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Finite element analysis of the vibration problem of a plate coupled with a fluid

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Summary. We consider the approximation of the vibration modes of an elastic plate in contact with a compressible fluid. The plate is modelled by Reissner-Mindlin equations while the fluid is described in terms of displacement variables. This formulation leads to a symmetric eigenvalue problem. Reissner-Mindlin equations are discretized by a mixed method, the equations for the fluid with Raviart-Thomas elements and a non conforming coupling is used on the interface. In order to prove that the method is locking free we consider a family of problems, one for each thickness $t > 0$, and introduce appropriate scalings for the physical parameters so that these problems attain a limit when $t \to 0$. We prove that spurious eigenvalues do not arise with this discretization and we obtain optimal order error estimates for the eigenvalues and eigenvectors valid uniformly on the thickness parameter t.

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Finally we present numerical results confirming the good performance of the method.

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1. Introduction

The approximation of the vibration modes of an elastic solid interacting with a fluid is an important problem which occurs in many engineering applications. During the last years a large amount of work has been devoted to this subject. A general overview can be found in the monographs by Morand and Ohayon[19] and Conca et al.[9], where numerical methods and further references are also given.

This paper deals with a particular fluid-solid interaction problem: the approximation of the small amplitude vibration modes of an elastic plate in contact with an ideal compressible fluid.

The vibration of a fluid alone is usually treated by choosing the pressure as primary variable. However, for coupled systems, such choice leads to non-symmetric eigenvalue problems (see for instance [24]). To avoid this drawback the fluid has been alternatively described by different variables: velocity potential, yielding a quadratic eigenvalue problem ([13]); both, pressure and displacement potentials, whose discretization leads to a symmetric but non-banded problem ([18]); etc.

On the other hand, the use of displacement variables to describe the fluid gives rise to symmetric banded eigenvalue problems. However, in this case, $\lambda = 0$ turns out to be an eigenvalue with an infinite dimensional eigenspace. Because of this, standard discretizations of this formulation suffer from the presence of non-zero frequency spurious modes with no physical entity ([15]).

An alternative discretization of this formulation has been introduced in [6]. It consists of using Raviart-Thomas elements for the fluid adequately coupled with piecewise linear elements for the structure. For two-dimensional problems non-existence of spurious modes and optimal error estimates have been proved in [4] and [21] by extending the spectral theory for non compact operators in [10] to non conforming methods. These results have been extended to 3D in [5].

The aim of this paper is to carry out a similar analysis for the interaction between a fluid and a particular thin structure: a plate, modelled by *Reissner-Mindlin* equations in order to allow for small as well as moderately large thickness.

Even for the plate alone, standard finite element discretizations of these equations lead to wrong results because of the so called *locking* phenomenon. In order to avoid this drawback, reduced integration or mixed methods are usually applied (see for instance [7]). A locking free method, the lowest order *MITC* one, has been recently analyzed in [11] in the context of vibration problems. Optimal order error estimates independent of the thickness have been established therein.

We consider a discretization of the coupled problem involving these elements for the bending of the plate and lowest order Raviart-Thomas elements for the fluid, coupled in a non conforming way.

To prove that this method is locking free, a family of problems (one for each thickness $t > 0$) attaining a finite limit as t goes to zero should be considered and approximation results valid uniformly on t should be sought. To perform such asymptotic analysis the physical constants of the plate and the fluid have to be adequately scaled. We characterize the spectrum of the continuous coupled problem for any $t > 0$ and introduce appropriate scalings of both densities which allow us to prove that this spectrum converge to that of a Kirchhoff plate in contact with a fluid, as the thickness becomes small.

The arguments in [4,21] could be in principle applied for the theoretical analysis of our finite element method; however they do not seem to provide optimal order error estimates for the approximation of the eigenvalues. Instead, we consider the restriction of the problem to the orthogonal complement of the eigenspace corresponding to $\lambda = 0$. This leads to an equivalent spectral problem for a compact operator. By proceeding in this way the discretization presents another variational crime, but we are able to prove, under mild assumptions, optimal order of convergence for the approximate eigenfunctions and eigenvalues and that non-zero spurious modes do not arise.

Finally, numerical experiments are presented confirming the theoretical results and showing the good performance of the method.

2. Statement of the problem

We consider as a model problem that of determining the natural vibration modes of a coupled system consisting of a compressible fluid contained in a three-dimensional cavity whose walls are all rigid, except for one of them which is an elastic plate.

Let Ω be a polyhedral convex three-dimensional domain which we assume completely filled with an inviscid compressible fluid. Its boundary $\partial\Omega$ is the union of the convex surfaces Γ_0, Γ_1, ... , Γ_J. We assume that Γ_0 is in contact with an elastic plate of thickness t. The remaining surfaces are assumed to be perfectly rigid walls. We denote by n the outer unit normal vector to $\partial\Omega$.

The bending of the plate in contact with the fluid is modelled by means of Reissner-Mindlin equations. We denote by Γ its middle surface and consider coordinates such that the 3D reference domain for the plate is $\Gamma \times \left(-\frac{t}{2}, \frac{t}{2}\right)$.

Throughout this paper we make use of the standard notation for Sobolev spaces $H^k(\Omega)$, $H_0^1(\Gamma)$, $H(\mathrm{div}, \Omega)$, $H_0(\mathrm{rot}, \Gamma)$, etc. and their respective norms (see for instance [14]). We also denote $\mathcal{H} := L^2(\Gamma) \times L^2(\Gamma)^2 \times L^2(\Omega)^3$, $\mathcal{X} := H_0^1(\Gamma) \times H_0^1(\Gamma)^2 \times H(\mathrm{div}, \Omega)$ and $\|\cdot\|$ the product norm of the latter.

Let $(u_1^{\mathrm{P}}, u_2^{\mathrm{P}}, u_3^{\mathrm{P}})$ denote the displacement of a point (x, y, z) of the plate. In the Reissner-Mindlin model the transversal displacement u_3^{P} is assumed to be independent of the z-coordinate:

$$u_3^{\mathrm{P}}(x, y, z) = w(x, y), \tag{2.1}$$

and the "in plane" displacements u_1^{P} and u_2^{P} are given by

$$u_1^{\mathrm{P}}(x, y, z) = -z\beta_1(x, y), \qquad u_2^{\mathrm{P}}(x, y, z) = -z\beta_2(x, y), \tag{2.2}$$

with $\beta := (\beta_1, \beta_2)$ being the rotations of the fibers normal to Γ. For the sake of simplicity we assume that the plate is clamped by its whole boundary; that is, $\beta_1 = \beta_2 = w = 0$ on $\partial\Gamma$.

Under the usual assumptions of this model the dynamic response of the plate to a pressure load q exerted on one of its faces is given by displacements of the form (2.1)-(2.2) with $(w, \beta) \in H_0^1(\Gamma) \times H_0^1(\Gamma)^2$ being such that

$$\begin{aligned} t^3 a(\beta, \eta) + \kappa t \int_\Gamma (\nabla w - \beta) \cdot (\nabla v - \eta) + t \int_\Gamma \rho_{\mathrm{P}} \ddot{w} v \\ + \frac{t^3}{12} \int_\Gamma \rho_{\mathrm{P}} \ddot{\beta} \cdot \eta = \int_\Gamma q v \qquad \forall (v, \eta) \in H_0^1(\Gamma) \times H_0^1(\Gamma)^2 \end{aligned} \tag{2.3}$$

(see for instance [16]). In the previous equation, the double dot means second derivatives with respect to time, ρ_{P} is the density of the plate, $\kappa := \frac{Ek}{2(1+\nu)}$, where E is the Young modulus, ν the Poisson ratio of the plate and k a correction factor which is usually taken as $5/6$ (see [1] for a justification of the use of this coefficient); finally, a is the bilinear form defined on $H_0^1(\Gamma)^2$ by

$$a(\beta, \eta) := \frac{E}{12(1-\nu^2)} \int_\Gamma \left[\sum_{i,j=1}^{2} (1-\nu) \varepsilon_{ij}(\beta) \varepsilon_{ij}(\eta) + \nu \, \mathrm{div}\, \beta \, \mathrm{div}\, \eta \right].$$

On the other hand, the governing equations for the free small amplitude motions of an inviscid compressible fluid contained in Ω are given by

$$p = -\rho_{\mathrm{F}} c^2 \, \mathrm{div}\, u \ \text{ in } \Omega, \tag{2.4}$$

$$\rho_{\mathrm{F}} \ddot{u} = -\nabla p \ \text{ in } \Omega, \tag{2.5}$$

where p is the pressure, u the displacement field, ρ_{F} the density and c the acoustic speed of the fluid. Since the fluid is considered inviscid, only the normal component of the displacement vanishes on the rigid part of the cavity boundary $\Gamma_{\mathrm{R}} := \Gamma_1 \cup \cdots \cup \Gamma_J$:

$$u \cdot n = 0 \quad \text{on } \Gamma_{\mathrm{R}}. \tag{2.6}$$

On the other hand, the normal displacement coincides with the transverse displacement of the plate on Γ_0. Since the latter do not depend on the z-coordinate, it can be considered that the midsurface Γ (instead of Γ_0) is one of the components of $\partial\Omega$ and hence

$$u \cdot n = w \quad \text{on } \Gamma. \tag{2.7}$$

Now, we multiply equation (2.5) by a test displacement field ϕ satisfying (2.6), we integrate by parts and use (2.4) to obtain

$$\int_\Omega \rho_{\mathrm{F}} \ddot{u} \cdot \phi + \int_\Omega \rho_{\mathrm{F}} c^2 \operatorname{div} u \operatorname{div} \phi = -\int_\Gamma p\, \phi \cdot n. \tag{2.8}$$

