The structure of minimizers of the frame potential on fusion frames

Pedro G. Massey, Mariano A. Ruiz and Demetrio Stojanoff*

Depto. de Matemática, FCE-UNLP, La Plata, Argentina and IAM-CONICET
e-mail: massey@mate.unlp.edu.ar , maruiz@mate.unlp.edu.ar and demetrio@mate.unlp.edu.ar

Abstract

In this paper we study the fusion frame potential that is a generalization of the Benedetto-Fickus (vectorial) frame potential to the finite-dimensional fusion frame setting. We stydy the structure of local and global minimizers of this potential, when restricted to suitable sets of fusion frames. These minimizers are related to tight fusion frames as in the classical vector frame case. Still, tight fusion frames are not as frequent as tight frames; indeed we show that there are choices of parameters involved in fusion frames for which no tight fusion frame can exist. We exhibit necessary and sufficient conditions for the existence of tight fusion frames with prescribed parameters, involving the so-called Horn-Klyachko's compatibility inequalities. The second part of the work is devoted to the study of the minimization of the fusion frame potential on a fixed sequence of subspaces, with a varying sequence of weights. We related this problem to the index of the Hadamard product by positive matrices and use it to give different characterizations of these minima.

Keywords and phrases: Fusion frames, frame potential, majorization, Hadamard product.

Mathematics subject classification (2000): 42C15, 15A60.

Contents

*Partially supported by CONICET (PIP 5272/05) and Universidad de La PLata (UNLP 11 X472).

1 Introduction

Fusion frames were introduced under the name of "frame of subspaces" in [8]. They are a generalization of the usual frames of vectors for a Hilbert space $\mathcal{H}$; indeed frames of vectors can be treated as "one-dimensional fusion frames". During the last years, the theory of fusion frames has been a fast-growing area. Several applications of fusion frames have been studied, for example, sensor networks [11], neurology [19], coding theory [4], [5] , [16], among others. In particular, applications which require distributed processing can be well described and studied using fusion frames. We refer the reader to [10] and the references therein, for a detailed treatment of the fusion frame theory. Further developments can be found in [7], [9] and [20].

In the finite dimensional setting, a fusion frame is a sequence of subspaces of $\mathbb{F}^n$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$) together with a set of positive weights such that the weighted sum of the orthogonal projections to these subspaces (called *fusion frame operator*) is a positive invertible operator (see Definition 2.1.1). As in the case of vector frames, it is usually desired that this invertible operator be a multiple of the identity. In this case, the frame is called a tight fusion frame (TFF). However, tight fusion frames might not exist for a fixed choice of dimensions $\mathbf{d} = (d_i)_{i\in\mathbb{I}_m}$ of the subspaces, for any sequence of weights $w = (w_i)_{i\in\mathbb{I}_m}$ (see the discussion following Proposition 3.1.1).

Inspired by the work in classical frame theory (see [2, 18]), we study a convex functional on fusion frame operators (the FF potential or FFP), also studied in [7], which generalizes the Benedetto-Fickus frame potential. In this paper we analyze its local and global minima. This study is motivated by the fact that local (global) minimizers characterize unit norm tight vector frames (see [2, 4, 5, 6, 18]). Since the FF potential can be seen as a "measure of orthogonality" of the frame vectors, it also provides an interesting geometrical description of fusion frames. These considerations motivate the study of this type of minimizations in the fusion frame context. It should be pointed out that a related study can be found in [7]. The authors became aware of this work in an advanced stage of writing this paper.

The main tool used in [18] for these problems in the case of vector frames, namely majorization of matrices, can be replaced in the context of fusion frames by the theory developed by Horn and Klyachko in order to have a spectral characterization of Hermitian matrices which are the sum of a set of Hermitian matrices. For example, this approach provides necessary and sufficient conditions for the existence of TFF's, summarized in a family of inequalities. Although this technique seems to be rather impractical due to the complex conditions involved, it becomes a useful tool in the study of the spectral structure of FF potential minimizers.

We first consider the problem of existence of TFF's. We show some dimensional restrictions on the subspaces regarding this problem, and then give equivalent conditions for the case of fixed dimensions and weights. We refer to [7] for further developments in this direction.

The rest of the paper deals with the minimization of the FF potential on some sets of Fusion Bessel sequences (i.e. sets of projections and weights whose fusion frame operator is not necessary invertible). Mainly, we work on Bessel sequences with fusion frame operator of trace one. This is a natural restriction in order to avoid scalar multiplications, and it allows an interpretation of the FF potential as a measure of the distance between the fusion frame operator and a suitable multiple of the identity corresponding to the (possibly non-existing) tight fusion frames with trace one. A detailed discussion of this approach can be found in Subsection 2.2.

The minimization of the FFP is done in three different settings: first, by fixing the weights w and the dimensions $\mathbf{d}$ of the subspaces. Then, by fixing only the dimensions $\mathbf{d}$. Finally, we consider a fixed sequence of subspaces $(W_i)_{i\in\mathbb{I}_m}$, and optimize it over the set of admissible weights. In the three cases, minimization is made under the previously mentioned "trace one" restriction.

For the first problem, a geometrical approach similar to that done in [18] allows us to obtain a characterization of local minimizers of the FF potential: they are orthogonal sums of tight frames on each eigenspace of the frame operator. Then, using Horn and Klyachko techniques, we prove that all minimizers (even those which are local minimizers) have the same eigenvalues, with the same multiplicities. Similar results are obtained for the second problem (by fixing only the dimensions of the subspaces).

The last section of the paper is devoted to the study of the optimization of the fusion frame potential of fusion frames obtained from a fixed sequence $\mathcal{V}$ of subspaces within $\mathbb{C}^n$ which generate the whole space. Since every sequence of weights makes $\mathcal{V}$ a fusion frame, we seek the best choice of weights, meaning those which minimize (globally) the FF potential. Then, we establish a connection between this optimization problem and the *Hadamard indexes* of a kind of Gram matrix associated to the fixed subspaces. These indexes are studied in [12], and they involve the Hadamard or entry-wise product of matrices. Using these tools, we get a characterization of the set of optimal weights, and a way to compute them under some reasonable assumptions on the initial sequence of subspaces. This analysis seems to be new even for the case of vector frames.

However, as it is shown by an example, minimizers could "erase" some of the initial subspaces (i.e. the set of optimal weights could have zeros). Moreover, it is possible to obtain a minimizer which is a Bessel sequence of subspaces which stops being generating, a phenomenon which does not happen in the previous settings. Motivated by this problem, we study the geometry of the set of all weights w which minimize the FF potential, and in particular their possible supports (namely, those sub-indexes i such that $w_i > 0$). At the end of the section, we present some examples which illustrate these type of anomalies.

The paper is organized as follows: Section 2 contains preliminary definitions on fusion frames and the basic notation used throughout the paper. This section also contains a brief exposition of Horn-Klyachko's compatibility inequalities. Section 3 is devoted to the study of minimizers of the FF potential, restricted to the sets of fusion frames detailed before. In Section 4 we analyze the problem of minimizing the FF potential for a fixed sequence of subspaces, varying the weights. The paper ends with an appendix containing definitions and several results concerning Hadamard indexes of positive matrices, which are related to the contents of Section 4.

We wish to thank to P. Casazza and M. Fickus for letting us know about their excellent work [7], which is closely related to the present paper.

2 Preliminaries and Notations.

In this paper $\mathcal{M}_n(\mathbb{C})$ denotes the algebra of complex $n\times n$ matrices, $\mathcal{G}l(n)$ the group of all invertible elements of $\mathcal{M}_n(\mathbb{C})$, $\mathcal{U}(n)$ the group of unitary matrices, $\mathcal{M}_n(\mathbb{C})_{sa}$ (resp. $\mathcal{M}_n(\mathbb{C})_{ah}$) denotes the real subspace of Hermitian (resp. anti-Hermitian) matrices, $\mathcal{M}_n(\mathbb{C})^+$ the set of positive semidefinite matrices, and $\mathcal{G}l(n)^+ = \mathcal{M}_n(\mathbb{C})^+ \cap \mathcal{G}l(n)$. Given $T \in \mathcal{M}_n(\mathbb{C})$, $R(T)$ denotes the image of T, $N(T)$ the null space of T, $\sigma(T)$ the spectrum of T, $\operatorname{tr} T$ the trace of T, and $\operatorname{rk} T$ the rank of T.

We write $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_{>0} = \mathbb{R}_+ \setminus \{0\}$. On the other hand, given $m \in \mathbb{N}$ we denote by $\mathbb{I}_m = \{1, \ldots, m\}$ and by $\mathbb{1} = \mathbb{1}_m \in \mathbb{R}^m$ the vector with all its entries equal to 1. Given a vector

$v \in \mathbb{R}^n$, $\operatorname{diag}(v) \in \mathcal{M}_n(\mathbb{C})$ is the diagonal matrix with v in its diagonal, and $v^\downarrow \in \mathbb{R}^n$ is the vector obtained by re-arrangement of the coordinates of v in non-increasing order. If $T \in \mathcal{M}_n(\mathbb{C})_{sa}$, we denote by $\lambda(T) \in \mathbb{R}^n$ the vector of eigenvalues of T, counted with multiplicities, in such a way that $\lambda(T) = \lambda(T)^\downarrow$.

Given a subspace $W \subseteq \mathbb{C}^n$, we denote by $P_W \in \mathcal{M}_n(\mathbb{C})^+$ the orthogonal projection onto W, i.e. $R(P_W) = W$ and $N(P_W) = W^\perp$. For vectors on $\mathbb{C}^n$ we shall use the Euclidean norm, but for matrices $T \in \mathcal{M}_n(\mathbb{C})$, we shall use both the spectral norm $\|T\| = \|T\|_{sp} = \max_{\|x\|=1} \|Tx\|$, and the Frobenius norm $\|T\|_2 = (\operatorname{tr} T^*T)^{\frac{1}{2}} = \big(\sum_{i,j\in\mathbb{I}_n} |T_{ij}|^2 \big)^{\frac{1}{2}}$. This norm is induced by the inner product $\langle A, \ B\rangle = \operatorname{tr}(B^*A)$, for $A, B \in \mathcal{M}_n(\mathbb{C})$.

2.1 Frames of subspaces, or fusion frames for $\mathbb{C}^n$

We begin by defining the basic notions of fusion frame theory in the finite-dimensional context. For an introduction to fusion frames for general Hilbert spaces, see [8], [10] or [20] . Briefly, a fusion frame for $\mathbb{C}^n$ is a generating sequence of subspaces, equipped with weights assigned to each subspace. Nevertheless, we prefer to give the "frame style" definition, which adjusts better to our purposes.

Definition 2.1.1. Let $\mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m}$ be closed subspaces of $\mathcal{H} \cong \mathbb{C}^n$, and $w = \{w_i\}_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{>0}$. The sequence $\mathcal{W}_w = (w_i\,, W_i)_{i\in\mathbb{I}_m}$ is a *fusion frame (FF) for* $\mathcal{H}$, if there exist $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{i\in\mathbb{I}_m} w_i^2\, \|P_{W_i} f\|^2 \le B\|f\|^2 \quad \text{for every } f \in \mathcal{H}\,. \tag{1}$$

If only the right-hand side inequality in Eq. (1) holds, then we say that $\mathcal{W}_w$ is a *Bessel sequence of subspaces* (BSS) for $\mathcal{H}$. The *frame operator* of $\mathcal{W}_w$ is defined by the formula

$$S_{\mathcal{W}_w} = \sum_{i\in\mathbb{I}_m} w_i^2\, P_{W_i} \in \mathcal{M}_n(\mathbb{C})^+\,. \tag{2}$$

Observe that $\mathcal{W}_w$ is a FF if and only if $S_{\mathcal{W}_w} \in \mathcal{G}l(n)^+$ and, in this case $A\, I_n \le S_{\mathcal{W}_w} \le B\, I_n$. We say that $\mathcal{W}_w$ is a *tight* FF (TFF) if $A = B$, in other words, if $S_{\mathcal{W}_w} = AI_n$.

Remark 2.1.2. Let $\mathcal{W}_w = (w_i\,, W_i)_{i\in\mathbb{I}_m}$ be a BSS for $\mathbb{C}^n$. For each $i \in \mathbb{I}_m$, we can take an orthonormal basis (ONB) $\mathcal{B}_i = \{e_j^{(i)}\}_{j\in J_i}$ of W_i. Hence, for every $f \in \mathcal{H}$, we have that

$$P_{W_i} f = \sum_{j\in J_i} \langle f, e_j^{(i)}\rangle\, e_j^{(i)}\,, \quad \text{for} \quad i \in \mathbb{I}_m \quad \Longrightarrow \quad S_{\mathcal{W}_w} f = \sum_{i\in\mathbb{I}_m} w_i^2 P_{W_i} f = \sum_{i\in\mathbb{I}_m} \sum_{j\in J_i} \langle f, w_i\, e_j^{(i)}\rangle\, w_i\, e_j^{(i)}\,.$$

Therefore, $\mathcal{W}_w$ induces a vector Bessel sequence $\mathcal{F} = \{w_i\, e_j^{(i)} : \ i \in \mathbb{I}_m\,, \ j \in J_i\}$ which has a very useful property: Its frame operator $S_{\mathcal{F}} = S_{\mathcal{W}_w}$.

2.2 Sets of fusion frames and the FF-potential

We shall establish some notation regarding sets of BSS's and FF's:

Definition 2.2.1. Fix $n, m \in \mathbb{N}$ and consider a Hilbert space $\mathcal{H} \cong \mathbb{C}^n$.

1. Let $\mathcal{B}_{m,n}$ the set of all BSS's of the form $\mathcal{W}_w = (w, \mathcal{W}) = (w_i\,, W_i)_{i\in\mathbb{I}_m}$, where $w \in \mathbb{R}^m_+$ and $\mathcal{W}$ a sequence of m subspaces of $\mathcal{H} \cong \mathbb{C}^n$.

2. Given a sequence $\mathbf{d} \in \mathbb{N}^m$ such that $\operatorname{tr} \mathbf{d} = \sum_{i \in \mathbb{I}_m} d_i \geq n$, let

$$\mathcal{B}_{m,n}(\mathbf{d}) = \left\{ \mathcal{W}_w \in \mathcal{B}_{m,n} \; : \; \dim W_i = d_i \quad \text{for every} \quad i \in \mathbb{I}_m \right\}, \tag{3}$$

3. Given $\mathbf{d} \in \mathbb{N}^m$ and $v \in \mathbb{R}^m_{>0}$, let

$$\mathcal{B}_{m,n}(\mathbf{d}, v) = \{ \mathcal{W}_w \in \mathcal{B}_{m,n} : \; \dim W_i = d_i \quad \text{for every} \quad i \in \mathbb{I}_m \quad \text{and} \quad w = v \} \tag{4}$$

   the subset of $\mathcal{B}_{m,n}(\mathbf{d})$ of BBS's with a fixed sequence of weights v.

4. Finally, let

$$\mathcal{B}^1_{m,n}(\mathbf{d}) = \left\{ \mathcal{W}_w \in \mathcal{B}_{m,n} : \; \dim W_i = d_i \text{ for every } i \in \mathbb{I}_m \quad \text{and} \quad \sum_{i \in \mathbb{I}_m} w_i^2 \, d_i = 1 \right\}, \tag{5}$$

   the set of those $\mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d})$ such that $\operatorname{tr} S_{\mathcal{W}_w} = 1$.

5. We say that a pair $(\mathbf{d}, w) \in \mathbb{N}^m \times \mathbb{R}^m_{>0}$ is *normalized* if $\operatorname{tr} \mathbf{d} \geq n$ and $\sum_{i \in \mathbb{I}_m} w_i^2 \, d_i = 1$. Observe that $(\mathbf{d}, w)$ is normalized if and only if $\mathcal{B}_{m,n}(\mathbf{d}, w) \subseteq \mathcal{B}^1_{m,n}(\mathbf{d})$.

