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GENERALIZED MEASURES OF QUANTUM CORRELATIONS FOR MIXED STATES

R. Rossignoli,[1,2*] N. Canosa,[1,3] and L. Ciliberti[1,3]

[1] *Departamento de Física - IFLP, Universidad Nacional de La Plata*
C.C.67, CP:1900, La Plata, Buenos Aires, Argentina
[2] *Comisión de Investigaciones Científicas (CIC)*
Calle 526, CP: 1900, La Plata, Buenos Aires, Argentina
[3] *Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET)*
Avenue Rivadavia 1917, C1033AAJ Buenos Aires, Argentina

*Corresponding author e-mail: rossigno @ fisica.unlp.edu.ar

Abstract

The exponential speedup achieved in certain quantum algorithms based on mixed states with negligible entanglement has renewed the interest on alternative measures of quantum correlations. Here we discuss a general measure of quantum correlations for composite systems based on generalized entropic functions, defined as the minimum information loss due to a local measurement. For pure states, the present measure becomes an entanglement entropy, i.e., it reduces to the generalized entropy of the reduced state. However, for mixed states it can be nonzero in separable states, vanishing just for states diagonal in a general product basis, like the quantum discord. Quadratic measures of quantum correlations can be derived as particular cases of the present formalism. The minimum information loss due to a joint local measurement is also considered. The evaluation of these measures in a simple yet relevant case is also discussed.

Keywords: quantum correlations, quantum entanglement, generalized entropic forms.

1. Introduction

The study of new measures of quantum correlations for mixed states, different from the usual entanglement measures, constitutes a recent and active research topic in the field of quantum information [1]. Among quantum correlations, quantum entanglement is recognized as an essential resource for the achievement of certain quantum-information-processing tasks such as superdense coding [2] and quantum teleportation [1, 3]. The presence of entanglement is also crucial in pure-state-based quantum computation for obtaining an exponential speedup over classical computation [4, 5]. However, this may not be the case for mixed-state-based quantum computation, such as the model proposed by Knill and Laflamme [6], where an exponential speedup over classical computation can, in principle, be obtained without a substantial presence of entanglement [7]. The occurrence of such speedup may indicate the

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existence of quantum correlations not captured by the entanglement measures for mixed states but still useful for quantum computation.

As a consequence, alternative measures of quantum correlations for mixed states such as the quantum discord, originally introduced by Ollivier and Zurek [8–10], have recently received much attention. When applied to pure bipartite entangled states, the quantum discord coincides with the usual entanglement entropy, although for mixed states substantial differences with entanglement measures do arise: The quantum discord is, in fact, nonzero for most separable mixed states, vanishing just for classically correlated states (at least, with respect to one of the constituents), i.e., separable states which are in addition diagonal in a certain standard, or conditional product basis, and which can then remain unaltered under a certain unread local measurement [8]. The circuit presented in [6] was, in fact, shown in [11] to exhibit non-negligible quantum correlations, as measured by the quantum discord, between the control qubit and the remaining qubits. This fact has enhanced the interest in the quantum discord. Since then, several properties and applications of the quantum discord have been investigated [12–22]. At the same time, other measures of quantum correlations for mixed states, with properties similar to those of the quantum discord, have also been introduced [23–27].

In this paper, we will discuss a general measure of quantum correlations [26] based on majorization concepts [28–30] and generalized entropic functions [31]. Defined as the minimum information loss due to a local measurement with unknown result, its basic general properties are similar to those of the quantum discord — for pure states, it reduces to the generalized entanglement entropy, while for mixed states it can be nonzero in separable states, vanishing just for states which are diagonal in a standard or conditional product basis. Since the present measure is not based on any particular entropy (the quantum discord is essentially based on the von Neumann entropy), it can be applied with entropic forms complying with minimum requirements [26, 31] and, in particular, with entropic forms which are just a quadratic function of the density matrix ρ, allowing then an easier evaluation which does not require its explicit diagonalization.