In our coupled problem, the unique load q exerted on the plate is the pressure p of the fluid. Therefore, by adding (2.8) to (2.3) and choosing test functions (v, η, ϕ) in the space

$$\mathcal{V} := \{(v, \eta, \phi) \in \mathcal{X} : \ \phi \cdot n = 0 \text{ on } \Gamma_{\mathrm{R}} \ \text{ and } \ \phi \cdot n = v \text{ on } \Gamma\},$$

we have that

$$\begin{aligned} t^3 a(\beta, \eta) + \kappa t \int_\Gamma (\nabla w - \beta) \cdot (\nabla v - \eta) + \int_\Omega \rho_{\mathrm{F}} c^2 \operatorname{div} u \operatorname{div} \phi \\ = -t \int_\Gamma \rho_{\mathrm{P}} \ddot{w} v - \frac{t^3}{12} \int_\Gamma \rho_{\mathrm{P}} \ddot{\beta} \cdot \eta - \int_\Omega \rho_{\mathrm{F}} \ddot{u} \cdot \phi. \end{aligned} \tag{2.9}$$

To obtain the free vibration modes of this coupled problem we seek harmonic in time solutions of (2.9). By so doing we obtain the following spectral problem (see for instance [19]):

Find $\lambda \in \mathbb{R}$ *and* $0 \neq (w, \beta, u) \in \mathcal{V}$ *such that*

$$\begin{aligned} t^3 a(\beta, \eta) + \kappa t \int_\Gamma (\nabla w - \beta) \cdot (\nabla v - \eta) + \int_\Omega \rho_{\mathrm{F}} c^2 \operatorname{div} u \operatorname{div} \phi \\ = \lambda \left(t \int_\Gamma \rho_{\mathrm{P}} w v + \frac{t^3}{12} \int_\Gamma \rho_{\mathrm{P}} \beta \cdot \eta + \int_\Omega \rho_{\mathrm{F}} u \cdot \phi \right) \\ \forall (v, \eta, \phi) \in \mathcal{V}, \end{aligned} \tag{2.10}$$

where λ is the square of the angular vibration frequency.

As usual, when a displacement formulation is used for the fluid, $\lambda = 0$ turns out to be an eigenvalue of this problem; its associated eigenspace is in this case

$$\mathcal{K} := \{(0,0,\phi) \in \mathcal{V} : \operatorname{div}\phi = 0 \text{ in } \Omega \text{ and } \phi \cdot n = 0 \text{ on } \partial\Omega\}. \tag{2.11}$$

Because of the symmetry of (2.10), the eigenfunctions corresponding to non-zero eigenvalues belong to the orthogonal complement of $\mathcal{K}$ in $\mathcal{V}$ with respect to the bilinear form in the right hand side of that equation. This orthogonal complement can be readily seen to coincide with

$$\mathcal{G} := \{(v,\eta,\phi) \in \mathcal{V} :\ \phi = \nabla q \text{ for some } q \in H^1(\Omega)\}. \tag{2.12}$$

Note that $\mathcal{K}$ and $\mathcal{G}$ are also orthogonal with respect to the bilinear form in the left hand side. So, the eigenpairs corresponding to non-zero eigenvalues are precisely the solutions of problem (2.10) restricted to $\mathcal{G}$ (i.e., with $\mathcal{V}$ substituted by $\mathcal{G}$) and this will be used below for the theoretical analysis.

We denote

$$\|(v,\eta,\phi)\|_{\bullet} := \left(\|v\|_{1,\Gamma}^2 + \|\eta\|_{1,\Gamma}^2 + \|\operatorname{div}\phi\|_{0,\Omega}^2\right)^{1/2} \tag{2.13}$$

which is a norm on $\mathcal{G}$ equivalent to $\|\cdot\|$. In fact, for $(v,\eta,\phi) \in \mathcal{G}$, $\phi = \nabla q$ with q being a solution of the compatible Neumann problem

$$\begin{aligned} \Delta q &= \operatorname{div}\phi \quad \text{in } \Omega, \\ \frac{\partial q}{\partial n} &= \begin{cases} 0 & \text{on } \Gamma_{\mathrm{R}}, \\ v & \text{on } \Gamma, \end{cases} \end{aligned}$$

and then, because of the standard a priori estimate, $\phi = \nabla q \in H^1(\Omega)^3$ and

$$\|\phi\|_{1,\Omega} \le C\left(\|v\|_{1/2,\Gamma} + \|\operatorname{div}\phi\|_{0,\Omega}\right). \tag{2.14}$$

A fortiori, $\|\phi\|_{0,\Omega} \le C\,(\|v\|_{1,\Gamma} + \|\operatorname{div}\phi\|_{0,\Omega})$, which yields the claimed equivalence.

Since we are interested in considering both, thin as well as moderately thick plates, the method to be used should remain stable as the thickness becomes small. To this goal, in static problems, the loads are typically assumed to depend adequately on the thickness in order to obtain a family of problems with uniformly bounded solutions: volumetric forces are supposed to be proportional to t^3 and surface loads to t^2 (see for instance [7]).

A simple way to do similar assumptions in our case is to consider densities for both, fluid and solid, depending on the thickness of the plate in the following way:

$$\rho_{\mathrm{F}} = \hat{\rho}_{\mathrm{F}} t^3, \qquad \rho_{\mathrm{P}} = \hat{\rho}_{\mathrm{P}} t^2.$$

Under these assumptions, the non-zero eigenvalues in (2.10) and their associated eigenfunctions are the solutions of the following rescaled problem:

Find $\lambda \in \mathrm{I\!R}$ *and* $0 \neq (w, \beta, u) \in \mathcal{G}$ *such that*

$$s_t\Big((w,\beta,u),(v,\eta,\phi)\Big) = \lambda\, r_t\Big((w,\beta,u),(v,\eta,\phi)\Big) \qquad \forall (v,\eta,\phi) \in \mathcal{G}, \tag{2.15}$$

with

$$s_t\Big((w,\beta,u),(v,\eta,\phi)\Big) := a(\beta,\eta) + \frac{\kappa}{t^2}\int_\Gamma (\nabla w - \beta)\cdot(\nabla v - \eta) + \int_\Omega \hat{\rho}_{\mathrm{F}} c^2 \operatorname{div} u \operatorname{div} \phi$$

and

$$r_t\Big((w,\beta,u),(v,\eta,\phi)\Big) := \int_\Gamma \hat{\rho}_{\mathrm{P}} w v + \frac{t^2}{12}\int_\Gamma \hat{\rho}_{\mathrm{P}} \beta\cdot\eta + \int_\Omega \hat{\rho}_{\mathrm{F}} u\cdot\phi.$$

As it is well known, $a(\beta,\eta) + \frac{\kappa}{t^2}\int_\Gamma (\nabla w - \beta)\cdot(\nabla v - \eta)$ is t-uniformly coercive on $H_0^1(\Gamma) \times H_0^1(\Gamma)^2$ (see for instance [7]). As a consequence of this and the equivalence of $\|\cdot\|$ and $\|\cdot\|_\bullet$, $s_t(\cdot,\cdot)$ turns out to be coercive on $\mathcal{G}$ with a coerciveness constant independent of t. So, if we endow $\mathcal{H}$ with the weighted L^2 norm $|\cdot|_t$ induced by r_t, the operator

$$\begin{array}{rccc} T_t : & \mathcal{H} & \longrightarrow & \mathcal{G} \\ & (f,\theta,g) & \longmapsto & (w,\beta,u) \end{array}$$

with $(w,\beta,u) \in \mathcal{G}$ being the solution of

$$s_t\Big((w,\beta,u),(v,\eta,\phi)\Big) = r_t\Big((f,\theta,g),(v,\eta,\phi)\Big) \qquad \forall (v,\eta,\phi) \in \mathcal{G}, \tag{2.16}$$

turns out to be uniformly bounded on t.

Because of (2.14), $\mathcal{G}$ is compactly included in $\mathcal{H}$ and, therefore, the operator $T_t : \mathcal{H} \longrightarrow \mathcal{H}$ is compact. Since s_t and r_t are symmetric and semipositive definite, the spectrum of T_t, apart from $\mu = 0$, consists of a sequence of positive finite multiplicity eigenvalues converging to zero. Clearly these eigenvalues are given by $\mu = \frac{1}{\lambda}$, for λ any eigenvalue of (2.15), with the same multiplicities and eigenfunctions.

3. A priori estimates and the limit problem

The purpose of this section is two-fold: on one hand, we will show that the operators T_t converge to a limit T_0 in norm and, on the other, we will prove that T_t are regularizing operators (even for $t = 0$).