6. The sets of fusion frames with the same restrictions will be denoted by replacing the letter $\mathcal{B}$ by $\mathcal{S}$. More precisely, we consider:

   (a) The set $\mathcal{S}_{m,n} = \{ \mathcal{W}_w \in \mathcal{B}_{m,n} : \mathcal{W}_w \text{ is a FF } \}$. In other words, a sequence $\mathcal{W}_w \in \mathcal{S}_{m,n}$ if $w \in \mathbb{R}^m_{>0}$ and $\mathcal{W}$ a generating sequence of subspaces of $\mathcal{H}$. In a similar fashion:
   (b) For a fixed $\mathbf{d} \in \mathbb{N}^m$, the set $\mathcal{S}_{m,n}(\mathbf{d}) = \{ \mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d}) : \mathcal{W}_w \text{ is a FF } \}$.
   (c) Given a pair $(\mathbf{d}, v) \in \mathbb{N}^m \times \mathbb{R}^m_{>0}$, $\mathcal{S}_{m,n}(\mathbf{d}, v) = \{ \mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d}, v) : \mathcal{W}_w \text{ is a FF } \}$.
   (d) If $\mathbf{d} \in \mathbb{N}^m$, the set $\mathcal{S}^1_{m,n}(\mathbf{d}) = \{ \mathcal{W}_w \in \mathcal{B}^1_{m,n}(\mathbf{d}) : \mathcal{W}_w \text{ is a FF } \}$.

The following definition is suggested by the classical Benedetto-Fickus potential, whose value in a vector frame $\mathcal{F}$ can be calculated as $\operatorname{FP}(\mathcal{F}) = \operatorname{tr} S^2_{\mathcal{F}}$.

Definition 2.2.2. Given $\mathcal{W}_w = (w_i, W_i)_{i \in \mathbb{I}_m} \in \mathcal{B}_{m,n}$, the Benedetto-Fickus *fusion frame potential* (FF-potential or FFP) of $\mathcal{W}_w$ is given by:

$$\operatorname{FFP}(\mathcal{W}_w) = \sum_{i,j=1}^{m} w_i^2 w_j^2 \operatorname{tr}(P_{W_i} P_{W_j}) = \operatorname{tr} \, S^2_{\mathcal{W}_w} \,. \tag{6}$$

Observe that, if $\dim W_i = 1$ for every $i \in \mathbb{I}_m$, then $\operatorname{FFP}(\mathcal{W}_w) = \operatorname{FP}(\mathcal{F})$, for any vector Bessel sequence $\mathcal{F}$ obtained from $\mathcal{W}_w$ as in Remark 2.1.2.

The scope of this paper is to study minimizers of the FF-potential. In order to avoid scalar multiplications (note that $\operatorname{FFP}(\mathcal{W}_{t \cdot w}) = t^4 \operatorname{FFP}(\mathcal{W}_w))$ we shall restrict ourselves to minimize the FF-potential on subsets of $\mathcal{B}^1_{m,n}(\mathbf{d})$, for $\mathbf{d}$ and m fixed. In other words, we shall minimize the FF-potential for those sequences $\mathcal{W}_w$ such that $\operatorname{tr} S_{\mathcal{W}_w} = 1$.

This specific restriction is justified because, if there exist tight FF's in $\mathcal{B}_{m,n}(\mathbf{d})$, then their FF-potential and frame bounds are determined exactly by the trace of their frame operators. Namely, if $\mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d})$ is tight, and $\operatorname{tr} S_{\mathcal{W}_w} = a$, then $S_{\mathcal{W}_w} = \frac{a}{n} I_n$ and $\operatorname{FFP}(\mathcal{W}_w) = \frac{a^2}{n}$.

Even in the case that there are no TFF's in $\mathcal{B}_{m,n}(\mathbf{d})$, this restriction seems to be quite natural. Indeed, for BSS's with fixed trace, the FF-potential can be seen as a measure of the (Frobenius) distance of their frame operators to a *fixed* multiple of the identity:

Proposition 2.2.3. *Let $\mathcal{W}_w \in \mathcal{B}^1_{m,\,n}(\mathbf{d})$. Then*

$$\left\| \tfrac{1}{n}\, I_n - S_{\mathcal{W}_w} \right\|_2^2 = \operatorname{tr} S^2_{\mathcal{W}_w} \; - \; \tfrac{1}{n} \; = \; FFP\,(\mathcal{W}_w) \; - \; \tfrac{1}{n}\,.$$

Proof. Since $\operatorname{tr} S_{\mathcal{W}_w} = 1$, a direct computation shows that

$$\left\| \tfrac{1}{n}\, I_n - S_{\mathcal{W}_w} \right\|_2^2 = \operatorname{tr}\left(\tfrac{1}{n^2}\, I_n - \tfrac{2}{n}\, S_{\mathcal{W}_w} + S^2_{\mathcal{W}_w} \right) = \operatorname{tr} S^2_{\mathcal{W}_w} \; - \; \tfrac{1}{n}\,. \qquad \square$$

The last result shows that if there exist tight FF's in $\mathcal{B}^1_{m,\,n}(\mathbf{d})$, then they are the unique global minimizers of the FF-potential on $\mathcal{B}^1_{m,\,n}(\mathbf{d})$. But in the case that there are no TFF's in $\mathcal{B}^1_{m,\,n}(\mathbf{d})$, the minimization of the FF-potential becomes more interesting: it provides the elements of $\mathcal{B}^1_{m,\,n}(\mathbf{d})$ that can be expected to have the best properties.

In this paper we deal mostly with these type of minimizations under two different further restrictions: we work in the set $\mathcal{B}_{m,\,n}(\mathbf{d}, w) \subseteq \mathcal{B}^1_{m,\,n}(\mathbf{d})$ for a fixed normalized pair $(\mathbf{d}\,,\, w)$, or we fix a generating sequence $\mathcal{W}$ of subspaces, and minimize the FF-potential over all sequences of weights $w \in \mathbb{R}^m_+$ such that $\mathcal{W}_w \in \mathcal{B}^1_{m,\,n}(\mathbf{d})$.

2.3 Klyachko-Fulton approach

Recall that given $x \in \mathbb{R}^n$, we denote by $x^\downarrow \in \mathbb{R}^n$ the vector obtained by rearranging the coordinates of x in non-increasing order. Given $x,\, y \in \mathbb{R}^n$ we say that x is *submajorized* by y, and write $x \prec_w y$, if $\sum\limits_{i=1}^{k} x^\downarrow_i \le \sum\limits_{i=1}^{k} y^\downarrow_i$ for every $k \in \mathbb{I}_n$. If we further have that $\operatorname{tr}(x) = \sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i$ then we say that x is *majorized* by y, and write $x \prec y$.

Example 2.3.1. As an elementary example, that we shall use in what follows, let $x \in \mathbb{R}^n_+$ and $0 \le a \le \operatorname{tr}(x) \le b$. The reader can easily verify that $\frac{a}{n}\, \mathbb{1}_n \prec_w x \prec_w b\, e_1$.

(Sub)majorization between vectors is extended by T. Ando in [1] to (sub)majorization between self-adjoint matrices as follows : given $A,\, B \in \mathcal{M}_n(\mathbb{C})_{sa}$, we say that A is *submajorized* by B, and write $A \prec_w B$, if $\lambda(A) \prec_w \lambda(B)$. If we further have that $\operatorname{tr}(A) = \operatorname{tr}(B)$ then we say that A is *majorized* by B and write $A \prec B$.

Although simple, submajorization plays a central role in optimization problems with respect to convex functionals and unitarily invariant norms, as the following result shows (for a detailed account in majorization see Bhatia's book [3]).

Theorem 2.3.2. Let $A,\, B \in \mathcal{M}_n(\mathbb{C})^{sa}$. Then, the following statements are equivalent:

1. $A \prec_w B$.
2. For every unitarily invariant norm $\|\cdot\|$ in $\mathcal{M}_n(\mathbb{C})$ we have $\|A\| \le \|B\|$.
3. For every increasing convex function $f : \mathbb{R} \to \mathbb{R}$ we have $\operatorname{tr} f(A) \le \operatorname{tr} f(B)$.

Moreover, if $A \prec_w B$ and there exists an increasing strictly convex function $f : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{tr} f(A) = \operatorname{tr} f(B)$ then there exists $U \in \mathcal{U}(n)$ such that $A = U^* B U$. $\square$

In what follows we describe the basic facts about the spectral characterization of the sums of Hermitian matrices obtained by Klyachko [15] and Fulton [14]. Let

$$\mathcal{K}^r_n = \left\{ (j_1, \ldots, j_r) \in \mathbb{N}^r : \; 1 \le j_1 < j_2 \ldots < j_r \le n \right\}.$$

For $J = (j_1, \ldots, j_r) \in \mathcal{K}_n^r$, define the associated partition

$$\lambda(J) = (j_r - r, \ldots, j_1 - 1)\,.$$

Denote by $LR_n^r(m)$ the set of $(m+1)$-tuples $(J_0, \ldots, J_m) \in (\mathcal{K}_n^r)^{m+1}$, such that the Littlewood-Richardson coefficient of the associated partitions $\lambda(J_0), \ldots, \lambda(J_m)$ is positive, i.e. one can generate the Young diagram of $\lambda(J_0)$ from those of $\lambda(J_1), \ldots, \lambda(J_m)$ according to the Littlewood-Richardson rule (see [13]). With this notation and terminology we have

Theorem 2.3.3 ([15, 14]). Let $\lambda_i = \lambda_i^\downarrow = (\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}) \in \mathbb{R}^n$ for $i = 0, \ldots, m$. Then, the following statements are equivalent:

1. There exists $A_i \in \mathcal{M}_n(\mathbb{C})_{sa}$ with $\lambda(A_i) = \lambda_i$ for $0 \le i \le m$ and such that
$$A_0 = A_1 + \ldots + A_m\,.$$
2. For each $r \in \{1, \ldots, n\}$ and $(J_0, \ldots, J_m) \in LR_n^r(m)$ we have
$$\sum_{j \in J_0} \lambda_j^{(0)} \le \sum_{i=1}^m \sum_{j \in J_i} \lambda_j^{(i)} \tag{7}$$
plus the condition $\sum\limits_{j=1}^n \lambda_j^{(0)} = \sum\limits_{i=1}^m \sum\limits_{j=1}^n \lambda_j^{(i)}$.

Moreover, if $(A_i)_{i=0}^m$ are as in item 1. above and $(J_0, \ldots, J_m) \in LR_n^r(m)$ satisfy equality in Eq. (7), then there exists a subspace $L \subseteq \mathbb{C}^n$ with $\dim L = r$, that simultaneously reduces A_i for $0 \le i \le m$ and such that $\lambda(P_L A_i) = (\lambda_j^{(i)})_{j \in J_i}$, where P_L denotes the orthogonal projection of $\mathbb{C}^n$ onto L. □

We shall refer to the inequalities in Eq. (7) as *Horn-Klyachko's compatibility inequalities.*

3 On the existence of tight fusion frames.

In this section we study the problem of the existence of TFF's in $\mathcal{B}_{m,n}^1(\mathbf{d})$, the set of BSS's $\mathcal{W}_w$ in $\mathbb{C}^n$ with fixed dimensions given by $\mathbf{d} \in \mathbb{N}^m$ which satisfy $\operatorname{tr} S_{\mathcal{W}_w} = 1$.

3.1 Dimensional restrictions

There are rather important differences between vector frames and frames of subspaces regarding the (fusion) frame potential. For example, in [2] (see also [6] and [18]) it is shown that the local minimizers of the frame potential on the set $(m \ge n)$

$$F_{m,n}^1 = \{\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_m} \ : \ \text{each } f_i \in \mathbb{C}^n \quad \text{and} \quad \operatorname{tr}(S_{\mathcal{F}}) = \sum_{i \in \mathbb{I}_m} \|f_i\|^2 = 1\,\}$$

are tight frames. Since the set $F_{m,n}^1$ is compact and the frame potential is a continuous function, there must be global (and hence local) minima of the frame potential. This was used to give an indirect proof of the existence of such frames in the vectorial case.

Let $\mathbf{d} \in \mathbb{N}^m$ with $\operatorname{tr} \mathbf{d} \ge n$ and consider the set $\mathcal{B}_{m,n}^1(\mathbf{d})$ defined in Eq. (5). Using Remark 2.1.2 it follows that if $\mathbf{d} = \mathbb{1}_m$, we can identify $\mathcal{B}_{m,n}^1(\mathbf{d})$ with $F_{m,n}^1$. Then the existence of TFF in $\mathcal{B}_{m,n}^1(\mathbb{1})$ is guaranteed, by the comments above. Hence it seems natural to ask whether there exist TFF's in $\mathcal{B}_{m,n}^1(\mathbf{d})$ for arbitrary $\mathbf{d}$. Observe that, in such case, the set of global minimizers of the FFP on $\mathcal{B}_{m,n}^1(\mathbf{d})$ would coincide with the set of TFF's, by Proposition 2.2.3. Nevertheless, the following results show that in general the answer to that question is no.

Proposition 3.1.1. *Let* $(\mathbf{d}, w) \in \mathbb{N}^m \times \mathbb{R}^m_{>0}$ *be a normalized pair, with* $M = \operatorname{tr} \mathbf{d} \geq n$*, and assume that* $\mathcal{W}_w \in \mathcal{B}^1_{m,n}(\mathbf{d})$ *is a TFF, so that* $S_{\mathcal{W}_w} = \frac{1}{n}\, I_n$ *. If there exists* $i \in \mathbb{I}_m$ *such that*

$$M - d_i = \sum_{k \neq i} d_k \leq n - 1 \quad \Longrightarrow \quad w_i^2 = \tfrac{1}{n} \quad \text{and} \quad P_{W_i} P_{W_j} = 0 \quad \text{for every} \quad j \in \mathbb{I}_m \setminus \{i\} \, .$$

Proof. Consider the tight vector frame $\mathcal{F} = \{w_i^2\, e_j^{(i)} \ : \ i \in \mathbb{I}_m \, , \ j \in \mathbb{I}_{d_i}\}$ associated to $\mathcal{W}_w$, as described in Remark 2.1.2. Let $G \in \mathcal{M}_M(\mathbb{C})^+$ denote the Gram matrix of the vector frame $\mathcal{F}$ and let $G_i = w_i^2\, I_{d_i}$ denote the Gram matrix of each subsequence $\{w_i^2\, e_j^{(i)} \ : \ j \in \mathbb{I}_{d_i}\}$. Then each G_i is a $d_i \times d_i$ principal sub-matrix of G. By Cauchy's interlacing principle [3] we get:

$$\lambda_j(G) \geq \lambda_j(G_i) \geq \lambda_{M-d_i+j}(G) \qquad \text{for} \qquad 1 \leq j \leq d_i \, ,$$

where $\lambda(G) = (\lambda_j(G)\,)_{j \in \mathbb{I}_M}$ (resp $\lambda(G_i) \in \mathbb{R}^{d_i}$) denotes the vector of eigenvalues of G (resp. G_i) counting multiplicities and with its entries arranged in non-increasing order. By assumption,

$$\lambda_j(G) = \tfrac{1}{n} \quad \text{for } j \in \mathbb{I}_n \quad , \quad \lambda_j(G) = 0 \quad \text{for} \quad n < j \leq M \ , \ \text{ and } \ \lambda_j(G_i) = w_i^2 \ \text{ for } \ j \in \mathbb{I}_{d_i} \, .$$

Thus if $\sum\limits_{k \neq i} d_k \leq n-1$, then $M - d_i + 1 \leq n$ and $\frac{1}{n} = \lambda_{M-d_i+1}(G) \leq \lambda_1(G_i) = w_i^2 \leq \lambda_1(G) = \frac{1}{n}$. It is known that, in this case, each of the vectors $e_j^{(i)}$, $1 \leq j \leq d_i$, must be orthogonal to every other vector in the $\frac{1}{n}$ -tight vector frame $\mathcal{F}$, which implies the last assertion of the theorem. □

Example 3.1.2 (About the existence of TFF's in $\mathcal{B}^1_{m,n}(\mathbf{d})$)**.** Consider now $\mathbf{d} = (2, 2)$ and assume that there exists $\mathcal{W}_w \in \mathcal{B}^1_{2,3}(\mathbf{d})$ that is a TFF. That is, we assume that there exist two subspaces $W_i \subset \mathbb{C}^3$ with $\dim W_i = 2$, $i = 1, 2$ and w_1 , $w_2 \in \mathbb{R}_{>0}$ such that $\frac{1}{3}\, I_3 = w_1^2\, P_{W_1} + w_2^2\, P_{W_2}$. Since d_1 , $d_2 \leq 3 - 1$ we conclude from Proposition 3.1.1 that $w_1^2 = w_2^2 = \frac{1}{3}$, and $P_{W_1} P_{W_2} = 0$, which is impossible. This argument can be extended to show that if the choices of $\mathbf{d} = (d_i)_{i \in \mathbb{I}_m}$ are such that each d_i is relatively small compared with n and $\sum\limits_{k \neq i} d_k$ then there are no TFF's in $\mathcal{B}^1_{m,n}(\mathbf{d})$. For example, in $\mathcal{B}^1_{k,\,2k-1}(2 \cdot \mathbb{1}_k)$, $\mathcal{B}^1_{3,\,7}(3, 3, 3)$, $\mathcal{B}^1_{3,\,9}(4, 4, 4)$, etc., there are no TFF's.