In Sec. 2, we discuss the basic concepts related to generalized entropic forms and their connection with majorization. The generalized information loss by measurement is discussed in Sec. 3, while the generalized measures of quantum correlations based on the previous concepts are presented in Sec. 4. In Sec. 5, we provide an illustration of our formalism, evaluating the present measures for a mixture of aligned states of two spins, which represents the reduced state in the definite parity ground state of an XY spin chain in the vicinity of the factorizing field [20]. Conclusions are finally drawn in Sec. 6.

2. Generalized Entropic Functions and Majorization

We consider a density matrix or operator $\rho = \sum_i p_i P_i$ describing the state of a quantum system, where $\{P_i\}$ denotes a complete set of one-dimensional orthogonal projectors ($p_i \geq 0$, $\mathrm{Tr}\,\rho = \sum_i p_i = 1$). We employ the generalized entropies [28, 31, 32]:

$$S_f(\rho) = \mathrm{Tr}\, f(\rho) = \sum_{i=1}^{n} f(p_i), \tag{1}$$

where $f(p) : [0,1] \to \Re$ is a smooth strictly concave function ($f(qp_i + (1-q)p_j) > qf(p_i) + (1-q)f(p_j)$ for $q \in (0,1)$ and $p_i \neq p_j$) satisfying $f(0) = f(1) = 0$. In this way, $S_f(\rho) \geq 0$ with $S_f(\rho) = 0$ if and only if ρ is a pure state ($p_i = \delta_{ij}$ for some j). Moreover, concavity implies that at fixed dimension n,

$S_f(\rho)$ is maximum for the uniform distribution $p_i = 1/n$ (see below). In what follows, we choose the "normalization" $2f(1/2) = 1$, such that $S_f(\rho) = 1$ for a maximum mixed single-qubit state.

Equation (1) is the most simple generalization of the von Neumann entropy:

$$S(\rho) = -\mathrm{Tr}\,\rho \log_2 \rho, \tag{2}$$

which corresponds obviously to $f(p) = -p \log_2 p$.

The so-called linear entropy

$$S_2(\rho) = 2(1 - \mathrm{Tr}\,\rho^2), \tag{3}$$

which is directly related to the purity $\mathrm{Tr}\,\rho^2$, corresponds to $f(p) = 2(p - p^2)$, and the Tsallis entropy [33, 34],

$$S_q(\rho) = \frac{1 - \sum_{i=1}^{n} p_i^q}{1 - 2^{1-q}}\,, \tag{4}$$

where $q > 0$, to $f(p) = (p - p^q)/(1 - 2^{1-q})$ for the present normalization. Equation (4) approaches the von Neumann entropy (2) for $q \to 1$ and reduces to (3) for $q = 2$, allowing then to change S_f continuously from Eq. (2) to Eq. (3).

All entropies (1) satisfy three of the four Khinchin axioms [35], i.e., (i) S_f is a continuous function of the probabilities; (ii) S_f is maximum for the uniform distribution; (iii) S_f remains unchanged if an event with zero probability is added. It is the fourth axiom of additivity (implying, in particular, $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$) which differentiates the von Neumann form (2) (or in general, $k\,S(\rho)$ with $k > 0$) from the rest. If this axiom is removed, a whole family of generalized entropies becomes available, Eq. (1) being the simplest generalization. One apparent advantage of the form (3) (or in general (4) for integer $q \geq 2$) over (2) is that just the power ρ^2 (or ρ^q) needs to be computed. In contrast, Eq. (2) (and (1) for a general f) requires the determination of the eigenvalues p_i of ρ.

In addition, all entropies (1) are concave functions of ρ, due to the concavity of f, and satisfy a fundamental property related to the theory of majorization [30]: They all increase with increasing mixedness [28,30,32]. A density ρ' is said to be more mixed than (or equivalently, majorized by) a second density ρ, which is written as $\rho' \prec \rho$, if the eigenvalues of ρ' and ρ, sorted both in decreasing order ($p_i \geq p_j$ if $i \leq j$), satisfy the inequalities

$$\rho' \prec \rho \Leftrightarrow \sum_{i=1}^{j} p'_i \leq \sum_{i=1}^{j} p_i, \qquad j = 1, \ldots, n-1. \tag{5}$$

