}\mathtt{I})^2$. In turn, it is enough to show that $P + P^{\mathrm{op}} \to \mathtt{I}^2$ is locally surjective in $\widehat{\mathbf{L}}$ (see III.7.6 in [14]). The intuition is clear: $\mathtt{I}^2$ is a 'unit square', $P \to \mathtt{I}^2$ is the 'closed triangle' above the diagonal and $P^{\mathrm{op}} \to \mathtt{I}^2$ is the closed triangle below the diagonal.

To prove local surjectivity let us consider a 'path in the square' $t = \langle \alpha, \beta \rangle : \mathtt{I} \to \mathtt{I}^2$. Such a 'path' is just an element $(\alpha, \beta) \in \mathtt{I}^2 I = \mathbf{L}(I, I) \times \mathbf{L}(I, I)$. For definiteness we fix $a, b, c, d \in \mathbb{R}$ such that $\alpha x = ax + b$ and $\beta x = cx + d$ for all $x \in I = [0, 1]$. If $\alpha x \leq \beta x$ for every $x \in I$ then t factors through P. On the other hand, if $\beta x \leq \alpha x$ for every $x \in I$ then t factors through P^{op}. So let us analyse in more detail what happens if t 'touches' the diagonal. That is, let us assume that there exists an $r \in I$ such that $\alpha r = \beta r$. Then $ar + b = cr + d$ and so $(a - c)r = d - b$. Now either $a = c$ or $a \neq c$. If $a = c$ then $0 = d - b$

so $b = d$ and in this case t is the diagonal, which factors through P and also through $P^{\rm op}$. In the case that, $a \neq c$ then $r = \frac{d-b}{a-c}$ and we can consider the K-cover on I determined by the partition $[0,r] \to \mathtt{I} \leftarrow [r,1]$. We claim that the restriction t_0 of t to $[0,r]$ factors through P and that the restriction t_1 of t to $[r,1]$ factors through $P^{\rm op}$. Indeed, if $x \leq r$ then $x(a-c) \leq d-b$ and so $ax + b \leq cx + d$ which means that $\alpha x \leq \beta x$ for every $x \leq r$, and hence that $t_0 : [0,r] \to \mathtt{I}^2$ factors through P. Similarly, $t_1 : [r,1] \to \mathtt{I}^2$ factors through $P^{\rm op}$. Altogether, $\mathbf{a}\mathtt{I} = \mathbf{L}_p(\_, I)$ in $\mathcal{F}$ can be equipped with the order structure described above and so there exists a geometric morphism $g : \mathcal{F} \to \widehat{\Delta}$ over $\mathbf{Set}$.

To prove the second part of the statement let $[1]$ in Δ be the total order with two elements. By Theorem VIII.8.5 in [14] the representable $\Delta(\_,[1])$ is the universal order and so we may assume that $g^*(\Delta(\_,[1])) = \mathbf{a}\mathtt{I} = \mathbf{L}_p(\_, I)$ in $\mathcal{F}$. By Example 8.10 $\Delta(\_,[1])$ is a connector for $p : \widehat{\Delta} \to \mathbf{Set}$ and by Lemma 8.13 $\mathbf{L}_p(\_, I)$ is a connector for $f : \mathcal{F} \to \mathbf{Set}$. So the two conspicuous horizontal forks below are coequalizers

$$\begin{tikzcd}
p_*((g_*X)^{\Delta(_,[1])}) \arrow[rr, shift left, "p_*ev_0"] \arrow[rr, shift right, "p_*ev_1"'] \arrow[d, "\cong"'] & & p_*(g_*X) \arrow[r, "\theta"] \arrow[d, "="'] & p_!(g_*X) \arrow[d] \\
f_*(X^{g^*\Delta(_,[1])}) \arrow[r, "="'] & f_*(X^{\mathbf{L}_p(_,I)}) \arrow[r, shift left, "f_*ev_0"] \arrow[r, shift right, "f_*ev_1"'] & f_*X \arrow[r, "\theta"] & f_!X
\end{tikzcd}$$

for every X in $\mathcal{F}$. Also, for general reasons, there is a canonical iso $p_*((g_*X)^T) \to f_*(X^{g^*T})$ for every X in $\mathcal{F}$ and T in $\widehat{\Delta}$. So the left rectangle above commutes and, since the vertical maps are isos, the induced right vertical map is also an iso and makes the right square commute. ∎

The cohesive $\mathcal{F} \to \mathbf{Set}$ and others suggested below should be contrasted with the *topological topos* $\mathcal{J}$ described in [4] and in Example 4.4. Although the details are still to be checked, Proposition 10.6 suggests that our topos $\mathcal{F}$ contains a reasonable category of polyhedra and continuous piecewise linear maps between them, and that the geometric morphism $g : \mathcal{F} \to \widehat{\Delta}$ behaves as a 'geometric-realization/singular-complex' adjunction. Of course, the topos $\mathcal{F}$ is not as fully related to classical topology as $\mathcal{J}$. On the other hand, $\mathcal{J} \to \mathbf{Set}$ is not pre-cohesive.

(The 'pieces preserving' geometric morphisms between pre-cohesive toposes hinted at in Proposition 10.6 are the theme of joint research with F. Marmolejo.)

As a third example, we may also consider the category $\mathbf{C}$ with the same objects as $\mathbf{L}$ (and $\mathbf{A}$) and, as morphisms, all the continuous $f : A \to B$ such that for every $(h_v : B_v \to B \mid v \in V) \in KB$ there exists a $(g_u : A_u \to A \mid u \in U) \in KA$ such that each fg_u factors through some h_v. The pair $(\mathbf{C}, K)$ is obviously a site and the induced $\mathbf{Shv}(\mathbf{C}, K) \to \mathbf{Set}$ is cohesive and sufficiently cohesive by Proposition 9.6. We don't know much about the continuous functions defining the site, but there it is. Facing these examples one naturally wonders about the existence of other intermediate monoids $\mathbf{A} \subseteq M \subseteq \mathbf{C}$ inducing cohesive toposes.

Also, the reader is invited to check if there are analogues of Proposition 10.6 for the toposes $\mathbf{Shv}(\mathbf{A}, K)$ and $\mathbf{Shv}(\mathbf{C}, K)$.

Finally, it would be good if (at least some of) the results in this paper could be proved over an arbitrary base. In particular, it should be possible to prove a version of Theorem 7.4. Of course, the general aim is to understand the Continuity condition over an 'arbitrary' base and without boundedness conditions. Recent progress to be discussed elsewhere shows that the codomain of a cohesive geometric morphism must satisfy internal choice. On the other hand, we don't know if the pre-cohesive examples over atomic toposes discussed in [15] satisfy Continuity.

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