Remark 3.1.3. The previous results show some dimensional restrictions for the existence of TFF's in $\mathcal{B}^1_{m,n}(\mathbf{d})$. In the paper [7], some sufficient conditions on n, m and $\mathbf{d}$ are given (particularly if $\mathbf{d}$ is a multiple of $\mathbb{1}_m$), which assure the existence in $\mathcal{B}^1_{m,n}(\mathbf{d})$ of such fusion frames. For further results in this direction, see also [16].

3.2 Characterizations for fixed weights

Recall that, if $\mathcal{W}_w = (w_i \, , W_i)_{i \in \mathbb{I}_m} \in \mathcal{B}_{m\,,n}$, the FFP of $\mathcal{W}_w$ is given by

$$\text{FFP}\,(\mathcal{W}_w) = \sum_{i,j=1}^{m} w_i^2 w_j^2 \operatorname{tr}(P_{W_i} P_{W_j}) = \operatorname{tr}\, S^2_{\mathcal{W}_w} \, .$$

Definition 3.2.1. Given $\mathcal{W}_w = (w_i \, , W_i)_{i \in \mathbb{I}_m} \in \mathcal{B}_{m\,,n}$, we define also the following matrix:

$$\mathbf{P}_q(\mathcal{W}_w) = \sum_{i,\,j=1}^{m} w_i^2 w_j^2 (P_{W_i} P_{W_j})^* P_{W_i} P_{W_j} = \sum_{i,\,j=1}^{m} w_i^2 w_j^2\, P_{W_j}\, P_{W_i}\, P_{W_j} \in \mathcal{M}_n(\mathbb{C})^+ \, . \tag{8}$$

The matrix $\mathbf{P}_q(\cdot)$ is related to the so-called q-potential [17] defined in the more general context of reconstruction systems. Notice that the FF-potential of the sequence $\mathcal{W}_w$ can be computed in terms of $\mathbf{P}_q(\mathcal{W}_w)$, since $\text{FFP}\,(\mathcal{W}_w) = \operatorname{tr}\, S^2_{\mathcal{W}_w} = \operatorname{tr} \mathbf{P}_q(\mathcal{W}_w)$.

The following theorem gives us some general bounds for the matrix $\mathbf{P}_q(\cdot)$ and states several conditions on $\mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d})$ which are equivalent to the assertion that $\mathcal{W}_w$ is a $\frac{1}{n}$-TFF.

Theorem 3.2.2. *Let $\mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d})$ such that $\operatorname{tr} S_{\mathcal{W}_w} = \sum_{i\in\mathbb{I}_m} w_i^2\, d_i \geq 1$. Then*

$$\tfrac{1}{n^2}\, I_n \prec_w \mathbf{P}_q(\mathcal{W}_w)\,, \tag{9}$$

for every UIN $\|\cdot\|$ on $\mathcal{M}_n(\mathbb{C})$ with associated symmetric gauge function ψ we have that

$$\tfrac{1}{n^2}\, \psi(\mathbb{1}) \leq \|\mathbf{P}_q(\mathcal{W}_w)\|\,, \tag{10}$$

and for every increasing convex function $f : \mathbb{R}_+ \to \mathbb{R}$ with $f(0) = 0$ we have

$$n \cdot f\big(\tfrac{1}{n^2}\big) \leq \operatorname{tr} f(\mathbf{P}_q(\mathcal{W}_w)\,)\,. \tag{11}$$

On the other hand, the following conditions are equivalent:

- *(a) $\mathcal{W}_w$ is a $\frac{1}{n}$-TFF.*
- *(b) $\mathbf{P}_q(\mathcal{W}_w) = \frac{1}{n^2}\, I_n$.*
- *(c) Majorization holds in Eq.* (9).
- *(d) There exists a UIN on $\mathcal{M}_n(\mathbb{C})$ such that equality holds in Eq.* (10).
- *(e) There exists an increasing strictly convex function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$ such that equality holds in Eq.* (11).

Proof. Since $\operatorname{tr}(S_{\mathcal{W}_w}) \geq 1$, it follows from Example 2.3.1 that $\frac{1}{n} I_n \prec_w S_{\mathcal{W}_w}$. By Theorem 2.3.2,

$$\tfrac{1}{n} = \operatorname{tr}\,(\tfrac{1}{n}\, I_n)^2 \leq \operatorname{tr} S_{\mathcal{W}_w}^2 = \operatorname{tr} \mathbf{P}_q(\mathcal{W}_w) \quad\Longrightarrow\quad \tfrac{1}{n^2} I_n \prec_w \mathbf{P}_q(\mathcal{W}_w\,)\,,$$

using Example 2.3.1 again. Notice that by Theorem 2.3.2, Eq. (10) and Eq. (11) are consequences of this last fact. Assume that $\mathcal{W}_w$ is a $\frac{1}{n}$-TFF. Then $S_{\mathcal{W}_w} = \sum_{j=1}^m w_j^2 P_{W_j} = \frac{1}{n}\, I_n$. Therefore

$$\begin{aligned}
\mathbf{P}_q(\mathcal{W}_w) &= \sum_{i,j=1}^m w_i^2 w_j^2 P_{W_i} P_{W_j} P_{W_i} = \sum_{i=1}^m w_i^2 P_{W_i}\Big(\sum_{j=1}^m w_j^2 P_{W_j} \Big) P_{W_i} \\
&= \sum_{i=1}^m w_i^2 P_{W_i}\big(\tfrac{1}{n}\, I_n \big) P_{W_i} = \tfrac{1}{n^2}\, I_n\,.
\end{aligned}$$

It is clear that this equality implies conditions (c), (d) (for every UIN) and (e) (for every f). Assume now that equality holds in Eq. (10) for some UIN. Then, using Eq. (9) we get

$$\tfrac{1}{n^2}\, \psi(\mathbb{1}) = \|\mathbf{P}_q(\mathcal{W}_w)\| = \psi(\lambda(P_q(\mathcal{W}_w)\,) \geq \tfrac{\operatorname{tr} \mathbf{P}_q(\mathcal{W}_w)}{n}\, \psi(\mathbb{1}) \geq \tfrac{1}{n^2}\, \psi(\mathbb{1})\,.$$

Hence, $\operatorname{tr} \mathbf{P}_q(\mathcal{W}_w) = \frac{1}{n} = \operatorname{tr}\,(\frac{1}{n^2}\, I_n)$, and majorization holds in Eq. (9). In this case, we have that

$$\operatorname{tr}\big(\tfrac{1}{n^2}\, I_n\big) = \operatorname{tr} \mathbf{P}_q(\mathcal{W}_w) = \operatorname{tr} S_{\mathcal{W}_w}^2\,.$$

Since $\frac{1}{n} I_n \prec_w S_{\mathcal{W}_w}$ and the function $f(x) = x^2$ is strictly convex, by Theorem 2.3.2 we conclude that there exists a unitary $U \in \mathcal{U}(n)$ such that $S_{\mathcal{W}_w} = U^*(\frac{1}{n}\, I_n)U = \frac{1}{n}\, I_n$. Finally, if there exists $f : \mathbb{R}_+ \to \mathbb{R}_+$ as in (e), then Theorem 2.3.2 and Eq. (9) imply that $S_{\mathcal{W}_w} = \frac{1}{n}\, I_n$. □

Theorem 3.2.3. *Let $(\mathbf{d}, w) \in \mathbb{N}^m \times \mathbb{R}_{>0}^m$ be a normalized pair. Then, the following statements are equivalent:*

1. *There exists* $\mathcal{W}_w = (w_i\,, W_i)_{i\in\mathbb{I}_m} \in \mathcal{B}_{m,\,n}(\mathbf{d}\,,\,w)$ *which is a* $\frac{1}{n}$ *-TFF.*
2. *For every* $1 \le r \le n-1$ *and every* $(J_0, \ldots, J_m) \in LR_n^r(m)$ *we have that*
$$\tfrac{r}{n} \le \sum_{i\in\mathbb{I}_m} w_i^2 \cdot |\, J_i \cap \{1, \ldots, d_i\}\,| .$$
3. *There exists an orthogonal projection* $P = (P_{ij})_{i,j\in\mathbb{I}_m} \in \mathcal{M}_m(\mathcal{M}_n(\mathbb{C})\,)$ *of rank* n *such that* $\frac{1}{n\,w_i^2}\, P_{ii}$ *is an orthogonal projection of rank* d_i *in* $\mathcal{M}_n(\mathbb{C})$*, for every* $i \in \mathbb{I}_m$ *.*

Proof. Notice that condition 1 is equivalent to the existence of orthogonal projections $\{P_i\}_{i\in\mathbb{I}_m}$ with $\operatorname{rk} P_i = d_i$ such that $\sum\limits_{i\in\mathbb{I}_m} w_i^2\, P_i = \frac{1}{n}\, I_n$. Hence, Theorem 2.3.3 assures that condition 1 implies condition 2, since these are Horn-Klyachko's compatibility inequalities for the spectra of $\{w_i^2\, P_i\}_{i\in\mathbb{I}_m}$ and $\frac{1}{n}\, I_n$. The converse of the previous implication also follows from Theorem 2.3.3 since a self adjoint operator A with $\lambda(A) = (\alpha, \ldots, \alpha, 0, \ldots, 0) \in \mathbb{R}^n$ is necessarily of the form $A = \alpha\, P$ for some projection $P \in \mathcal{M}_n(\mathbb{C})$.

Assume now condition 1 and let $\mathcal{W}_w = (w_i, W_i)_{i\in\mathbb{I}_m} \in \mathcal{B}^1_{m,\,n}(\mathbf{d})$ be a $\frac{1}{n}$-TFF. Let us consider a partial isometry $V_i \in \mathcal{M}_n(\mathbb{C})$ such that $V_i^* V_i = P_{W_i}$, for every $i \in \mathbb{I}_m$. Define
$$V^* = \big[w_1 V_1^* \,|\, w_2 V_2^* \,|\, \cdots \,|\, w_m V_m^*\big] \in \mathcal{M}_{n\,,\,mn}(\mathbb{C})\,.$$
Observe that $V^*V = \sum\limits_{i\in\mathbb{I}_m} w_i^2 V_i^* V_i = \frac{1}{n}\, I_n$. Hence $VV^* = (w_i w_j V_i V_j^*)_{i,j=1}^m \in \mathcal{M}_m(\mathcal{M}_n(\mathbb{C})\,)$ satisfies that $VV^* = \frac{1}{n}\, P$ for an orthogonal projection $P = (P_{ij})_{i,j\in\mathbb{I}_m} \in \mathcal{M}_m(\mathcal{M}_n(\mathbb{C})\,)$. By comparing the diagonal blocks we get that
$$\tfrac{1}{n}\, P_{ii} = w_i^2\, V_i V_i^* \quad \Longrightarrow \quad \tfrac{1}{n\,w_i^2}\, P_{ii} = \,V_i V_i^* \;, \qquad \text{for every} \qquad i \in \mathbb{I}_m \;.$$
Conversely, assume that $P = (P_{ij})_{i,j\in\mathbb{I}_m} \in \mathcal{M}_m(\mathcal{M}_n(\mathbb{C})\,)$ is a projection with $\operatorname{rk} P = n$ as in (3). Then there exist a matrix $U = [U_1|\cdots|U_m] \in \mathcal{M}_{n\,,\,mn}(\mathbb{C})$ (where each $U_i \in \mathcal{M}_n(\mathbb{C})\,$) such that $UU^* = I_n$ and $P = U^*U$. Comparing the block diagonal entries, for every $i \in \mathbb{I}_m$ we get that $\frac{1}{n\,w_i^2}\, P_{ii} = \frac{1}{n\,w_i^2}\, U_i^* U_i$ and $Q_i = \frac{1}{n\,w_i^2}\, U_i U_i^*$ is also an orthogonal projection of rank d_i . On the other hand, we have that
$$\tfrac{1}{n}\, I_n = \tfrac{1}{n}\, UU^* = \sum_{i\in\mathbb{I}_m} w_i^2\, \tfrac{1}{n\,w_i^2}\, U_i U_i^* = \sum_{i\in\mathbb{I}_m} w_i^2\, Q_i\,. \tag{12}$$
Taking $W_i = R(U_i) = R(Q_i)$, we get that $P_{W_i} = Q_i$ for every $i \in \mathbb{I}_m$. By this fact and Eq. (12), we conclude that the sequence $\mathcal{W} = (w_i, W_i)_{i\in\mathbb{I}_m} \in \mathcal{B}^1_{m,\,n}(\mathbf{d})$ is a $\frac{1}{n}$ -TFF for $\mathbb{C}^n$. $\square$

4 Minimization for fixed weights

Recall that a pair $(\mathbf{d}, w) \in \mathbb{N}^m \times \mathbb{R}^m_{>0}$ is called normalized if $\operatorname{tr}\mathbf{d} \ge n$ and $\sum_{i\in\mathbb{I}_m} w_i^2\, d_i = 1$. In this section we study, for such a pair $(\mathbf{d}, w)$, the structure of minimizers for the FFP in the set $\mathcal{B}_{m,\,n}(\mathbf{d}, w)$ of BSS's $\mathcal{W}_w$ in $\mathbb{C}^n$ with fixed dimensions given by $\mathbf{d}$ and fixed weights w.

4.1 Lower bound for the potential

In this subsection we translate, using Remark 2.1.2, some well known results about vector frames (see [6] or [20]) to the FF context. An interesting fact is that there is a notion of irregularity, defined in terms of the parameters of a given FF, which agree with the vectorial n-irregularity of their associated vector frames. Nevertheless, the lower bound obtained for the FF-potential is not always attained in the set $\mathcal{B}_{m,\,n}(\mathbf{d}, w)$ (see Example 4.1.3) .

Definition 4.1.1. Given a pair $\mathbf{d} \in \mathbb{N}^m$ and $w = w^\downarrow \in \mathbb{R}^m_{>0}$, consider its *q-irregularity* defined as

$$J_0(\mathbf{d}, w) = \max \left\{ j \in \mathbb{I}_m \ : \ \big(n - \sum_{i=1}^{j} d_i\big)\, w_j^2 > \sum_{i=j+1}^{m} w_i^2\, d_i \right\} ,$$

if this set is not empty empty. In the case where it is empty, let $J_0(\mathbf{d}, w) = 0$.

