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Derivation of a Darcy's Law for a Porous Medium Composed of Two Solid Phases Saturated by a Single-Phase Fluid: A Homogenization Approach

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Abstract The objective of this article is to derive a macroscopic Darcy's law for a fluid-saturated moving porous medium whose matrix is composed of two solid phases which are not in direct contact with each other (*weakly coupled solid phases*). An example of this composite medium is the case of a solid matrix, unfrozen water, and an ice matrix within the pore space. The macroscopic equations for this type of saturated porous material are obtained using two-space homogenization techniques from microscopic periodic structures. The pore size is assumed to be small compared to the macroscopic scale under consideration. At the microscopic scale the two *weakly coupled* solids are described by the linear elastic equations, and the fluid by the linearized Navier–Stokes equations with appropriate boundary conditions at the solid–fluid interfaces. The derived Darcy's law contains three permeability tensors whose properties are analyzed. Also, a formal relation with a previous macroscopic fluid flow equation obtained using a phenomenological approach is given. Moreover, a constructive proof of the existence of the three permeability tensors allows for their explicit computation employing finite elements or analogous numerical procedures.

Keywords Fluid-saturated composite porous solids · Homogenization · Darcy's law

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Nomenclature

script f	The fluid phase
script s_j	The solid phase j, $j = 1, 2$
Ω_j	The solid phase s_j for $j = s_1, s_2$ or the fluid phase for $j = f$ in Ω
Y_j	The solid phase s_j for $j = s_1, s_2$ or the fluid phase for $j = f$ in Y
Γ_{jf}	The interface in one period Y between the fluid phase and the solid phase s_j for $j = s_1, s_2$
ν_{jf}	The unit outward normal to the solid phase j at the interface between the fluid and the solid phase, $j = s_1, s_2$
$\boldsymbol{\nu}_{jk}$	The unit outer normal at the interface Γ_{jk} for $j, k = s_1, s_2, f, j \neq k$
$\mathbf{e}^k$	The standard basis in $\mathbb{R}^3$
$\chi_{\Gamma_{jf}}(x, y)$	The characteristic function of Γ_{jf}
x	The macroscopic spatial variable
y	The microscopic spatial variable; $y = \frac{x}{\epsilon}$
η, κ	The fluid viscocities
$\rho_f, \rho_{s_1}, \rho_{s_2}$	The fluid density and mass densities of solids s_1 and s_2
$\mathbf{e}$	The linear strain tensor
$\mathbf{a}_{s_1}, \mathbf{a}_{s_2}$	Fourth-order positive-definite elastic tensors associated with the solid phases
ω	The angular frequency
$\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_j(\omega)$	The time Fourier transform of the stress tensor in the phase j, j=s_1, s_2, f at the angular frequency ω
$\mathbf{u}_j = \mathbf{u}_j(\omega)$	The time Fourier transform of the displacement vector in the phase j, $j = s_1, s_2, f$ at the angular frequency ω
$p_f = p_f(\omega)$	The time Fourier transform of the pressure in the fluid at the angular frequency ω
$g_{j,k}$	The k-component of $i\omega\mathbf{u}_j^{(0)}$ on Γ_{jf}
ϵ	The ratio between the microscopic and macroscopic spatial variables
$\mathbf{v}_j$	$= i\omega\mathbf{u}_j \quad$ for $j = s_1, s_2, f$
$\boldsymbol{\gamma}_j^k(x, y)$	$= \chi_{\Gamma_{jf}}(x, y)\mathbf{e}^k, \quad j = s_1, s_2, \quad k = 1, 2, 3$
$(\cdot, \cdot)_S$	The complex $L^2(S)$ inner product of functions in S
$\langle \cdot, \cdot \rangle_\gamma$	The complex $L^2(\gamma)$ inner product of functions in γ
$L^2(S)$	The space of square-integrable functions in S
$[H^1(S)]^3$	The complex vector valued Sobolev space of square-integrable functions with its first-order derivatives are also square-integrable in S
$H^1(\mathrm{div}\, 0; Y_f)$	$= \left\{\boldsymbol{\varphi} \in \left[H^1(Y_f)\right]^3 : \nabla_y \cdot \boldsymbol{\varphi} = 0, \quad \boldsymbol{\varphi} \text{ is } Y\text{-periodic}\right\}$
$\mathcal{W}$	$= \left\{q \in H^1(Y_f) : \int_{Y_f} q\, \mathrm{d}y = 0, \quad q \text{ is } Y\text{-periodic}\right\}$
$\mathcal{W}_{Y_j}$	$= \left\{\boldsymbol{\varphi} \in [H^1(Y_j)]^3 \colon \boldsymbol{\varphi} \text{ is } Y\text{-periodic}, \int_{Y_j} \boldsymbol{\varphi}\, \mathrm{d}y = 0, \int_{Y_j} \nabla \times \boldsymbol{\varphi}\, \mathrm{d}y = 0\right\},$ $j = s_1, s_2.$
$\mathcal{V}_{Y_f}^2$	$= \left\{\boldsymbol{\varphi} \in \left[H^2(Y_f)\right]^3 : \nabla_y \cdot \boldsymbol{\varphi} = 0 \text{ in } Y_f,\ \boldsymbol{\varphi} = 0 \text{ on } \Gamma_{sf},\ \boldsymbol{\varphi} \text{ is } Y\text{-periodic}\right\}$
$\mathbf{V}^k$	$= (V_l^k)_{1 \le l \le 3} \in \mathcal{V}_{Y_f}^1$ the solution of Problem (2.21)

$\mathbf{v}_f^{(0),B}$, $p_f^{(1),B}$	The Y-periodic solutions of Problem (2.19)
$\mathbf{Z}^{j,k,(m)}$	The solution of Problem (3.6)
$\mathbf{Z}^{j,k}$	$= (Z_l^{j,k})_{1\le l\le 3}$ the solution of (2.27) with $\mathbf{g}$ replaced by $\boldsymbol{\gamma}_j^k$, $j = s_1, s_2$, $k = 1, 2, 3$
$\mathbf{K}^{j,(m)}$	$= \left(\mathbf{K}^{j,(m)}\right)_{kl} = Z_k^{j,l,(m)}$
$\ll \mathbf{K} \gg$	Macroscopic tensors averaged over the period Y; see (2.33)
$\ll \mathbf{K}^{s_j} \gg$	Macroscopic tensors averaged over the period Y for $j = 1, 2$; see (2.33)

1 Introduction

The study of fluid flow in porous saturated media is a subject of interest in many fields such as geophysics, rock physics, and materials science.

The fundamental concepts about the stress–strain relations and the dynamics of deformable porous single-phase solids fully saturated by a fluid were established in the works of Biot (1956a,b, 1962). This formulation assumes that the quantities measured at the macroscopic scale can be described using the concepts of continuum mechanics.

When the porous matrix is composed of two (or more) different solid phases, more complicated models are required. Based on Biot's theory, Leclaire et al. (1994) developed a phenomenological model to describe wave propagation in a porous solid matrix, where the pore space is filled with ice and water, assuming no contact between the solid and ice particles. This assumption is valid for example in finely dispersed frozen media, for which there exists a layer of unfrozen water around the solid particles isolating them from the ice, as explained in Leclaire et al. (1994).

This model, valid for uniform porosity, predicts the existence of three compressional and two shear waves; these additional (slow) waves were first observed in laboratory experiments by Leclaire et al. (1995). Later, Leclaire's model was generalized by Carcione and Tinivella (2000) to include the static and viscodynamic interaction between the solid and ice particles and grain cementation with decreasing temperature, used as a parameter to determine the bulk water content. Also, under the assumption that sand and clay are *non-welded* and form a continuous and inter-penetrating porous composite skeleton, Carcione et al. (2000) applied this theory to study the acoustic properties of shaley sandstones.

The Leclaire et al. (1994) and Carcione and Tinivella (2000) models were later generalized and analyzed to the case of variable porosity (Santos et al. 2004; Rubino et al. 2006a), and applied in Rubino et al. (2006b) for the analysis of wave propagation in gas hydrate-bearing sediments. Gas hydrates are crystalline molecular complexes composed of water and gas, mainly methane, which form under certain conditions of low temperature, high pressure, and gas concentration. They are ice-like structures within the pore space which are present in permafrost and seafloor along continental margins and are a potential energy resource.

Partially frozen porous media, gas hydrate-bearing sediments, and shaley sandstones are examples of porous materials where the two solid phases are *weakly coupled* or *non-welded*, i.e., both solids form a continuous and interacting composite structure, interchanging mechanical energy. Similar *weakly coupled* formulations have previously been proposed. For instance, McCoy (1991) has proposed a mixture theory appropriate for the combination of two *acoustic phases*.

As a consequence of the models in Leclaire et al. (1994) and Santos et al. (2004) a generalized Darcy's law for this type of materials is obtained. The macroscopic description of porous media can also be obtained by means of the *homogenization method*, which consists of passing from the microscopic description at the pore and grain scales to the macroscopic scale. Important contributions to the solution of this problem were given by Sanchez-Palencia (1980) and Bensoussan et al. (1978), who developed the so-called two-space homogenization technique. This method provides a systematic procedure for deriving macroscopic dynamical equations starting from the governing equations for the medium valid at the microscale. It was successfully applied by different authors to obtain theoretical justifications of Darcy's law and Biot's equations for single-phase porous media (Auriault et al. 1985; Burridge and Keller 1981; Levy 1979; Sanchez-Palencia 1987). The procedure was recently applied to derive the equations of motion for saturated composite porous media for the special case when only one of the solid phases is in contact with the fluid phase (Santos et al. 2005).

Following these ideas, the aim of this paper is to apply the homogenization procedure to obtain a description of the macroscopic fluid flow within a fluid-saturated porous medium where the porous matrix is composed of two *weakly coupled* solid phases as assumed in Leclaire et al. (1994).

The analysis is restricted to the range of small deformations and for Newtonian fluids under the assumption of spatial periodicity. As a result, a generalized Darcy's law for the composite material is obtained, in which the macroscopic fluid flow represents the contributions from the moving boundaries of the two solid phases as well as the gradient of the fluid pressure. The argument employs the concept of *very weak* solutions of the local Stokes problems to obtain explicit forms of the permeability tensors in terms of the non-homogeneous boundary data. The derived Darcy's law is formally in agreement with those derived in Leclaire et al. (1994) and Santos et al. (2004) using phenomenological arguments.

The organization of the paper is as follows. In Sect. 2 we state the local equations and apply the homogenization procedure to obtain our form of Darcy's law containing three permeability tensors whose properties are analyzed in Sect. 3. Also Sect. 3 contains a formal relation of our Darcy's law with previously derived forms using phenomenological arguments. In Sect. 4 we give a set of conclusions. Finally, in Appendix A we prove existence and uniqueness results for the local Stokes problems with non-homogeneous boundary data using the concept of *very weak* solutions.