In such a case, the probabilities p'_i are more "spread out" than the p_i's and can be written as a convex combination of permutations of the latter [30]. This entails that $\rho' \prec \rho$ if and only if ρ' is a convex combination of unitaries of ρ, i.e., $\rho' = \sum_\alpha q_\alpha U_\alpha^\dagger \rho U_\alpha$, with $q_\alpha > 0$ and $U_\alpha^\dagger U_\alpha = I$ [1,28,30]. The state ρ' is then more random or disordered than ρ. Correspondingly, the full random state $\rho = I/n$ (with I the identity) is more mixed than any other state of the same dimension, whereas any state ρ is more mixed than a pure state, i.e., $I/n \prec \rho \prec |\Phi\rangle\langle\Phi|$ $\forall$ state ρ and pure state $|\Phi\rangle$. If the dimensions of ρ and ρ' differ, the same definition can be applied after completing with zeros the smallest set of eigenvalues.

The connection with the generalized entropies (1) is as follows.

It can be shown that

$$\rho' \prec \rho \Rightarrow S_f(\rho') \geq S_f(\rho) \tag{6}$$

for any f of the previous form [28, 36] (the same property actually holds for any Schur concave function of ρ [30]). Increasing mixedness implies the increase in any entropy. However, for a given f, the converse of Eq. (6) is not necessarily true, so that the concept of disorder implied by Eqs. (5) is stronger than that based on any particular choice of f. Nonetheless, the converse relation holds as follows:

$$S_f(\rho') \geq S_f(\rho) \qquad \forall\ f \qquad \Rightarrow \rho' \prec \rho, \tag{7}$$

where $\forall\ f$ indicates for all concave functions of the form specified in (1) (a smooth sufficient set was provided in [36]). Hence, the characteristic trait of an increased mixedness is a universal entropy increase, and such universal increase warrants as well an increased mixedness. The generalized entropies (1) allow one to capture the rigorous concept of disorder implied by majorization through the generalized entropic inequalities (6)–(7). It should be noted, however, that Eqs. (5) define a partial-order relationship, in the sense that it may well happen that $\rho' \not\prec \rho$ and $\rho \not\prec \rho'$, i.e., they can be incomparable according to the majorization criterion.

3. Generalized Information Loss by Measurement

An important general process which can be rigorously characterized by majorization is that of a measurement with unknown result.

Let us consider a set of orthogonal projectors P_k ($\sum_k P_k = 1$, $P_k P_{k'} = \delta_{kk'} P_k$) determining a projective measurement M on the system [1]. After an unread measurement, the state of the system becomes

$$\rho' = \sum_k q_k \rho_k = \sum_k P_k \rho P_k, \tag{8}$$

where $q_k = \mathrm{Tr}\,\rho P_k$ is the probability of outcome k and $\rho_k = P_k\,\rho\,P_k\,/q_k$, the state after such outcome. Equation (8) is just the "diagonal" of ρ in a particular basis $\{|j'\rangle\}$ (determined by the eigenvectors of the blocks $P_k \rho P_k$). It is well known that such diagonals are more mixed than the original state, i.e., they satisfy $\rho' \prec \rho$ [28, 29], a property which is sometimes referred to as Schur's theorem. According to Eq. (6), this implies that the generalized entropies (1) can never decrease under such measurement:

$$S_f(\rho') \geq S_f(\rho) \tag{9}$$

for any S_f of the previous form. Moreover, $S_f(\rho') = S_f(\rho)$ if and only if $\rho' = \rho$.

We can now define the information loss due to this measurement [26],

$$I_f^M(\rho) = S_f(\rho') - S_f(\rho), \tag{10}$$

which quantifies, according to S_f, the information associated with the off-diagonal elements of ρ in the basis $\{|j'\rangle\}$, lost in the measurement. It satisfies $I_f^M(\rho) \geq 0$, with $I_f^M(\rho) = 0$ if and only if $\rho' = \rho$.