Proposition 4.1.2. *Let $(\mathbf{d}, w) \in \mathbb{N}^m \times \mathbb{R}^m_{>0}$ be a normalized pair, where $w = w^\downarrow$. Recall that $M = \sum_{i \in \mathbb{I}_m} d_i \geq n$. Let $j_0 = J_0(\mathbf{d}, w)$ and $c = \dfrac{\sum_{i=j_0+1}^{m} w_i^2\, d_i}{n - \sum_{i=1}^{j_0} d_i} < w_{j_0}^2$. If $\mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d}, w)$ then*

$$FFP(\mathcal{W}_w) \geq \sum_{i=1}^{j_0} d_i\, w_i^4 + (n - \sum_{i=j_0+1}^{m} d_i)\ c^2 . \tag{13}$$

Moreover, equality holds in Eq. (13) if and only if the following two conditions hold:

1. *$P_{W_i}\, P_{W_j} = 0$ for $1 \leq i \neq j \leq j_0$ and*
2. *$\{w_i\,, W_i\}_{i=j_0+1}^m$ is a TFF for $\operatorname{span}\{W_i :\ 1 \leq i \leq j_0\}^\perp$.*

Proof. 1. Let $\mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d}, w)$ and let $\mathcal{F} = \{w_i^2\, e_j^{(i)} \ : \ i \in \mathbb{I}_m\,,\ j \in \mathbb{I}_{d_i}\}$ be an associated vector frame, as described in Remark 2.1.2. Let $\mathbf{a} \in \mathbb{R}^M$ denote the vector whose coordinates are the norms of the elements of $\mathcal{F}$ arranged in non-increasing order. Then,

$$w_k = a_{M(k,j)}\ , \quad \text{where} \quad M(k,j) = \sum_{i=1}^{k-1} d_i + j \quad , \ \text{for} \quad k \in \mathbb{I}_m \ \text{ and } \ j \in \mathbb{I}_{d_k} .$$

We now consider the n-irregularity $r_n(\mathbf{a})$ of the vector $\mathbf{a}$:

$$r_n(\mathbf{a}) = \max \Big\{ j \ \in \ \mathbb{I}_{n-1} \ : \ (n-j)a_j > \sum_{i=j+1}^{M} a_i \Big\} ,$$

if the set on the right is non-empty, and $r_n(\mathbf{a}) = 0$ otherwise. It is straightforward that $r_n(\mathbf{a}) = \sum_{i=1}^{j_0} d_i$ in the first case. Therefore, inequality (13) can be deduced from [6, Theorem 10] (see also [20]). The same result of [6] shows that equality in Eq. (13) implies that $S_1 = \{e_j^{(i)} : 1 \leq i \leq j_0\,,\ j \in \mathbb{I}_{d_i}\}$ is an orthonormal system in $\mathbb{C}^n$, and $S_2 = \{e_j^{(i)} : j_0 + 1 \leq i \leq m\,,\ j \in \mathbb{I}_{d_i}\}$ is a tight frame for $\mathcal{S}^\perp$, where $\mathcal{S} = \operatorname{span}\{S_1\} = \operatorname{span}\{W_i :\ 1 \leq i \leq j_0\}$. □

The following example shows that the lower bound in Eq. (13) is not sharp in general.

Example 4.1.3. If we set $n = 3$, $\mathbf{d} = (2, 2)$ and $w_1 = w_2 = \frac{1}{2}$ then $\sum_{i \in \mathbb{I}_m} w_i^2\, d_i = 1$ and $J_0(\mathbf{d}, w) = 0$. Therefore, by Theorem 4.1.2 the equality (13) holds only in tight fusion frames. Still, there are no TFF's in $\mathcal{B}_{2,3}(\mathbf{d}, w) \subseteq \mathcal{B}_{2,3}(\mathbf{d})$ since the Example 3.1.2 shows that there are no TFF's in the (bigger set) $\mathcal{B}_{2,3}(\mathbf{d})$.

4.2 Structure of local minima: The geometrical approach

In what follows we consider a perturbation result for Bessel sequences of subspaces. We begin by considering some well known facts from differential geometry that we shall need below. In what follows we consider the unitary group $\mathcal{U}(n)$ together with its natural differential geometric (Lie) structure. It is well known that the tangent space $\mathcal{T}_{I_n}\mathcal{U}(n)$ at the identity can be naturally identified with the real vector space

$$\mathcal{M}_n(\mathbb{C})_{ah} = i \cdot \mathcal{M}_n(\mathbb{C})_{sa} = \big\{ X \in \mathcal{M}_n(\mathbb{C}) : X^* = -X \big\} \, ,$$

of anti-Hermitian matrices. Given $G \in \mathcal{M}_n(\mathbb{C})^+$ we consider the smooth map

$$\Psi_G : \mathcal{U}(n) \to \mathcal{U}(G) \subseteq \mathcal{M}_n(\mathbb{C}) \quad \text{ given by } \quad \Psi_G(U) = U^*GU \ , \quad U \in \mathcal{U}(n) \, , \tag{14}$$

where $\mathcal{U}(G)$ is the unitary orbit of G. Under the previous identification, the differential of Ψ_G at a the point $I_n \in \mathcal{U}(n)$ in the direction given by $X \in \mathcal{M}_n(\mathbb{C})_{ah}$ is given by

$$(D\Psi_G)_{I_n}(X) = XG - GX = [X, G] \, . \tag{15}$$

It is well known that the map Ψ_G is a submersion of $\mathcal{U}(n)$ onto $\mathcal{U}(G)$. Therefore, the differential $(D\Psi_G)_{I_n}$ is an epimorphism, and hence Eq. (15) gives us a description of the tangent space of the manifold $\mathcal{U}(G)$ at the point G: We have that $\mathcal{T}_G\mathcal{U}(G) = \big\{ \, [X, G] : X \in \mathcal{M}_n(\mathbb{C})_{ah} \, \big\}$.

We now adopt the notation $\mathcal{M}_n^1(\mathbb{C})_{sa} = \{A \in \mathcal{M}_n(\mathbb{C})_{sa} : \operatorname{tr} A = 1\}$. Observe that $\mathcal{M}_n^1(\mathbb{C})_{sa}$ is an affine manifold contained in the real vector space $\mathcal{M}_n(\mathbb{C})_{sa}$, and its tangent space is the subspace $\mathcal{M}_n^0(\mathbb{C})_{sa} = \{A \in \mathcal{M}_n(\mathbb{C})_{sa} : \operatorname{tr} A = 0\}$. On the other hand, given $X, Y \in \mathcal{M}_n(\mathbb{C})_{sa}$, it is easy to see that $\operatorname{tr}(XY) \in \mathbb{R}$. Therefore the inner product $\langle A, \ B \rangle = \operatorname{tr}(B^*A)$ of $\mathcal{M}_n(\mathbb{C})$ still works as a real inner product on $\mathcal{M}_n(\mathbb{C})_{sa}$.

Given a set $\{P_j : j \in \mathbb{I}_m\} \subseteq \mathcal{M}_n(\mathbb{C})_{sa}$ of projections, we denote by

$$\{P_j : j \in \mathbb{I}_m\}' = \{A \in \mathcal{M}_n(\mathbb{C}) : AP_j = P_j\, A \quad \text{for every} \quad j \in \mathbb{I}_m\} \, . \tag{16}$$

Note that $\{P_j : j \in \mathbb{I}_m\}'$ is a closed selfadjoint subalgebra of $\mathcal{M}_n(\mathbb{C})$. Therefore, the algebra $\{P_j : j \in \mathbb{I}_m\}' \neq \mathbb{C}\, I_n \iff$ there exists a non-trivial orthogonal projection $Q \in \{P_j : j \in \mathbb{I}_m\}'$.

Theorem 4.2.1. *Let* $(\mathbf{d}, w)$ *be a normalized pair, and fix* $\mathcal{W}_w = (w_i \, , \, W_i)_{i \in \mathbb{I}_m} \in \mathcal{B}_{m,n}(\mathbf{d}, w)$. *Denote* $P_j = P_{W_j}$ *for every* $j \in \mathbb{I}_m$. *Let* $\Psi : \mathcal{U}(n)^m \to \mathcal{M}_n^1(\mathbb{C})_{sa} \subseteq \mathcal{M}_n(\mathbb{C})_{sa}$ *be the smooth function given by*

$$\Psi(U_1, \dots, U_m) = \sum_{j=1}^m w_j^2 \, U_j^* P_j \, U_j = \sum_{j=1}^m w_j^2 \, \Psi_{P_j}(U_j) \quad , \qquad \textit{for} \qquad (U_1, \dots, U_m) \in \mathcal{U}(n)^m \, .$$

Then the following conditions are equivalent:

1. *The differential of* Ψ *at* $I = (I_n, \dots, I_n) \in \mathcal{U}(n)^m$ *is* **surjective**.
2. $\{P_j : j \in \mathbb{I}_m\}' = \mathbb{C}\, I_n$.

In this case, the image of Ψ *contains an open neighborhood of* $\Psi(I) = \sum\limits_{j=1}^m w_j^2 \, P_j$ *in* $\mathcal{M}_n^1(\mathbb{C})_{sa}$, *and* Ψ *admits smooth (and hence continuous) local cross sections around* $\Psi(I)$.

Proof. It is clear from its definition that Ψ is a smooth function. Moreover, under the previous identification $\mathcal{T}_{I_n}\mathcal{U}(n) = \mathcal{M}_n(\mathbb{C})_{ah}$, and using Eq. (15), we can see that

$$D\Psi_I(X_1,\ldots,X_m) = \sum_{j=1}^{m} w_j^2\,[X_j,P_j] \quad , \qquad \text{for} \qquad (X_1,\ldots,X_m)\in\mathcal{M}_n(\mathbb{C})_{ah}^m\,. \tag{17}$$

The tangent space of $\mathcal{M}_n^1(\mathbb{C})_{sa}$ is the real vector space $\mathcal{M}_n^0(\mathbb{C})_{sa}$, which has a natural inner product given by $\langle Y,Z\rangle = \operatorname{tr}(YZ)$. Denote by $T = D\Psi_I$ and assume that T is not surjective. Then there exists $0\neq Y\in\mathcal{M}_n^0(\mathbb{C})_{sa}$ which is orthogonal to the image of T. Using Eq. (17) we deduce that, for every $(X_1,\ldots,X_m)\in\mathcal{M}_n(\mathbb{C})_{ah}^m$, it holds that

$$0 = \big\langle T(X_1,\ldots,X_m),Y\big\rangle = \sum_{j=1}^{m} w_j^2 \operatorname{tr}\big([X_j\,,P_j]\,Y\big) = \sum_{j=1}^{m} w_j^2 \operatorname{tr}\big(X_j\,[P_j\,,Y]\big)\,. \tag{18}$$

Since each $[P_j\,,Y]\in\mathcal{M}_n(\mathbb{C})_{ah}$, we can choose each $X_j = [P_j\,,Y]$, and so Eq. (18) implies that $[P_j\,,Y] = 0$ for every $j\in\mathbb{I}_m$. In other words, that $Y\in\{P_j : j\in\mathbb{I}_m\}'$. On the other hand, since $0\neq Y\in\mathcal{M}_n(\mathbb{C})_{sa}$ and $\operatorname{tr} Y = 0$, then $Y\notin\mathbb{C}\,I_n$. The converse follows from the previous argument, by taking $Y\in\{P_j : j\in\mathbb{I}_m\}'$ such that $0\neq Y = Y^*$ and $\operatorname{tr} Y = 0$. □

Let $(\mathbf{d},w)\in\mathbb{N}^m\times\mathbb{R}_{>0}^m$ be a normalized pair. We shall consider on $\mathcal{B}_{m,\,n}(\mathbf{d},w)$ the distance

$$d_P(\mathcal{W}_w\,,\mathcal{W}'_w) = \max_{i\in\mathbb{I}_m}\|P_{W_i} - P_{W'_i}\|$$

(recall that the weights are fixed), called *punctual*, and the pseudo-distance

$$d_S(\mathcal{W}_w\,,\mathcal{W}'_w) = \|S_{\mathcal{W}_w} - S_{\mathcal{W}'_w}\|\,,$$

called *operatorial*. The problem of finding *local* minimizers for the FF-potential can be posed with respect to either of these metrics.

Corollary 4.2.2. *Let $(\mathbf{d},w)\in\mathbb{N}^m\times\mathbb{R}_{>0}^m$ be a normalized pair. Assume that $\mathcal{W}_w\in\mathcal{B}_{m,\,n}(\mathbf{d},w)$ satisfies that $\{P_{W_j} : j\in\mathbb{I}_m\}' = \mathbb{C}\,I_n$. Then $\mathcal{W}_w$ is a FF and the map*

$$S:\mathcal{B}_{m,\,n}(\mathbf{d},w)\to\mathcal{M}_n^1(\mathbb{C})_{sa} \quad \text{given by} \quad S(\mathcal{V}_w) = S_{\mathcal{V}_w} = \sum_{i\in\mathbb{I}_m} w_i^2 P_{V_i}\,,$$

for $\mathcal{V}_w = (w_i,V_i)_{i\in\mathbb{I}_m}\in\mathcal{B}_{m,\,n}(\mathbf{d},w)$, satisfies that:

1. *The image of S contains an open neighborhood of $S_{\mathcal{W}_w}$ in $\mathcal{M}_n^1(\mathbb{C})_{sa}$.*
2. *S has d_P-continuous local cross sections around $S_{\mathcal{W}_w}$.*

Proof. Notice that the condition $\{P_{W_j} : j\in\mathbb{I}_m\}' = \mathbb{C}\,I_n$ implies that $\mathcal{W}$ is a generating sequence of subspaces. To prove the properties of the map S, compose a local cross section for the map Ψ of Theorem 4.2.1 (which is open in I) with the map $\Phi:\mathcal{U}(n)^m\to\mathcal{B}_{m,\,n}(\mathbf{d})$ given by

$$\Phi(U_1\,,\ldots,U_m) = (w_i\,,U_i(W_i)\,)_{i\in\mathbb{I}_m} = (w_i\,,R(U_iP_{W_i}U_i^*)\,)_{i\in\mathbb{I}_m}\,.$$

Observe that $S\circ\Phi = \Psi$, so that S is open in $\Phi(I) = \mathcal{W}_w$. □

Remark 4.2.3. Here is an alternative statement of Corollary 4.2.2: Under the same assumptions on $\mathcal{W}_w$, it holds that $S_{\mathcal{W}_w} \in \mathcal{G}l(n)^+$ and, for every sequence $(S_k)_{k\in\mathbb{N}}$ in $\mathcal{M}_n^1(\mathbb{C})_{sa}$ such that $S_k \xrightarrow[k\to\infty]{} S_{\mathcal{W}_w}$, there exists a sequence $(\mathcal{V}_k)_{k\geq k_0}$ in $\mathcal{S}_{m,n}(\mathbf{d}, w)$ such that

$$d_P(\mathcal{V}_k , \mathcal{W}_w) \xrightarrow[k\to\infty]{} 0 \quad \text{and} \quad S_{\mathcal{V}_k} = S_k \quad \text{for every} \quad k_0 \leq k \in \mathbb{N}.$$

This formulation of Corollary 4.2.2 generalizes [18, Thm 5.3] to the context of fusion frames with fixed weights.

It is not clear that a d_P-local minimizer for the FF-potential on $\mathcal{B}_{m,n}(\mathbf{d}, w)$ must be a fusion frame, i.e. that its frame operator is an invertible operator. The following lemma shows that this is true.