2 The Homogenization Procedure and Darcy's Law

2.1 The Local Description and Formal Expansion

Let us consider a composite porous medium consisting of a porous solid matrix, an ice matrix, and unfrozen water, i.e., two solid phases and one single-phase fluid. It will be assumed that there is no contact between the solid matrix and the ice, or equivalently there exists a layer of unfrozen water around the solid particles isolating them from the ice. The solid matrix and the ice will be referred to by the subscripts or superscripts s_1 and s_2, while the fluid phase will be indicated by the subscript or superscript f. The porous medium will be considered to be periodic and composed of a large number of periods with l and L denoting the length of the period and the macroscopic length, respectively, so that $\epsilon = \frac{l}{L} \ll 1$. The microscopic and macroscopic behaviors will be described by the two dependent spatial variables $\mathbf{x}$ and $\mathbf{y} = \frac{\mathbf{x}}{\epsilon}$.

Let Ω denote a periodic porous medium consisting of the solid and the ice matrices, Ω_{s_1} and Ω_{s_2}, and the fluid phase Ω_f. Also let Y denote one period in Ω so that

$$Y = Y_{s_1} \cup Y_{s_2} \cup Y_f, \quad Y_j = Y \cap \Omega_j, \quad j = s_1, s_2, f.$$

Also, let

$$\Gamma_{jf} = \partial Y_j \cap \partial Y_f, \quad \Gamma_{je} = \partial Y_j \cap \partial Y, \quad j = s_1, s_2,$$
$$\Gamma_{sf} = \Gamma_{s_1 f} \cup \Gamma_{s_2 f}, \quad \Gamma_{fe} = \partial Y_f \cap \partial Y,$$

so that

$$\partial Y_f = \Gamma_{s_1 f} \cup \Gamma_{s_2 f} \cup \Gamma_{fe}, \quad \partial Y_{s_1} = \Gamma_{s_1 f} \cup \Gamma_{s_1 e}, \quad \partial Y_{s_2} = \Gamma_{s_2 f} \cup \Gamma_{s_2 e}.$$

Figure 1 displays a 2D realization of this type of periodic structure.

We assume that all phases are connected and that at the local level the two solid phases are linear elastic and the fluid is viscous Newtonian. We further assume that the transient Reynolds number is $O(1)$ at the local level so that the fluid viscosities η and κ are scaled by ϵ^2. Let $\mathbf{u}_j = \mathbf{u}_j(\omega)$ and $\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_j(\omega)$, $j = s_1, s_2, f$ denote the time Fourier transforms at the angular frequency ω of the displacement vectors and stress tensors of the three phases, respectively, let $p_f = p_f(\omega)$ be the fluid pressure and set $\mathbf{v}_j = i\omega\mathbf{u}_j$. The local variables are defined in their domain of definition and taken to be zero elsewhere. In what follows, to avoid cumbersome notation the explicit dependence on the frequency ω of the field variables will be omitted except when it is desired to emphasize this dependence. The local equations are given by

$$\text{solid 1:} \quad \nabla \cdot \boldsymbol{\sigma}_{s_1} = -\omega^2 \rho_{s_1} \mathbf{u}_{s_1}, \quad Y_{s_1}, \tag{2.1a}$$
$$\boldsymbol{\sigma}_{s_1} = \mathbf{a}_{s_1} : \mathbf{e}(\mathbf{u}_{s_1}), \quad Y_{s_1}, \tag{2.1b}$$
$$\text{solid 2:} \quad \nabla \cdot \boldsymbol{\sigma}_{s_2} = -\omega^2 \rho_{s_2} \mathbf{u}_{s_2}, \quad Y_{s_2}, \tag{2.1c}$$
$$\boldsymbol{\sigma}_{s_2} = \mathbf{a}_{s_2} : \mathbf{e}(\mathbf{u}_{s_2}), \quad Y_{s_2}, \tag{2.1d}$$
$$\text{fluid:} \quad \nabla \cdot \boldsymbol{\sigma}_f = i\omega \rho_f \mathbf{v}_f, \quad Y_f, \tag{2.1e}$$
$$\boldsymbol{\sigma}_f = -p_f I + \boldsymbol{\tau}_f, \quad Y_f, \tag{2.1f}$$
$$\boldsymbol{\tau}_f = 2\eta\epsilon^2 \mathbf{e}(\mathbf{v}_f) + \epsilon^2 \left(\kappa - \frac{2}{3}\eta\right) \nabla \cdot \mathbf{v}_f, \quad Y_f, \tag{2.1g}$$
$$i\omega p_f = B_f \nabla \cdot \mathbf{v}_f, \quad Y_f. \tag{2.1h}$$

Here ρ_f, ρ_{s_1}, ρ_{s_2}, $\mathbf{a}_{s_1}$, and $\mathbf{a}_{s_2}$ are, respectively, the fluid density, mass densities, and fourth-order positive-definite elastic tensors associated with the two solid phases, depending

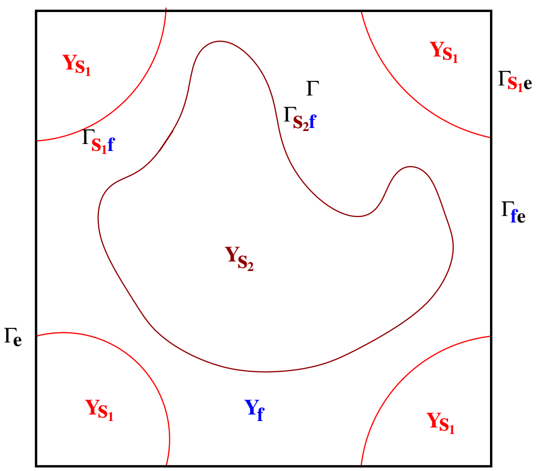

Fig. 1 A 2D realization of one period Y of the composite medium Ω

on the space variable and Y-periodic. Also, $\mathbf{e}$ denotes the linear strain tensor, i.e.,

$$e_{lm}(\mathbf{v}_f) = \frac{1}{2}\left(\frac{\partial v_{f,l}}{\partial x_m} + \frac{\partial v_{f,m}}{\partial x_l}\right).$$

Here, and in what follows, if $\mathbf{a}$ and $\mathbf{b}$ are, respectively, fourth- and second-order tensors, then $\mathbf{a} : \mathbf{b}$ denotes the index contraction operation $a_{klst}b_{st}$ with the usual Einstein's convention of summing on repeated indices.

Also, with $\boldsymbol{\nu}_{jk}$, $j, k = s_1, s_2, f$, $j \neq k$, denoting the unit outer normal at the interface Γ_{jk}, the boundary conditions among the different solid and fluid phases are

$$\boldsymbol{\nu}_{s_1 f} \cdot \boldsymbol{\sigma}_{s_1} = \boldsymbol{\nu}_{s_1 f} \cdot \boldsymbol{\sigma}_f, \quad \Gamma_{s_1 f}, \tag{2.2a}$$
$$\boldsymbol{\nu}_{s_2 f} \cdot \boldsymbol{\sigma}_{s_2} = \boldsymbol{\nu}_{s_2 f} \cdot \boldsymbol{\sigma}_f, \quad \Gamma_{s_2 f}, \tag{2.2b}$$
$$\mathbf{v}_{s_1} = \mathbf{v}_f, \quad \Gamma_{s_1 f}, \tag{2.2c}$$
$$\mathbf{v}_{s_2} = \mathbf{v}_f, \quad \Gamma_{s_2 f}. \tag{2.2d}$$

Next, following Sanchez-Palencia (1980, 1987) and Auriault et al. (1985), we expand the unknowns $\mathbf{u}_{s_1}, \mathbf{u}_{s_2}, \mathbf{u}_f$ in the form

$$\mathbf{u}_j^{\epsilon} = \mathbf{u}_j(x, y) = \mathbf{u}_j^{(0)}(x, y) + \epsilon \mathbf{u}_j^{(1)}(x, y) + \epsilon^2 \mathbf{u}_j^{(2)}(x, y) + \cdots, \quad j = s_1, s_2, f, \tag{2.3}$$

where the functions $\mathbf{u}_j^{(n)}(x, y)$, $n = 0, 1, \ldots,$ are Y-periodic. Then we substitute the expansions (2.3) into Eqs. (2.1)–(2.2) describing the local behavior recalling that for the spatial derivatives we have that $\frac{\mathrm{d}}{\mathrm{d}x}$ becomes

$$\frac{\partial}{\partial x} + \epsilon^{-1}\frac{\partial}{\partial y}.$$

Similarly,

$$\mathbf{e} = \mathbf{e}_x + \epsilon^{-1}\mathbf{e}_y, \quad \nabla = \nabla_x + \epsilon^{-1}\nabla_y, \quad \text{etc.}$$

2.2 Solution of the Local Equations for the Solid Phases

Let us consider the local equations for the solid phases at the lowest order. First, from (2.1b) and (2.1d),

$$\begin{aligned}\boldsymbol{\sigma}_j &= \mathbf{a}_j : (\mathbf{e}_x + \epsilon^{-1}\mathbf{e}_y)(\mathbf{u}_j^{(0)} + \epsilon \mathbf{u}_j^{(1)} + \cdots) \\ &= \epsilon^{-1}\mathbf{a}_j : \mathbf{e}_y(\mathbf{u}_j^{(0)}) + \mathbf{a}_j : \left(\mathbf{e}_x(\mathbf{u}_j^{(0)}) + \mathbf{e}_y(\mathbf{u}_j^{(1)})\right) + \epsilon\, \mathbf{a}_j : \left(\mathbf{e}_x(\mathbf{u}_j^{(1)}) + \mathbf{e}_y(\mathbf{u}_j^{(2)})\right) + \cdots \\ &= \epsilon^{-1}\boldsymbol{\sigma}_j^{(-1)} + \boldsymbol{\sigma}_j^{(0)} + \epsilon\,\boldsymbol{\sigma}_j^{(1)} + \cdots, \quad Y_j,\ j = s_1, s_2.\end{aligned} \tag{2.4}$$

Next, from (2.1a) and (2.4)

$$\begin{aligned}&\epsilon^{-2}\nabla_y \cdot \boldsymbol{\sigma}_j^{(-1)} + \epsilon^{-1}\left(\nabla_x \cdot \boldsymbol{\sigma}_j^{(-1)} + \nabla_y \cdot \boldsymbol{\sigma}_j^{(0)}\right) + \epsilon^0\left(\nabla_x \cdot \boldsymbol{\sigma}_j^{(0)} + \nabla_y \cdot \boldsymbol{\sigma}_j^{(1)}\right) + \cdots \\ &\quad = -\rho_j\omega^2\left(\mathbf{u}_j^{(0)} + \epsilon \mathbf{u}_j^{(1)} + \cdots\right), \quad Y_j,\ j = s_1, s_2.\end{aligned} \tag{2.5}$$