By introducing the shear strain $\gamma := \frac{\kappa}{t^2}(\nabla w - \beta) \in H_0(\mathrm{rot}, \Gamma)$, for any $t > 0$, problem (2.16) can be rewritten as:

$$
(3.1)\quad \begin{cases} a(\beta,\eta) + \displaystyle\int_\Gamma \gamma\cdot(\nabla v - \eta) + \int_\Omega \hat\rho_{\mathrm{F}} c^2 \operatorname{div} u \operatorname{div}\phi \\ \quad = \displaystyle\int_\Gamma \hat\rho_{\mathrm{P}} f v + \frac{t^2}{12}\int_\Gamma \hat\rho_{\mathrm{P}}\theta\cdot\eta + \int_\Omega \hat\rho_{\mathrm{F}} g\cdot\phi \qquad \forall (v,\eta,\phi)\in\mathcal{G}, \\ \gamma = \dfrac{\kappa}{t^2}(\nabla w - \beta). \end{cases}
$$

In absence of the fluid, these are the standard Reissner-Mindlin equations whose solutions are known to converge to that of the mixed formulation of Kirchhoff model (see [7]). In our case we will show below that the limit of T_t is

$$
\begin{array}{rccc} T_0: & \mathcal{H} & \longrightarrow & \mathcal{G} \\ & (f,\theta,g) & \longmapsto & (w_0,\beta_0,u_0) \end{array}
$$

with $(w_0,\beta_0,u_0)\in\mathcal{G}$ such that there exists $\gamma_0\in H_0(\mathrm{rot},\Gamma)'$ satisfying

$$
(3.2)\quad \begin{cases} a(\beta_0,\eta) + \langle\gamma_0, \nabla v - \eta\rangle + \displaystyle\int_\Omega \hat\rho_{\mathrm{F}} c^2 \operatorname{div} u_0 \operatorname{div}\phi \\ \qquad\qquad = \displaystyle\int_\Gamma \hat\rho_{\mathrm{P}} f v + \int_\Omega \hat\rho_{\mathrm{F}} g\cdot\phi \qquad \forall (v,\eta,\phi)\in\mathcal{G}, \\ \nabla w_0 - \beta_0 = 0, \end{cases}
$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing in $H_0(\mathrm{rot},\Gamma)$. Notice that, since $\beta_0 = \nabla w_0$, by taking $\eta = \nabla v$ for $v\in H_0^2(\Gamma)$, we obtain the classical variational formulation of Kirchhoff equations coupled with those of the fluid, namely:

$$
\int_\Gamma \frac{E}{12(1-\nu^2)}\,\Delta w_0\,\Delta v + \int_\Omega \hat\rho_{\mathrm{F}} c^2 \operatorname{div} u_0 \operatorname{div}\phi = \int_\Gamma \hat\rho_{\mathrm{P}} f v + \int_\Omega \hat\rho_{\mathrm{F}} g\cdot\phi,
$$

for all $(v,\phi)\in H_0^2(\Gamma)\times H(\mathrm{div},\Omega)$ such that $\phi\cdot n = v$ on Γ and $\phi\cdot n = 0$ on Γ_{R}.

The arguments used for the plate alone (see [7]) can be easily extended to show that problem (3.2) satisfies both classical Brezzi's conditions. This ensures the existence of a unique solution of this problem and its continuous dependence on the data $(f,g)\in L^2(\Gamma)\times L^2(\Omega)^3$. Therefore T_0 is a well defined bounded linear operator.

Now we prove further regularity of the solutions of problems (3.1) and (3.2):

Theorem 3.1 *Let* $(f,\theta,g) \in \mathcal{H}$ *and* $(w,\beta,u) = T_t(f,\theta,g)$. *Let* $\gamma = \frac{\kappa}{t^2}(\nabla w - \beta)$ *for* $t > 0$, *or* $\gamma = \gamma_0$ *(as defined in (3.2)) for* $t = 0$. *Then,* $(w,\beta,u) \in H^2(\Gamma) \times H^2(\Gamma)^2 \times H^1(\operatorname{div},\Omega)$, $\gamma \in L^2(\Gamma)$ *and there holds*

$$\|w\|_{2,\Gamma} + \|\beta\|_{2,\Gamma} + \|u\|_{H^1(\operatorname{div},\Omega)} + \|\gamma\|_{0,\Gamma} \leq C|(f,\theta,g)|_t,$$

with $C > 0$ *independent of* $t \in [0, t_{\max}]$.

Proof. We give the proof for $t > 0$, but the arguments extend trivially to $t = 0$.

Since $(w,\beta,u) \in \mathcal{G}$, from (2.14) we have that $u \in H^1(\Omega)^3$ and

$$\|u\|_{1,\Omega} \leq C\left(\|w\|_{1,\Gamma} + \|\operatorname{div} u\|_{0,\Omega}\right) \leq C|(f,\theta,g)|_t. \tag{3.3}$$

(Here and throughout this section C stands for a positive constant, not necessarily the same at each occurrence but always independent of t.)

By taking $(0,0,\nabla\xi)$ as a test function in (2.16), with $\nabla\xi$ being the gradient part of a Helmholtz decomposition of any $\phi \in \mathcal{D}(\Omega)^3$, it is simple to show that

$$-\hat{\rho}_{\mathrm{F}} c^2 \nabla \operatorname{div} u = \hat{\rho}_{\mathrm{F}} \nabla q \qquad \text{in } \Omega,$$

where ∇q is the gradient part of the analogous Helmholtz decomposition of g. Hence $\operatorname{div} u \in H^1(\Omega)$ with

$$\|\operatorname{div} u\|_{1,\Omega} \leq C|(f,\theta,g)|_t. \tag{3.4}$$

Now, given $(v,\eta,\phi) \in \mathcal{G}$, by integrating by parts in (2.16) we obtain

$$\begin{aligned} a(\beta,\eta) + \frac{\kappa}{t^2}\int_\Gamma (\nabla w - \beta)\cdot(\nabla v - \eta) &= \int_\Gamma (\hat{\rho}_{\mathrm{P}} f - \hat{\rho}_{\mathrm{F}} c^2 \operatorname{div} u) v \\ &+ \frac{t^2}{12}\int_\Gamma \hat{\rho}_{\mathrm{P}} \theta\cdot\eta \qquad \forall (v,\eta) \in H_0^1(\Gamma) \times H_0^1(\Gamma)^2. \end{aligned}$$

This is the classical Reissner-Mindlin equation for which a priori estimates are known (see for instance [2]):

$$\begin{aligned} \|w\|_{2,\Gamma} + \|\beta\|_{2,\Gamma} + \|\gamma\|_{0,\Gamma} + t\,\|\gamma\|_{1,\Gamma} &\leq C\left(\|\hat{\rho}_{\mathrm{P}} f - \hat{\rho}_{\mathrm{F}} c^2 \operatorname{div} u\|_{0,\Gamma} + t^2\|\hat{\rho}_{\mathrm{P}}\theta\|_{0,\Gamma}\right) \\ &\leq C\,|(f,\theta,g)|_t. \end{aligned} \tag{3.5}$$

This, together with (3.3) and (3.4), allow us to conclude the proof. □

Now we are able to make precise in what sense T_t converge to T_0 as t goes to zero.

Theorem 3.2 *There exists a constant* C, *independent of* t, *such that, for all* $(f,\theta,g) \in \mathcal{H}$,

$$\|(T_t - T_0)(f,\theta,g)\| \leq Ct\,|(f,\theta,g)|_t.$$

Proof. Let $(w, \beta, u) := T_t(f, \theta, g)$ and $(w_0, \beta_0, u_0) := T_0(f, \theta, g)$. Subtracting (3.2) from (3.1) we have

$$\begin{cases} a(\beta - \beta_0, \eta) + \displaystyle\int_\Gamma (\gamma - \gamma_0) \cdot (\nabla v - \eta) + \int_\Omega \hat{\rho}_{\mathrm{F}} c^2 \operatorname{div}(u - u_0) \operatorname{div} \phi \\ \quad = \dfrac{t^2}{12} \displaystyle\int_\Gamma \hat{\rho}_{\mathrm{P}} \theta \cdot \eta \qquad \forall (v, \eta, \phi) \in \mathcal{G}, \\ \gamma = \dfrac{\kappa}{t^2} \Big[\nabla (w - w_0) - (\beta - \beta_0) \Big]. \end{cases}$$

(Notice that $\int_\Gamma (\gamma - \gamma_0) \cdot (\nabla v - \eta)$ makes sense, since (3.5) implies that $\gamma_0 \in L^2(\Gamma)^2$.) By taking $v = w - w_0,\ \eta = \beta - \beta_0$ and $\phi = u - u_0$ we obtain

$$\begin{aligned} a(\beta - \beta_0, \beta - \beta_0) &+ \int_\Omega \hat{\rho}_{\mathrm{F}} c^2 [\operatorname{div}(u - u_0)]^2 \\ &= \frac{t^2}{12} \int_\Gamma \hat{\rho}_{\mathrm{P}} \theta \cdot (\beta - \beta_0) - \frac{t^2}{\kappa} \int_\Gamma (\gamma - \gamma_0) \cdot \gamma. \end{aligned}$$