Lemma 4.2.4. *Let $(\mathbf{d}, w) \in \mathbb{N}^m \times \mathbb{R}^m_{>0}$ be a normalized pair. Let $\mathcal{W}_w$ be a d_P-local minimizer for the FF-potential in $\mathcal{B}_{m,n}(\mathbf{d}, w)$. Then $S_{\mathcal{W}_w}$ is invertible (equivalently, $\mathcal{W}_w \in \mathcal{S}_{m,n}(\mathbf{d}, w)$).*

Proof. Suppose that $S_{\mathcal{W}_w}$ has nontrivial nullspace $N(S_{\mathcal{W}_w})$. If $x \in N(S_{\mathcal{W}_w})$

$$0 = \langle S_{\mathcal{W}_w} x , x\rangle = \sum_{j\in\mathbb{I}_m} \langle w_j^2 P_{W_j} x , x\rangle = \sum_{j\in\mathbb{I}_m} w_j^2 \, \|P_{W_j} x\|^2 . \tag{19}$$

In other words, $W_i \subseteq R(S_{\mathcal{W}_w})$ for every $i \in \mathbb{I}_m$. Since $\operatorname{tr}(\mathbf{d}) \geq n > \dim R(S_{\mathcal{W}_w})$, we deduce that there exists $i \neq j$ in $\mathbb{I}_m$ such that $P_{W_j} P_{W_i} \neq 0$. Fix that pair i, j. Fix also $f \in W_i \setminus W_j^\perp$ and $g \in N(S_{\mathcal{W}_w})$ two unit vectors. For every $t \in [0, \pi/2]$, take the unit vector $g(t) = \cos t \cdot f + \sin t \cdot g$. Let $V_i = W_i \ominus \operatorname{span}\{f\}$, $W_i(t) = V_i \oplus \operatorname{span}\{g(t)\}$ and let $\mathcal{W}_w(t) \in \mathcal{B}_{m,n}(\mathbf{d}, w)$ be the sequence obtained by replacing W_i by $W_i(t)$ in $\mathcal{W}_w$. As $g \in W_k^\perp$ for every $k \in \mathbb{I}_m$, for every $t \in (0, \pi/2]$,

$$\begin{aligned}
\tfrac{1}{2}\left(\mathrm{FFP}(\mathcal{W}_w) - \mathrm{FFP}(\mathcal{W}_w(t))\right) &= \sum_{k=1}^m w_i^2 w_k^2 \Big(\operatorname{tr}(P_{W_i} P_{W_k}) - \operatorname{tr}(P_{W_i(t)} P_{W_k}) \Big) \\
&= \sum_{k=1}^m w_i^2 w_k^2 \Big(\operatorname{tr}(P_{W_i} P_{W_k}) - \operatorname{tr}\left(g(t)g(t)^* P_{W_k}\right) - \operatorname{tr}(P_{V_i} P_{W_k}) \Big) \\
&= \sum_{k=1}^m w_i^2 w_k^2 \Big(\operatorname{tr}(P_{W_i} P_{W_k}) - \cos^2 t \operatorname{tr}\left(ff^* P_{W_k}\right) - \operatorname{tr}(P_{V_i} P_{W_k}) \Big) \\
&> \sum_{k=1}^m w_i^2 w_k^2 \Big(\operatorname{tr}(P_{W_i} P_{W_k}) - \operatorname{tr}\left(ff^* P_{W_k}\right) - \operatorname{tr}(P_{V_i} P_{W_k}) \Big) \\
&= 0 ,
\end{aligned}$$

because $\operatorname{tr}\left(ff^* P_{W_j}\right) = \|P_{W_j} f\|^2 \neq 0$. Hence $\mathrm{FFP}(\mathcal{V}_w(t)) < \mathrm{FFP}(\mathcal{W}_w)$ for every $t \in (0, \pi/2]$. Taking $t \to 0$, we have that $\mathcal{W}_w(t) \xrightarrow{d_P} \mathcal{W}_w$, and this contradicts the minimality of $\mathcal{W}_w$. □

Given $S \in \mathcal{M}_n(\mathbb{C})_{sa}$ with $\sigma(S) = \{\mu_1, \ldots, \mu_r\}$, we denote by $P_{\mu_k}(S) = P_{N(S-\mu_k I_n)} \in \mathcal{M}_n(\mathbb{C})^+$, the spectral projection of S relative to μ_k, for $k \in \mathbb{I}_r$. These projections satisfy that

1. $P_{\mu_k}(S)\, P_{\mu_j}(S) = 0$ if $k \neq j$, and $\sum_{k=1}^p P_{\mu_k}(S) = I_n$ (i.e., they are a system of projectors).

2. For every $k \in \mathbb{I}_r$, it holds that $S\, P_{\mu_k}(S) = \mu_k\, P_{\mu_k}(S)$, so that $S = \sum_{k=1}^p \mu_k\, P_{\mu_k}(S)$.

The following theorem generalizes a similar result given in [7, Theorem 4], for the case $w = \mathbb{1}_m$. Nevertheless, our approach is based on completely different techniques.

Theorem 4.2.5. *Let $(\mathbf{d}, w) \in \mathbb{N}^m \times \mathbb{R}^m_{>0}$ be a normalized pair. Let $\mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d}, w)$ be a local minimizer of the FF-potential with respect to the distance d_P. If $\sigma(S_{\mathcal{W}_w}) = \{\mu_1, \dots, \mu_r\}$, then*

$$P_{\mu_k}(S_{\mathcal{W}_w}) \in C_{\mathcal{W}_w} = \{P_{W_j} : j \in \mathbb{I}_m\}' \quad \textit{for every} \quad k \in \mathbb{I}_r . \tag{20}$$

The same property holds whenever $\mathcal{W}_w$ is a d_P-local minimizer in $\mathcal{B}^1_{m,n}(\mathbf{d})$.

Proof. Recall from Lemma 4.2.4 that $\mathcal{W}_w \in \mathcal{S}_{m,n}(\mathbf{d}, w)$, in other words that $0 \notin \sigma(S_{\mathcal{W}_w})$. Consider the set $\mathcal{Q}$ of finite systems of projectors $\{Q_k\}_{k\in\mathbb{I}_p}$ such that each $Q_k \in C_{\mathcal{W}_w}$. Observe that $\mathcal{Q}$ is not empty because $\{I_n\} \in \mathcal{Q}$. Then $\mathcal{Q}$ has a maximal element $\{Q_k\}_{k\in\mathbb{I}_p}$ with respect to the order induced by refinement. Fix $k \in \mathbb{I}_p$. For each $i \in \mathbb{I}_m$ put $\mathcal{M}_i = W_i \cap R(Q_k)$ and $\mathcal{N}_i = W_i \cap R(Q_k)^\perp$. Using that $Q_k \in C_{\mathcal{W}_w}$, we get that each $W_i = \mathcal{M}_i \oplus \mathcal{N}_i$. Set $r_i = \dim \mathcal{M}_i$ and $\mathbf{r} = (r_1, \dots, r_m)$. Then, the sequence $\mathcal{W}_{k,w} = (w_i, \mathcal{M}_i)_{i\in\mathbb{I}_m}$ is a FF for $R(Q_k)$. We claim that $\mathcal{W}_{k,w}$ is a local minimizer of the FF-potential in

$$\mathcal{B}(Q_k, \mathbf{r}, w) = \big\{ \mathcal{V}_w = (w_i, V_i)_{i\in\mathbb{I}_m} \in \mathcal{B}_{m,n}(\mathbf{r}, w) \ : \ V_i \subseteq R(Q_k) \quad \text{for every} \ \ i \in \mathbb{I}_m \big\} .$$

Indeed, given $\mathcal{V}_w \in \mathcal{B}(Q_k, \mathbf{r}, w)$, put $\tilde{\mathcal{V}}_w = (w_i, V_i \oplus \mathcal{N}_i)_{i\in\mathbb{I}_m} \in \mathcal{B}_{m,n}(\mathbf{d}, w)$. Observe that the map $\mathcal{V}_w \mapsto \tilde{\mathcal{V}}_w$ preserves the distance d_P. Moreover, since $Q_k \in C_{\mathcal{W}_w}$, then each $P_{\mathcal{N}_i} = (I_n - Q_k)P_{W_i}$ so that, by Eq. (6), $\text{FFP}\,(\tilde{\mathcal{V}}_w) = \text{FFP}\,(\mathcal{V}_w) + \text{FFP}\,\big((w_i, \mathcal{N}_i)_{i\in\mathbb{I}_m}\big)$, and the second summand does not depend on $\mathcal{V}_w$. Then, the claim follows from the fact that $\widetilde{\mathcal{W}_{k,w}} = \mathcal{W}_w$.

Observe that $S_{\mathcal{W}_w}$ commutes with Q_k. We now show that $S_{\mathcal{W}_w} Q_k = \alpha_k Q_k$ for some $\alpha_k \in \sigma(S_{\mathcal{W}_w})$. Indeed, by the maximality of $\{Q_i\}_{i=1}^p$ in $\mathcal{Q}$, it follows that there is no non-trivial sub-projection Q' of Q_k such that $Q' \in \{P_{\mathcal{M}_j} : j \in \mathbb{I}_m\}'$. Then we can apply Corollary 4.2.2 (taking $\mathcal{H} = R(Q_k)$ and renormalizing the traces) to show that every positive operator (with the correct trace) near $S_{\mathcal{W}_w} Q_k$ has the form $S_{\mathcal{V}_w}$ for some $\mathcal{V}_w \in \mathcal{B}(Q_k, \mathbf{r}, w)$ close to $\mathcal{W}_k$. But if $S_{\mathcal{W}_w} Q_k$ is not a scalar multiple of Q_k, then we can choose $S_{\mathcal{V}_w}$ in such a way that

$$\text{FFP}\,(\mathcal{V}_w) = \operatorname{tr} S^2_{\mathcal{V}_w} < \operatorname{tr}\, \big(S^2_{\mathcal{W}_w} Q_k \big) = \text{FFP}\,(\mathcal{W}_k) .$$

This contradicts the fact that $\mathcal{W}_k$ is a local minimizer of the FF-potential in $\mathcal{B}(Q_k, \mathbf{r}, w)$. Hence, $S_{\mathcal{W}_w} Q_k = \alpha_k Q_k$ and $Q_k \le P_{\alpha_k}(S_{\mathcal{W}_w})$. Using that $\sum\limits_{k=1}^{p} Q_k = I_n$, it is easy to see that each

$$P_{\mu_i}(S_{\mathcal{W}_w}) = \sum_{k\in J_i} Q_k \in C_{\mathcal{W}_w} , \qquad \text{where} \qquad J_i = \{ k \in \mathbb{I}_p \ : \ \alpha_k = \mu_i \}. \qquad \square$$

Remark 4.2.6. Next we give two reinterpretations of Theorem 4.2.5. Under the assumptions of the theorem, the following properties hold:

1. For each $i \in \mathbb{I}_m$, there exists an ONB $\mathcal{B}_i = \{e^{(i)}_j : j \in \mathbb{I}_{d_i}\}$ of W_i, consisting of eigenvectors of $S_{\mathcal{W}_w}$. Indeed, observe that each $P_{W_i} = \sum_{k\in\mathbb{I}_r} P_{W_i} P_{\mu_k}(S_{\mathcal{W}_w})$ and the fact that, for a fixed $i \in \mathbb{I}_m$, the projections $P_{W_i} P_{\mu_k}(S_{\mathcal{W}_w})$ are pairwise orthogonal.

2. For each $\mu_k \in \sigma(S_{\mathcal{W}_w})$, denote by $\mathcal{M}_{k,i} = N(S_{\mathcal{W}_w} - \mu_k I_n) \cap W_i$. Then, it follows that the sequence $\mathcal{W}_k = (w_i, \mathcal{M}_{k,i})_{i\in\mathbb{I}_m}$ is a **tight** FF for $N(S_{\mathcal{W}_w} - \mu_k I_n)$. This follows because its frame operator $S_{\mathcal{W}_k} = P_{\mu_k}(S_{\mathcal{W}_w})\, S_{\mathcal{W}_w} = \mu_k P_{\mu_k}(S_{\mathcal{W}_w})$.

4.3 The eigenvalues of all d_S-minimizers coincide

Recall that, given $S \in \mathcal{M}_n(\mathbb{C})_{sa}$, we denote by $\lambda(S) \in \mathbb{R}^n$ the vector of the n eigenvalues of S, counted with multiplicities, in such a way that $\lambda(S) = \lambda(S)^{\downarrow}$.

Lemma 4.3.1. *Let $(\mathbf{d}, w) \in \mathbb{N}^m \times \mathbb{R}^m_{>0}$ be a normalized pair. Then*

1. *$\mathcal{B}_{m,n}(\mathbf{d}, w)$ is d_P-compact.*
2. *The set $\Lambda_{m,n}(\mathbf{d}, w) = \{\lambda(S_{\mathcal{W}_w}) : \mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d}, w)\}$ is a convex and compact subset of $\mathbb{R}^n$.*
3. *The set $\{S_{\mathcal{W}_w} : \mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d}, w)\}$ is compact and closed under unitary conjugation.*

Proof. Let $\mathcal{W}_w, \mathcal{W}'_w \in \mathcal{B}_{m,n}(\mathbf{d}, w)$. For $t \in [0,1]$ consider $\lambda = t\,\lambda(S_{\mathcal{W}_w}) + (1-t)\,\lambda(S_{\mathcal{W}'_w})$ and notice that $\lambda = \lambda^{\downarrow}$. Therefore, for every admissible $(m+1)$-tuple $(J_0, \ldots, J_m) \in LR^r_n(m)$, $1 \le r \le n-1$ we have

$$\sum_{j \in J_0} \lambda_j = \sum_{i=1}^m \sum_{j \in J_i} t\,\lambda_j(S_{\mathcal{W}_w}) + (1-t)\,\lambda_j(S_{\mathcal{W}'_w}) \le \sum_{i=1}^m \sum_{j \in J_i} w_j^2\ |\{1, \ldots, d_j\} \cap J_i|\,,$$

since both $\lambda(S_{\mathcal{W}_w})$ and $\lambda(S_{\mathcal{W}'_w})$ satisfy Horn-Klyachko's inequalities. Hence, by Theorem 2.3.3, there exists $\mathcal{V}_w = (w_i, V_i)_{i \in \mathbb{I}_m} \in \mathcal{B}_{m,n}(\mathbf{d}, w)$ such that $\lambda(S_{\mathcal{V}_w}) = \lambda$. This shows that $\lambda \in \Lambda_{m,n}(\mathbf{d}, w)$, so that $\Lambda_{m,n}(\mathbf{d}, w)$ is convex. The fact that the set $\{S_{\mathcal{W}} : \mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d}, w)\}$ is closed under unitary conjugation is apparent. Finally $\mathcal{B}_{m,n}(\mathbf{d}, w)$ is d_P-compact because each Grassmann manifold

$$\mathcal{P}_{d_i}(n) = \{P = P^*P \in \mathcal{M}_n(\mathbb{C})^+ : \operatorname{tr} P = d_i\} = \{UP_iU^* : U \in \mathcal{U}(n)\} = \mathcal{U}(P_i)\,,$$

for every fixed $P_i \in \mathcal{P}_{d_i}(n)$, is compact. This follows because $\mathcal{U}(n)$ is compact. By continuity, the other sets involved are also compact. □

Theorem 4.3.2. *Let $(\mathbf{d}, w) \in \mathbb{N}^m \times \mathbb{R}^m_{>0}$ be a normalized pair. Then,*

1. *The spectra (with multiplicities) of all the frame operators of global minimizers of the FF-potential in $\mathcal{B}_{m,n}(\mathbf{d}, w)$ coincide.*
2. *The local minimizers of the FF-potential in $\mathcal{B}_{m,n}(\mathbf{d}, w)$ with respect to the pseudo-distance d_S lie in $\mathcal{S}_{m,n}(\mathbf{d}, w)$ and are also global minimizers.*

Proof. Let $\mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d}, w)$ and notice that $\operatorname{FFP}(\mathcal{W}_w) = \|\lambda(S_{\mathcal{W}_w})\|^2$. Since $\Lambda_{m,n}(\mathbf{d}, w)$ is a compact convex set, then there exists a unique $\lambda_0 \in \Lambda_{m,n}(\mathbf{d}, w)$ which minimizes the Euclidean norm on $\Lambda_{m,n}(\mathbf{d}, w)$. Hence, if $\mathcal{W}_w$ is a global minimizer of the FF-potential in $\mathcal{B}_{m,n}(\mathbf{d}, w)$, we can conclude that $\|\lambda(S_{\mathcal{W}_w})\|^2 \le \|\lambda_0\|^2$, which implies that $\lambda(S_{\mathcal{W}_w}) = \lambda_0$.