If S_f is the von Neumann entropy (2), Eq. (10) becomes the relative entropy [1, 28, 37] between ρ and ρ' [26]:

$$I^M(\rho) = \mathrm{Tr}\,\rho\,(\log_2 \rho - \log_2 \rho') = S(\rho||\rho'), \tag{11}$$

whereas in the case of the linear entropy (3), Eq. (10) becomes twice the squared Hilbert–Schmidt distance

$$I_2^M(\rho) = 2\mathrm{Tr}(\rho^2 - \rho'^2) = 2||\rho - \rho'||^2, \tag{12}$$

where $||A|| = \sqrt{\mathrm{Tr} A^\dagger A}$ denotes the Hilbert–Schmidt norm. While the positivity of Eqs. (11)–(12) is apparent (the relative entropy is well known to be a nonnegative quantity [1, 28]), their positivity in the present case is just a consequence of the general positivity of Eq. (10) [26].

4. Information Loss by Unread Local Measurement

Let us now focus on a composite system $A+B$ in the initial state ρ_{AB} and consider the information loss due to an unread local measurement M_B defined by one-dimensional local projectors $P_j^B = |j_B\rangle\langle j_B|$. The state after this measurement, [Eq. (8), with $P_k \to I_A \otimes P_j^B$] is

$$\rho'_{AB} = \sum_j q_j \, \rho_{A/j} \otimes P_j^B, \tag{13}$$

where $q_j = \mathrm{Tr}[\rho_{AB} I_A \otimes P_j^B]$ is the probability of outcome j and $\rho_{A/j} = \mathrm{Tr}_B[\rho_{AB} I_A \otimes P_j^B]/q_j$, the reduced state of A after such outcome. The corresponding loss of information is

$$I_f^{M_B}(\rho_{AB}) = S_f(\rho'_{AB}) - S_f(\rho_{AB}). \tag{14}$$

The minimum over all possible local measurements of the previous form,

$$I_f^B(\rho_{AB}) = \underset{M_B}{\mathrm{Min}} \, I_f^{M_B}(\rho_{AB}), \tag{15}$$

depends just on ρ_{AB} and defines a generalized entropic measure of quantum correlations. It satisfies $I_f^B(\rho_{AB}) \geq 0$, with $I_f^B(\rho_{AB}) = 0$ if and only if there is a complete local measurement in B which leaves ρ_{AB} unchanged, i.e., if ρ_{AB} is already of the form (13). Such form corresponds to a classically correlated state with respect to B, i.e., it is a state for which there is a local measurement in B which leaves it invariant. Full classically correlated states

$$\rho_{AB}^c = \sum_{ij} p_{ij} P_i^A \otimes P_j^B, \tag{16}$$

where $P_i^{A,B}$ denote orthogonal one-dimensional projectors, are a particular case of Eq. (13) (that where the $\rho_{A/j}$ are mutually commuting), so that Eq. (15) will vanish in all these states.

Product states $\rho_{AB} = \rho_A \otimes \rho_B$ are, of course, a particular case of (16) ($p_{ij} = p_i^A p_j^B \; \forall \; i,j$).

On the other hand, separable states defined, in general, as convex combinations of product states $\rho_{AB}^s = \sum_\alpha q_\alpha \rho_A^\alpha \otimes \rho_B^\alpha$, $q_\alpha > 0$ [38] are not necessarily of the form (13), although they certainly include the states (13) or (16). Hence, Eq. (15) will be positive not only in entangled states (states which are not separable), but also in all separable states not of form (13), detecting therefore the quantum correlations arising from the mixture of noncommuting product states. It has then the same basic properties as the original quantum discord [8] (see below).

A most fundamental property of Eq. (15) is that, in the case of pure states ($\rho_{AB}^2 = \rho_{AB}$), it can be shown that [26]

$$I_f^B(\rho_{AB}) = I_f^A(\rho_{AB}) = S_f(\rho_A) = S_f(\rho_B), \tag{17}$$

the minimizing local measurement being that determined by the Schmidt basis [1]. Hence, Eq. (15) reduces, for pure states, to the generalized entanglement entropy, which measures the entanglement between A and B according to the selected measure S_f. Thus, for the von Neumann entropy (2), Eq. (17) becomes the standard entanglement entropy $E_{AB} = S(\rho_A) = S(\rho_B)$, while for the linear entropy (3), Eq. (17) becomes the tangle [39, 40], i.e., the square of the pure-state concurrence [41] $C_{AB}^2 = S_2(\rho_A) = S_2(\rho_B)$. The pure-state entanglement can then be seen as the minimum information loss due to an unread local measurement.

