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Helly EPT graphs on bounded degree trees: Characterization and recognition

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ABSTRACT

The edge-intersection graph of a family of paths on a host tree is called an *EPT* graph. When the tree has maximum degree h, we say that the graph is $[h, 2, 2]$. If, in addition, the family of paths satisfies the Helly property, then the graph is Helly $[h, 2, 2]$. In this paper, we present a family of *EPT* graphs called gates which are forbidden induced subgraphs for $[h, 2, 2]$ graphs. Using these we characterize by forbidden induced subgraphs the Helly $[h, 2, 2]$ graphs. As a byproduct we prove that in getting a Helly *EPT*-representation, it is not necessary to increase the maximum degree of the host tree. In addition, we give an efficient algorithm to recognize Helly $[h, 2, 2]$ graphs based on their decomposition by maximal clique separators.

1. Introduction and class definitions

A graph is called **EPT** if it is the edge-intersection graph of a family of paths in a tree. The class of *EPT* graphs was first investigated by Golumbic and Jamison. The recognition and coloring problems restricted to *EPT* graphs are NP-complete, whereas the maximum clique and maximum stable set problems are polynomially solvable [4,5].

In the last decades many papers were devoted to the study of *EPT* graphs and their generalizations; for an overview refer to [3,8,11]. In [9], Jamison and Mulder introduced the notation $[\mathbf{h}, \mathbf{s}, \mathbf{t}]$ to denote the class of graphs G for which there exist a representation on a host tree T with maximum degree h and a family of subtrees $(T_v)_{v \in V(G)}$ of T, all of maximum degree at most s, such that $uv \in E(G)$ if and only if the subtrees T_u and T_v have at least t vertices in common. Therefore, $[\mathbf{h}, \mathbf{2}, \mathbf{2}]$ is the class of *EPT* graphs that admit a representation on a host tree with maximum degree h. Clearly, $[2, 2, 2]$ is the class of interval graphs. It is known that $[3, 2, 2]$ is precisely the class of chordal *EPT* graphs [9], while $[4, 2, 2]$ is the class of weakly chordal *EPT* graphs [7]. A complete hierarchy of related graph classes emerging by imposing different restrictions on the tree representation is published in [6].

For a fixed $h > 4$, the time complexity of deciding if an *EPT* graph belongs to the class $[h, 2, 2]$ is open; it is known to be polynomial time solvable for $h \in \{2, 3, 4\}$. In [6] and [7], Golumbic et al. wonder if, for $h > 4$, the only obstructions for an *EPT* graph to belong to $[h, 2, 2]$ are the chordless cycles of size greater than h. In [1], we give a negative answer to this question and present a family of forbidden induced subgraphs called prisms. In this paper, we generalize the class of prisms and present a wider family of *EPT* graphs called k-gates which are forbidden induced subgraphs for the classes $[h, 2, 2]$ when $h < k$.

Formally, an **EPT-representation** of a graph G is a pair $\langle \mathcal{P}, T \rangle$ where $\mathcal{P}$ is a family $(P_v)_{v \in V(G)}$ of subpaths of the **host tree** T satisfying that two vertices u and w of G are adjacent if and only if $E(P_u) \cap E(P_w) \neq \emptyset$. When the maximum degree of the

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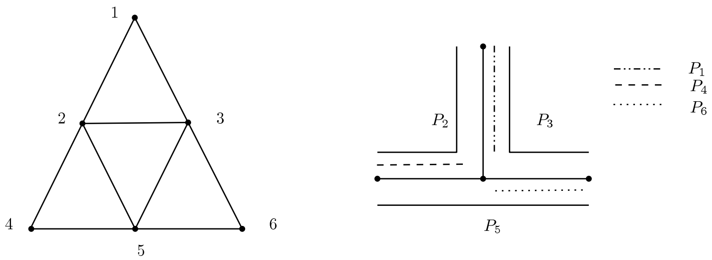

Fig. 1. An *EPT*-representation of the sun S_3. In this representation, the central triangle $\{2, 3, 5\}$ is a claw-clique; the other three triangles are edge-cliques.

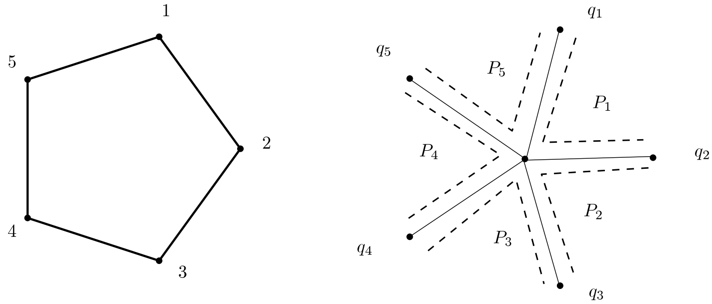

Fig. 2. The cycle C_5 and its *EPT*-representation: a pie of size 5.

host tree T is h, the *EPT*-representation of G is called an **(h, 2, 2)-representation** of G. Hence, [**h, 2, 2**] is the class of graphs that admit an $(h, 2, 2)$-representation.

A set family $(S_i)_{i\in I}$ satisfies the **Helly property** if every pairwise intersecting subfamily $(S_i)_{i\in I'}$ with $\emptyset \neq I' \subseteq I$ has non-empty total intersection, i.e. $\bigcap_{i\in I'} S_i \neq \emptyset$. A **Helly EPT-representation** is an *EPT* representation $\langle \mathcal{P}, T\rangle$ such that the family $(E(P))_{P\in\mathcal{P}}$ satisfies the Helly property. For instance, the *EPT*-representation depicted in Fig. 1 is not a Helly *EPT*-representation.