Observe that the map $\sigma : \mathcal{B}_{m,n}(\mathbf{d}, w) \to \Lambda_{m,n}(\mathbf{d}, w)$ given by $\sigma(\mathcal{W}_w) = \lambda(S_{\mathcal{W}_w})$ is continuous with respect to the pseudo-distance d_S of $\mathcal{B}_{m,n}(\mathbf{d}, w)$. Moreover, σ is an open map. Indeed, fix $\mathcal{W}_w \in \mathcal{B}_{m,n}(\mathbf{d}, w)$, $\lambda = \lambda(S_{\mathcal{W}_w}) \in \Lambda_{m,n}(\mathbf{d}, w)$ and take $\mu \in \Lambda_{m,n}(\mathbf{d}, w)$ close to λ. Take $U \in \mathcal{U}(n)$ such that $S_{\mathcal{W}_w} = U \operatorname{diag}(\lambda)\, U^*$. By Lemma 4.3.1, there exists $\mathcal{V}_w \in \mathcal{B}_{m,n}(\mathbf{d}, w)$ such that $S_{\mathcal{V}_w} = U \operatorname{diag}(\mu)\, U^*$. Now observe that $d_S(\mathcal{W}_w, \mathcal{V}_w) = \|S_{\mathcal{W}_w} - S_{\mathcal{V}_w}\| = \|\lambda - \mu\|_\infty$.

Therefore, if $\mathcal{W}_w$ is a d_S-local minimizer of the FF-potential in $\mathcal{B}_{m,n}(\mathbf{d}, w)$, then $\lambda(S_{\mathcal{W}_w})$ is a local minimum for the Euclidean norm in the set $\Lambda_{m,n}(\mathbf{d}, w) \subseteq \mathbb{R}^n$. By a standard computation, the convexity of $\Lambda_{m,n}(\mathbf{d}, w)$ implies that $\lambda(S_{\mathcal{W}_w})$ must be *the* global minimizer λ_0, and therefore $\mathcal{W}_w$ is a global minimizer in $\mathcal{B}_{m,n}(\mathbf{d}, w)$. □

Conjecture 4.3.3. Local minimizers of the frame potential in $\mathcal{B}_{m,n}(\mathbf{d},w)$ (resp. $\mathcal{S}^1_{m,n}(\mathbf{d})$) with respect to punctual distance d_P are also global minimizers.

In some particular cases (i.e. for particular choices of the parameters n, m, $\mathbf{d}$ and w), there is an affirmative answer for this conjecture (see [7, Theorem 5]).

Remark 4.3.4. All the previous results remain true if one replaces $\mathcal{B}_{m,n}(\mathbf{d},w)$ by $\mathcal{B}^1_{m,n}(\mathbf{d})$, the set of those $\mathcal{W}_w \in \mathcal{B}_{m,n}$ such that $\operatorname{tr} S_{\mathcal{W}_w} = 1$. In other words, minimizing the FFP without fixing the sequence of weights. We present some of the new statements without proofs, since they are based on techniques that are similar to those already developed.

1. As in the proof Lemma 4.3.1, Horn-Klyachko's compatibility inequalities (7) show that
$$\Lambda^1_{m,n}(\mathbf{d}) = \{\lambda(S_{\mathcal{W}_w}) : \ \mathcal{W}_w \in \mathcal{B}^1_{m,n}(\mathbf{d})\} \quad \text{is compact and convex.}$$
2. This fact implies that the (ordered) spectra of the frame operators of global minimizers of the FF-potential in $\mathcal{B}^1_{m,n}(\mathbf{d})$ coincide.
3. Finally, the argument of the proof of Theorem 4.3.2 can be adapted to yield that local minimizers of the FF-potential in $\mathcal{B}^1_{m,n}(\mathbf{d})$, with respect to the operator distance d_S , are also global.

As in the case of fixed weights (Lemma 4.2.4), the global minimizers for the FF-potential in $\mathcal{B}^1_{m,n}(\mathbf{d})$ are fusion frames.

Proposition 4.3.5. *Let $\mathcal{W}_w$ be a d_S-local (and hence global) minimizer for the FF-potential in $\mathcal{B}^1_{m,n}(\mathbf{d})$. Then its frame operator $S_{\mathcal{W}_w} \in \mathcal{G}l(n)^+$. In other words, $\mathcal{W}_w \in \mathcal{S}^1_{m,n}(\mathbf{d})$.*

Proof. Let $J = \{i \in \mathbb{I}_m : w_i \neq 0\}$, and $k = \sum\limits_{i\in J} d_i$. Note that, if $k \geq n$, we can apply Lemma 4.2.4 (fixing the weight w_J) and we are done.

We assume that $k < n$, and will obtain a contradiction. Without loss of generality, we can suppose that $J = \mathbb{I}_r$. It follows immediately that $W_i \perp W_j$ for $1 \leq i \neq j \leq r$ and $w_i^2 = \frac{1}{k}$ for every $1 \leq i \leq r$. Thus, $\operatorname{FFP}(\mathcal{W}_w) = \frac{1}{k}$. Moreover, if $d = d_{r+1}$, then $k + d > n$. Otherwise, if we take a subspace $W_{r+1} \subseteq \big(\bigcup\limits_{i\in\mathbb{I}_r} W_i\big)^\perp$ with $\dim W_{r+1} = d$ and we set $w_i^2 = \frac{1}{k+d}$, for $i \in \mathbb{I}_{r+1}$, then we get a BSS in $\mathcal{B}^1_{m,n}(\mathbf{d})$ with FF-potential $\frac{1}{k+d} < \frac{1}{k}$.

Therefore, we can construct $\mathcal{V}_v = (v_j\,, V_j)_{j\in\mathbb{I}_{r+1}} \in \mathcal{B}^1_{m,n}(\mathbf{d})$ in the following way:

$$v_j^2 = \begin{cases} a & \text{if } 1 \leq j \leq r \\ b & \text{if } j = r+1 \end{cases} \quad \text{and} \quad V_j = \begin{cases} W_j & \text{if } 1 \leq j \leq r \\ \Big[\bigoplus\limits_{i=1}^{r} W_i\Big]^\perp \oplus T & \text{if } j = r+1\,, \end{cases}$$

where $T \subseteq \bigoplus_{i=1}^r W_i$ is a subspace with $\dim T = d + k - n$ (so that $\dim V_{r+1} = d$), and $ka + db = 1$. It is easy to see, by taking an orthonormal basis of each subspace V_i , that

$$f(a) \ = \operatorname{FFP}(\mathcal{V}_v) = ka^2 + db^2 + 2(d+k-n)ab \quad \text{with} \quad ka + db = 1 \ \ , \quad a \in \big[\,0\,,\,\tfrac{1}{k}\,\big]\,. \qquad (21)$$

Easy computations show that $f'(\frac{1}{k}) = \frac{2(n-k)}{d} > 0$. Since $f(\frac{1}{k}) = \frac{1}{k}$, there exist pairs (a,b) such that the FF-potential is lower that $\frac{1}{k}$, which contradicts the minimality of $\mathcal{W}_w$. □

Next, we summarize the facts described in Remark 4.3.4 and Proposition 4.3.5.

Theorem 4.3.6. *Let $\mathbf{d} \in \mathbb{N}^m$. Then, local minimizers of the frame potential in $\mathcal{B}^1_{m,n}(\mathbf{d})$ with respect to the pseudo-distance d_S lie in $\mathcal{S}^1_{m,n}(\mathbf{d})$ and are also global minimizers. Moreover, the spectra of the frame operators of these local minimizers coincide.* □

5 Minimization for fixed subspaces

In this section we shall characterize the sequences of weights which minimize the potential of a fixed sequence of subspaces. The main tools are some results about Hadamard indexes of [12], which we shall state in some detail in the appendix. Recall that, for $A, B \in \mathcal{M}_n(\mathbb{C})$, their Hadamard product is the matrix $A \circ B = (A_{ij}\, B_{ij})_{i\,,\,j\in\mathbb{I}_n} \in \mathcal{M}_n(\mathbb{C})$.

5.1 Minimal weights

In this section we fix $m \in \mathbb{N}$, a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = n$, and $\mathbf{d} \in \mathbb{N}^m$ with $M = \operatorname{tr} \mathbf{d} \geq n$. We also fix a sequence $\mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m}$ of subspaces which spans $\mathcal{H}$, such that $\dim W_i = d_i$ for all $i \in \mathbb{I}_m$. Our aim is to minimize the FFP over all sequences $w \in \mathbb{R}_+^m$ such that $\mathcal{W}_w \in \mathcal{B}^1_{m,\,n}(\mathbf{d})$.

Recall that the Benedetto-Fickus FF-potential of $\mathcal{W}_w$ is given by:

$$\operatorname{FFP}(\mathcal{W}_w) = \sum_{i,j=1}^{m} w_i^2 w_j^2 \operatorname{tr}(P_{W_i} P_{W_j}) = \operatorname{tr} S^2_{\mathcal{W}_w} \,. \tag{22}$$

Definition 5.1.1. Given $\mathcal{W}_w = (w_i\,,\, W_i)_{i\in\mathbb{I}_m} \in \mathcal{B}^1_{m,\,n}(\mathbf{d})$, let $B = B_{\mathcal{W}_w} \in \mathcal{M}_m(\mathbb{R})$ be defined by

$$B_{ij} = w_i^2 w_j^2 \operatorname{tr}(P_{W_i} P_{W_j}) \ , \quad \text{for every} \quad i\,,\, j \in \mathbb{I}_m \,.$$

Lemma 5.1.2. *Let $\mathcal{W}_w = (w_i\,,\, W_i)_{i\in\mathbb{I}_m} \in \mathcal{B}^1_{m,\,n}(\mathbf{d})$. Then, $B_{\mathcal{W}_w} \in \mathcal{M}_m(\mathbb{R})^+$.*

Proof. Indeed, $B_{\mathcal{W}_w} \in \mathcal{M}_m(\mathbb{R})^+$ because it is the Gram matrix for the vectors $\{w_i^2 P_{W_i}\}_{i\in\mathbb{I}_m}$ in the Euclidean space $\mathcal{M}_n(\mathbb{C})$ with the inner product defined as $\langle X\,,\, Y\rangle = \operatorname{tr}(Y^*X)$. ☐

5.1.3. We shall fix some notations and assumptions:

1. We begin with a fixed normalized sequence of weights, in the sense that

$$\mathbf{w} = (w_i)_{i\in\mathbb{I}_m} \quad \text{is given by} \quad w_i = d_i^{-\frac{1}{2}} \quad \text{for every} \quad i \in \mathbb{I}_m \,.$$

   Observe that the condition $w_i = d_i^{-\frac{1}{2}}$ means that each "vector" $w_i\, P_{W_i}$ of $\mathcal{W}_w$ has size $\|w_i\, P_{W_i}\|_2 = w_i \operatorname{tr} P_{W_i} = 1$. This justifies the word "normalized" for $\mathcal{W}_w$.

2. Given a sequence of weights $\mathbf{a} = (a_i)_{i\in\mathbb{I}_m} \in \mathbb{R}_+^m$, we denote by $\mathbf{a}\cdot\mathcal{W}_w$ the Bessel sequence of subspaces $\mathbf{a}\cdot\mathcal{W}_w = (a_i\, w_i\, W_i)_{i\in\mathbb{I}_m}$.

3. If $\mathbf{a} \in \mathbb{R}_+^m$, then $\operatorname{tr}(S_{\mathbf{a}\cdot\mathcal{W}_w}) = \sum\limits_{i\in\mathbb{I}_m} a_i^2 w_i^2 d_i$. Therefore, as we start with normalized weights,

$$\mathbf{a}\cdot\mathcal{W}_w \in \mathcal{B}^1_{m,\,n}(\mathbf{d}) \iff \sum_{i\in\mathbb{I}_m} a_i^2 w_i^2 d_i = \sum_{i\in\mathbb{I}_m} a_i^2 = 1 \iff \|\mathbf{a}\| = 1 \,. \tag{23}$$

4. Let $A = A_{\mathcal{W}_w} \in \mathcal{M}_m(\mathbb{R})$ be the matrix given by $A_{ij} = (B_{\mathcal{W}_w})_{ij}^{\frac{1}{2}} = w_i\, w_j \operatorname{tr}(P_{W_i} P_{W_j})^{\frac{1}{2}}$, for $i, j \in \mathbb{I}_m$. Observe that A is selfadjoint, but possibly $A \notin \mathcal{M}_m(\mathbb{R})^+$. On the other hand,

$$\begin{aligned} \operatorname{FFP}(\mathcal{W}_w) &= \sum_{i,j=1}^{m} w_i^2\, w_j^2 \operatorname{tr}(P_{W_i} P_{W_j}) = \|A_{\mathcal{W}_w}\|_2^2 \ , \\ \operatorname{FFP}(\mathbf{a}\cdot\mathcal{W}_w) &= \sum_{i,j=1}^{m} a_i^2\, a_j^2\, w_i^2\, w_j^2 \operatorname{tr}(P_{W_i} P_{W_j}) = \|\mathbf{a}\mathbf{a}^* \circ A_{\mathcal{W}_w}\|_2^2 \,. \end{aligned} \tag{24}$$

5. Using the previous identities, we can now define the main notion of this section:

$$I(\mathcal{W}_w) := \min_{\|a\|=1} \text{FFP}\,(\mathbf{a}\cdot\mathcal{W}_w) = \min_{\|a\|=1} \|\mathbf{a}\mathbf{a}^*\circ A_{\mathcal{W}_w}\|_2^2 \,.$$

In order to compute $I(\mathcal{W}_w)$ as well as to describe the set of weights $\mathbf{a}\in\mathbb{R}_+^m$, with $\|\mathbf{a}\|=1$ for which $I(\mathcal{W}_w) = \text{FFP}\,(\mathbf{a}\cdot\mathcal{W}_w)$, the main tools are some results about Hadamard indexes of [12], which we shall state in some detail in the appendix. Here we just give the basic definitions.

Definition 5.1.4. Let $B\in\mathcal{M}_m(\mathbb{C})^+$. The *minimal*-Hadamard index of G is

$$I(B) = \max\,\{\ \lambda\geq 0\ :\ B\circ C\geq\lambda C \quad \text{for every} \quad C\in\mathcal{M}_m(\mathbb{C})^+\}\,.$$

For $A\in\mathcal{M}_m(\mathbb{C})_{sa}$, we define the spectral and the $\|\cdot\|_2$ Hadamard indexes:

$$I_{sp}(A) = \min_{\|x\|=1} \|A\circ x\,x^*\|_{sp} \quad \text{and} \quad I_2(A) = \min_{\|x\|=1} \|A\circ x\,x^*\|_2\,.$$

For a matrix $G\in\mathcal{M}_n(\mathbb{C})$, we write $0\underset{\text{ij}}{\leqslant} G$ if all entries $G_{i,j}\geq 0$. Given $J\subseteq\mathbb{I}_m$ with $|J|=k$ we denote by $G_J\in\mathcal{M}_k(\mathbb{C})$ the submatrix of G given by $G_J = \big(G_{ij}\big)_{i,j\in J}$. Similarly, if $x\in\mathbb{C}^n$, we write $x\geqslant 0$ if $x\in\mathbb{R}_+^m$ and $x_J = (x_j)_{j\in J}\in\mathbb{C}^k$.