Hence, from the coerciveness of a and the a priori estimate (3.5) for $\|\gamma\|_{0,\Gamma}$ and $\|\gamma_0\|_{0,\Gamma}$, we have

$$\begin{aligned} \|\beta - \beta_0\|_{1,\Gamma}^2 &+ \|\operatorname{div}(u - u_0)\|_{0,\Omega}^2 \\ &\le C t^2 \|\theta\|_{0,\Gamma} \|\beta - \beta_0\|_{0,\Gamma} + C t^2 \left(\|\gamma\|_{0,\Gamma} + \|\gamma_0\|_{0,\Gamma} \right) \|\gamma\|_{0,\Gamma} \\ &\le C t \, |(f, \theta, g)|_t \, \|\beta - \beta_0\|_{0,\Gamma} + C t^2 |(f, \theta, g)|_t^2 \end{aligned}$$

and therefore

$$\|\beta - \beta_0\|_{1,\Gamma} + \|\operatorname{div}(u - u_0)\|_{0,\Omega} \le C t \, |(f, \theta, g)|_t. \tag{3.6}$$

Finally observe that

$$\nabla (w - w_0) = (\beta - \beta_0) + \frac{t^2}{\kappa} \gamma$$

and so, using again the a priori estimate (3.5) for $\|\gamma\|_{0,\Gamma}$, we obtain

$$\|w - w_0\|_{1,\Gamma} \le C \left[\|\beta - \beta_0\|_{0,\Gamma} + t^2 |(f, \theta, g)|_t \right],$$

which, together with (3.6) and the equivalence between $\|\cdot\|_\bullet$ and $\|\cdot\|$ on $\mathcal{G}$, allow us to conclude the proof. $\square$

4. Discretization

Let $\{\mathcal{T}_h\}$ be a regular family of partitions of Ω in tetrahedra; h stands for the maximum diameter of the elements. Each $\mathcal{T}_h$, induces a triangulation on Γ:

$$\mathcal{T}_h^\Gamma := \{T \subset \Gamma : \ T \text{ is a face of a tetrahedron } K \in \mathcal{T}_h\}.$$

To approximate the fluid displacements, we use lowest order Raviart-Thomas elements (see [20]):

$$R_h := \{\phi_h \in H(\mathrm{div}, \Omega) : \ \phi_h|_K \in \mathcal{P}_0^3 \oplus (x, y, z)\mathcal{P}_0 \ \ \forall K \in \mathcal{T}_h\}. \tag{4.1}$$

For the plate we consider a method analyzed in [12] and [8]. It is based on different finite element spaces for the rotations, the transverse displacement and the shear strain. For the former we take piecewise linear functions augmented in such a way that they have quadratic tangential components on the boundary of each element. Namely, for each $T \in \mathcal{T}_h^\Gamma$, let n be a unit normal on ∂T and define

$$\mathcal{Q}(T) := \{\eta \in \mathcal{P}_2(T)^2 : \ \eta \cdot n|_\ell \in \mathcal{P}_1(\ell) \ \text{ for each edge } \ell \text{ of } T\};$$

then, the finite element space for the rotations is defined by

$$H_h := \{\eta_h \in H_0^1(\Gamma)^2 : \ \eta_h|_T \in \mathcal{Q}(T) \ \ \forall T \in \mathcal{T}_h^\Gamma\}.$$

For the transverse displacements we take standard piecewise linear elements, namely,

$$W_h := \{v_h \in H_0^1(\Gamma) : \ v_h|_T \in \mathcal{P}_1(T) \ \ \forall T \in \mathcal{T}_h^\Gamma\}.$$

Finally, to discretize the shear strain we use the lowest order rotated Raviart-Thomas space

$$Z_h := \{\psi_h \in H_0(\mathrm{rot}, \Gamma) : \ \psi_h|_T \in \mathcal{P}_0^2 \oplus (-y, x)\mathcal{P}_0 \ \ \forall T \in \mathcal{T}_h^\Gamma\}$$

and the reduction operator

$$\Pi : H^1(\Gamma)^2 \cap H_0(\mathrm{rot}, \Gamma) \longrightarrow Z_h,$$

locally defined for each $\psi \in H^1(\Gamma)^2$ by (see [7,20])

$$\int_\ell \Pi\psi \cdot \tau = \int_\ell \psi \cdot \tau, \tag{4.2}$$

for every edge ℓ of the triangulation (τ being a unit tangent vector along ℓ). It can be shown that this operator satisfies ([7,20])

$$\|\psi - \Pi\psi\|_{0,\Gamma} \leq Ch\|\psi\|_{1,\Gamma}. \tag{4.3}$$

We impose weakly the interface condition (2.7); in fact, we take as discrete space for the coupled problem

$$\mathcal{V}_h := \{(v_h, \eta_h, \phi_h) \in W_h \times H_h \times R_h : \\ \phi_h \cdot n = 0 \text{ on } \Gamma_{\mathrm{R}} \text{ and } \int_T \phi_h \cdot n = \int_T v_h \; \forall T \in \mathcal{T}_h^{\Gamma}\}.$$

Note that, for elements in $\mathcal{V}_h$, the equality $\phi_h \cdot n = v_h$ must hold only at the baricenter of the triangles in $\mathcal{T}_h^{\Gamma}$. So $\mathcal{V}_h \not\subset \mathcal{V}$, giving rise to a variational crime for our method. Let us remark that if the interface condition were imposed strongly (namely, $\phi_h \cdot n = v_h$ on Γ) it would imply $v_h \equiv 0$.

The discrete eigenvalue problem reads:

Find $\lambda_h \in \mathbb{R}$ *and* $0 \neq (w_h, \beta_h, u_h) \in \mathcal{V}_h$ *such that*

$$(4.4)\quad \begin{cases} a(\beta_h, \eta_h) + \displaystyle\int_\Gamma \gamma_h \cdot (\nabla v_h - \Pi \eta_h) + \int_\Omega \hat{\rho}_{\mathrm{F}} c^2 \operatorname{div} u_h \operatorname{div} \phi_h \\ \quad = \lambda_h \left(\displaystyle\int_\Gamma \hat{\rho}_{\mathrm{P}} w_h v_h + \frac{t^2}{12} \int_\Gamma \hat{\rho}_{\mathrm{P}} \beta_h \cdot \eta_h + \int_\Omega \hat{\rho}_{\mathrm{F}} u_h \cdot \phi_h \right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall (v_h, \eta_h, \phi_h) \in \mathcal{V}_h, \\ \gamma_h = \dfrac{\kappa}{t^2} (\nabla w_h - \Pi \beta_h). \end{cases}$$

Note that the use of the reduction operator Π leads to a second variational crime.

For this problem, $\lambda_h = 0$ turns out to be an eigenvalue with corresponding eigenspace

$$\mathcal{K}_h := \{(0, 0, \phi_h) \in \mathcal{V}_h : \ \operatorname{div} \phi_h = 0 \text{ in } \Omega \text{ and } \phi_h \cdot n = 0 \text{ on } \partial\Omega\}.$$

As in the continuous case, for the theoretical analysis we may restrict the discrete eigenvalue problem to the orthogonal complement of $\mathcal{K}_h$ in $\mathcal{V}_h$ with respect to r_t. We denote it $\mathcal{G}_h$ and write

Find $\lambda_h \in \mathbb{R}$ *and* $0 \neq (w_h, \beta_h, u_h) \in \mathcal{G}_h$ *such that*

$$(4.5)\quad s_{th}\Big((w_h, \beta_h, u_h), (v_h, \eta_h, \phi_h)\Big) \\ = \lambda_h r_t\Big((w_h, \beta_h, u_h), (v_h, \eta_h, \phi_h)\Big) \qquad \forall (v_h, \eta_h, \phi_h) \in \mathcal{G}_h,$$

with

$$s_{th}\Big((w_h, \beta_h, u_h), (v_h, \eta_h, \phi_h)\Big) := a(\beta_h, \eta_h) \\ + \frac{\kappa}{t^2} \int_\Gamma (\nabla w_h - \Pi \beta_h) \cdot (\nabla v_h - \Pi \eta_h) + \int_\Omega \hat{\rho}_{\mathrm{F}} c^2 \operatorname{div} u_h \operatorname{div} \phi_h.$$

Let us remark that, although $\mathcal{V}_h \not\subset \mathcal{V}$, $\mathcal{K}_h \subset \mathcal{K} \subset \mathcal{V}$. In contrast, $\mathcal{G}_h \not\subset \mathcal{G}$; in fact, for $(v_h, \eta_h, \phi_h) \in \mathcal{G}_h$, ϕ_h is not necessarily a gradient and this yields another variational crime.

The bilinear form s_{th} turns out to be coercive on $\mathcal{G}_h$ with a coerciveness constant independent of t and h. In fact, it is known that $a(\beta_h, \eta_h) + \frac{\kappa}{t^2}\int_\Gamma(\nabla w_h - \Pi\beta_h)\cdot(\nabla v_h - \Pi\eta_h)$ is uniformly coercive on $H_h \times W_h$ (see [12]), whereas $\|\cdot\|_{\bullet}$ and $\|\cdot\|$ are equivalent on $\mathcal{G}_h$ (with constants not depending on h), as follows from the lemma below which provides a Helmholtz decomposition for the discrete fluid displacements.