We can as well consider the minimum information loss $I_f^{M_{AB}}(\rho_{AB})$ due to a joint local measurement M_{AB}, based on products of one-dimensional local projectors $P_i^A \otimes P_j^B$, which leads to a post-measurement state of the classically correlated form (16), with $p_{ij} = \langle ij|\rho_{AB}|ij\rangle$. Since this joint measurement is a sequence of local measurements where the results are unknown, i.e., a local measurement in B followed by a measurement in A, then $I_f^{M_{AB}}(\rho_{AB}) \geq I_f^{M_B}(\rho_{AB})$ ($M_B = \{P_j^B\}$ denotes the measurement in B). The ensuing minimum

$$I_f^{AB}(\rho_{AB}) = \underset{M_{AB}}{\text{Min}}\, I_f^{M_{AB}}(\rho_{AB}) \tag{18}$$

is a measure of all nonclassical correlations and satisfies, in general,

$$I_f^{AB}(\rho_{AB}) \geq I_f^{B}(\rho_{AB}), \tag{19}$$

with $I_f^{AB}(\rho_{AB}) = 0$ if and only if ρ_{AB} is already of the classically correlated form (16). It is worth mentioning that for pure states, there is no difference between both previous quantities, i.e.,

$$I_f^{AB}(\rho_{AB}) = I_f^{B}(\rho_{AB}) = S_f(\rho_A) = S_f(\rho_B), \tag{20}$$

since, after a measurement in the Schmidt basis, the state ρ'_{AB} is already of the form (16). Then, the pure-state entanglement can be also identified with the minimum information loss due to a joint local measurement [26].

4.1. Connection with the Quantum Discord

In the case of von Neumann entropy, Eq. (15) becomes

$$I^B(\rho_{AB}) = \underset{M_B}{\text{Min}}\, I^{M_B}(\rho_{AB}) = \underset{\rho^d_{AB}}{\text{Min}}\, S(\rho_{AB}||\rho^d_{AB}), \tag{21}$$

which is the minimum relative entropy between ρ_{AB} and any state ρ^d_{AB} of the form (13), diagonal in a conditional product basis [26].

The original quantum discord [8, 9, 11] is closely related to (21) and can be written as

$$D^B(\rho_{AB}) = \underset{M_B}{\text{Min}}\, D^{M_B}(\rho_{AB}) \tag{22}$$

with

$$D^{M_B}(\rho_{AB}) = S(\rho'_{AB}) - S(\rho'_B) - [S(\rho_{AB}) - S(\rho_B)], \tag{23}$$

or

$$D^{M_B}(\rho_{AB}) = I^{M_B}(\rho_{AB}) - I^{M_B}(\rho_B), \tag{24}$$

where ρ'_{AB} is the measured state (13) and ρ'_B/ ρ_B the reduced state after/before the measurement. Thus, $D^B(\rho_{AB}) \leq I^B(\rho_{AB})$, both coinciding when the optimum local measurement is the same for (14) and (23) and corresponds to the basis where ρ_B is diagonal.

In the case of entropy (3), Eq. (14) leads to a quadratic measure

$$I_2^B(\rho_{AB}) = \underset{M_B}{\text{Min}}\, I_2^{M_B}(\rho_{AB}) = 2\, \underset{\rho^d_{AB}}{\text{Min}}\, ||\rho_{AB} - \rho^d_{AB}||^2, \tag{25}$$

which is twice the minimum squared Hilbert–Schmidt distance between ρ_{AB} and any state ρ^d_{AB} of the form (13) (see [26]). This quantity coincides with twice the geometric discord of [18] and provides the simplest measure (in relation with its evaluation) of the information loss.