From the previous definitions and Eq. (24), we get the fundamental equality: $I(\mathcal{W}_w) = I_2(A_{\mathcal{W}_w})$. Now it is clear why the results of the appendix can be useful for computing $I(\mathcal{W}_w)$.

Remark 5.1.5. Let $\mathbf{a}\in\mathbb{R}^m$. Then

$$\text{FFP}\,(\mathbf{a}\cdot\mathcal{W}_w) = \sum_{i,j=1}^m a_i^2\,a_j^2\,w_i^2 w_j^2 tr(P_{W_i}P_{W_j}) = \big\langle (B\circ\mathbf{a}\,\mathbf{a}^*)\,\mathbf{a},\mathbf{a}\big\rangle = \big\langle B(\mathbf{a}\circ\mathbf{a}^*),(\mathbf{a}\circ\mathbf{a}^*)\big\rangle \tag{25}$$

where, as before, $B = B_{\mathcal{W}_w}\in\mathcal{M}_m(\mathbb{R})^+$. Moreover, by Eq. (31) in Proposition A.1.2,

$$I(\mathcal{W}_w) = \min_{\|\mathbf{a}\|=1} \|A_{\mathcal{W}_w}\circ\mathbf{a}\,\mathbf{a}^*\|_2^2 = \min_{\|\mathbf{a}\|=1} \|B\circ\mathbf{a}\,\mathbf{a}^*\| = I_{sp}(B)\,. \tag{26}$$

This identity is useful because $0\underset{\text{ij}}{\leqslant} B\in\mathcal{M}_m(\mathbb{R})^+$ and its spectral index is easier to compute. Indeed, observe that $\text{tr}(P_{W_i}\,P_{W_j})\geq 0$ for every $i,j\in\mathbb{I}_m$, since each $P_{W_i}\in\mathcal{M}_n(\mathbb{C})^+$.

5.2 Critical Points and local minimizers.

In what follows, we shall use all the assumptions and notations of the previous subsection, but we need the following extra notations:

1. Given $\mathbf{a}\in\mathbb{R}^m$, we write $z = \mathbf{a}\circ\mathbf{a} = (a_1^2,\dots,a_m^2)$ and $J = \text{supp}\{\mathbf{a}\} = \{j\in\mathbb{I}_n : a_j\neq 0\}$.
2. Let $L:\mathbb{R}^m\to\mathbb{R}^+$ be given by $L(\mathbf{a}) = \text{FFP}\,(\mathbf{a}\cdot\mathcal{W}_w) = \|A_{\mathcal{W}_w}\circ\mathbf{a}\mathbf{a}^*\|_2^2 = \langle B_{\mathcal{W}_w}\,z\,,\,z\rangle$.
3. We consider the affine manifold $\Delta_0 = \{x\in\mathbb{R}^m : \text{tr}\,x = 0\}$ and the compact convex simplex $\Delta = \{x\in\Delta_0\ :\ x\geqslant 0\}$.
4. $S^{m-1} = \{x\in\mathbb{R}^m : \|x\| = 1\}$ is the unit sphere of $\mathbb{R}^m$.

Lemma 5.2.1. *Let $0\leqslant\mathbf{a}\in S^{m-1}$. Denote $B = B_{\mathcal{W}_w}$. The following conditions are equivalent:*

1. $\mathbf{a}$ *is a critical point for the map* L *restricted to the sphere* S^{m-1}.

2. $B_J z_J = I(B_J)\mathbb{1}_J$ *, where* $J = \operatorname{supp}\{\mathbf{a}\}$.

Proof. Observe that $L(\mathbf{a}) = \langle Bz, z\rangle = \sum\limits_{i,j=1}^{m} b_{ij}\, a_i^2\, a_j^2$. Since $B = B^* = B^T$, we have that

$$0 = \nabla L(\mathbf{a}) = 4\, \big(a_1 \sum_{j=1}^{m} b_{1j}\, a_j^2\,, \ldots,\ a_m \sum_{j=1}^{m} b_{mj}\, a_j^2\big) = 4\ Bz \circ \mathbf{a}\,.$$

The tangent space of S^{m-1} at $\mathbf{a}$ is $\{\mathbf{a}\}^\perp$. Hence, $\mathbf{a}$ is a critical point for S^{m-1} if and only if

$$0 = \langle \nabla L(\mathbf{a}), y\rangle = 4\ \langle Bz \circ \mathbf{a}, y\rangle \quad \text{for every} \quad y \in \{\mathbf{a}\}^\perp \iff Bz \circ \mathbf{a} \in \operatorname{span}\{\mathbf{a}\}\ .$$

This is clearly equivalent to the equation $B_J z_J = \lambda \mathbb{1}_J$, for some $\lambda \in \mathbb{R}$. In this case, since $0 \underset{\text{ij}}{\leqslant} B$ with $0 < B_{ii}$ for every $i \in \mathbb{I}_n$ and $0 \leqslant z$, we can conclude that $\lambda > 0$. By Proposition A.1.2 applied to the matrix B_J , we have that $B_J\ z_J = \lambda \mathbb{1}_J$. Hence,

$$B_J\ \tfrac{z_J}{\lambda} = \mathbb{1}_J \implies I(B_J)^{-1} = \operatorname{tr}\ \tfrac{z_J}{\lambda} = \langle \mathbb{1}_J\,,\ \tfrac{z_J}{\lambda}\rangle = \lambda^{-1}\ \langle \mathbb{1}\,,\ z\rangle = \lambda^{-1}\,.$$

Therefore $\lambda = I(B_J)$ and $B_J z_J = I(B_J)\mathbb{1}_J$. ☐

Theorem 5.2.2. *Let* $0 \leqslant \mathbf{a} \in S^{m-1}$ *such that* $B_J z_J = I(B_J)\mathbb{1}_J$ *, i.e., the vector* $\mathbf{a}$ *is a critical point of* L *restricted to* S^{m-1}. *Then, the following conditions are equivalent:*

1. $I(B_J) = I_{sp}(B) = I(\mathcal{W}_w\,)$.

2. $\mathbf{a}$ *is a global minimum of* L *restricted to* S^{m-1}.

3. $\mathbf{a}$ *is a local minimum of* L *restricted to* S^{m-1}.

4. $Bz \geqslant I(B_J)\mathbb{1}$. *In other words, that* $(Bz)_j \ge I(B_J)$ *for every* $j \notin J$.

Proof. Denote $A = A_{\mathcal{W}_w}$. Recall that $I_2(A)^2 = I_{sp}(B)$, by Eq. (26). By Lemma 5.2.1,

$$\|A \circ \mathbf{a}\mathbf{a}^*\|_2^2 \ = \ \langle Bz\,,\ z\rangle \ = \ I(B_J)\ \langle \mathbb{1}\,,\ z\rangle = \ I(B_J)\,.$$

This gives the equivalence $1 \leftrightarrow 2$. Observe that, if $\mathbf{b} \in S^{m-1}$, then $w = \mathbf{b} \circ \mathbf{b} \in \Delta$. For each $w \in \Delta$, consider the line $\gamma_w : [0,1] \to \Delta$ joining z and w, given by the formula $\gamma_w(t) = (1-t)z + tw$, for every $t \in [0,1]$. Consider the map $\rho_w : [0,1] \to \mathbb{R}^+$ given by

$$\rho_w(t) = \langle B\gamma_w(t)\,,\ \gamma_w(t)\rangle = (1-t)^2\ \langle Bz\,,\ z\rangle + t^2\ \langle Bw\,,\ w\rangle + 2t(1-t)\ \langle Bz\,,\ w\rangle \tag{27}$$

for every $t \in [0,1]$. Since $\langle Bz\,,\ z\rangle = I(B_J)$, the derivative of ρ_w evaluated at zero is

$$\dot\rho_w(0) = -2\,\langle Bz\,,\ z\rangle + 2\,\langle Bz\,,\ w\rangle = 2\Big(\ \langle Bz\,,\ w\rangle - I(B_J)\,\Big)\,.$$

On the other hand, for every $t \in \mathbb{R}$,

$$\ddot\rho_w(t) = 2\,\langle Bz\,,\ z\rangle + 2\,\langle Bw\,,\ w\rangle - 4\,\langle Bz\,,\ w\rangle = 2\,\langle B(z-w)\,,\ z-w\rangle \ge 0\,. \tag{28}$$

Since ρ_w is a second degree polynomial, its leading coefficient is $\frac{1}{2}\ \ddot\rho_w(t) \ge 0$. Suppose now that $Bz \geqslant I(B_J)\mathbb{1}$. Using that $w \in \Delta$, we get that

$$\langle Bz\,,\ w\rangle \ge I(B_J)\,\langle \mathbb{1}\,,\ w\rangle = I(B_J) \implies \dot\rho_w(0) \ge 0 \implies \dot\rho_w(t) \ge 0 \quad \text{for every} \quad t \ge 0\,.$$

Therefore $\rho_w(1) \geq \rho_w(0)$. In other words, we have proved that $\langle Bw\,,\,w\rangle \geq \langle Bz\,,\,z\rangle$ for every $w \in \Delta$. This implies that $\mathbf{a}$ is a global minimum of L restricted to S^{m-1}.

Suppose now that $\mathbf{a} \in S^{m-1}$, $z = \mathbf{a} \circ \mathbf{a}$, and that there exists $k \in \mathbb{I}_m$ such that $(Bz)_k < I(B_J)$. Observe that $\mathbf{a} \neq e_k$, because $(Be_k)_k = b_{kk} = I(B_{\{k\}})$.

Let $w = e_k \in \Delta$, and consider the curves γ_w and ρ_w defined before. By the previous computations, we have that $\dot\rho_w(0) = 2(\,(Bz)_k - I(B_J)\,) < 0$. Therefore, for every $t > 0$ small enough, we have that $\gamma_w(t) \in \Delta$ and $\langle B\gamma_w(t)\,,\,\gamma_w(t)\rangle < \langle Bz\,,\,z\rangle$. Taking the vectors $\mathbf{a}(t) = \operatorname{sgn}(\mathbf{a})\,\gamma_w(t)^{\frac{1}{2}} \in S^{m-1}$, we conclude that $\mathbf{a}$ fails to be a local minimum. $\square$

Remark 5.2.3. Given $\mathbf{a} \in S^{m-1}$, then $z = \mathbf{a} \circ \mathbf{a} \in \Delta$ and $L(\mathbf{a}) = \langle Bz, z\rangle$. Therefore

$$I(\mathcal{W}_w) = I_2(A) = \min_{\mathbf{a}\in S^{m-1}} \|A \circ \mathbf{a}\mathbf{a}^*\|_2^2 = \min_{0\,\leqslant\,\mathbf{a}\in\,S^{m-1}} \|A \circ \mathbf{a}\mathbf{a}^*\|_2^2 = \min_{z\in\Delta} \langle Bz, z\rangle\,,$$

since every $z \in \Delta$ produces a unit vector $0 \leqslant \mathbf{a} = z^{\frac{1}{2}} \in S^{m-1}$. Then in order to get the unit vectors $\mathbf{a}$ which attain this minimum, it suffices to characterize the sets $\mathcal{S}(\mathcal{W}_w) = \arg\min_{z\in\Delta}\{\langle Bz\,,\,z\rangle\}$ and

$$J(\mathcal{W}_w) = \big\{J \subseteq \mathbb{I}_m \,:\, J = \operatorname{supp}\{z\} \ \text{ for some } \ z \in \mathcal{S}(\mathcal{W}_w)\big\}\,.$$

If $\mathbb{I}_m \notin J(\mathcal{W}_w)$, it is possible to obtain minimizers $\mathbf{a}\cdot\mathcal{W}_w$ which are not fusion frames, because $S_{\mathbf{a}\cdot\mathcal{W}_w} \notin \mathcal{G}l(n)^+$ (see Example 5.2.7). Still, if $I(\mathcal{W}_w) < \frac{\sqrt{1+n}}{n}$ then $S_{\mathbf{a}\cdot\mathcal{W}_w} \in \mathcal{G}l(n)^+$ for any minimizer $\mathbf{a}$, since in such case Proposition 2.2.3 implies that $\|I - n\,S_{\mathbf{a}\cdot\mathcal{W}_w}\| < 1$. Otherwise, the characterization of the set $J(\mathcal{W}_w)$ is useful in order to discern if there are optimal sequences of weights $\mathbf{a}$ such that $\mathbf{a}\cdot\mathcal{W}_w$ remains being a FF. Item 4 of Theorem 5.2.2 gives a description of the elements of $\mathcal{S}(\mathcal{W}_w)$. But its proof gives more information:

Corollary 5.2.4. *Consider $\mathcal{W}_w \in \mathcal{S}_{m\,,\,n}(\mathbf{d})$, $A = A_{\mathcal{W}_w}$ and $B = B_{\mathcal{W}_w}$ as before. Then:*

1. *The set $\mathcal{S}(\mathcal{W}_w) = \arg\min_{z\in\Delta}\{\langle Bz\,,\,z\rangle\}$ is convex. Moreover,*

$$\mathcal{S}(\mathcal{W}_w) = \big(z_0 + N(B)\,\big) \cap \Delta \qquad \textit{for any point } z_0 \in \mathcal{S}(\mathcal{W}_w)\,.$$

2. *$J(\mathcal{W}_w)$ is closed under taking unions, so that $J_{\mathcal{W}_w} = \bigcup J(\mathcal{W}_w) = \bigcup_{z\in\,\mathcal{S}(\mathcal{W}_w)} \operatorname{supp}\{z\}$ is an element of $J(\mathcal{W}_w)$, and there exists $z_1 \in \mathcal{S}(\mathcal{W}_w)$ with maximal support.*

Proof. 1. Let $z, w \in \Delta$, and consider the function, defined in Eq. (27):

$$\rho_{z,\,w}(t) = \big\langle B(\,(1-t)z + tw)\,,\,(1-t)z + tw\big\rangle\ , \quad t \in \mathbb{R}\,.$$

Suppose now that $w, z \in \mathcal{S}(\mathcal{W}_w)$ and $w \neq z$. Using that $\rho_{z,\,w}$ is of second degree, the equality $\ddot\rho_{z,\,w}(t) = 2\,\langle B(z-w)\,,\,z-w\rangle \geq 0$ given by Eq. (28), and the fact $\rho_{z,\,w}(t) \geq 0$ for every $t \in \mathbb{R}$, we can conclude that

$$\rho_{z,\,w} \quad \text{is constant} \quad \iff \ddot\rho_{z,\,w}(t) = 0 \iff z - w \in N(B)\,.$$

On the other hand, we have that $\rho_{z,\,w}(1) = \rho_{z,\,w}(0) = \min_{t\in[0,1]} \rho_{z,\,w}(t)$. This implies that the map $\rho_{z,\,w}$ is constant, so that $\gamma(t) \in \mathcal{S}(\mathcal{W}_w)$ for every $t \in [0,1]$, and $z - w \in N(B)$. The proof of the fact that $(z_0 + N(B)\,) \cap \Delta \subseteq \mathcal{S}(\mathcal{W}_w)$ for every $z_0 \in \mathcal{S}(\mathcal{W}_w)$ is similar.