Lemma 4.1 *For any $(v_h, \eta_h, \phi_h) \in \mathcal{G}_h$, ϕ_h can be written as*

$$\phi_h = \nabla\xi + \chi,$$

with ξ and χ satisfying $(v_h, \eta_h, \nabla\xi) \in \mathcal{G}$ and $\operatorname{div}\chi = 0$. Moreover, there exists a constant C, independent of h, such that

$$\|\nabla\xi\|_{1,\Omega} \le C\Big(\|\operatorname{div}\phi_h\|_{0,\Omega} + \|v_h\|_{1,\Gamma}\Big), \tag{4.6}$$

$$\|\chi\|_{0,\Omega} \le Ch\Big(\|\operatorname{div}\phi_h\|_{0,\Omega} + \|v_h\|_{1,\Gamma}\Big). \tag{4.7}$$

Proof. We do not include it here since it is essentially identical to those of decomposition (5.5) in [4] and its corresponding estimates, which are given in the proofs of Theorem 5.4 and Lemma 5.5 of that reference. □

Since s_{th} are uniformly coercive on $\mathcal{G}_h$, if we define

$$\begin{aligned} T_{th}:\quad \mathcal{H} \quad &\longrightarrow \quad \mathcal{G}_h \\ (f,\theta,g) &\longmapsto (w_h,\beta_h,u_h) \end{aligned}$$

with $(w_h, \beta_h, u_h) \in \mathcal{G}_h$ being the solution of

$$\begin{aligned} s_{th}\Big((w_h,\beta_h,u_h),(v_h,\eta_h,\phi_h)\Big) \\ = r_t\Big((f,\theta,g),(v_h,\eta_h,\phi_h)\Big) \qquad \forall (v_h,\eta_h,\phi_h) \in \mathcal{G}_h, \end{aligned} \tag{4.8}$$

operators T_{th} turn out to be uniformly bounded in t and h. Clearly, as in the continuous case, the non-zero eigenvalues of T_{th} are exactly those of the form $\mu_h = \frac{1}{\lambda_h}$, for λ_h the eigenvalues of Problem (4.5).

5. Convergence of the discrete operators

The goal of this section is to show that $\|(T_t - T_{th})(f,\theta,g)\| \le Ch\,|(f,\theta,g)|_t$ (here and thereafter C denotes a strictly positive constant, not necessarily the same at each occurrence, but always independent of $t \in [0, t_{\max}]$ and

h). Such result will allow us to prove in the next section the convergence of the eigenfunctions of the discrete problem to those of the continuous one.

Throughout this section, we consider $(f, \theta, g) \in \mathcal{H}$ fixed and denote

$$\begin{aligned} &(w, \beta, u) := T_t(f, \theta, g), && (w_h, \beta_h, u_h) := T_{th}(f, \theta, g), \\ &\gamma := \frac{\kappa}{t^2}(\nabla w - \beta), && \gamma_h := \frac{\kappa}{t^2}(\nabla w_h - \Pi \beta_h). \end{aligned}$$

As mentioned above, our method involves two kinds of variational crimes: $s_{th} \neq s_t$ (because of the use of the reduction operator Π) and $\mathcal{G}_h \not\subset \mathcal{G}$ (because of the weak imposition of the interface condition and the fact that, for $(v_h, \eta_h, \phi_h) \in \mathcal{G}_h$, ϕ_h is not necessarily a gradient). Therefore, two consistency terms appear in the error equation; in fact, from (4.8) and the definition of s_t we have

$$\begin{aligned} a(\beta - \beta_h, \eta_h) &+ \int_\Gamma (\gamma - \gamma_h) \cdot (\nabla v_h - \Pi \eta_h) \\ &+ \int_\Omega \hat{\rho}_{\mathrm{F}} c^2 \operatorname{div}(u - u_h) \operatorname{div} \phi_h = \int_\Gamma \gamma \cdot (\eta_h - \Pi \eta_h) \\ &+ M_h(v_h, \eta_h, \phi_h) \qquad \forall (v_h, \eta_h, \phi_h) \in \mathcal{G}_h, \end{aligned} \tag{5.1}$$

where

$$M_h(v_h, \eta_h, \phi_h) := s_t\Big((w, \beta, u), (v_h, \eta_h, \phi_h)\Big) - r_t\Big((f, \theta, g), (v_h, \eta_h, \phi_h)\Big).$$

The first consistency term $\int_\Gamma \gamma \cdot (\eta_h - \Pi \eta_h)$ will be easily bounded by using (4.3). For the second one we have the following estimate:

Lemma 5.1 *There holds*

$$|M_h(v_h, \eta_h, \phi_h)| \leq Ch \, \|g\|_{0,\Omega} \|(v_h, \eta_h, \phi_h)\|_\bullet \quad \forall (v_h, \eta_h, \phi_h) \in \mathcal{G}_h.$$

Proof. Given $(v_h, \eta_h, \phi_h) \in \mathcal{G}_h$, consider the Helmholtz decomposition in Lemma 4.1:

$$\phi_h = \nabla \xi + \chi,$$

with $(v_h, \eta_h, \nabla \xi) \in \mathcal{G}$ and $\operatorname{div} \chi = 0$. Hence $M_h(v_h, \eta_h, \nabla \xi)$ vanishes and

$$M_h(v_h, \eta_h, \phi_h) = M_h(0, 0, \chi) = -\int_\Omega \hat{\rho}_{\mathrm{F}} g \chi. \tag{5.2}$$

This, together with (4.7), allow us to conclude the proof. $\square$

The following lemma shows that the spaces $\mathcal{G}_h$ provide suitable approximations for (w, β, u):

Lemma 5.2 *There exists* $(\hat{w}, \hat{\beta}, \hat{u}) \in \mathcal{G}_h$ *such that*

$$\|(\hat{w}, \hat{\beta}, \hat{u}) - (w, \beta, u)\| \leq Ch\, |(f, \theta, g)|_t.$$

Moreover, if $\hat{\gamma} := \frac{\kappa}{t^2}(\nabla \hat{w} - \Pi \hat{\beta})$*, there also holds*

$$t\, \|\hat{\gamma} - \gamma\|_{0,\Gamma} \leq Ch\, |(f, \theta, g)|_t.$$

Proof. Let $(\hat{w}, \hat{\beta}) \in W_h \times H_h$ be such that $\|\hat{\beta} - \beta\|_{1,\Gamma} \leq Ch\, \|\beta\|_{2,\Gamma}$, $\|\hat{w} - w\|_{1,\Gamma} \leq Ch\, \|w\|_{2,\Gamma}$, and $t\, \|\hat{\gamma} - \gamma\|_{0,\Gamma} \leq Ch\, (t\, \|\gamma\|_{1,\Gamma} + \|\gamma\|_{0,\Gamma})$ (such $(\hat{w}, \hat{\beta})$ has been proved to exist in example 4.1 of [12], with $\hat{w}$ being the Lagrange interpolant of w).

Arguing as in Theorem 5.2 of [4] we can find $u^{\mathrm{I}} \in R_h$ such that $(\hat{w}, \hat{\beta}, u^{\mathrm{I}}) \in V_h$ and $\|u^{\mathrm{I}} - u\|_{H(\mathrm{div},\Omega)} \leq Ch\, \left[\|w\|_{2,\Gamma} + \|u\|_{H^1(\mathrm{div},\Omega)}\right]$.

Now, let $(0, 0, u_{\mathcal{K}_h})$ be the r_t projection of $(\hat{w}, \hat{\beta}, u^{\mathrm{I}})$ onto $\mathcal{K}_h$. Hence, for $\hat{u} := u^{\mathrm{I}} - u_{\mathcal{K}_h}$, $(\hat{w}, \hat{\beta}, \hat{u}) \in \mathcal{G}_h$. Moreover, since $u_{\mathcal{K}_h}$ and $(\hat{u} - u)$ are orthogonal in $H(\mathrm{div}, \Omega)$, we have

$$\|\hat{u} - u\|_{H(\mathrm{div},\Omega)} \leq \|(\hat{u} - u) + u_{\mathcal{K}_h}\|_{H(\mathrm{div},\Omega)} = \|u^{\mathrm{I}} - u\|_{H(\mathrm{div},\Omega)}.$$

Therefore, by applying the a priori estimate in Theorem 3.1 we conclude the proof. □

In order to attain the goal of this section, we first prove convergence for the discrete operators in $\|\cdot\|_{\bullet}$. As a by product we obtain convergence of the shear strains, which will be used in the next section.

