A COMMENT ON THE EQUATION OF STATE FOR A DEGENERATED GAS

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In current literature on degenerated stellar configurations there seems to exist a strange misconception concerning the equations for relativistic degeneracy.

As it is well known in a non-relativistic degenerated gas we can neglect the partial pressure of heavy particles due to the fact that in the equation of state for non-relativistic degeneracy the mass of the particles is contained in the denominator: with the usual notation,

$$P = \frac{1}{20}\left(\frac{6}{\pi}\right)^{2/3}\frac{h^2}{m g^{2/3}}\, n^{5/3}$$

for a non-relativistic completely degenerated Fermi gas. The statistical weight g is equal to 2 for electrons, protons, etc. As the mass of electrons is, of course, much smaller than that of protons, practically P is equal to the partial pressure of free electrons and one can put, as usual,

$$P = P_e \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_e}\, n_e^{5/3}$$

Defining μ_e by means of the equation

$$n_e = \frac{\rho}{\mu_e H}$$

equation (2) is written

$$P = K_1 \rho^{5/3}$$

where

$$K_1 = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_e H^{5/3}} \quad \frac{1}{\mu_e^{5/3}} = 1{,}0042 \times 10^{13} \mu_e^{-\frac{5}{3}}$$

However, the equation for a relativistic completely degenerated gas does not contain the mass of the particles. This is physically understandable because in the relativistic case the energy of the particles is much larger than their rest energy. For a fully degenerated relativistic Fermi gas, we have, in fact, taking again $g = 2$,

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hc\, n^{4/3}$$

As a consequence, we can no longer neglect the partial pressure of the heavy particles and if we wish to use the density ρ we ought to write equation (6) in the form

$$P = K_2 \rho^{4/3}$$

where

$$K_2 = \frac{1}{8} \cdot \frac{3^{1/3}}{\pi} \frac{hc}{H^{4/3}} \cdot \frac{1}{\mu^{4/3}} = 1{,}2441 \times 10^{15} \mu^{-4/3}$$

where the mean molecular weight μ is now defined by

$$n = \frac{\rho}{\mu H}$$

The point is that, as far as I know, in all books on stellar structure one finds written μ_e instead of μ in equation (8), which is certainly incorrect.

On the other hand, the pressure of a Bose gas is zero at $T = 0$, whatever its density, which is, of course, quite clear from the fact that the particles of a Bose gas do not obey Pauli's exclusion principle.

I do not think that the above misconception may have any important consequence upon the current ideas on the structure of white dwarfs. However, it may be worth mentioning that if white dwarfs contained a large amount of hydrogen, then it should be necessary to take into account the correct form of the constant K_2. For instance, the well known limiting mass for a completely degenerated star should be considerably increased.

If, on the other hand, it is assumed that white dwarfs are practically devoid of H, then it is correct to use μ_e in equation (8), since the most abundant nuclei (H_e^4, C^{12}, N^{14}, O^{16},...) obey Bose statistics and therefore their partial pressure in a fully degenerated gas is zero. In this latter case the current theory of white dwarfs still holds true.

It is, however, worth mentioning that this theory can in no way be used to compute the hydrogen content of white dwarfs, since it already supposes that these stars do not contain any hydrogen at all. Of course, other arguments, concerning for instance nuclear reactions, are perfectly valid.