A graph G is **Helly EPT** if G admits a Helly *EPT*-representation. In [10], Monma and Wei characterize Helly *EPT* graphs via decomposing the graph by maximal clique separators and prove that the class can be recognized efficiently. Finding a characterization by forbidden induced subgraphs of *EPT* and of Helly *EPT* graphs is long standing open problem.

We say that G is **Helly** [**h**, **2**, **2**] if G admits a **Helly** (**h**, **2**, **2**)**-representation**, that is, a Helly *EPT*-representation on a host tree with maximum degree h. We characterize by forbidden induced subgraphs the Helly *EPT* graphs that belong to Helly $[h, 2, 2]$, for $h > 2$. As a byproduct, we prove that in getting a Helly EPT representation from a given EPT representation, it is not necessary to increase the maximum degree of the host tree. In addition, we characterize Helly $[h, 2, 2]$ graphs by their atoms in the decomposition by maximal clique separators. We give an efficient algorithm to recognize Helly $[h, 2, 2]$ graphs.

The paper is organized as follows: in Section 2, we provide basic definitions and known results. In Section 3, we introduce the graphs named k-gates and focus on their main properties; we show that k-gates are Helly *EPT* but do not admit an *EPT*-representation on a host tree with maximum degree less than k. In Section 4, we show that a Helly *EPT* graph G belongs to the class Helly $[h, 2, 2]$ if and only if G does not have a k-gate as induced subgraph for any $k > h$. Finally, in Section 5, we use the Monma and Wei decomposition by maximal clique separator to obtain an efficient algorithm for the recognition of Helly $[h, 2, 2]$ graphs for $h > 2$. Section 6 contains the conclusions.

2. Preliminaries and general results for *EPT* graphs

In this paper all graphs are finite and simple. A **clique** of a graph G is a subset of vertices pairwise adjacent. A **maximal clique** is a clique contained in no other clique of the graph.

The complete bipartite graph $K_{1,n}$ is also called **star of size n**. The edges of a star are called **spokes**. The star $K_{1,3}$ is named the **claw graph**.

For an integer $k > 2$, a **pie of size k** in an *EPT*-representation $\langle \mathcal{P}, T\rangle$ is a star subgraph of T with center q and neighbors $q_1, \ldots, q_k$, and a subfamily of paths $P_1, \ldots, P_k$ such that $\{q_i, q, q_{i+1}\} \subseteq V(P_i)$ for $i \in [1, k-1]$ (the natural interval $\{1, 2, \ldots, k-1\}$), and $\{q_k, q, q_1\} \subseteq V(P_k)$. In such a case, we also say that the paths $P_1, \ldots, P_k$ of $\mathcal{P}$ form a pie; and in particular when $k = 3$, we say that the three paths form a claw. An example is offered in Fig. 2.

Golumbic and Jamison introduced the notion of a pie in order to describe *EPT*-representations of chordless cycles.

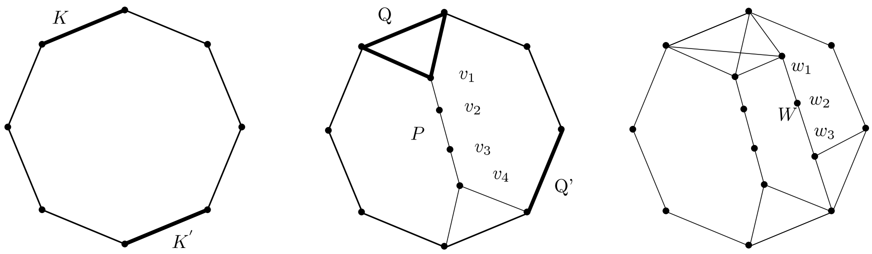

Fig. 3. Some examples of gates. From left to right, the second gate is obtained from the first using the bold maximal cliques K and K' and the path $P : (v_1, v_2, v_3, v_4)$. The third gate is obtained from the second using the bold maximal cliques Q and Q' and the path $W : (w_1, w_2, w_3)$.

Theorem 1 ([5]). *Let $\langle \mathcal{P}, T\rangle$ be an EPT-representation of a graph G. If G contains a chordless cycle C_k with $k > 3$, then $\langle \mathcal{P}, T\rangle$ contains a pie of size k whose paths are in one-to-one correspondence with the vertices of C_k.*

We say that a path P of $\mathcal{P}$ contains an edge e of T if $e \in E(P)$. Let $\mathcal{P} = (P_v)_{v\in V(G)}$. For every claw Y in T, $\mathbf{K_Y}$ denotes the clique $\{v \in V(G) : P_v$ contains two spokes of $Y\}$ of G. For every edge e of T, we let $\mathbf{K_e}$ be the clique $\{v \in V(G) : e \in E(P_v)\}$ of G. A maximal clique K of G is an **edge-clique** if there exists $e \in E(T)$ such that $K = K_e$. Inspired by the following result, a maximal clique that is not an edge-clique is called a **claw-clique**. See an example in Fig. 1.

Theorem 2 ([4]). *Let $\langle \mathcal{P}, T\rangle$ be an EPT-representation of G. If K is a maximal clique of G then either there is an edge $e \in E(T)$ such that $K = K_e$ or there is a claw Y in T such that $K = K_Y$.*

Notice that the condition of being an edge-clique or a claw-clique depends on the given representation.

Clearly, there is a claw-clique if and only if three paths form a claw. Thus, in a Helly *EPT*-representation every maximal clique is an edge-clique.