2. Let z and w in $\mathcal{S}(\mathcal{W}_w)$, with supports J_1 and J_2 respectively. Then, since the entire line $tz+(1-t)w \in \mathcal{S}(\mathcal{W}_w)$ $(t \in [0,1])$, if we take $u = tz+(1-t)w$ for any $t \in (0,1)$, it is easy to see that $u \in \mathcal{S}(\mathcal{W}_w)$ and $\operatorname{supp}\{u\} = J_1 \cup J_2$. Since $J(\mathcal{W}_w)$ is finite, then the set

$$J_{\mathcal{W}_w} = \bigcup_{z \in \mathcal{S}(\mathcal{W}_w)} \operatorname{supp}\{z\} \in J(\mathcal{W}_w)\,.$$

Hence $J_{\mathcal{W}_w}$ is the support of some $z_1 \in \mathcal{S}(\mathcal{W}_w)$. □

Corollary 5.2.5. Let $B = B_{\mathcal{W}_w}$ and $A = A_{\mathcal{W}_w}$ as before. Assume that there exists $v \in \mathbb{R}^m$ such that $v \geqslant 0$ and $B\,v = \mathbb{1}$. Denote $\mathbf{a} = (\operatorname{tr} v)^{-\frac{1}{2}}\,(v_1^{\frac{1}{2}}, \dots, v_m^{\frac{1}{2}})$. Then

$$\|\mathbf{a}\| = 1 \quad \text{and} \quad (\operatorname{tr} v)^{-1} = \operatorname{FFP}(\mathbf{a} \cdot \mathcal{W}_w) = I_2(A)^2 = I(\mathcal{W}_w)\,. \tag{29}$$

Proof. The fact that $v \geqslant 0$ and $B\,v = \mathbb{1}$ implies, by Propositions A.1.2 and A.1.3, that

$$I_2(A)^2 = I_{sp}(B) = I(B) = I(B_J) = (\operatorname{tr} v)^{-1}\,,$$

where $J = \operatorname{supp}\{v\} = \operatorname{supp}\{\mathbf{a}\}$. Since $z = \frac{v}{\operatorname{tr} v} \in \Delta$, then $\mathbf{a} = z^{\frac{1}{2}} \in S^{m-1}$ and $Bz = I(B_J)\mathbb{1}$. Hence, by Theorem 5.2.2, we have that $z \in \mathcal{S}(\mathcal{W}_w)$ and $\mathbf{a} \in S^{m-1}$ satisfies Eq. (29). □

Remark 5.2.6. The results of this section seems to be unknown still for the case of vector frames. In this case our restrictions translate to the following: Let $\mathcal{F} = (f_i)_{i \in \mathbb{I}_m}$ be a frame for $\mathcal{H}$ such that each $\|f_i\| = 1$ (i.e., $d_i = 1 \implies w_i = d_i^{-\frac{1}{2}} = 1$). For $\mathbf{a} \in \mathbb{R}^m$, we consider the sequence $\mathbf{a} \cdot \mathcal{F} = (a_i\, f_i)_{i \in \mathbb{I}_m}$, and we define $I(\mathcal{F}) = \min\limits_{\|\mathbf{a}\|=1} \operatorname{FP}(\mathbf{a} \cdot \mathcal{F})$. Then, all the results of the section remain true if one consider the matrices

$$A_{\mathcal{F}} = \Big(|\langle f_j, f_i\rangle|\Big)_{i,j \in \mathbb{I}_m} \in \mathcal{M}_m(\mathbb{C})_{sa} \quad \text{and} \quad B_{\mathcal{F}} = \Big(|\langle f_j, f_i\rangle|^2\Big)_{i,j \in \mathbb{I}_m} \in \mathcal{M}_m(\mathbb{C})^+\,.$$

Some proofs are slightly easier in this case, because $I(\mathcal{F}) = I_2(A_{\mathcal{F}}) = I_2(G_{\mathcal{F}})$, where $G_{\mathcal{F}}$ is the Gram matrix of $\mathcal{F}$: $G_{\mathcal{F}} = \big(\langle f_j\,,\, f_i\rangle\big)_{i,j \in \mathbb{I}_m} \in \mathcal{M}_m(\mathbb{C})^+$. Observe that the diagonal entries of the three matrices involved are equal to 1.

Example 5.2.7. Let $B = \frac{1}{4}\begin{pmatrix} 4 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 4 \end{pmatrix}$. Since $A = (B_{ij}^{\frac{1}{2}})_{ij \in \mathbb{I}_3} \in \mathcal{G}l(3)^+$, we deduce that A is the Gram matrix of a Riesz basis $\mathcal{F}$ of $\mathbb{C}^3$. Let $v = (\,\frac{4}{5}\,,\,\frac{4}{5}\,,0) \geqslant 0$. Observe that by Corollary 5.2.5,

$$Bv \;= \mathbb{1}_3 \implies z = (\operatorname{tr} v)^{-1}\,v = \Big(\,\tfrac{1}{2}\,,\tfrac{1}{2}\,,0\,\Big) \in \mathcal{S}(\mathcal{F})\,.$$

Since $N(B) = \{0\}$, Corollary 5.2.4 assures that $\mathcal{S}(\mathcal{F}) = \{z\}$, and $J_0 = \{1,2\}$ is the maximal support for $\mathcal{S}(\mathcal{F})$. Taking $\mathbf{a} = z^{\frac{1}{2}}$, we have that $\mathbf{a} \cdot \mathcal{F}$ is the unique scaled sequence of $\mathcal{F}$ with minimal frame potential, but it fails to be a frame for $\mathbb{C}^3$, because it has just two nonzero elements.

Example 5.2.8. It can be proved that every $G \in \mathcal{M}_3(\mathbb{C})^+$ such that $\operatorname{rk} G = 2$ and $G_{ii} = 1$ for every $i \in \mathbb{I}_3$ (considered as the Gram matrix of a frame $\mathcal{F}$ for $\mathbb{C}^2$ with three unitary elements), satisfies that the minimizers $\mathbf{a} \cdot \mathcal{F}$ of the BF-potential are frames for $\mathbb{C}^2$. Indeed, given $z \in \mathcal{S}(\mathcal{F})$, it is easy to see that $J = \operatorname{supp}\{z\}$ has more than one element (otherwise $z = e_i$ for some $i \in \mathbb{I}_3$). If $J = \mathbb{I}_3$ there is nothing to prove. Assume that $\operatorname{supp}\{z\} = J$ with $|J| = 2$. If $\operatorname{rk} G_J < 2$ we must have $B_J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. In this case, $I(B_J) = I_{sp}(B) = 1$. But the unique matrix $B \in \mathcal{M}_3(\mathbb{C})^+$

which satisfies that $0 \underset{\text{ij}}{\leqslant} B$, $I_{sp}(B) = 1$ and $B_{ii} = 1$ for every $i \in \mathbb{I}_3$ is $B = \mathbb{1} \cdot \mathbb{1}^*$. Indeed, if some $B_{ij} < 1$, then $I_{sp}(B) \le I_{sp} \begin{pmatrix} 1 & B_{ij} \\ B_{ij} & 1 \end{pmatrix} = \frac{1+B_{ij}}{2} < 1$. Finally, since $1 = \text{rk } \mathbb{1} \cdot \mathbb{1}^* = \text{rk } B = \text{rk } G \circ \overline{G} \ge \text{rk } G = 2$, we have a contradiction.

Remark 5.2.9. Let $\mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m}$ be a generating set of subspaces. Given a partition $\{J_k\}_{k\in\mathbb{I}_p}$ of the set $\mathbb{I}_m$, we say that the sequence $\{\mathcal{W}_k\}_{k\in\mathbb{I}_p}$ of $\mathcal{W}$ given by $\mathcal{W}_k = (W_i)_{i\in J_k}$ is a *partition in orthogonal components* (POC) of $\mathcal{W}$ if $W_i \perp W_j$ for every pair $i \in J_k$, $j \in \mathbb{I}_m \setminus J_k$.

Note that by definition the trivial partition given by $J_1 = \mathbb{I}_m$ produces a POC of $\mathcal{W}$. If $\{\mathcal{W}_k\}_{k\in\mathbb{I}_p}$ is a POC of $\mathcal{W}$, we say that it is *maximal* if the only POC of each $\mathcal{W}_k$ is the trivial one. It is clear that there always exits such a maximal POC for $\mathcal{W}$.

Let $\{\mathcal{W}_k\}_{k\in\mathbb{I}_p}$ be a maximal POC of $\mathcal{W}$ with $|J_k| = m_k$ for $1 \le k \le p$. Let $\mathbf{a}_k \in \mathbb{R}^{m_k}$ be such that $\|a_k\| = 1$ and $I(\mathcal{W}_k) = \text{FFP}\,(\mathbf{a}_k \cdot \mathcal{W}_k)$ for each $1 \le k \le p$. Then, there exists $\gamma = (\gamma_k)_{k\in\mathbb{I}_p} \in \mathbb{R}^p_{>0}$ with $\|\gamma\| = 1$ and such that

$$I(\mathcal{W}) = \sum_{k=1}^{p} \text{FFP}\,(\gamma_k\, \mathbf{a}_k \cdot \mathcal{W}_k)\,.$$

Conversely, if $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_p)$ with $0 \leqslant \mathbf{a}_k \in \mathbb{R}^{m_k}$ and $\|\mathbf{a}\| = 1$ is such that $I(\mathcal{W}) = \text{FFP}\,(\mathbf{a} \cdot \mathcal{W})$ then $\mathbf{a}_k \neq 0$ and $I(\mathcal{W}_k) = \text{FFP}\,(\|\mathbf{a}_k\|^{-1}\, \mathbf{a}_k \cdot \mathcal{W}_k)$, for $1 \le k \le p$. Hence, we can restrict our study of the optimal weight of sequence of subspaces to each of the components of the maximal partition. This in turn implies that we can reduce the problem of describing the optimal weights to the case where the matrix B (which has non-negative entries and is positive semi-definite) is irreducible i.e., none of its symmetric permutations can be written as the direct sum of two matrices. This last property is relevant in the theory of matrices with non-negative entries.

A Hadamard products and indexes.

In this section we recall some definitions and results from [12] which are closely related with the problems of Section 4. The exposition is done with some detail for several reasons: a) Most results we state are explicitly used in the previous section. b) The formulation of these results given in [12] is quite technical and intricate, so we intend here to give a clarified version. c) Although some results in the appendix are not directly applied, they are included since they give effective criteria for computing the indexes and the vectors that realize them. This is relevant since we have identified these objects as the optimal weights and the minimal potential for fusion frames.

A.1 Basic definitions and properties

We begin with an extended version of Definition 5.1.4

Definition A.1.1. Let $G \in \mathcal{M}_m(\mathbb{C})^+$.

1. The *minimal*-Hadamard index of G is the number

$$I(G) = \max\{\lambda \ge 0 \,:\, G \circ B \ge \lambda B \quad \text{for every} \quad B \in \mathcal{M}_m(\mathbb{C})^+\}\,.$$

2. Given an u.i.n N in $\mathcal{M}_m(\mathbb{C})$, the N-Hadamard index of G is

$$\begin{aligned} I_N(G) &= \max\, \left\{\lambda \ge 0 \,:\, N(G \circ B) \ge \lambda N(B) \quad \text{for every} \quad B \in \mathcal{M}_m(\mathbb{C})^+\right\} \\ &= \min\, \left\{N(G \circ B) \,:\, B \in \mathcal{M}_m(\mathbb{C})^+ \text{ and } N(B) = 1\right\}\,. \end{aligned}$$

The index of G associated with the spectral norm $\|\cdot\| = \|\cdot\|_{sp}$ is denoted by $I_{sp}(G)$, and the one associated with the Frobenius norm $\|\cdot\|_2$ is denoted by $I_2(G)$.

Proposition A.1.2. Let $G \in \mathcal{M}_m(\mathbb{C})^+$, $\mathbb{1} = (1, 1, \ldots, 1) \in \mathbb{C}^m$ and $\mathbb{E} = \mathbb{1} \cdot \mathbb{1}^T$.

1. $I(G) \neq 0$ if and only if $\mathbb{1} \in R(G)$. If there exists $y \in \mathbb{C}^n$ such that $Gy = \mathbb{1}$, then
$$I(G)^{-1} = \sum_{i \in \mathbb{I}_m} y_i = \langle y, \mathbb{1} \rangle = \rho(G^\dagger \mathbb{E}) = \min \{ \langle Gz, z \rangle \ : \ \sum_{i \in \mathbb{I}_m} z_i = 1 \} . \qquad (30)$$
If $G > 0$, then also $I(G) = \Big(\sum_{i,j=1}^{n} (G^{-1})_{ij} \Big)^{-1} = \frac{\det G}{\det(G+\mathbb{E}) - \det G}$.

2. $I(G) \leq I_N(G)$ for every UIN N.

3. If $J \subseteq \mathbb{I}_m$, then $I(G_J) \geq I(G)$ and $I_N(G_J) \geq I_N(G)$.

4. If $D = \operatorname{diag}(d) \in \mathcal{M}_m(\mathbb{C})^+$ is diagonal, then $I_N(D) = N'(d^{-1})^{-1}$. In particular,
$$I(D) = I_{sp}(D) = \Big(\sum_{i \in \mathbb{I}_m} d_i^{-1} \Big)^{-1} \quad \text{and} \quad I_2(D) = \Big(\sum_{i \in \mathbb{I}_m} d_i^{-2} \Big)^{-\frac{1}{2}} .$$

5. Both indexes I_2 e I_{sp} are attained by matrices $B \in \mathcal{M}_m(\mathbb{C})^+$ of rank one. This means that
$$I_2(G) = \min_{\|x\|=1} \|G \circ xx^*\|_2 \quad \text{and} \quad I_{sp}(G) = \min_{\|y\|=1} \|G \circ yy^*\| .$$
Moreover, the minima are also attained at vectors $x \geqslant 0$ (or $y \geqslant 0$).

6. Let $B = G \circ \overline{G} \in \mathcal{M}_m(\mathbb{R})^+$. Then $I_2(G) = I_{sp}(B)^{\frac{1}{2}} = I_{sp}(G \circ \overline{G})^{\frac{1}{2}}$.

7. Moreover, if $0 \underset{\text{ij}}{\leqslant} B \in \mathcal{M}_m(\mathbb{R})^+$ and $A \in \mathcal{M}_n(\mathbb{R})_{sa}$ is given by $A_{ij} = B_{ij}^{1/2}$ for $1 \leq i, j \leq m$ then, even if $A \notin \mathcal{M}_m(\mathbb{R})^+$, the index $I_2(A)$ of Definition 5.1.4 still satisfies
$$I_2(A) = \min_{\|x\|=1} \|A \circ x\,x^*\|_2 = \min_{\|x\|=1} \|B \circ x\,x^*\|^{\frac{1}{2}} . \qquad (31)$$

8. It holds that $I_{sp}(G) = \inf \{ \, I_{sp}(D) : G \leq D$ and D is diagonal $\}$. Therefore
$$I_2(G) = \inf \{ \, \big(\sum_1^n d_{ii}^{-2} \big)^{-\frac{1}{2}} : \ 0 < D \ \text{ is diagonal and } \ G \circ \overline{G} \leq D^2 \, \} . \qquad \square$$

Proposition A.1.3. Let $G \in \mathcal{M}_m(\mathbb{R})^+$ such that $0 \underset{\text{ij}}{\leqslant} G$. Then $I_{sp}(G) = I(G) \neq 0 \iff$ there exists $u \geqslant 0$ such that $Gu = \mathbb{1}$. $\square$

Proposition A.1.4. Let $G \in \mathcal{M}_m(\mathbb{C})^+$. Denote by $P = G \circ \overline{G}$. If $x \in \mathbb{R}_*^m$, then
$$\|G \circ xx^*\|_2^2 = \sum_{i,j} |G_{ij}|^2 |x_i|^2 |x_j|^2 = \langle P(x \circ x), x \circ x \rangle = \langle (P \circ xx^*)x, x \rangle \leq \|P \circ xx^*\| .$$
Take $x \geqslant 0$ such that $\|x\| = 1$ and $\|G \circ xx^*\|_2 = I_2(G)$. Then
$$(G \circ \overline{G} \circ xx^*)x = I(P_J)x \ , \quad \text{where} \quad J = \{ i \in \mathbb{I}_m \ : \ x_i \neq 0 \} .$$
In this case, they hold that

1. The vector $u = I(P_J)^{-1}(x_J \circ x_J) \in \mathbb{C}^J$ has strict positive entries and $P_J u = \mathbb{1}_J$.

2. $I_{sp}(P) = I_{sp}(P_J) = I(P_J)$.

3. $I_{sp}(P) = \|P \circ xx^*\|$. $\square$

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