Also, from (2.1f)–(2.1g),

$$\begin{aligned}\boldsymbol{\sigma}_f^{(0)} + \epsilon\boldsymbol{\sigma}_f^{(1)} + \cdots &= -p_f^{(0)}I + \epsilon\left(-p_f^{(1)}I + 2\eta\mathbf{e}(\mathbf{v}_f^{(0)})\right) + \cdots \\ &= -p_f^{(0)}I + \epsilon\left(-p_f^{(1)}I + \boldsymbol{\tau}_f^{(1)}\right) + \cdots, \quad Y_f.\end{aligned} \tag{2.6}$$

Next we use (2.2a)–(2.2d) to obtain the boundary conditions for the local problems. First, from (2.2a), (2.2b), and (2.6),

$$\begin{aligned}&\boldsymbol{\nu}_{jf}\cdot\left(\epsilon^{-1}\boldsymbol{\sigma}_j^{(-1)}+\boldsymbol{\sigma}_j^{(0)}+\epsilon\boldsymbol{\sigma}_j^{(1)}+\cdots\right)\\&\quad=\boldsymbol{\nu}_{jf}\cdot\left(-p_f^{(0)}I+\epsilon\left(-p_f^{(1)}I+2\eta\mathbf{e}(\mathbf{v}_f^{(0)})\right)+\cdots\right),\ \Gamma_{jf},\ j=s_1,s_2.\end{aligned}\tag{2.7}$$

From (2.5) at ϵ^{-2} and (2.4) and (2.7) at ϵ^{-1} we obtain the following elliptic system for $\mathbf{u}_{s_1}^{(0)}$:

$$\nabla_y\cdot\boldsymbol{\sigma}_{s_1}^{(-1)}=0,\quad Y_{s_1},\tag{2.8a}$$
$$\boldsymbol{\sigma}_{s_1}^{(-1)}=\mathbf{a}_{s_1}:\mathbf{e}_y(\mathbf{u}_{s_1}^{(0)}),\quad Y_{s_1},\tag{2.8b}$$
$$\boldsymbol{\nu}_{s_1f}\cdot\boldsymbol{\sigma}_{s_1}^{(-1)}=0,\quad \Gamma_{s_1f},\tag{2.8c}$$
$$\mathbf{u}_{s_1}^{(0)}\ \text{is}\quad Y\text{-periodic}.\tag{2.8d}$$

For an open set $S\subset\mathbb{R}^3$ and a two-dimensional manifold γ, let $L^2(S)$ and $L^2(\gamma)$ be the space of square-integrable functions with the complex inner-products $(\cdot,\cdot)_S$ and $\langle\cdot,\cdot\rangle_\gamma$, respectively. Denote by $[H^1(S)]^3$ the complex vector valued Sobolev space of square-integrable functions with its first-order derivatives are also square-integrable in S; the norm in $[H^1(S)]^3$ will be denoted by $\|\cdot\|_{1,S}$.

Let us formulate (2.8) in variational form. Set

$$\mathcal{W}_{Y_j}=\left\{\boldsymbol{\varphi}\in[H^1(Y_j)]^3:\ \boldsymbol{\varphi}\ \text{is}\ Y\text{-periodic},\ \int_{Y_j}\boldsymbol{\varphi}\,\mathrm{d}y=0,\ \int_{Y_j}\nabla\times\boldsymbol{\varphi}\,\mathrm{d}y=0\right\},\quad j=s_1,s_2.$$

Then a weak form of (2.8) is given to find $\mathbf{u}_{s_1}^{(0)}\in\mathcal{W}_{Y_{s_1}}$ such that

$$\left(\mathbf{a}_{s_1}:\mathbf{e}_y(\mathbf{u}_{s_1}^{(0)}),\mathbf{e}_y(\boldsymbol{\varphi})\right)_{Y_{s_1}}=0,\quad \boldsymbol{\varphi}\in\mathcal{W}_{Y_{s_1}}.\tag{2.9}$$

Note that thanks to Korn's second inequality (Duvaut and Lions 1976; Nitsche 1981) and the fact that $\mathbf{a}_{s_1}$ is positive-definite the sesquilinear form $\left(\mathbf{a}_{s_1}:\mathbf{e}_y(\mathbf{u}),\mathbf{e}_y(\mathbf{v})\right)_{Y_{s_1}}$ defines an inner product in the Hilbert space $\mathcal{W}_{Y_{s_1}}$ equivalent to the H^1-inner product (Brenner and Sung 1992). Thus the Lax–Milgram lemma implies that $\mathbf{u}_{s_1}^{(0)}=0\in\mathcal{W}_{Y_{s_1}}$ is the unique solution of (2.9), or equivalently, the solution of (2.8a)–(2.8c) is independent of the y-variable, so that

$$\mathbf{u}_{s_1}^{(0)}(x,y)=\mathbf{u}_{s_1}^{(0)}(x),\quad Y_{s_1},\tag{2.10a}$$
$$\boldsymbol{\sigma}_{s_1}^{(-1)}=0,\quad Y_{s_1},\tag{2.10b}$$

where (2.10b) follows from (2.8b). With an identical argument, for the solid phase 2 we get

$$\mathbf{u}_{s_2}^{(0)}(x,y)=\mathbf{u}_{s_2}^{(0)}(x),\quad Y_{s_2},\tag{2.11a}$$
$$\boldsymbol{\sigma}_{s_2}^{(-1)}=0,\quad Y_{s_2}.\tag{2.11b}$$

2.3 Solution of the Local Equations for the Fluid Phase: A Generalized Darcy's Law

We now consider the local equations for the fluid at the lowest order. First, from the fluid equations (2.1e)–(2.1h) it follows that

$$\eta\epsilon^2\Big(\Delta_x+\epsilon^{-1}\left(\nabla_x\cdot\nabla_y+\nabla_y\cdot\nabla_x\right)+\epsilon^{-2}\Delta_y\Big)\left(\mathbf{v}_f^{(0)}+\epsilon\mathbf{v}_f^{(1)}+\epsilon^2\mathbf{v}_f^{(2)}+\cdots\right)$$
$$+\epsilon^2\left(\kappa-\frac{1}{2}\eta\right)\left[\nabla_x\nabla_x+\epsilon^{-1}\left(\nabla_x\nabla_y+\nabla_y\nabla_x\right)+\epsilon^{-2}\nabla_y\nabla_y\right]$$
$$\cdot\left(\mathbf{v}_f^{(0)}+\epsilon\mathbf{v}_f^{(1)}+\epsilon^2\mathbf{v}_f^{(2)}+\cdots\right)=\left(\nabla_x+\epsilon^{-1}\nabla_y\right)\Big(p_f^{(0)}+\epsilon p_f^{(1)}$$
$$+\epsilon^2 p_f^{(2)}+\cdots\Big)+i\omega\rho_f\left(\mathbf{v}_f^{(0)}+\epsilon\mathbf{v}_f^{(1)}+\cdots\right), \tag{2.12}$$

and

$$i\omega\left(p_f^{(0)}+\epsilon p_f^{(1)}+\cdots\right)=B_f\left(\nabla_x+\epsilon^{-1}\nabla_y\right)\cdot\left(\mathbf{v}_f^{(0)}+\epsilon\mathbf{v}_f^{(1)}+\epsilon^2\mathbf{v}_f^{(2)}+\right). \tag{2.13}$$

Thus from (2.12) at ϵ^{-1} we get

$$p_f^{(0)}(x,y)=p_f^{(0)}(x), \tag{2.14}$$

and then it follows from (2.13) at ϵ^{-1} that

$$\nabla_y\cdot\mathbf{v}_f^{(0)}=0,\quad Y_f. \tag{2.15}$$

Hence, from (2.15) and (2.12) at ϵ^0 we get

$$\eta\Delta_y\mathbf{v}_f^{(0)}=\nabla_y\cdot\boldsymbol{\tau}_f^{(1)}(\mathbf{v}_f^{(0)})=\nabla_y p_f^{(1)}+\nabla_x p_f^{(0)}+i\omega\rho_f\mathbf{v}_f^{(0)},\quad Y_f. \tag{2.16}$$

Also, it follows from (2.2c)–(2.2d) that

$$\mathbf{v}_f^{(0)}=i\omega\mathbf{u}_{s_1}^{(0)}(x),\quad \Gamma_{s_1f}, \tag{2.17a}$$
$$\mathbf{v}_f^{(0)}=i\omega\mathbf{u}_{s_2}^{(0)}(x),\quad \Gamma_{s_2f}. \tag{2.17b}$$

Let us split Problem (2.15), (2.16), and (2.17a)–(2.17b) into two subproblems as follows. First, let $\mathbf{v}_f^{(0),I}$ and $p_f^{(1),I}$ be Y-periodic such that

$$i\omega\rho_f\mathbf{v}_f^{(0),I}-\eta\Delta_y\mathbf{v}_f^{(0),I}+\nabla_y p_f^{(1),I}=-\nabla_x p_f^{(0)},\quad Y_f, \tag{2.18a}$$
$$\nabla_y\cdot\mathbf{v}_f^{(0),I}=0,\quad Y_f, \tag{2.18b}$$
$$\mathbf{v}_f^{(0),I}=0,\quad \Gamma_{sf}. \tag{2.18c}$$

Second, let $\mathbf{v}_f^{(0),B}$ and $p_f^{(1),B}$ be the Y-periodic solutions of

$$i\omega\rho_f\mathbf{v}_f^{(0),B}-\eta\Delta_y\mathbf{v}_f^{(0),B}+\nabla_y p_f^{(1),B}=0,\quad Y_f, \tag{2.19a}$$
$$\nabla_y\cdot\mathbf{v}_f^{(0),B}=0,\quad Y_f, \tag{2.19b}$$
$$\mathbf{v}_f^{(0),B}=i\omega\mathbf{u}_{s_1}^{(0)}(x),\quad \Gamma_{s_1f}, \tag{2.19c}$$
$$\mathbf{v}_f^{(0),B}=i\omega\mathbf{u}_{s_2}^{(0)}(x),\quad \Gamma_{s_2f}. \tag{2.19d}$$