3. Gates and multipies

A clear corollary of Theorem 1 is that if $k > 3$ then the chordless cycle C_k is an obstruction for every class $[h, 2, 2]$ with $h \in [2, k-1]$. In [6] and [7] (see Section Open questions in both papers), Golumbic et al. wonder if besides cycles there are other *EPT* forbidden induced subgraphs for the class $[h, 2, 2]$ when $h > 4$. In [1], answering negatively the previous question, we describe for every $h > 2$ an *EPT* graph F_h without chordless cycles of size greater than h, which does not admit an *EPT*-representation on a host tree with maximum degree h. The graphs introduced in the following definition generalize the graphs F_h. In Section 4, for every $h > 2$, we obtain a total characterization of Helly $[h, 2, 2]$ graphs using them.

Definition 3. A **gate** is defined recursively as follows.

- The chordless cycles C_n for $n > 3$ are gates.
- If G is a gate, K and K' are disjoint maximal cliques of G, and $P : (v_1, \ldots, v_l)$ with $l > 1$ is a chordless path disjoint from G, then the graph obtained from the union of G and P by adding all the edges between v_1 and the vertices of K and all the edges between v_l and the vertices of K' is a gate.

If the number of maximal cliques of a gate G is k then we say that G is a **k-gate**.

In Fig. 3 we offer some examples of gates. Notice that the chordless cycle C_4 is the only gate on 4 vertices. Moreover, C_4 is the gate with minimum number of vertices and also the one with minimum number of maximal cliques. Therefore, whenever we refer to a k-gate, we assume $k > 3$.

Lemma 4. *If G is a k-gate then $G \in$ Helly $[k, 2, 2]$. Furthermore, G admits a Helly $(k, 2, 2)$-representation on a host tree that is a star.*

Proof. First notice that, by Theorem 1 and the fact that $k > 3$, the statement holds when G is a cycle.

We will proceed by induction on k. If $k = 4$ then $G = C_4$ and the proof follows. If $k > 4$ and G is not a cycle, then G is obtained from an m-gate H using disjoint maximal cliques K and K' of H and a chordless path $P : (v_1, v_2, \ldots, v_l)$ with $l > 1$ disjoint from H (see Fig. 4). Notice that $m = k - (l - 1) < k$. By the inductive hypothesis, there exists a Helly $(m, 2, 2)$-representation $\langle \mathcal{P}, T\rangle$ of H, where T is a star with m spokes. By Theorem 2 and the fact that the representation is Helly, there exist two spokes of T, say e and e', such that $K = K_e$ and $K' = K_{e'}$. Denote by T' the star that is obtained by adding $l - 1$ spokes $e_1, \ldots, e_{l-1}$ to T. Let P_{v_1} be the subpath of T' defined by the edges e and e_1. For $i \in [2, l-1]$ let P_{v_i} be the subpath of T' defined by the edges e_{i-1} and e_i; and let P_{v_l} the one defined by the edges e_{l-1} and e'. It is simple to see that $\langle \mathcal{P}', T'\rangle$ is a $(k, 2, 2)$-representation of G, where $\mathcal{P}'$ is the family $\mathcal{P}$ plus the paths P_{v_i} for $i \in [1, l]$. To finish the proof observe that $\langle \mathcal{P}', T'\rangle$ is Helly because $\langle \mathcal{P}, T\rangle$ is Helly and $l > 1$. □

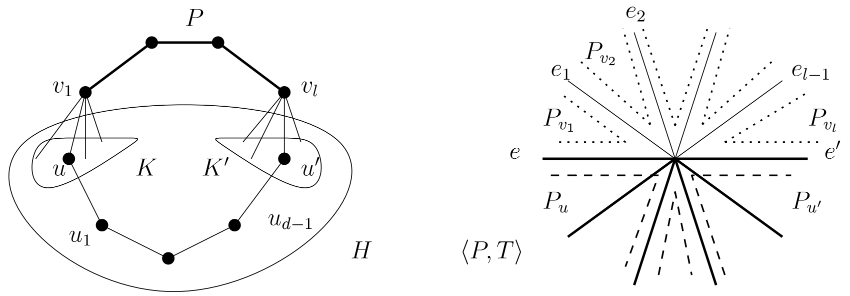

Fig. 4. On the left the k-gate G obtained from the $(k-(l-1))$-gate H using the maximal cliques K and K' and the disjoint bold path $P : (v_1, \dots, v_l)$. On the right, in bold the star T of the $\langle P, T\rangle$ representation of H, which together with the spokes $e_1, \dots, e_{l-1}$ forms the star T' for the representation of G. The paths P_{v_i} representing the vertices of P are dotted, while the paths corresponding to the vertices of the path $(u, u_1, \dots, u')$ are dashed.

Lemma 5. *If G is a gate and $v \in V(G)$, then v belongs to exactly two maximal cliques of G. Furthermore, if C_1 and C_2 are those maximal cliques then $C_1 \cap C_2 = \{v\}$.*

Proof. Clearly the statement holds for chordless cycles. We will proceed by induction on the number of vertices of the gate G. If $G = C_4$, the proof is trivial. If $|V(G)| > 4$ and G is not a cycle, then G is a gate obtained from another gate H, using disjoint maximal cliques K and K' of H and a chordless path $P : (v_1, \dots, v_l)$ with $l > 1$ disjoint from H. By the inductive hypothesis, the statement holds for any vertex of H.