Lemma 5.3 *There holds*

$$\|w - w_h\|_{1,\Gamma} + \|\beta - \beta_h\|_{1,\Gamma} + t\, \|\gamma - \gamma_h\|_{0,\Gamma} + \|\operatorname{div}(u - u_h)\|_{0,\Omega} \leq Ch\, |(f, \theta, g)|_t.$$

Proof. Let $(\hat{w}, \hat{\beta}, \hat{u})$ and $\hat{\gamma}$ be as in Lemma 5.2. From the error equation (5.1), by proceeding as in the proof of Lemma 3.1 of [12], we obtain

$$\begin{aligned}
&\|\hat{\beta} - \beta_h\|_{1,\Gamma}^2 + t\, \|\hat{\gamma} - \gamma_h\|_{0,\Gamma}^2 + \|\operatorname{div}(\hat{u} - u_h)\|_{0,\Omega}^2 \\
&\quad \leq C \Big[\|\hat{\beta} - \beta\|_{1,\Gamma}^2 + t\, \|\hat{\gamma} - \gamma\|_{0,\Gamma}^2 + \|\operatorname{div}(\hat{u} - u)\|_{0,\Omega}^2 \\
&\qquad + \|\gamma\|_{0,\Gamma} \|(\hat{\beta} - \beta_h) - \Pi(\hat{\beta} - \beta_h)\|_{0,\Gamma} \\
&\qquad + |M_h(\hat{w} - w_h, \hat{\beta} - \beta_h, \hat{u} - u_h)|\Big] \\
&\quad \leq C \Big[\|\hat{\beta} - \beta\|_{1,\Gamma}^2 + t\, \|\hat{\gamma} - \gamma\|_{0,\Gamma}^2 + \|\operatorname{div}(\hat{u} - u)\|_{0,\Omega}^2\Big] \\
&\qquad + Ch \Big[\|\gamma\|_{0,\Gamma} \|(\hat{\beta} - \beta_h)\|_{1,\Gamma} \\
&\qquad + \|g\|_{0,\Omega} \|(\hat{w} - w_h, \hat{\beta} - \beta_h, \hat{u} - u_h)\|_{\bullet}\Big],
\end{aligned}$$

where we have used (4.3) and Lemma 5.1 for the last inequality.

Now, using that $\nabla(\hat{w}-w_h) = \frac{t^2}{\kappa}(\hat{\gamma}-\gamma_h) - \Pi(\hat{\beta}-\beta_h)$ and (4.3), simple algebra yields

$$\begin{aligned}
&\|\hat{\beta}-\beta_h\|_{1,\Gamma} + t\,\|\hat{\gamma}-\gamma_h\|_{0,\Gamma} + \|\operatorname{div}(\hat{u}-u_h)\|_{0,\Omega} \\
&\quad\le C\left[\|\hat{\beta}-\beta\|_{1,\Gamma} + t\,\|\hat{\gamma}-\gamma\|_{0,\Gamma} + \|\operatorname{div}(\hat{u}-u)\|_{0,\Omega}\right] \\
&\qquad + Ch\Big(\|\gamma\|_{0,\Gamma} + \|g\|_{0,\Omega}\Big).
\end{aligned}$$

So, triangle inequality leads to

$$\begin{aligned}
&\|\beta-\beta_h\|_{1,\Gamma} + t\,\|\gamma-\gamma_h\|_{0,\Gamma} + \|\operatorname{div}(u-u_h)\|_{0,\Omega} \\
&\quad\le C\left[\|\hat{\beta}-\beta\|_{1,\Gamma} + t\,\|\hat{\gamma}-\gamma\|_{0,\Gamma} + \|\operatorname{div}(\hat{u}-u)\|_{0,\Omega}\right] \\
&\qquad + Ch\Big(\|\gamma\|_{0,\Gamma} + \|g\|_{0,\Omega}\Big).
\end{aligned} \tag{5.3}$$

Furthermore, by using this and (4.3) in $\nabla(w-w_h) = \frac{t^2}{\kappa}(\gamma-\gamma_h) + (\beta-\Pi\beta) + \Pi(\beta-\beta_h)$, we obtain

$$\begin{aligned}
\|w-w_h\|_{1,\Gamma} \le C\Big[&\|\hat{\beta}-\beta\|_{1,\Gamma} + t\,\|\hat{\gamma}-\gamma\|_{0,\Gamma} \\
&+ \|\operatorname{div}(\hat{u}-u)\|_{0,\Omega}\Big] + Ch\Big(\|\beta\|_{1,\Gamma} + \|\gamma\|_{0,\Gamma} + \|g\|_{0,\Omega}\Big).
\end{aligned} \tag{5.4}$$

Finally (5.3), (5.4), Lemma 5.2 and Theorem 3.1 allow us to conclude the proof. □

Now, it only remains to estimate the L^2 norm of $(u-u_h)$:

Lemma 5.4 *There holds*

$$\|u-u_h\|_{0,\Omega} \le Ch\,|(f,\theta,g)|_t.$$

Proof. Let $q \in H^1(\Omega)$ such that $u = \nabla q$ and let $u_h = \nabla\xi + \chi$ as in Lemma 4.1. Then, $u - u_h = \nabla(q-\xi) - \chi$, with $\|\chi\|_{0,\Omega} \le Ch\,|(f,\theta,g)|_t$ because of (4.7) and the uniform boundedness of T_{th}. On the other hand,

$$\begin{aligned}
\Delta(q-\xi) &= \operatorname{div}(u-u_h) \quad \text{in } \Omega, \\
\frac{\partial}{\partial n}(q-\xi) &= \begin{cases} 0 & \text{on } \Gamma_{\mathrm{R}}, \\ w-w_h & \text{on } \Gamma. \end{cases}
\end{aligned}$$

So, we have $\|\nabla(q-\xi)\|_{0,\Omega} \le C\Big[\|\operatorname{div}(u-u_h)\|_{0,\Omega} + \|w-w_h\|_{1/2,\Gamma}\Big] \le Ch\,|(f,\theta,g)|_t$, the latter inequality because of Lemma 5.3. □

Summing up, we may prove the claimed convergence:

Theorem 5.1 *There exists a constant C such that, for any $(f, \theta, g) \in \mathcal{H}$, there holds*

$$\|(T_t - T_{th})(f, \theta, g)\| \leq Ch\,|(f, \theta, g)|_t. \tag{5.5}$$

Proof. The theorem is an immediate consequence of Lemmas 5.3 and 5.4. □

6. Spectral approximation

In this section we show that the eigenpairs of the discrete operators T_{th} provide optimal order approximations for those of the continuous one T_t. Under mild assumptions, the obtained estimates are shown to be independent of the thickness of the plate.

For $t > 0$ fixed, as a consequence of Theorem 5.1, if μ_t is an eigenvalue of T_t with multiplicity m, then exactly m eigenvalues $\mu_{th}^{(1)}, \ldots, \mu_{th}^{(m)}$ of T_{th} (repeated according to their respective multiplicities) converge to μ_t as h goes to zero (see [17]). The spectral theory for compact operators in [3] can be directly applied to obtain error estimates.

However, further considerations are needed to show that they do not deteriorate as t becomes small. According to Theorem 3.2, T_0 is the limit in norm of the compact operators T_t. Hence we have the following result:

Theorem 6.1 *Let $\mu_0 > 0$ be an eigenvalue of T_0 of multiplicity m. Let D be any disc in the complex plane centered at μ_0 and containing no other element of the spectrum of T_0. Then, for t small enough, D contains exactly m eigenvalues of T_t (repeated according to their respective multiplicities). Consequently, each eigenvalue $\mu_0 > 0$ of T_0 is a limit of eigenvalues μ_t of T_t, as t goes to zero.*

Proof. It is a consequence of standard properties of separation of isolated parts of the spectra (see for instance [17]). □

For the sake of simplicity we state our results for eigenvalues of T_t converging to a simple eigenvalue of T_0 (at the end of this section we will discuss this assumption). Firstly, we establish error estimates for the eigenfunctions:

Theorem 6.2 *Let μ_t be an eigenvalue of T_t converging to a simple eigenvalue μ_0 of T_0 as t goes to zero. Let μ_{th} be the eigenvalue of T_{th} that converges to μ_t as h goes to zero. Let (w, β, u) and (w_h, β_h, u_h) be the eigenfunctions corresponding to μ_t and μ_{th}, respectively, both normalized in the same manner. Then, for t and h small enough, there holds*

$$\|(w, \beta, u) - (w_h, \beta_h, u_h)\| \leq Ch. \tag{6.1}$$

Proof. Because of Theorem 5.1, $T_{th}|_{\mathcal{X}}$ converges to $T_t|_{\mathcal{X}}$ in norm. Then $T_t|_{\mathcal{X}}$ is compact and (6.1) is a direct consequence of Theorem 7.1 in [3], with a constant C depending on the constant in (5.5) (which is independent of t) and on the inverse of the distance of μ_t to the rest of the spectrum of T_t. Now, Theorem 6.1 implies that, for t small enough, this distance can be bounded below in terms of the distance of μ_0 to the rest of the spectrum of T_0, which obviously does not depend on t. □

Since s_t, s_{th} and r_t are symmetric, then T_t and T_{th} are self-adjoint with respect to r_t. Therefore, we may use Remark 7.5 in [3] to show a double order of convergence for the approximation of the eigenvalues.