Let us solve the cell problem (2.18)–(2.18c) for $\mathbf{v}_f^{(0),I}$. Let

$$\mathcal{V}^1_{Y_f} = \left\{ \boldsymbol{\varphi} \in \left[H^1(Y_f)\right]^3 : \quad \nabla_y \cdot \boldsymbol{\varphi} = 0 \text{ in } Y_f, \quad \boldsymbol{\varphi} = 0 \text{ on } \Gamma_{sf}, \quad \boldsymbol{\varphi} \text{ is } Y\text{-periodic} \right\},$$

provided with the natural (complex) inner product in $\left[H^1(Y_f)\right]^3$. Then a variational formulation of (2.18), (2.18c) can be stated as follows: Find $\mathbf{v}_f^{(0),I} \in \mathcal{V}^1_{Y_f}$ such that

$$i\omega \left(\rho_f \mathbf{v}_f^{(0),I}, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta \nabla_y \mathbf{v}_f^{(0),I}, \nabla_y \boldsymbol{\varphi}\right)_{Y_f} = -\nabla_x p_f^{(0)}(x) \cdot \int_{Y_f} \boldsymbol{\varphi} \mathrm{d}y, \quad \boldsymbol{\varphi} \in \mathcal{V}^1_{Y_f}. \quad (2.20)$$

It is known that (2.20) has a unique solution, which can be found as usual by solving the following set of problems (Sanchez-Palencia 1980). For $k = 1, 2, 3$ let $\mathbf{V}^k = (V_l^k)_{1 \le l \le 3} \in \mathcal{V}^1_{Y_f}$ be the solution of

$$i\omega \left(\rho_f \mathbf{V}^k, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta \nabla_y \mathbf{V}^k, \nabla_y \boldsymbol{\varphi}\right)_{Y_f} = \mathbf{e}^k \cdot \int_{Y_f} \boldsymbol{\varphi} \mathrm{d}y, \quad \boldsymbol{\varphi} \in \mathcal{V}^1_{Y_f}, \quad (2.21)$$

where $\mathbf{e}^k$ denotes the standard basis in $\mathbb{R}^3$ and set

$$\mathbf{K}(x, y, \omega) = (\mathbf{K}(x, y, \omega))_{tk} = V_t^k(x, y, \omega). \quad (2.22)$$

Then,

$$\mathbf{v}_f^{(0),I}(x, y, \omega) = -\mathbf{K}(x, y, \omega) \nabla p_f^{(0)}(x, \omega). \quad (2.23)$$

We turn to analyze the second subproblem (2.19a)–(2.19d). First, notice that it follows from (2.10a) and (2.11a) that

$$\begin{aligned} 0 &= \int_{Y_j} \nabla_y \cdot \mathbf{u}_j^{(0)}(x) \mathrm{d}y = \mathbf{u}_j^{(0)}(x) \cdot \int_{\partial Y_j} \boldsymbol{\nu}_j \mathrm{d}y \\ &= \mathbf{u}_j^{(0)}(x) \cdot \int_{\Gamma_{jf}} \boldsymbol{\nu}_j \mathrm{d}y + \mathbf{u}_j^{(0)}(x) \cdot \int_{\Gamma_{je}} \boldsymbol{\nu}_j \mathrm{d}y = \mathbf{u}_j^{(0)}(x) \cdot \int_{\Gamma_{jf}} \boldsymbol{\nu}_j \mathrm{d}y, \quad j = s_1, s_2 \end{aligned} \quad (2.24)$$

since $\int_{\Gamma_{je}} \boldsymbol{\nu}_j \mathrm{d}y = 0$ due to the periodicity of the boundary Γ_{je}. Thus the boundary data function defined by

$$\mathbf{g}(x, y, \omega) = \begin{cases} i\omega \mathbf{u}_{s_1}^{(0)}(x, \omega), & \Gamma_{s_1 f}, \\ i\omega \mathbf{u}_{s_2}^{(0)}(x, \omega), & \Gamma_{s_2 f}, \\ \text{periodic in} & \Gamma_{fe}, \end{cases} \quad (2.25)$$

satisfies the consistency condition

$$\int_{\partial Y_f} \mathbf{g} \cdot \boldsymbol{\nu}_f \mathrm{d}y = 0. \quad (2.26)$$

Remark 2.1 Notice that our boundary data take constant values on each boundary component of Y_f, which implies that the solution of Problem (2.19) is smooth. Therefore a classical abstract theory of existence of solutions of Stokes problems (e.g., Temam 1984) can be applied to our case. However, it is our intention to derive a form of the permeability tensor depending explicitly on the boundary data, and consequently we analyze the problem in the *very weak* sense (Conca 1987; Marusic-Paloka 2000).

Set

$$\mathcal{V}^2_{Y_f} = \left\{\boldsymbol{\varphi} \in \left[H^2(Y_f)\right]^3 : \quad \nabla_y \cdot \boldsymbol{\varphi} = 0 \text{ in } Y_f, \quad \boldsymbol{\varphi} = 0 \text{ on } \Gamma_{sf}, \quad \boldsymbol{\varphi} \text{ is } Y\text{-periodic}\right\},$$

$$\mathcal{W} = \left\{q \in H^1(Y_f) : \quad \int_{Y_f} q \, \mathrm{d}y = 0, \quad q \text{ is } Y\text{-periodic}\right\}.$$

Now we state the existence and uniqueness results on the solution of Problem (2.19).

Theorem 2.1 *There exists a unique* Y*-periodic very weak solution* $\mathbf{v}_f^{(0),B} \in \left[L^2(Y_f)\right]^3$ *and* $\mathbf{v}_f^{(0),B}|_{\Gamma_{sf}} \in \left[L^2(\Gamma_{sf})\right]^3$ *of Problem* (2.19) *in the following sense:*

$$i\omega\left(\rho_f \mathbf{v}_f^{(0),B}, \boldsymbol{\varphi}\right)_{Y_f} - \left(\eta \mathbf{v}_f^{(0),B}, \Delta_y \boldsymbol{\varphi}\right)_{Y_f} = -\left\langle \eta \mathbf{g}, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}}\right\rangle_{\Gamma_{sf}}, \quad \boldsymbol{\varphi} \in \mathcal{V}^2_{Y_f}, \tag{2.27a}$$

$$\left(\mathbf{v}_f^{(0),B}, \nabla q\right)_{Y_f} = \langle \mathbf{g} \cdot \boldsymbol{\nu}, q\rangle_{\Gamma_{sf}}, \quad q \in \mathcal{W}. \tag{2.27b}$$

The proof of Theorem 2.1 is given in Appendix A.
Set

$$\boldsymbol{\gamma}_j^k(x, y) = \chi_{\Gamma_{jf}}(x, y)\mathbf{e}^k, \quad j = s_1, s_2, \quad k = 1, 2, 3, \tag{2.28}$$

where $\chi_{\Gamma_{jf}}(x, y)$ denotes the characteristic function of Γ_{jf}. Also on Γ_{sf} write the boundary data vector $\mathbf{g}$ in (2.25) in the form

$$\mathbf{g} = \sum_k \left(g_{s_1,k}\boldsymbol{\gamma}_{s_1}^k + g_{s_2,k}\boldsymbol{\gamma}_{s_2}^k\right), \quad y \in \Gamma_{sf}, \tag{2.29}$$

where $g_{j,k}$ denotes the k-component of $i\omega\mathbf{u}_j^{(0)}$ on Γ_{jf}. Let $\mathbf{Z}^{j,k} = (Z_l^{j,k})_{1\le l\le 3}$ be the solution of (2.27) with $\mathbf{g}$ replaced by $\boldsymbol{\gamma}_j^k$, $j = s_1, s_2, k = 1, 2, 3$.

$$\mathbf{K}^j(x, y, \omega) = \left(\mathbf{K}^j(x, y, \omega)\right)_{lk} = Z_l^{j,k}(x, y, \omega), \quad j = s_1, s_2. \tag{2.30}$$

Then, by linearity the solution of (2.27) is given by

$$\mathbf{v}_f^{(0),B}(x, y, \omega) = \mathbf{K}^{s_1}(x, y, \omega)\left[i\omega\mathbf{u}_{s_1}^{(0)}(x, \omega)\right] + \mathbf{K}^{s_2}(x, y, \omega)\left[i\omega\mathbf{u}_{s_2}^{(0)}(x, \omega)\right]. \tag{2.31}$$

Combining (2.23) and (2.31), we conclude that

$$\begin{aligned}\mathbf{v}_f^{(0)}(x, y, \omega) &= \mathbf{v}_f^{(0),I}(x, y, \omega) + \mathbf{v}_f^{(0),B}(x, y, \omega)\\ &= -\mathbf{K}(x, y, \omega)\nabla p_f^{(0)}(x, \omega) + \mathbf{K}^{s_1}(x, y, \omega)\left[i\omega\mathbf{u}_{s_1}^{(0)}(x, \omega)\right]\\ &\quad +\mathbf{K}^{s_2}(x, y, \omega)\left[i\omega\mathbf{u}_{s_2}^{(0)}(x, \omega)\right].\end{aligned} \tag{2.32}$$

Let

$$\ll \theta \gg = \frac{1}{|Y|}\int_Y \theta(y)\mathrm{d}y,$$

denote the average of $\theta(y)$ over Y, where θ is defined to be zero outside its domain of definition. Then, averaging (2.32) over Y yields

$$\begin{aligned} \ll \mathbf{v}_f^{(0)} \gg (x,\omega) = &- \ll \mathbf{K} \gg (x,\omega)\nabla_x p_f^{(0)}(x,\omega) \\ &+ \ll \mathbf{K}^{s_1} \gg (x,\omega)\left[i\omega \mathbf{u}_{s_1}^{(0)}(x,\omega)\right] \\ &+ \ll \mathbf{K}^{s_2} \gg (x,\omega)\left[i\omega \mathbf{u}_{s_2}^{(0)}(x,\omega)\right]. \end{aligned} \tag{2.33}$$

which is a generalized Darcy's law for our composite system.

Remark 2.2 The derived Darcy's law (2.33) contains three frequency-dependent permeability tensors: the tensor $\ll \mathbf{K} \gg$ associated with the gradient of pressure as in the classical Darcy's law and two additional tensors $\ll \mathbf{K}^{s_1} \gg$ and $\ll \mathbf{K}^{s_2} \gg$ which carry over to the macroscale information from the two solid phases' microscopic surface geometry.

Remark 2.3 In Subsect. 3.3 the coefficients in (2.33) are identified with those appearing in (3.17), which is a form of Darcy's law derived by Leclaire et al. (1994) for this type of composite materials using a phenomenological approach. To relate the permeability tensors in (2.33) with the coefficients appearing in (3.17), further research is needed to obtain numerical solutions of the local problems (2.20) and (2.27) for specific geometry configurations.

3 Properties of the Permeability Tensors

In this section we analyze some properties of the permeability tensors $\ll \mathbf{K} \gg$, $\ll \mathbf{K}^{s_1} \gg$, and $\ll \mathbf{K}^{s_2} \gg$ which appear in the Darcy's law (2.33).