Observe that the maximal cliques of G are:

the maximal cliques of H other than K and K';
the maximal cliques of P, i.e. $\{v_i, v_{i+1}\}$ for $i \in [1, l-1]$;
$K \cup \{v_1\}$; and
$K' \cup \{v_l\}$.

If v is the vertex v_1 (or the vertex v_l) then the proof follows with $C_1 = K \cup \{v_1\}$ and $C_2 = \{v_1, v_2\}$ ($C_1 = K' \cup \{v_l\}$ and $C_2 = \{v_{l-1}, v_l\}$, resp.). If v is any other vertex v_i of the path P, then the proof follows with $C_1 = \{v_{i-1}, v_i\}$ and $C_2 = \{v_i, v_{i+1}\}$. Otherwise v is a vertex of H. Hence, by the inductive hypothesis v is in exactly two maximal cliques of H, even more the intersection of those two maximal cliques contains only the vertex v; thus the proof follows easily from the previous description of the maximal cliques of G. □

Lemma 6. *Let v be a vertex of a gate G, C_1 and C_2 maximal cliques of G such that $C_1 \cap C_2 = \{v\}$, and $W : (w_1, \dots, w_t)$ a chordless path disjoint from G with $t > 1$. Then, the graph G' obtained from the union of $G - v$ and W by adding all edges between w_1 and the vertices of $C_1 - \{v\}$ and all edges between w_t and the vertices of $C_2 - \{v\}$ is a gate.*

Proof. Clearly the statement holds for chordless cycles. We will proceed by induction on the number of vertices of G. If $G = C_4$, the proof is trivial. If $|V(G)| > 4$ and G is not a cycle, then G is a gate obtained from another gate H, using disjoint maximal cliques K and K' of H and a chordless path $P : (v_1, \dots, v_l)$ with $l > 1$ disjoint from H. By the inductive hypothesis, the statement holds for H.

Observe that if v is one of the vertices of P then G' is isomorphic to the gate obtained from H using the same cliques K and K' and (instead of the path P) a path with $l + t$ vertices.

If v is a vertex of K (see Fig. 5), we can assume that $C_1 = K \cup \{v_1\}$ and C_2 is a maximal clique of G different from $K' \cup \{v_l\}$, which means that in H the vertex v is the intersection between the maximal cliques K and C_2. Thus, by the inductive hypothesis, the graph H' obtained from the union of $H - v$ and W plus all edges between w_1 and the vertices of $K - \{v\} = C_1 - \{v, v_1\}$ and all edges between w_t and the vertices of $C_2 - \{v\}$ is a gate. Since the path P is disjoint from H', and $(C_1 - \{v_1, v\}) \cup \{w_1\}$ and K' are disjoint cliques of H', thus, by the recursive definition of a gate, the union of H' and P plus all edges between v_1 and the vertices of $(C_1 - \{v_1, v\}) \cup \{w_1\}$, and all edges between v_l and the vertices of K' is a gate. The proof follows from the fact that this is the same graph G' depicted in the statement of the theorem.

If v is a vertex of K' or if $v \in V(H) - (K \cup K')$ the proof is analogous. □

Golumbic and Jamison proved that (see Theorem 1) chordless cycles of size at least four admit a *unique EPT*-representation called a pie. In what follows, generalizing that result, we introduce the definition of multipie and prove that also gates admit a *unique EPT*-representation.

Definition 7. A **multipie of size k** in an *EPT*-representation $\langle \mathcal{P}, T\rangle$ is a star subgraph of T with central vertex q and neighbors $q_1, \dots, q_k$ and a subfamily $\mathcal{P}'$ of $\mathcal{P}$ such that:

(1) if $P \in \mathcal{P}'$ then $|V(P) \cap \{q_1, q_2, \dots, q_k\}| = 2$ (every path contains exactly two spokes of the star);
(2) if $i \neq j$ then $|\{P \in \mathcal{P}' : \{q_i, q_j\} \subseteq V(P)\}| < 2$ (no two paths contain the same two spokes);

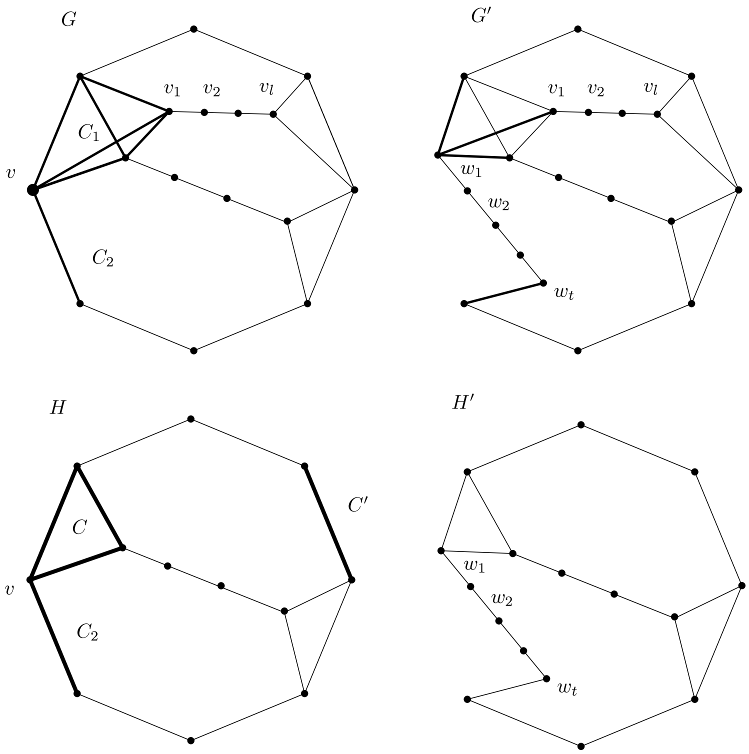

Fig. 5. An example following the proof of Lemma 6.