To this goal we will make use of suitable estimates for the expression $r_t\Big((T_t-T_{th})(f,\theta,g),(f,\theta,g)\Big)$, for $(f,\theta,g)\in\mathcal{G}$. Throughout the remainder of this section we consider $(f,\theta,g)\in\mathcal{G}$ fixed and denote again

$$
\begin{aligned}
(w,\beta,u) &:= T_t(f,\theta,g), &\qquad (w_h,\beta_h,u_h) &:= T_{th}(f,\theta,g),\\
\gamma &:= \frac{\kappa}{t^2}(\nabla w-\beta), &\qquad \gamma_h &:= \frac{\kappa}{t^2}(\nabla w_h-\Pi\beta_h).
\end{aligned}
$$

We consider the Helmholtz decomposition of u_h given by Lemma 4.1:

$$u_h = \nabla\xi+\chi.$$

Then $(w-w_h,\beta-\beta_h,u-\nabla\xi)$ belongs to $\mathcal{G}$ and hence it can be used as a test function (v,η,ϕ) in (2.16); by so doing, simple algebra yields

$$
\begin{aligned}
&a(\beta,\beta-\beta_h)+\frac{t^2}{\kappa}\int_\Gamma\gamma\cdot(\gamma-\gamma_h)+\int_\Omega\hat\rho_{\mathrm F}c^2\operatorname{div}u\,\operatorname{div}(u-\nabla\xi)\\
&\quad=\int_\Gamma\hat\rho_{\mathrm P}f(w-w_h)+\frac{t^2}{12}\int_\Gamma\hat\rho_{\mathrm P}\theta\cdot(\beta-\beta_h)\\
&\qquad+\int_\Omega\hat\rho_{\mathrm F}g\cdot(u-\nabla\xi)-\int_\Gamma\gamma\cdot(\beta_h-\Pi\beta_h).
\end{aligned}
$$

Now, subtracting from this the error equation (5.1) with $(v_h,\eta_h,\phi_h)=(w_h,\beta_h,u_h)$, and using (5.2) for M_h, by simple calculations we obtain

$$
\begin{aligned}
r_t\Big((T_t-T_{th})(f,\theta,g),(f,\theta,g)\Big) &= a(\beta-\beta_h,\beta-\beta_h)\\
&\quad+\frac{t^2}{\kappa}\int_\Gamma|\gamma-\gamma_h|^2+\int_\Omega\hat\rho_{\mathrm F}c^2(\operatorname{div}u-\operatorname{div}u_h)^2\\
&\quad+2\int_\Omega\hat\rho_{\mathrm F}g\cdot\chi-2\int_\Gamma\gamma\cdot(\beta_h-\Pi\beta_h).
\end{aligned}\tag{6.2}
$$

So, it only remains to estimate the two last terms in the right hand side of (6.2). The latter has been recently analyzed in [11] in order to establish optimal L^2 error estimates for the finite element method we are using for the bending of the plate:

Lemma 6.1 *There holds*

$$\left| \int_\Gamma \gamma \cdot (\beta_h - \Pi \beta_h) \right| \leq Ch^2 |(f, \theta, g)|_t^2.$$

Proof. The proofs in Lemmas 3.3 and 3.4 of [11] can be easily adapted to our case. □

Finally, we estimate the remaining term in (6.2):

Lemma 6.2 *There holds*

$$\left| \int_\Omega \hat{\rho}_{\mathrm{F}} g \cdot \chi \right| \leq Ch^2 \|(f, \theta, g)\| \, |(f, \theta, g)|_t.$$

Proof. Since $(f, \theta, g) \in \mathcal{G}$, then $g = \nabla q$ and, because of (2.14), $q \in H^2(\Omega)^2$ with

$$\|q\|_{2,\Omega} \leq C \left(\|f\|_{1/2,\Gamma} + \| \operatorname{div} g\|_{0,\Omega} \right) \leq C \|(f, \theta, g)\|.$$

Now, since $\operatorname{div} \chi = 0$ and $\chi = u_h - \nabla \xi$, we have

$$\int_\Omega \hat{\rho}_{\mathrm{F}} g \cdot \chi = \int_{\partial\Omega} \hat{\rho}_{\mathrm{F}} q \,(u_h - \nabla \xi) \cdot n = \int_\Gamma \hat{\rho}_{\mathrm{F}} q \,(u_h \cdot n - w_h),$$

the latter because of $(w_h, \beta_h, \nabla \xi) \in \mathcal{G}$. Since $(w_h, \beta_h, u_h) \in \mathcal{G}_h$, then $u_h \cdot n = P(w_h)$, with P being the $L^2(\Gamma)$-projection onto the piecewise constant functions on $\mathcal{T}_h^\Gamma$. Hence,

$$\begin{aligned} \left| \int_\Omega \hat{\rho}_{\mathrm{F}} g \cdot \chi \right| &= \left| \int_\Gamma \hat{\rho}_{\mathrm{F}} [q - P(q)] \, [P(w_h) - w_h] \right| \\ &\leq Ch^2 \|q\|_{1,\Gamma} \|w_h\|_{1,\Gamma} \\ &\leq Ch^2 \|q\|_{2,\Omega} \|w_h\|_{1,\Gamma} \\ &\leq Ch^2 \|(f, \theta, g)\| \, |(f, \theta, g)|_t, \end{aligned}$$

concluding the proof. □

Summing up we obtain an optimal order of convergence for the approximation of the eigenvalues:

Theorem 6.3 *Let μ_t and μ_{th} be as in Theorem 6.2. Then, for t and h small enough, there holds*

$$|\mu_t - \mu_{th}| \leq Ch^2. \tag{6.3}$$

Proof. Let (f, θ, g) be an eigenfunction corresponding to μ_t normalized in $|\cdot|_t$. We apply Remark 7.5 in [3] which, in our case, reads

$$|\mu_t - \mu_{th}| \le C\bigg[\Big|r_t\Big((T_t - T_{th})(f,\theta,g),(f,\theta,g)\Big)\Big| \\ + |(T_t - T_{th})(f,\theta,g)|_t^2\bigg],$$

with a constant C depending only on the distance of μ_t to the rest of the spectrum of T_t. By repeating the arguments in the proof of Theorem 6.2 we observe that, for t small enough, this constant can be chosen independent of t.

Now, since $T_t(f,\theta,g) = \mu_t(f,\theta,g)$ and T_t is t-uniformly bounded, then

$$\|(f,\theta,g)\| \le \frac{C}{\mu_t}|(f,\theta,g)|_t \le C,$$

the latter because $\mu_t \to \mu_0$. Thus, using (6.2), Lemmas 5.3, 6.1 and 6.2 and Theorem 5.1, we conclude the proof. $\square$

Theorems 6.2 and 6.3 have been stated for eigenvalues of T_t (i.e., the equations of the fluid coupled with Reissner-Mindlin equations for the plate) converging to simple eigenvalues of T_0 (i.e., idem with Kirchhoff equations for the plate). A multiple eigenvalue of the latter arises usually because of symmetries of the geometry; in such a case, the eigenvalue of the former converging to it is also multiple and with the same multiplicity. The proofs of both theorems above extend trivially to cover this case.

Instead, if T_0 had a multiple eigenvalue not due to symmetry reasons, it could split into different eigenvalues of T_t. In this case, the proofs of the theorems do not provide estimates independent of the thickness, since the constants therein blow up as the distance between the eigenvalues of T_t becomes smaller.

For conforming methods, the *minimum-maximum* principle yields estimates not involving this distance (see for instance Sect. 8 of [3]). This approach has been partially extended by Vanmaele and Ženíšek [22,23] to certain non conformities: namely, the approximation of the spectral problem for divergence type elliptic operators on curved domains. However, to the best of our knowledge, estimates of this kind have not been proved for general non conforming methods.

On the other hand, by combining Theorems 3.2 and 5.1 we have that

$$\|(T_{th} - T_0)(f,\theta,g)\| \le C(t+h)\,|(f,\theta,g)|_t \qquad \forall (f,\theta,g) \in \mathcal{H}.$$

This estimate can be used to prove spectral convergence as t and h both converge to zero. In fact, if μ_0 is an eigenvalue of T_0 with multiplicity m,

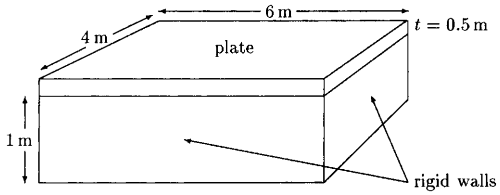

Fig. 1. 3D cavity filled with fluid

there exist exactly m eigenvalues $\mu_{th}^{(1)}, \ldots, \mu_{th}^{(m)}$ of T_{th} (repeated according to their respective multiplicities) converging to μ_0 as t and h goes to zero (see once more [17]). Let $\mathcal{E}_0$ be the eigenspace of T_0 corresponding to μ_0 and $\mathcal{E}_{th}$ be the direct sum of the eigenspaces of T_{th} corresponding to $\mu_{th}^{(1)}, \ldots, \mu_{th}^{(m)}$. Then, by proceeding as in Theorem 6.2 we have that

$$\widehat{\delta}\,(\mathcal{E}_0, \mathcal{E}_{th}) \leq C(t + h),$$

where $\widehat{\delta}$ denotes the *gap* or symmetric distance between both subspaces (see for instance [3]).

This can be also used to prove that $\mu_{th}^{(j)}$, $j = 1, \ldots, m$, all converge to μ_0 with order at least $\mathcal{O}(t + h)$. However, the arguments in the proof of Theorem 6.3 cannot be used in this case to prove a double order of convergence (indeed, r_t degenerates as t goes to zero and thus it does not induce a norm on $\mathcal{H}$ for $t = 0$). Nevertheless, the numerical experiments show that such double order estimates also hold.

7. Numerical experiments

We present in this section some numerical results obtained with a FORTRAN implementation of the finite element method described above. The code was previously validated by computing the vibration modes of moderately thick plates in contact with fluids and comparing the obtained results with those yielded by the method described in [5] (which is a code to compute the vibration modes of three-dimensional structures coupled with fluids that was applied to the 3D elasticity equations of the plate).