3.1 Properties of $\ll \mathbf{K} \gg$

Note that defining on $\mathcal{V}^1_{Y_f}$ the sesquilinear form

$$B(\mathbf{u},\mathbf{v}) = i\omega\left(\rho_f \mathbf{u},\mathbf{v}\right)_{Y_f} + (\eta\nabla\mathbf{u},\nabla\mathbf{v})_{Y_f}, \quad \mathbf{u},\mathbf{v}\in\mathcal{V}^1_{Y_f}, \tag{3.1}$$

and the continuous linear functional

$$L_{\mathbf{f}}(\boldsymbol{\varphi}) = \mathbf{f}\cdot\int_{Y_{\mathbf{f}}}\boldsymbol{\varphi}\mathrm{d}y, \tag{3.2}$$

with $\mathbf{f} = \mathbf{f}(x,\omega) = -\nabla p_f^{(0)}(x,\omega)$, Problem (2.20) can be stated in the form: find $\mathbf{u}\in\mathcal{V}^1_{Y_f}$ such that

$$B(\mathbf{u},\mathbf{v}) = L_{\mathbf{f}}(\mathbf{v}), \quad \mathbf{v}\in\mathcal{V}^1_{Y_f}. \tag{3.3}$$

Note that $B(\mathbf{u},\mathbf{v})$ is continuous and coercive in $\mathcal{V}^1_{Y_f}$ since

$$\begin{aligned} |B(\mathbf{u},\mathbf{u})| &\geq \frac{1}{2}\Big(|\mathrm{Re}(B(\mathbf{u},\mathbf{u}))| + |\mathrm{Im}(B(\mathbf{u},\mathbf{u}))|\Big) \\ &= \frac{1}{2}\left[(\eta\nabla\mathbf{u},\nabla\mathbf{u}) + \omega\left(\rho_f\mathbf{u},\mathbf{u}\right)\right] \geq C(\omega)\|\mathbf{u}\|_1^2. \end{aligned} \tag{3.4}$$

Thus, by Lax–Milgram Lemma, Problem (3.3) has a unique solution, which implies that the solution operator

$$\mathbf{f}\to\mathbf{u} = T_{\mathbf{f}}$$

where $\mathbf{u}$ solves (3.3) is injective. Thus if $\{\mathbf{e}^k : k = 1, 2, 3\}$ denotes the standard basis in $\mathbb{R}^3$, $\{\mathbf{V}^k = T_{\mathbf{f}}^{-1}(\mathbf{e}^k), k = 1, 2, 3\}$ forms a linearly independent set in $\mathcal{V}^1_{Y_f}$. Thus the tensor $\mathbf{K}$ is invertible. Set $\mathbf{V}^k = \mathrm{Re}(\mathbf{V}^k) + i\,\mathrm{Im}(\mathbf{V}^k) = \mathbf{V}^k_R + i\mathbf{V}^k_I$ and recall that $\ll \mathrm{Re}\,(\mathbf{K})_{kl} \gg = \frac{1}{|Y|}\left(\mathbf{e}^k, \mathbf{V}^l_R\right)_{Y_f}$.

Assuming $\boldsymbol{\varphi}$ to be real, take the real and imaginary parts in (2.21) to obtain

$$-\omega\left(\rho_f \mathbf{V}^k_I, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta\nabla_y \mathbf{V}^k_R, \nabla_y\boldsymbol{\varphi}\right)_{Y_f} = \mathbf{e}^k \cdot \int_{Y_f} \boldsymbol{\varphi}\,\mathrm{d}y, \quad \boldsymbol{\varphi} \in \mathcal{V}^1_{Y_f}, \tag{3.5a}$$

$$\omega\left(\rho_f \mathbf{V}^k_R, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta\nabla_y \mathbf{V}^k_I, \nabla_y\boldsymbol{\varphi}\right)_{Y_f} = 0, \quad \boldsymbol{\varphi} \in \mathcal{V}^1_{Y_f}, \quad k = 1, 2, 3. \tag{3.5b}$$

Choose $\boldsymbol{\varphi} = \mathbf{V}^l_R$ in (3.5a) and $\boldsymbol{\varphi} = \mathbf{V}^k_I$ in (3.5b) with k replaced by l and add the resulting equations to get

$$\ll \mathrm{Re}\,(\mathbf{K})_{kl} \gg = \frac{1}{|Y|}\left[\left(\eta\, \nabla_y \mathbf{V}^k_R, \nabla_y \mathbf{V}^l_R\right)_{Y_f} + \left(\eta\nabla_y \mathbf{V}^l_I, \nabla_y \mathbf{V}^k_I\right)_{Y_f}\right],$$

which shows that $\ll \mathrm{Re}\,(\mathbf{K}) \gg$ is symmetric. Observe that for any $\boldsymbol{\xi} \in \mathbb{R}^3$,

$$\boldsymbol{\xi}^T \ll \mathrm{Re}\,(\mathbf{K}) \gg \boldsymbol{\xi} = \frac{1}{|Y|}\left[\|\eta^{\frac{1}{2}}\nabla_y \mathbf{V}^k_R \xi_k\|^2_{0,Y_f} + \|\eta^{\frac{1}{2}}\nabla_y \mathbf{V}^k_I \xi_k\|^2_{0,Y_f}\right] \geq 0,$$

where the equality holds if and only if $\nabla_y \mathbf{V}^k_R \xi_k = 0$ and $\nabla_y \mathbf{V}^k_I \xi_k = 0$. Since $\mathbf{V}^k = 0$ on Γ_{sf}, by Poincaré inequality, $\mathbf{V}^k_R \xi_k = \mathbf{V}^k_I \xi_k = 0$, and therefore, $\ll \mathrm{Re}\,(\mathbf{K}) \gg$ is positive-definite.

Next, recalling that $\ll \mathrm{Im}\,(\mathbf{K})_{kl} \gg = \frac{1}{|Y|}\left(\mathbf{e}^k, \mathbf{V}^l_I\right)_{Y_f}$ we analyze the properties of $\ll \mathrm{Im}\,(\mathbf{K}) \gg$. For this, we choose $\boldsymbol{\varphi} = \mathbf{V}^l_I$ in (3.5a) and $\boldsymbol{\varphi} = \mathbf{V}^k_R$ in (3.5b) with k replaced by l. Then

$$\ll \mathrm{Im}\,(\mathbf{K})_{kl} \gg = -\frac{\omega}{|Y|}\left[\left(\rho_f^{\frac{1}{2}} \mathbf{V}^l_R, \mathbf{V}^k_R\right)_{Y_f} + \left(\rho_f^{\frac{1}{2}} \mathbf{V}^k_I, \mathbf{V}^l_I\right)_{Y_f}\right],$$

which implies that $\ll \mathrm{Im}\,(\mathbf{K}) \gg$ is symmetric and negative-definite. This in turn implies that both the real and imaginary parts of $(\ll \mathbf{K} \gg)^{-1}$ are symmetric and positive-definite.

3.2 Properties of $\ll \mathbf{K}^{s_1} \gg$ and $\ll \mathbf{K}^{s_2} \gg$

Let us turn to analyze the $\ll \mathbf{K}^j \gg$-tensors, $j = s_1, s_2$, having the contribution from the boundaries Γ_{jf}, $j = s_1, s_2$. For this purpose, it is convenient to analyze the properties of the solution $\mathbf{Z}^{j,k,(m)}$ of (A.5) in Sect. Proof of Theorem 2.1, with right-hand side $\mathbf{g} = \mathbf{e}^k$, $k = 1, 2, 3$. Thus, set

$$H^1(\mathrm{div}\,0; Y_f) = \left\{\boldsymbol{\varphi} \in \left[H^1(Y_f)\right]^3 : \quad \nabla_y \cdot \boldsymbol{\varphi} = 0, \quad \boldsymbol{\varphi} \text{ is } Y\text{-periodic}\right\}$$

and let $\mathbf{Z}^{j,k,(m)} \in H^1(\mathrm{div}\,0; Y_f)$ be the solution of

$$i\omega\left(\rho_f \mathbf{Z}^{j,k,(m)}, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta\nabla_y \mathbf{Z}^{j,k,(m)}, \nabla_y\boldsymbol{\varphi}\right)_{Y_f} + m\left\langle \mathbf{Z}^{j,k,(m)}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{jf}}$$
$$= m\left\langle \mathbf{e}^k, \boldsymbol{\varphi}\right\rangle_{\Gamma_{jf}}, \boldsymbol{\varphi} \in H^1(\mathrm{div}\,0; Y_f),\ j = s_1, s_2. \tag{3.6}$$

Set $\mathbf{Z}^{j,k,(m)} = \text{Re}\left(\mathbf{Z}^{j,k,(m)}\right) + i\ \text{Im}\left(\mathbf{Z}^{j,k,(m)}\right) = \mathbf{Z}_R^{j,k,(m)} + i\mathbf{Z}_I^{j,k,(m)}$. Assuming $\boldsymbol{\varphi}$ to be real, take the real and imaginary parts in (3.6) to obtain

$$-\omega\left(\rho_f \mathbf{Z}_I^{j,k,(m)}, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta\nabla_y \mathbf{Z}_R^{j,k,(m)}, \nabla_y\boldsymbol{\varphi}\right)_{Y_f} + m\left\langle \mathbf{Z}_R^{j,k,(m)}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{jf}} = m\left\langle \mathbf{e}^k, \boldsymbol{\varphi}\right\rangle_{\Gamma_{jf}}, \quad (3.7a)$$

$$\omega\left(\rho_f \mathbf{Z}_R^{j,k,(m)}, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta\nabla_y \mathbf{Z}_I^{j,k,(m)}, \nabla_y\boldsymbol{\varphi}\right)_{Y_f} + m\left\langle \mathbf{Z}_I^{j,k,(m)}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{jf}} = 0,$$
$$\boldsymbol{\varphi} \in H^1(\text{div}0 : Y_f). \quad (3.7b)$$

Next, with the choice $\boldsymbol{\varphi} = \mathbf{Z}_I^{j,l,(m)}$ in (3.7a) and $\boldsymbol{\varphi} = \mathbf{Z}_R^{j,k,(m)}$ in (3.7b) with k replaced by l, take the difference in the resulting equations to obtain

$$-\omega\left[\left(\rho_f \mathbf{Z}_R^{j,l,(m)}, \mathbf{Z}_R^{j,k,(m)}\right)_{Y_f} + \left(\rho_f \mathbf{Z}_I^{j,k,(m)}, \mathbf{Z}_I^{j,l,(m)}\right)_{Y_f}\right] = m\left\langle \mathbf{e}^k, \mathbf{Z}_I^{j,l,(m)}\right\rangle_{\Gamma_{jf}}. \quad (3.8)$$