(3) if $i \in [1, k]$ then $|\{P \in \mathcal{P}' : \{q, q_i\} \subseteq V(P)\}| > 1$ (every spoke of the star is contained by at least two paths);
(4) no three paths of $\mathcal{P}'$ form a claw.

Observe that every pie is a multipie. The following theorem generalizes Theorem 1.

Theorem 8. *Let $\langle \mathcal{P}, T\rangle$ be an EPT-representation of G. If G contains as induced subgraph a k-gate then $\langle \mathcal{P}, T\rangle$ contains a multipie of size k whose paths are in one-to-one correspondence with the vertices of the gate.*

Proof. Let $\langle \mathcal{P}, T\rangle$ be an *EPT*-representation of G with $\mathcal{P} = (P_v)_{v\in V(G)}$. We can assume, without loss of generality, that G is a k-gate.

Notice that if G is any chordless cycle then, by Theorem 1 and the fact that every pie is a multipie, the proof follows.

We will proceed by induction on k. If $k = 4$ then $G = C_4$ and the proof follows. If $k > 4$ and G is not a cycle, then G is obtained from an m-gate H using disjoint maximal cliques K and K' of H and a path $P : (v_1, v_2, \ldots, v_l)$ with $l > 1$ disjoint from H. Notice that $m = k - (l - 1) < k$ and that H is an induced subgraph of G; thus, by the inductive hypothesis, $\langle \mathcal{P}, T\rangle$ contains a multipie of size m formed by a star subgraph S of T and the path subfamily $\mathcal{P}' = (P_v)_{v\in V(H)}$.

Let S be the star with center q and leaves $q_1, \ldots, q_m$. By condition (4) in Definition 7, no three paths of $\mathcal{P}'$ form a claw, then there exists a spoke of S, say $e_1 = qq_1$, such that $K \subseteq K_{e_1}$; and there exists another spoke, without loss of generality say $e_m = qq_m$, such that $K' \subseteq K_{e_m}$. Furthermore, by condition (2), e_1 and e_m are the only spokes of S satisfying the described property.

The distance between two vertices is the length (number of edges) of a shortest path between the two vertices. Let d be the minimum distance in H between a vertex of K and a vertex of K'. Clearly, $d > 0$. Choose vertices $u \in K$ and $u' \in K'$ such that the distance between them in H is d. Let $(u, u_1, u_2, \ldots, u_{d-1}, u')$ be a shortest path in H between u and u'. Notice that $u, u_1, u_2, \ldots, u_{d-1}, u', v_l, v_{l-1}, \ldots, v_2, v_1$ induce a cycle in G of size $d + l + 1 > 3$. By Theorem 1, $\langle \mathcal{P}, T\rangle$ contains a pie corresponding to this cycle. Let S' be the star subgraph of T used by this pie. Notice that the center of S' must be the same vertex q of T. Furthermore, since the vertex v_1 of P is adjacent to all vertices in K, the vertex v_l is adjacent to all vertices in K', and there are no other adjacencies between vertices of P and H, we have that S' has $l - 1$ spokes, say $e'_1, \ldots, e'_{l-1}$, that are not spokes of S. The remaining $(d + l + 1) - (l - 1) = d + 2$ spokes of S' are also spokes of S. Therefore the union of S and S' is a star subgraph of T with center q and $m + l - 1 = k$ spokes. In addition, we can assume that $E(P_{v_1}) = \{e_1, e'_1\}$, $E(P_{v_i}) = \{e'_{i-1}, e'_i\}$ for $i \in [2, l - 1]$ and $E(P_{v_l}) = \{e'_{l-1}, e_m\}$. Since the members of $\mathcal{P}$ are exactly the members of $\mathcal{P}'$ and the paths P_{v_i} for $i \in [1, l]$, it is easy to check (following Definition 7) that $\mathcal{P}$ forms a multipie on the star $S \cup S'$, and the proof is complete. □

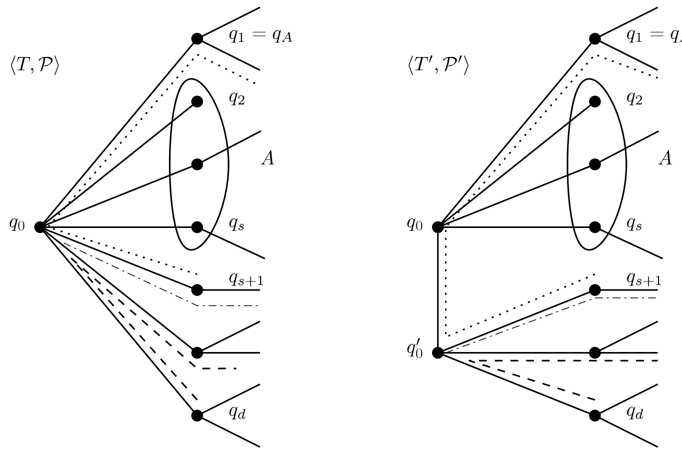

Fig. 6. An example following the proof of Lemma 9.

4. Forbidden induced subgraphs for Helly EPT graphs on bounded degree trees

The goal of this section is Theorem 10 which proves that gates are the only subgraphs that force the use of a host tree with large enough degree in any Helly *EPT*-representation of a graph. Next lemma is a crucial tool in the proof.