As a first test of the performance of our method we have considered a 3D cavity completely filled with water with all of its walls being perfectly rigid, except for one of them which has been taken as a moderately thick plate clamped by its whole boundary. We have used the geometrical parameters for the cavity and the plate as given in Fig. 1.

We have considered the following values for the physical parameters of the plate and the fluid which correspond to steel and water, respectively,

- density of the plate: $\rho_{\mathrm{P}} = 7700\,\mathrm{kg/m}^3$,

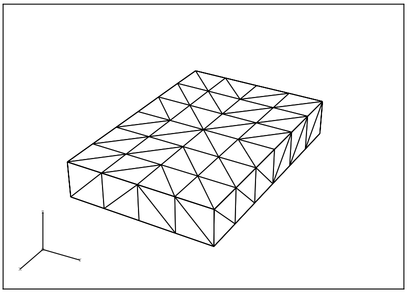

Fig. 2. Mesh on the fluid for $N = 1$

- Young modulus: $E = 1.44 \times 10^{11}$ Pa,
- Poisson coefficient: $\nu = 0.35$,
- density of the fluid: $\rho_{\mathrm{F}} = 1000\,\mathrm{kg/m^3}$,
- sound speed: $c = 1430\,\mathrm{m/s}$,

We have used succesive uniform refinements of the initial mesh which is depicted in Fig. 2. The refinement parameter N is the number of layers of elements for the fluid domain in the vertical direction; it is related with the meshize by $h = \sqrt{3}/N$.

We denote by ω_m^h the m-th lowest computed approximate angular vibration frequency (i.e., the square root of the m-th strictly positive eigenvalue of the discrete problem (4.4)) and ω_m the corresponding exact vibration frequency to which it converges. We have observed that the relative error of ω_m^h behaves roughly like

$$\frac{\omega_m^h - \omega_m}{\omega_m} \approx C_m h^\alpha, \tag{7.1}$$

with an order of convergence α very close to 2 and constants C_m which depend on the particular mode but are almost independent of the meshsize h. Then, for each mode, we have estimated the exact vibration frequencies ω_m, the value of the constants C_m and the order of convergence α by means of a least square fitting of the model

$$\omega_m^h \approx \omega_m \left(1 + C_m h^\alpha\right)$$

to the approximate frequencies computed on four different meshes ($N = 3, 4, 5, 6$).

Table 1 shows the six lowest vibration frequencies of the coupled system computed with our method for each of these meshes. We also include for

Table 1. Angular vibration frequencies of a moderately thick steel plate in contact with water

Mode	$N=3$	$N=4$	$N=5$	$N=6$	α	ω_m	C_m
ω_1^h	696.880	697.166	697.302	697.377	1.92	697.555	-0.0028
ω_2^h	1019.075	1017.201	1016.310	1015.819	1.91	1014.635	0.0125
ω_3^h	1081.299	1081.559	1081.682	1081.750	1.93	1081.911	-0.0016
ω_4^h	1317.326	1317.121	1317.037	1316.995	2.44	1316.921	0.0012
ω_5^h	1470.968	1464.565	1461.561	1459.916	1.95	1456.063	0.0129
ω_6^h	1504.253	1505.621	1506.242	1506.574	2.07	1507.298	-0.0063

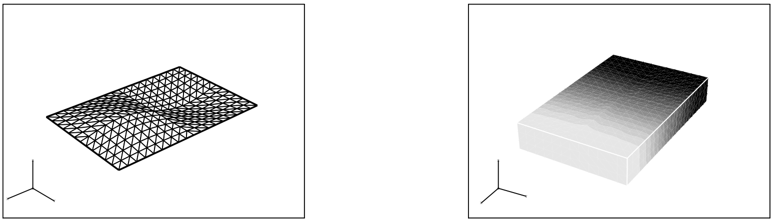

Fig. 3. Deformed plate and fluid pressure. Mode ω_1

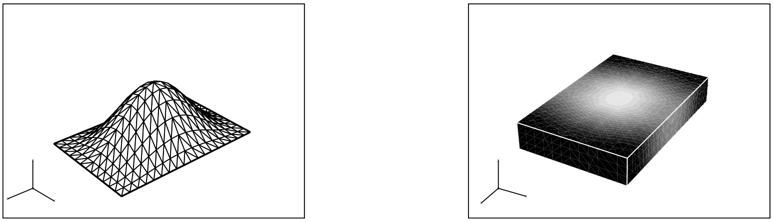

Fig. 4. Deformed plate and fluid pressure. Mode ω_2

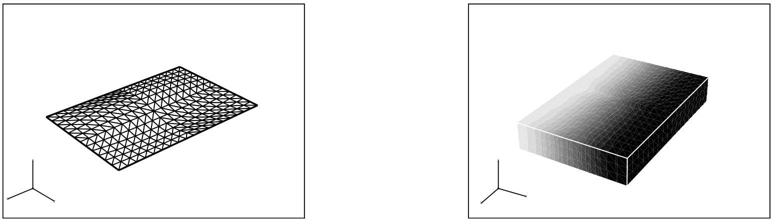

Fig. 5. Deformed plate and fluid pressure. Mode ω_3

each mode the estimated values of the exact vibration frequency ω_m, the order of convergence α and the constant C_m. It can be observed that the approximation is excellent even for rather coarse meshes.

Figs. 3 to 8 show the deformed plate and the fluid pressure for each of these six vibration modes.

Secondly we have checked the stability of the method as the thickness becomes small. To this goal we have applied our method to a sequence of

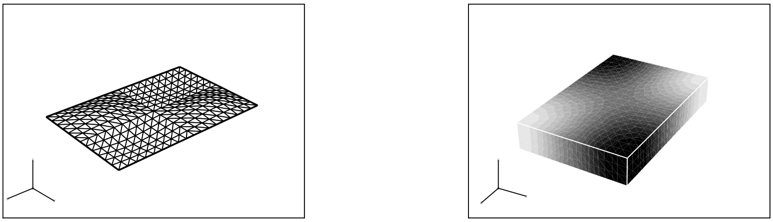

Fig. 6. Deformed plate and fluid pressure. Mode ω_4

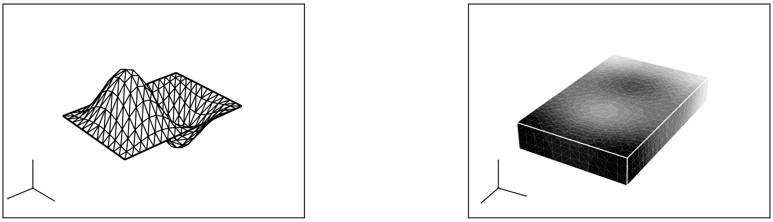

Fig. 7. Deformed plate and fluid pressure. Mode ω_5

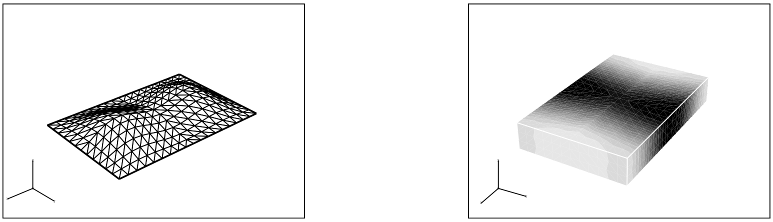

Fig. 8. Deformed plate and fluid pressure. Mode ω_6

model problems, the first one being that of the previous test (i.e., with the same physical and geometrical parameters) and the others being obtained by succesively reducing the thickness t of the plate; the densities of the plate and the fluid has been taken accordingly to the assumptions made in Sect. 2; namely, $\rho_{\mathrm{F}} = \hat{\rho}_{\mathrm{F}} t^3$ and $\rho_{\mathrm{P}} = \hat{\rho}_{\mathrm{P}} t^2$.

For each vibration mode and each thickness t we have estimated the order of convergence α, the constants C_m in estimate (7.1) and the exact vibration frequencies ω_m by a least square fitting of the computed approximate frequencies ω_h^m similar to that of the previous experiment.

We summarize the results obtained for the two lowest frequency vibration modes in Table 2 and Table 3, respectively. It can be clearly observed in both tables that the constants C_m do not depend on the thickness and, hence, that the method is free of locking.

Table 2. First vibration frequency ω_1^h for plates of different thickness coupled with fluid

t	$N=3$	$N=4$	$N=5$	$N=6$	α	ω_1	C_1
0.5	696.880	697.167	697.302	697.377	1.92	697.555	−0.0028
0.05	715.219	715.726	715.965	716.095	1.95	716.403	−0.0048
0.005	715.413	715.924	716.164	716.296	1.95	716.605	−0.0048
0.0005	715.415	715.926	716.166	716.298	1.95	716.607	−0.0048

Table 3. Second vibration frequency ω_2^h for plates of different thickness coupled with fluid

t	$N=3$	$N=4$	$N=5$	$N=6$	α	ω_2	C_2
0.5	1019.075	1017.201	1016.310	1015.819	1.91	1014.635	0.0125
0.05	1112.976	1111.292	1110.494	1110.054	1.91	1108.998	0.0102
0.005	1114.178	1112.502	1111.706	1111.267	1.91	1110.206	0.0102
0.0005	1114.190	1112.515	1111.718	1111.279	1.91	1110.219	0.0102

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