Also, add (3.7a) with the choice $\boldsymbol{\varphi} = \mathbf{Z}_R^{j,l,(m)}$ with (3.7b) with the choice $\boldsymbol{\varphi} = \mathbf{Z}_I^{j,k,(m)}$ with k replaced by l to get

$$\left(\eta\nabla \mathbf{Z}_R^{j,k,(m)}, \nabla \mathbf{Z}_R^{j,l,(m)}\right)_{Y_f} + \left(\eta\nabla \mathbf{Z}_I^{j,l,(m)}, \nabla \mathbf{Z}_I^{j,k,(m)}\right)_{Y_f}$$
$$+m\left[\left\langle \mathbf{Z}_R^{j,k,(m)}, \mathbf{Z}_R^{j,l,(m)}\right\rangle_{\Gamma_{jf}} + \left\langle \mathbf{Z}_I^{j,l,(m)}, \mathbf{Z}_I^{j,k,(m)}\right\rangle_{\Gamma_{jf}}\right] = m\left\langle \mathbf{e}^k, \mathbf{Z}_R^{j,l,(m)}\right\rangle_{\Gamma_{jf}}. \quad (3.9)$$

Similarly, take $\boldsymbol{\varphi} = \mathbf{e}^l$ in (3.7a) and $\boldsymbol{\varphi} = \mathbf{e}^k$ in (3.7b) with k replaced by l to have

$$-\omega\left(\rho_f Z_{I,l}^{j,k,(m)}, 1\right)_{Y_f} + m\left\langle Z_{R,l}^{j,k,(m)}, 1\right\rangle_{\Gamma_{jf}} = m\delta_{kl}\left|\Gamma_{jf}\right|, \quad (3.10a)$$

$$\omega\left(\rho_f Z_{R,k}^{j,l,(m)}, 1\right)_{Y_f} + m\left\langle Z_{I,k}^{j,l,(m)}, 1\right\rangle_{\Gamma_{jf}} = 0. \quad (3.10b)$$

Using (3.10b) in (3.8) and (3.10a) (interchanging k and l) in (3.9) we have

$$\left(Z_{R,k}^{j,l,(m)}, 1\right)_{Y_f} = \left(\mathbf{Z}_R^{j,l,(m)}, \mathbf{Z}_R^{j,k,(m)}\right)_{Y_f} + \left(\mathbf{Z}_I^{j,k,(m)}, \mathbf{Z}_I^{j,l,(m)}\right)_{Y_f} \quad (3.11)$$

and

$$\omega\rho_f\left(Z_{I,k}^{j,l,(m)}, 1\right)_{Y_f} = \eta\left[\left(\nabla \mathbf{Z}_R^{j,k,(m)}, \nabla \mathbf{Z}_R^{j,l,(m)}\right)_{Y_f} + \left(\nabla \mathbf{Z}_I^{j,l,(m)}, \nabla \mathbf{Z}_I^{j,k,(m)}\right)_{Y_f}\right]$$
$$+m\left[\left\langle \mathbf{Z}_R^{j,k,(m)}, \mathbf{Z}_R^{j,l,(m)}\right\rangle_{\Gamma_{jf}} + \left\langle \mathbf{Z}_I^{j,l,(m)}, \mathbf{Z}_I^{j,k,(m)}\right\rangle_{\Gamma_{sf}} - \delta_{kl}\left|\Gamma_{jf}\right|\right]. \quad (3.12)$$

Let

$$\mathbf{K}^{j,(m)} = \left(\mathbf{K}^{j,(m)}\right)_{kl} = Z_k^{j,l,(m)},$$

so that

$$\ll \text{Re}(\mathbf{K}^{j,(m)})_{kl} \gg = \frac{1}{|Y|}\left(Z_{R,k}^{j,l,(m)}, 1\right)_{Y_f}, \quad \ll \text{Im}(\mathbf{K}^{j,(m)})_{kl} \gg = \frac{1}{|Y|}\left(Z_{I,k}^{j,t,(m)}, 1\right)_{Y_f}.$$

It follows from (3.11) and (3.12) that, for each m, $\ll \text{Re}(\mathbf{K}^{j,(m)}) \gg$ is symmetric positive-definite and $\ll \text{Im}(\mathbf{K}^{j,(m)}) \gg$ is symmetric. Due to the weak convergence of $\mathbf{Z}^{j,l,(m)}$ to $\mathbf{Z}^{j,l}$

in $[L^2(Y_f)]^3$, we have (c.f. (A.18)),

$$\lim_{m\to\infty}\left(Z_{R,k}^{j,l,(m)},1\right)_{Y_f}=\left(Z_{R,k}^{j,l},1\right)_{Y_f},$$
$$\lim_{m\to\infty}\left(Z_{I,k}^{j,l,(m)},1\right)_{Y_f}=\left(Z_{I,k}^{j,l},1\right)_{Y_f},\quad j=s_1,s_2. \tag{3.13}$$

Consequently, $\ll \text{Re}\left(\mathbf{K}^j\right)\gg$ is symmetric, positive semi-definite and $\ll \text{Im}\left(\mathbf{K}^j\right)\gg$ is symmetric. Also, thanks to (3.13) the first term in the left-hand side in (3.10b) is bounded. Thus taking the limit in (3.8) as m tends to ∞, it follows that

$$\left\langle Z_{I,k}^{j,l},1\right\rangle_{\Gamma_{jf}}=0. \tag{3.14}$$

Next note that from (3.10a)

$$\left\langle Z_{R,l}^{j,k,(m)},1\right\rangle_{\Gamma_{jf}}-\delta_{kl}\left|\Gamma_{jf}\right|=\frac{1}{m}\omega\rho_f\left(Z_{I,l}^{j,k,(m)},1\right)_{Y_f}.$$

In the above equation, the first term in left-hand side converges due to the weak convergence of $\mathbf{Z}^{j,k,(m)}$ to $\mathbf{Z}^{j,k}$ in $[L^2(\Gamma_{sf}]^3$, and the right-hand side tends to zero as m goes to infinity thanks to (3.13). Thus,

$$\left\langle Z_{R,l}^{j,k},1\right\rangle_{\Gamma_{jf}}=\delta_{kl}\left|\Gamma_{jf}\right|. \tag{3.15}$$

3.3 A Formal Relation of Darcy's Law (2.33) with a Previous Phenomenological Derivation

Consider Eq. (13) in Leclaire et al. (1994) for the fluid part in the steady-state case (i.e., the velocities are time-independent). In terms of our notation it can be stated as follows:

$$-\phi_w\nabla p_f^{(0)}(x,t)=b_{12}\left[\mathbf{v}_f^{(0)}(x,t)-\mathbf{v}_{s_1}^{(0)}(x,t)\right]+b_{23}\left[\mathbf{v}_f^{(0)}(x,t)-\mathbf{v}_{s_2}^{(0)}(x,t)\right], \tag{3.16}$$

where ϕ_w, b_{12}, and b_{23} are positive coefficients independent of time as defined in Leclaire et al. (1994). For the sake of convenience, the above equation is written in the form:

$$\mathbf{v}_f^{(0)}(x,t)=-\frac{\phi_w}{b_{12}+b_{23}}\nabla p_f^{(0)}(x,t)+\frac{b_{12}}{b_{12}+b_{23}}\mathbf{v}_{s_1}^{(0)}(x,t)+\frac{b_{23}}{b_{12}+b_{23}}\mathbf{v}_{s_2}^{(0)}(x,t). \tag{3.17}$$

Next, note that the inverse Fourier transform of the solutions $\mathbf{V}^k$ and $\mathbf{Z}^k$ of the local Stokes problems (2.21) and (2.27) vanishes for negative times (i.e. they are causal functions of time t). Consequently, the inverse Fourier transforms of the permeability tensors $\ll \mathbf{K}\gg$, $\ll \mathbf{K}^j\gg$, $j=s_1,s_2$ are causal, and Darcy's Law (2.33) can be restated in the space–time domain using convolutions as follows. Let

$$\ll \mathbf{S} \gg (x, \omega) = \frac{\ll \mathbf{K} \gg (x, \omega)}{i\omega},$$
$$\ll \mathbf{S}^j \gg (x, \omega) = \frac{\ll \mathbf{K}^j \gg (x, \omega)}{i\omega}, \quad j = s_1, s_2, \tag{3.18}$$

and for any function $f(\omega)$ let $\widehat{f}(t)$ denote its inverse Fourier transform. Then, (2.33) in the space–time domain becomes

$$\begin{aligned} \ll \mathbf{v}_f^{(0)} \gg (x, t) &= -\int_0^t \widehat{\ll \mathbf{S} \gg}(x, t-\tau) \frac{\partial}{\partial \tau} \nabla_x \widehat{p_f^{(0)}}(x, \tau) d\tau \\ &\quad - \sum_{j=s_1,s_2} \int_0^t \widehat{\ll \mathbf{S}^j \gg}(x, t-\tau) \frac{\partial}{\partial \tau} \widehat{\mathbf{v}_j^{(0)}}(x, \tau) d\tau \\ &= -\widehat{\ll \mathbf{S} \gg}(x, 0+) \nabla_x \widehat{p_f^{(0)}}(x, t) + \int_0^t \widehat{\ll \mathbf{K} \gg}(x, t-\tau) \nabla_x \widehat{p_f^{(0)}}(x, \tau) d\tau \\ &\quad - \sum_{j=s_1,s_2} \left[\widehat{\ll \mathbf{S}^j \gg}(x, 0+) \widehat{\mathbf{v}_j^{(0)}}(x, t) + \int_0^t \widehat{\ll \mathbf{K}^j \gg}(x, t-\tau) \widehat{\mathbf{v}_j^{(0)}}(x, \tau) d\tau \right]. \end{aligned} \tag{3.19}$$

Thus in the isotropic case we can identify the coefficients in (3.17) with the $\widehat{\ll \mathbf{S} \gg}(x, 0+)$ and $\widehat{\ll \mathbf{S}^j \gg}(x, 0+)$ terms in (3.19).

4 Conclusions

The homogenization procedure was applied to derive the generalized macroscopic Darcy's (2.33) for a porous medium composed of two *weakly coupled* solid phases saturated by a single-phase fluid. The derived Darcy's contains three frequency-dependent permeability tensors: the tensor $\ll \mathbf{K} \gg (x, \omega)$ associated with the gradient of pressure and the tensors $\ll \mathbf{K}^{s_1} \gg (x, \omega)$ and $\ll \mathbf{K}^{s_2} \gg (x, \omega)$ related to the surface geometries of the moving boundaries of the solid phases.