Lemma 9. *Let $\langle \mathcal{P}, T\rangle$ be an EPT-representation of G and let q_0 be a vertex of T with degree $d > 3$. Assume there exists $A \subseteq N_T(q_0)$ with $|A| \in [1, d-3]$ and a vertex $q_A \in N_T(q_0) \setminus A$ such that if a path $P \in \mathcal{P}$ contains an edge q_0q with $q \in A$ and an edge q_0q' with $q' \notin A$ then $q' = q_A$. Then there exists an EPT-representation $\langle \mathcal{P}', T'\rangle$ of G satisfying the following three statements:*

(1) *the maximum degree of T' is less than or equal to the maximum degree of T;*
(2) *the number of vertices with degree d of T' is one less than the number of vertices with degree d of T;*
(3) *$\langle \mathcal{P}', T'\rangle$ is Helly if $\langle \mathcal{P}, T\rangle$ is Helly.*

Proof. Let T' be the tree obtained from T by adding a new vertex q_0' adjacent to q_0, and replacing, for every $q \in N_T(q_0) \setminus (A \cup \{q_A\})$, the edge q_0q by the edge $q_0'q$. See Fig. 6.

Notice that $d_{T'}(q_0') = 1 + (d - (|A| + 1)) = d - |A| < d$; $d_{T'}(q_0) = 1 + (|A| + 1) < 2 + d - 2 = d$ and $d_{T'}(q) = d_T(q)$ for any other vertex q.

The path family $\mathcal{P}'$ is obtained by replacing in $\mathcal{P}$:

– each path P containing both an edge q_0q with $q \in N_T(q_0) \setminus (A \cup \{q_A\})$ and the edge q_0q_A (recall that no path of $\mathcal{P}$ contains an edge q_0q with $q \in N_T(q_0) \setminus (A \cup \{q_A\})$ together with an edge q_0q' with $q' \in A$) by the path P' obtained from P by removing the edge q_0q and adding the edges $q_0'q$ and $q_0'q_0$ (see the dotted paths in Fig. 6);

– each path P containing two different edges q_0q with $q \in N_T(q_0) \setminus (A \cup \{q_A\})$ by the path P' obtained from P by removing the edge q_0q and adding the edge $q_0'q$ (see the dashed paths in Fig. 6);

– each path P containing exactly one edge q_0q with $q \in N_T(q_0) \setminus (A \cup \{q_A\})$ and no other edge incident in q_0 by the path P' obtained from P by removing the edge q_0q and adding the edge $q_0'q$ (see the dash dotted paths in Fig. 6).

Finally, in order to prove (3), observe that if $\langle \mathcal{P}, T\rangle$ is Helly but $\langle \mathcal{P}', T'\rangle$ is not, then there must exist in $\langle \mathcal{P}', T'\rangle$ a claw-clique on a claw using the edge q_0q_0', which is not possible since any path of $\mathcal{P}'$ containing the edge q_0q_0' also contains the edge q_0q_A. □

Theorem 10. *Let G be a Helly EPT graph and $h > 2$. Then, $G \notin$ Helly $[h, 2, 2]$ if and only if there exists $k > h$ such that G has a k-gate as induced subgraph.*

Proof. We will prove the direct implication, the converse follows from Theorem 8 and the fact that Helly $[h, 2, 2] \subseteq [h, 2, 2]$.

Assume that G is a Helly *EPT* graph which does not admit a Helly $(h, 2, 2)$-representation. Let d be the smallest positive integer such that $G \in$ Helly $[d, 2, 2]$. Clearly, $d > h$. Let $\langle \mathcal{P}, T\rangle$ be a Helly $(d, 2, 2)$-representation of G minimizing the number of vertices with degree d of the host tree T.

Claim 11. *We can assume that if $q \in V(T)$ is the end vertex of a path $P \in \mathcal{P}$ then $d_T(q) < 3$.*

Proof. If it is not the case then we can obtain the desired representation by subdividing every edge of T (and consequently every edge of every path of $\mathcal{P}$) in three parts and then shortening every path of $\mathcal{P}$ by removing its two end vertices. Notice that the maximum degree of the host tree does not change. □

Let $q_0 \in V(T)$ be a vertex with degree d and let $q_1, \ldots, q_d$ be its neighbors. Denote by H the subgraph of G induced by the vertices v such that $q_0 \in V(P_v)$.

Claim 12. *The subgraph H contains a chordless cycle of length at least* 4.

Proof. Let $P : (v_1, \ldots, v_l)$ be the longest induced path in H and assume, without loss of generality, that $\{q_i, q_0, q_{i+1}\} \subseteq V(P_{v_i})$, for all $i \in [1, l]$. Observe that $l \in [2, d-1]$.

If every path of $\mathcal{P}$ containing q_0q_{l+1} also contains q_0q_l then, by Lemma 9 with $A = \{q_1, q_2, \ldots, q_d\} \setminus \{q_l, q_{l+1}\}$ and $q_A = q_l$, there is a Helly $(d, 2, 2)$-representation of G on a host tree with less vertices of degree d, contrary to our assumption. Therefore, there exist $j \in [1, d] \setminus \{l, l+1\}$ and a vertex x of H such that $\{q_{l+1}, q_0, q_j\} \subseteq V(P_x)$. Clearly, $x \notin V(P)$.

If $j > l+1$, then $V(P) \cup \{x\}$ induces in H a path longer than P, which contradicts the choice of P. If $j = l-1$, then $P_x, P_{v_{l-1}}$ and P_{v_l} violate the Helly property, which contradicts the fact that $\langle \mathcal{P}, T\rangle$ is a Helly *EPT*-representation of G. Thus $l > 2$ and $j < l-1$ which implies that the vertices $\{v_j, \ldots, v_{l-1}, v_l, x\}$, induce in H a chordless cycle, as we wanted to prove. □

It follows from the previous Claim 12 and Definition 3 that H has a gate as induced subgraph. Let R be a gate induced in H of maximum size, say k. By Theorem 8, there is a multipie of size k on the star with center q_0 and, without loss of generality, spokes $q_0q_1, \ldots, q_0q_k$. We will prove that $k = d$.

Suppose, in order to derive a contradiction, that $k < d$.

Since G is connected there must exist a vertex y such that the path P_y contains one of the edges $q_0q_1, \ldots, q_0q_k$ and one of the edge $q_0q_{k+1}, \ldots, q_0q_d$. Without loss of generality, we assume that $\{q_k, q_0, q_{k+1}\} \subseteq V(P_y)$.

If all paths containing the edge q_0q_{k+1} also contain the edge q_0q_k, then, by Lemma 9 with $A = \{q_i, i \in [1, d] \setminus \{k, k+1\}\}$ and $q_A = q_k$, we can obtain a new Helly *EPT*-representation of G on a host tree with fewer vertices of degree d, contrary to our assumption. Hence, there exist $j \neq k, k+1$ and a vertex z of H such that $\{q_j, q_0, q_{k+1}\} \subseteq V(P_z)$ (notice that y and z are adjacent and do not belong to the gate R). We claim that $j > k-1$. Indeed, assume, in order to derive a contradiction, that $j < k$. Let C_k and C_j be the maximal cliques of R corresponding to the edges q_0q_k and q_0q_j of T, respectively. Notice that C_k and C_j are disjoint, otherwise P_y, P_z and P_v violate the Helly property, where v is a vertex in the intersection. Thus, using maximal cliques C_k and C_j, and the path $P : (y, z)$ disjoint from R, we obtain a $(k+1)$-gate induced in H, which contradicts that k is the maximum size of a gate in H. Therefore, $j > k-1$, and, $j \neq k, k+1$, we can assume without loss of generality $j = k+2$.

Denote by V' the set of vertices $v \in V(H)$ such that P_v contains an edge q_0q_i for some $i \leq k$ and an edge $q_0q_{i'}$ for some $i' > k$. Notice that $y \in V'$ and $z \notin V'$. Let G_z be the connected component of $G - V'$ containing the vertex z. Because of the definition of V' and Claim 11, if $v \in V(G_z) \cap V(H)$ then P_v must contain a pair of edges q_0q_i and $q_0q_{i'}$ of T with $k+1 \leq i < i' \leq d$. Thus, without loss of generality, we can assume there exists $s \in [k+2, d]$ such that

$$V(G_z) \cap V(H) = \bigcup_{k+1 \leq i < i' \leq s} \{v \in V(G) : \{q_i, q_0, q_{i'}\} \subseteq V(P_v)\}, \tag{1}$$

and

$$\text{for every } i \in [k+1, s],\ q_0q_i \in E(P_v) \text{ for some } v \in V(G_z) \cap V(H). \tag{2}$$

In order to apply Lemma 9 one more time, we will prove the following claim.

Claim 13. *Let A be the set $\{q_i \in N_T(q_0)$ with $i \in [k+1, s]\}$. If a path $P \in \mathcal{P}$ contains an edge q_0q with $q \in A$ and an edge q_0q' with $q' \notin A$ then $q' = q_k$.*

Proof. Let w be the vertex of H such that $P = P_w$ and let $q_r \in A$ such that $q = q_r$. By (1) and the fact that q_r is the only element of A in $V(P_w)$, we have that $w \in V'$; which implies $q' = q_t$ for some $t \in [1, k]$. Assume, in order to derive a contradiction, that $t < k$.

On the other hand, since $q_0q_r \in E(P_w)$, by (2), there exists $z' \in V(G_z) \cap V(H)$ adjacent to w. Let $Q : (z, z_1, \ldots, z')$ be a z, z'-path in G_z. Notice that $V(Q) \cap V(R) = \emptyset$, and none of the vertices of Q is adjacent to a vertex of R. Let Q' be a chordless subpath of $(y, z, z_1, \ldots, z', w)$ joining y and w.

Denote by K and K' to the maximal cliques of R corresponding to de edges q_0q_k and q_0q_t of T, respectively. Observe that y is adjacent to all the vertices of K and w is adjacent to all the vertices of K'; neither y nor w has other neighbors in R.

Therefore, either by Definition 3 if K and K' are disjoint, or by Lemma 6 if K and K' are non-disjoint, we have that a gate bigger than R can be obtained in H using the path Q'. It contradicts the fact that R is the biggest gate induced in H. Thus, it follows $t = k$ and the proof is complete. □

Then, in view of the previous Claim 13, by Lemma 6, we can obtain a Helly $(d, 2, 2)$-representation of G on a host tree with fewer vertices of degree d. This contradicts the fact that $\langle \mathcal{P}, T\rangle$ is a representation minimizing the number of vertices with degree d.

The contradiction comes from supposing $k < d$, thus $k = d$. Since k is the size of a gate induced in G and $k > h$, the proof is completed. □

By [**h**, **2**, **2**] ∩ **Helly EPT** we denote the class of graphs which admit both an $(h, 2, 2)$-representation and a Helly *EPT*-representation.

Corollary 14. $[h, 2, 2] \cap$ *Helly EPT* $=$ *Helly* $[h, 2, 2]$ *for any* $h > 2$.