It was shown that the real and imaginary parts of the permeability tensor $\ll \mathbf{K} \gg (x, \omega)$ are positive-definite and negative-definite, respectively, and both parts are symmetric as it happens in the classical Darcy's law. It was also demonstrated that $\ll \mathrm{Re}(\mathbf{K}^j) \gg (x, \omega)$, $j = s_1, s_2$ are symmetric, positive semi-definite and $\ll \mathrm{Im}(\mathbf{K}^j) \gg (x, \omega)$, $j = s_1, s_2$ are symmetric.

A formal relation with a macroscopic frequency independent Darcy's law previously derived by Leclaire et al. (1994) for the isotropic case using a phenomenological approach is also given.

In Appendix A a constructive proof of the existence of the three permeability tensor components was given applying the concept of *very weak* solutions. Therefore these tensors are explicitly computable from given microscopic configurations employing finite elements or similar numerical procedures. Future research includes the numerical calculation of the three permeability tensors for some specific geometry configurations, to relate them to the coefficients appearing in the phenomenological formulation in Leclaire et al. (1994).

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Appendix

A Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1. To avoid cumbersome notations we restate Problem (2.19) in the following form: find Y-periodic functions $\mathbf{v}$ and p such that

$$i\omega\rho_f\mathbf{v} - \eta\Delta_y\mathbf{v} + \nabla_y p = 0, \quad Y_f, \tag{A.1a}$$

$$\nabla_y \cdot \mathbf{v} = 0, \quad Y_f, \tag{A.1b}$$

$$\mathbf{v} = \mathbf{g}, \quad \Gamma_{sf}, \tag{A.1c}$$

where $\mathbf{g} \in \left[L^2(\Gamma_{sf})\right]^3$ is Y-periodic and satisfies the consistency condition

$$\int_{\Gamma_{sf}} \mathbf{g} \cdot \boldsymbol{\nu}_f \mathrm{d}y = 0. \tag{A.2}$$

For the proof of Theorem 2.1, we prove first three auxiliary lemmas and then employ a compactness argument to show the existence and uniqueness of Problem (A.1).

As in Marusic-Paloka (2000), for $m \in \mathbf{Z}^+$, consider the sequence of penalized problems: find Y-periodic functions $\mathbf{v}^m$ and p^m such that

$$i\omega\rho_f\mathbf{v}^m - \eta\Delta_y\mathbf{v}^m + \nabla_y p^m = 0, \quad Y_f, \tag{A.3a}$$

$$\nabla_y \cdot \mathbf{v}^m = 0, \quad Y_f, \tag{A.3b}$$

$$\mathbf{v}^m + \frac{1}{m}\left(\eta\frac{\partial\mathbf{v}^m}{\partial\boldsymbol{\nu}_f} - p^m\boldsymbol{\nu}_f\right) = \mathbf{g}, \quad \Gamma_{sf}. \tag{A.3c}$$

Define the sesquilinear form $A_{\omega,m} : H^1(\mathrm{div}\,0; Y_f) \times H^1(\mathrm{div}\,0; Y_f) \to \mathbb{C}$ by the rule

$$A_{\omega,m}(\mathbf{v}, \boldsymbol{\varphi}) = i\omega\left(\rho_f\mathbf{v}, \boldsymbol{\varphi}\right)_{Y_f} + \left(\eta\nabla_y\mathbf{v}, \nabla_y\boldsymbol{\varphi}\right)_{Y_f} + m\left\langle\mathbf{v}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{sf}},$$
$$\mathbf{v}, \boldsymbol{\varphi} \in H^1(\mathrm{div}\,0; Y_f). \tag{A.4}$$

Then, testing (A.3a) against $\boldsymbol{\varphi} \in H^1(\mathrm{div}\,0; Y_f)$, we obtain the following variational formulation of Problem (A.3): find $\mathbf{v}^m \in H^1(\mathrm{div}\,0; Y_f)$ such that

$$A_{\omega,m}(\mathbf{v}^m, \boldsymbol{\varphi}) = m\left\langle\mathbf{g}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{sf}}, \quad \boldsymbol{\varphi} \in H^1(\mathrm{div}\,0; Y_f). \tag{A.5}$$

Lemma A.1 *For each positive integer m there exists a unique solution $\mathbf{v}^m \in H^1(\mathrm{div}\,0; Y_f)$ of (A.5) such that*

$$\|\mathbf{v}^m\|_{L^2(\Gamma_{sf})} \le \|\mathbf{g}\|_{L^2(\Gamma_{sf})}. \tag{A.6}$$

Proof First note that

$$|A_{\omega,m}(\mathbf{v}^m, \mathbf{v}^m)| \ge \frac{1}{2}\left(\left(\omega\rho_f\mathbf{v}^m, \mathbf{v}^m\right)_{Y_f} + \left(\eta\nabla_y\mathbf{v}^m, \nabla_y\mathbf{v}^m\right)_{Y_f} + m\left\langle\mathbf{v}^m, \mathbf{v}^m\right\rangle_{\Gamma_{sf}}\right)$$
$$\ge C(\omega)\|\mathbf{v}^m\|^2_{1,Y_f} + \frac{m}{2}\|\mathbf{v}^m\|^2_{L^2(\Gamma_{sf})}, \qquad \mathbf{v} \in H^1(\mathrm{div}\,0; Y_f),$$

where $C(\omega) = \frac{1}{2}\min(\rho_f\omega, \eta)$. Thus the Lax–Milgram lemma implies the existence and uniqueness of the solution of (A.5). Next, taking the real part in the equation

$$A_{\omega,m}(\mathbf{v}^m, \mathbf{v}^m) = m\left\langle\mathbf{g}, \mathbf{v}^m\right\rangle_{\Gamma_{sf}}$$

leads to

$$\eta\|\nabla_y \mathbf{v}^m\|_{1,Y_f}^2 + m\|\mathbf{v}^m\|_{L^2(\Gamma_{sf})}^2 \le |A_{\omega,m}(\mathbf{v}^m, \mathbf{v}^m)| = m\left|\langle \mathbf{g}, \mathbf{v}^m\rangle_{\Gamma_{sf}}\right|$$
$$\le m\|\mathbf{g}\|_{L^2(\Gamma_{sf})}\|\mathbf{v}^m\|_{L^2(\Gamma_{sf})}.$$

Thus, (A.6) follows.This completes the proof. □

Next we proceed to get an estimate for $\|\mathbf{v}^m\|_{L^2(Y_f)}$. First we prove an auxiliary result which is an extension of a regularity estimate given in Galdi (1994).

Lemma A.2 *Let $\mathbf{f} \in \left[L^2(Y_f)\right]^3$ be Y-periodic. Then there exists a Y-periodic unique solution $(\mathbf{u}, \pi) \in H^2(Y_f) \times H^1(Y_f)$ such that*

$$i\omega\rho_f \mathbf{u} - \eta\Delta_y \mathbf{u} + \nabla_y \pi = \mathbf{f}, \quad Y_f, \tag{A.7a}$$
$$\nabla_y \cdot \mathbf{u} = 0, \quad Y_f, \tag{A.7b}$$
$$\mathbf{u} = 0, \quad \Gamma_{sf}, \tag{A.7c}$$

satisfying

$$\|\mathbf{u}\|_{H^2(Y_f)} + \|\pi\|_{H^1(Y_f)} \le C_2\|\mathbf{f}\|_{L^2(Y_f)}, \tag{A.8}$$

where C_2 is independent of ω.

Proof Let $\mathbf{F} \in \left[L^2(Y_f)\right]^3$ be Y-periodic and $(\mathbf{U}, P)$ be the Y-periodic solution of

$$-\eta\Delta_y \mathbf{U} + \nabla_y P = \mathbf{F}, \quad Y_f, \tag{A.9a}$$
$$\nabla_y \cdot \mathbf{U} = 0, \quad Y_f, \tag{A.9b}$$
$$\mathbf{U} = 0, \quad \Gamma_{sf}. \tag{A.9c}$$

According to Galdi (1994) (Theorem 6.1, pp. 225), the following regularity estimate holds:

$$\|\mathbf{U}\|_{H^2(Y_f)} + \|P\|_{H^1(Y_f)} \le \widetilde{C}_1\|\mathbf{F}\|_{L^2(Y_f)}. \tag{A.10}$$

Consequently, applying (A.10) in (A.7) we get

$$\|\mathbf{u}\|_{H^2(Y_f)} + \|\pi\|_{H^1(Y_f)} \le \widetilde{C}_1\left(\|\mathbf{f}\|_{L^2(Y_f)} + \omega\|\mathbf{u}\|_{L^2(Y_f)}\right). \tag{A.11}$$

By testing (A.7a) against $\mathbf{u}$ and taking the imaginary part in the resulting equation, it follows that

$$\omega\|\mathbf{u}\|_{L^2(Y_f)} \le \frac{1}{\rho_f}\|\mathbf{f}\|_{L^2(Y_f)}, \tag{A.12}$$

which combined with (A.11) proves the validity of (A.8).

Using the result in Lemma A.2, we now obtain an estimate for $\|\mathbf{v}^m\|_{L^2(Y_f)}$. □

Lemma A.3 *The solution $\mathbf{v}^m$ of (A.5) satisfies the estimate*

$$\|\mathbf{v}^m\|_{L^2(Y_f)} \le C_3\|\mathbf{g}\|_{L^2(\Gamma_{sf})}, \tag{A.13}$$

where $C_3 > 0$ is a constant independent of m and ω.