Proof. Clearly, Helly $[h, 2, 2] \subseteq [h, 2, 2] \cap$ Helly *EPT*. Assume, in order to derive a contradiction, that $G \in [h, 2, 2] \cap$ Helly *EPT* and $G \notin$ Helly $[h, 2, 2]$. By Theorem 10, G contains a k-gate as induced subgraph for some $k > h$. Thus by Theorem 8, any *EPT*-representation of G contains a multipie of size k. This contradicts the fact that $G \in [h, 2, 2]$. □

5. Decomposition by clique separators and complexity

A maximal clique K of a connected graph G is a **separator** of G if $G - K$ is not connected. An **atom** is a connected graph with no separators. Given a separator K, the vertices of G can be uniquely partitioned into $K, V_1, \ldots, V_s$ with $s > 1$ so that each V_i is the vertex set of a connected component of $G - K$. Let G_i be the subgraph of G induced by $K \cup V_i$ for $i \in [1, s]$. In [10], by applying this process recursively on every connected graph G_i until they are all atoms (i.e. they have no separators), G is progressively decomposed to obtain a **(maximal) clique decomposition tree** with each leaf node being associated with an atom and each internal node being associated with a maximal clique separator of G. The atoms at the leaves of the clique decomposition tree are called the **atoms of** G; they are invariants of G in the sense that they are independent of the separators chosen to perform the successive steps of the decomposition process.

The algorithm of Tarjan in [12] (see also [2]) computes the clique decomposition tree of any graph G, the running time is $\mathcal{O}(n \cdot m)$ where $n = |V(G)|$ and $m = |E(G)|$.

The goal of [10] is to use this approach to characterize several graph classes of intersection graphs arising from families of paths in a tree. In particular, the characterization leads to an efficient algorithm to recognize Helly *EPT* graphs.

In this section, we focus on the time complexity of the following two problems, the first one is posed for any given fixed $h > 2$.

RECOGNIZING HELLY [h, 2, 2] GRAPHS
Input: A connected graph G.
Question: Does G belong to Helly $[h, 2, 2]$?

CHEAPEST REPRESENTATION
Input: A connected graph G.
Goal: Determine the minimum $h > 1$ such that $G \in$ Helly $[h, 2, 2]$.

The efficient solution of the latter given by Theorem 18 implies an efficient solution of the former.

Lemma 15. *If a gate H is an induced subgraph of a graph G then H is an induced subgraph of some atom of G.*

Proof. It is enough to prove that a gate has no maximal clique separators which follows trivially from the recursive definition of gates. □

Lemma 16. *If H is a Helly EPT atom with exactly $k > 3$ maximal cliques then H has a k-gate as induced subgraph.*

Proof. Since the size of a gate is the number of its maximal cliques, we have that H has no t-gates for any $t > k$. Assume in order to derive a contradiction that H has no k-gates. Thus, by Theorem 10, $H \in$ Helly $[h, 2, 2]$ for some $h < k$. Let $\langle \mathcal{P}, T \rangle$ be a Helly $(h, 2, 2)$-representation of H minimizing the number of edges of the host tree T. Since H is an atom with k maximal cliques then T is a star and $|E(T)| = k$. Hence the maximum degree of T equals $|E(T)|$, this is $h = k$, in contradiction with the fact that $h < k$. □

Theorem 17. *Let G be a Helly EPT graph and $h > 2$. Then, $G \in$ Helly $[h, 2, 2]$ if an only if every atom of G has at most h maximal cliques.*

Proof. If $G \in$ Helly $[h, 2, 2]$ then, by Theorem 10, G has no gates of size greater than h as induced subgraphs. Thus, by Lemma 16, G has no atoms with more than h maximal cliques.

Conversely, assume, in order to obtain a contradiction, that $G \notin$ Helly $[h, 2, 2]$. Thus, by Theorem 10, G has a k-gate H as induced subgraph, for some $k > h$. By Lemma 15, H is an induced subgraph of some atom of G. It implies that the atom has at least k maximal cliques, which contradicts the fact that every atom of G has at most h maximal cliques. □

Theorem 18. *The problem CHEAPEST REPRESENTATION is polynomial time solvable.*

Proof. Using the efficient algorithm described in [10], determine whether the given graph G belongs to Helly *EPT* or not. If it does not, answer that such an h does not exist. If it does, then determine the number of maximal cliques of each atom G_i of G and called it k_i. Notice that it can be done efficiently since the total number of maximal cliques of a Helly *EPT* graph G is at most $\lfloor \frac{3|V(G)|-4}{2} \rfloor$ [10]. Let k be the maximum of k_i over all atoms G_i.

If $k < 4$, then every atom is chordal which implies $G \in Chordal \cap EPT = [3, 2, 2]$ (see [10] and [4]). Now test whether G is an interval graph or not and answer $h = 2$ in an affirmative case and $h = 3$ otherwise.

If $k > 3$, then, by Theorem 17, $G \in$ Helly $[k, 2, 2]$ and $G \notin$ Helly $[k-1, 2, 2]$, thus answer $h = k$. □

6. Conclusions and future work

In this paper, we prove that gates are the only obstructions for a Helly *EPT* graph to admit a Helly *EPT*-representation on a host tree with given maximum degree. We characterize Helly $[h, 2, 2]$ graphs by their atoms in the decomposition by maximal clique separators. We give an efficient algorithm to recognize Helly $[h, 2, 2]$ graphs.

We conjecture that gates are also the only forbidden induced subgraphs for an *EPT* graph to belong to the class $[h, 2, 2]$.

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