Proof Consider the auxiliary Y-periodic problem to find $\mathbf{u}$ and π satisfying

$$-i\omega\rho_f\mathbf{u} - \eta\Delta_y\mathbf{u} + \nabla_y\pi = \mathbf{v}^m, \quad Y_f, \tag{A.14a}$$

$$\nabla_y \cdot \mathbf{u} = 0, \quad Y_f, \tag{A.14b}$$

$$\mathbf{u} = 0, \quad \Gamma_{sf}. \tag{A.14c}$$

Take $\boldsymbol{\varphi} = \mathbf{u}$ in (A.5), use integration by parts in the η-term, and apply (A.14) in the resulting equation. After integrating by parts the term $(\mathbf{v}^m, \nabla\pi)$, owing to the fact that $\mathbf{v}^m$ is divergence free, we obtain

$$\left(\mathbf{v}^m, \mathbf{v}^m\right)_{Y_f} + \left\langle \mathbf{v}^m, \left(\eta\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} - \boldsymbol{\nu}\pi\right)\right\rangle_{\Gamma_{sf}} = 0. \tag{A.15}$$

Next, recall the continuity of the trace operators

$$\begin{array}{lll} H^2(Y_f) \longrightarrow L^2(\Gamma_{sf}) & & H^1(Y_f) \longrightarrow L^2(\Gamma_{sf}) \\ \mathbf{u} \qquad\quad \longmapsto \frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} & \text{and} & \mathbf{u} \qquad\quad \longmapsto \mathbf{u}_{|\Gamma_{sf}} \end{array}. \tag{A.16}$$

Then from (A.6) and (A.8) it follows that

$$\begin{aligned} \left|\left\langle \mathbf{v}^m, \left(\eta\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} - \boldsymbol{\nu}\pi\right)\right\rangle_{\Gamma_{sf}}\right| &\le \|\mathbf{v}^m\|_{L^2(\Gamma_{sf})} \left\|\left(\eta\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}} - \boldsymbol{\nu}\pi\right)\right\|_{L^2(\Gamma_{sf})} \\ &\le C\|\mathbf{g}\|_{L^2(\Gamma_{sf})}\left(\|\mathbf{u}\|_{H^2(Y_f)} + \|\pi\|_{H^1(Y_f)}\right) \\ &\le C_3\|\mathbf{g}\|_{L^2(\Gamma_{sf})}\|\mathbf{v}^m\|_{L^2(Y_f)}. \end{aligned} \tag{A.17}$$

Next, using (A.17) in (A.15) we get

$$\|\mathbf{v}^m\|^2_{L^2(Y_f)} \le C_3\|\mathbf{g}\|_{L^2(\Gamma_{sf})}\|\mathbf{v}^m\|_{L^2(Y_f)},$$

which shows the validity of (A.13). This completes the proof. □

Now we proceed to derive the desired existence and uniqueness result on the solution of Problem (A.1). Note that the bounds in Lemmas A.1 and A.3 imply that there exists a subsequence of $\mathbf{v}^m$, that we denote again $\mathbf{v}^m$, such that

$$\mathbf{v}^m \rightharpoonup \mathbf{v}^0 \quad \text{weakly in} \quad \left[L^2(Y_f)\right]^3, \tag{A.18a}$$

$$\mathbf{v}^m \rightharpoonup \mathbf{z}^0 \quad \text{weakly in} \quad \left[L^2(\Gamma_{sf})\right]^3. \tag{A.18b}$$

We wish to show that $\mathbf{z}^0 = \mathbf{g}$ on Γ_{sf}. First, since $\nabla\cdot\mathbf{v}^m = 0$ in Y_f, we notice that $\int_{\Gamma_{sf}} \mathbf{z}^0\cdot\boldsymbol{\nu}\,\mathrm{d}S = 0$. Then, take a Y-periodic function $\boldsymbol{\varphi} \in \left[C^2(\overline{Y}_f)\right]^3$ with $\nabla\cdot\boldsymbol{\varphi} = 0$ as a test function in (A.5) and use (A.6) and (A.13) to obtain

$$\begin{aligned} \left|m\left\langle \mathbf{v}^m - \mathbf{g}, \boldsymbol{\varphi}\right\rangle_{\Gamma_{sf}}\right| &= \left|i\omega\left(\rho_f\mathbf{v}^m, \boldsymbol{\varphi}\right) - \left(\eta\mathbf{v}^m, \Delta\boldsymbol{\varphi}\right) + \left\langle \eta\mathbf{v}^m, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\nu}}\right\rangle_{\Gamma_{sf}}\right| \\ &\le C\Bigg(\omega\|\mathbf{v}^m\|_{L^2(Y_f)}\|\boldsymbol{\varphi}\|_{L^2(Y_f)} + \|\mathbf{v}^m\|_{L^2(Y_f)}\|\Delta\boldsymbol{\varphi}\|_{L^2(Y_f)} \\ &\qquad + \|\mathbf{v}^m\|_{L^2(\Gamma_{sf})}\left\|\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{\nu}}\right\|_{L^2(\Gamma_{sf})}\Bigg) \\ &\le C(\boldsymbol{\varphi})\max\{1,\omega\}\|\mathbf{g}\|_{L^2(\Gamma_{sf})} \end{aligned} \tag{A.19}$$

Taking limit as $m \to \infty$ in (A.19), we get that

$$\langle \mathbf{z}^0 - \mathbf{g}, \boldsymbol{\varphi} \rangle_{\Gamma_{sf}} = 0,$$

for all such test functions $\boldsymbol{\varphi}$. Since the traces of functions $\boldsymbol{\varphi} \in \left[C^2(\overline{Y}_f)\right]^3$ are dense in $L^2(\Gamma_{sf})$, we conclude that

$$\mathbf{z}^0 = \mathbf{g}, \quad \text{a.e. on } \Gamma_{sf}.$$

Again, take $\boldsymbol{\varphi} \in \mathcal{V}^2_{Y_f}$ in (A.5) and use integration by parts in the η-term to obtain

$$i\omega \left(\rho_f \mathbf{v}^m, \boldsymbol{\varphi}\right)_{Y_f} - \left(\eta \mathbf{v}^m, \Delta_y \boldsymbol{\varphi}\right)_{Y_f} + \left\langle \eta \mathbf{v}^m, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}} \right\rangle_{\Gamma_{sf}} = 0. \tag{A.20}$$

Next, using (A.18), take limit when $m \to \infty$ in (A.20) to obtain

$$i\omega \left(\rho_f \mathbf{v}^0, \boldsymbol{\varphi}\right)_{Y_f} - \left(\eta \mathbf{v}^0, \Delta_y \boldsymbol{\varphi}\right)_{Y_f} = -\left\langle \eta \mathbf{g}, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\nu}} \right\rangle_{\Gamma_{sf}}, \quad \boldsymbol{\varphi} \in \mathcal{V}^2_{Y_f}.$$

Also, note that for $\boldsymbol{\psi} \in \mathcal{W}$, since $\nabla \cdot \mathbf{v}^m = 0$,

$$\left(\mathbf{v}^m, \nabla \boldsymbol{\psi}\right)_{Y_f} = \left\langle \mathbf{v}^m \cdot \boldsymbol{\nu}, \boldsymbol{\psi} \right\rangle_{\Gamma_{sf}}. \tag{A.21}$$

Due to $\nabla \boldsymbol{\psi} \in \left[L^2(Y_f)\right]^3$ and (A.18a), by taking limit in (A.21) as $m \to \infty$, we get

$$\left(\mathbf{v}^0, \nabla \boldsymbol{\psi}\right)_{Y_f} = \langle \mathbf{g} \cdot \boldsymbol{\nu}, \boldsymbol{\psi} \rangle_{\Gamma_{sf}}, \quad \boldsymbol{\psi} \in \mathcal{W}.$$

Thus $\mathbf{v}^0$ is a *very weak* solution of (A.1) in the sense defined in (2.27). This completes the proof of Theorem 2.1.

References

Auriault, J.L., Borne, L., Chambon, R.: Dynamics of porous saturated media, checking of the generalized law of Darcy. J. Acoust. Soc. Am. **77**, 1641–1650 (1985)

Bensoussan, A., Lions, J.L., Papanicolaou, A.G.: Asymptotic Analysis for Periodic Structures, Volume 5 of Studies in Mathematics and its Applications. North-Holland, Amsterdam (1978)

Biot, M.A.: Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low frequency range. J. Acoust. Soc. Am. **28**, 168–178 (1956a)

Biot, M.A.: Theory of propagation of elastic waves in a fluid-saturated porous solid. II. High frequency range. J. Acoust. Soc. Am. **28**, 179–191 (1956b)

Biot, M.A.: Mechanics of deformation and acoustic propagation in porous media. J. Appl. Phys. **33**, 1482–1498 (1962)

Brenner, S., Sung, L.: Linear finite element methods for planar elasticity. Math. Comp. **59**, 321–338 (1992)

Burridge, R., Keller, J.B.: Poroelasticity equations derived from microstructure. J. Acoust. Soc. Am. **70**(4), 1140–1146 (1981)

Carcione, J.M., Tinivella, U.: Bottom-simulating reflectors: seismic velocities and AVO effects. Geophysics **65**(1), 54–67 (2000)

Carcione, J.M., Gurevich, B., Cavallini, F.: A generalized Biot–Gassmann model for the acoustic properties of shaley sandstones. Geophys. Prospect. **48**, 539–557 (2000)

Conca, C.: Etude d'un fluide traversant une paroi perforée, I, II, III. J. Math. Pures Appl. **66**, 1–69 (1987)

Duvaut, G., Lions, J.L.: Inequalities in Mechanics and Physics. Springer, Berlin (1976)

Galdi, G.P.: An Introduction to the Mathematical Theory of the Navier–Stokes Equations, I, II. Springer-Verlag, Berlin (1994)

Leclaire, P., Cohen-Tenoudji, F., Aguirre Puente, J.: Extension of Biot's theory of wave propagation to frozen porous media. J. Acoust. Soc. Am. **96**(6), 3753–3767 (1994)

Leclaire, P., Cohen-Tenoudji, F., Aguirre Puente, J.: Observation of two longitudinal and two transverse waves in a frozen porous medium. J. Acoust. Soc. Am. **97**, 2052–2055 (1995)

Levy, T.: Propagation of waves in a fluid-saturated porous elastic solid. Int. J. Eng. Sci. **17**, 1005–1014 (1979)

Marusic-Paloka, E.: Solvability of the Navier–Stokes system with L^2 boundary data. Appl. Math. Optim. **41**, 365–375 (2000)

McCoy, J.J.: Conditionally averaged response formulation for two-phase random mixtures. J. Appl. Mech. **58**, 973–981 (1991)

Nitsche, J.A.: On Korn's second inequality. RAIRO Anal. Numer. **15**, 237–248 (1981)

Rubino, J.G., Ravazzoli, C.L., Santos, J.E.: Reflection and transmission of waves in composite porous media: a quantification of slow waves conversions. J. Acoust. Soc. Am. **120**(5), 2425–2436 (2006a)

Rubino, J.G., Ravazzoli, C.L., Santos, J.E.: The role of Biot slow waves in heterogeneous gas hydrates bearing sediments. J. Geophys. Res. Solid Earth (2006b) (submitted)

Sanchez-Palencia, E.: Non-Homogeneous Media and Vibration Theory. Lecture Notes in Physics, vol. 127. Springer-Verlag, Berlin (1980)

Sanchez-Palencia, E.: Homogenization Techniques for Composite Media. Lecture Notes in Physics, vol. 272. Springer-Verlag, New York (1987)

Santos, J.E., Ravazzoli, C.L., Carcione, J.M.: A model for wave propagation in a composite solid matrix saturated by a single-phase fluid. J. Acoust. Soc. Am. **115**(6), 2749–2760 (2004)

Santos, J.E., Ravazzoli, C.L., Geiser, J.: On the static and dynamic behavior of fluid saturated composite porous solids; a homogenization approach. Int. J. Solids Struct. **43**, 1224–1238 (2005)

Temam, R.: Navier–Stokes Equations: Theory and Numerical Analysis, Volume 2 (revised edition) of Studies in Mathematics and its Applications. North Holland, Amsterdam (1984)