5. Applications

As an illustration, we will discuss the important case of a pair of qubits in a statistical mixture of two aligned states along two different directions, not necessarily opposite, such that the corresponding local states are not necessarily orthogonal. If we choose the z axis as the bisector of the angle between both directions, we can write the state as

$$\rho_{AB} = \frac{1}{2}(|\theta\theta\rangle\langle\theta\theta| + |-\theta-\theta\rangle\langle-\theta-\theta|)\,, \tag{26}$$

or

$$\rho_{AB} = \frac{1}{4}\begin{pmatrix} \alpha_+ & 0 & 0 & \beta \\ 0 & \beta & \beta & 0 \\ 0 & \beta & \beta & 0 \\ \beta & 0 & 0 & \alpha_- \end{pmatrix}, \qquad \alpha_\pm = (1 \pm \cos\theta)^2, \qquad \beta = \sin^2\theta, \tag{27}$$

where $|\theta\rangle = \exp[-i\frac{\theta}{2}\sigma_y]|0\rangle$ is the single qubit state aligned along an axis forming an angle θ with the z axis in the x, z plane. Equation (27) is the representation in the standard product basis of σ_z eigenstates. This type of state was carefully discussed in [20]. For a ferromagnetic-type-XY spin-1/2 chain in a uniform transverse field B, this state represents essentially the exact reduced state of any pair in the definite parity ground state in the vicinity of the factorizing field [42].

This state is separable, i.e., it has no entanglement since it is a convex superposition of product states. The concurrence C_{AB} correspondingly vanishes identically. Nonetheless, it has nonzero quantum discord D^B [20] if $\theta \in (0, \pi/2)$. It can be easily shown that it also has nonzero values of any I_f^B or I_f^{AB} in this interval. For $\theta = 0$, it is obviously a pure product state, while for $\theta = \pi/2$ it is a classically correlated state, i.e., a state of the form (16) diagonal in the standard product basis, since $\langle -\theta|\theta\rangle = \cos^2\theta$ vanishes for $\theta = \pi/2$. This implies that $I_f(\theta) \equiv I_f(\rho_{AB}(\theta)) = 0$ for both $\theta = 0$ and $\theta = \pi/2$ for any S_f.

In this case, the local measurement is represented by a spin measurement along a unit vector **k**. The minimizing vector will be located in the plane generated by the two-spin directions determining the state $\rho(\theta)$, i.e., in the x, z plane for the previous choice of axes.

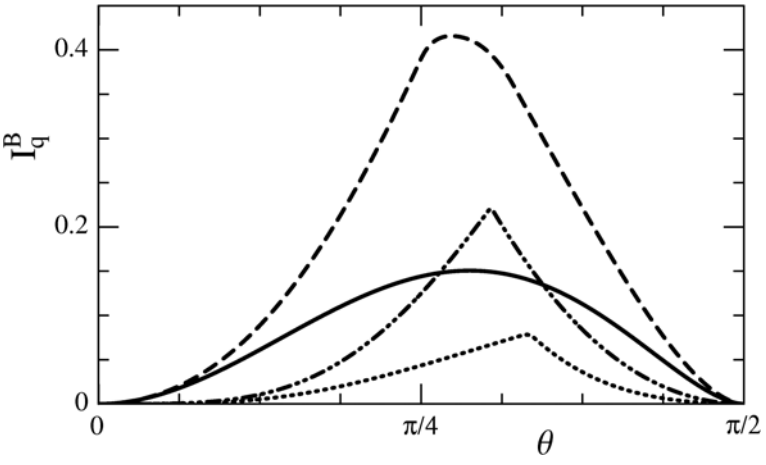

Fig. 1. Generalized entropic measure I_f^B given by expression (14) as a function of the angle θ determining the state (26), evaluated for three different entropic functions: I_1^B (dashed curve) corresponds to the case where $S_f(\rho)$ is the von Neumann entropy [$q \to 1$ in (4)], I_2^B (dashed–dotted curve) to the entropy (3), and I_4^B (dotted curve) to the entropy (4) for $q = 4$. The quantum discord (22) is shown by the solid curve.

As seen in Fig. 1, where we plot $I_f(\theta)$ for the case of different q values in the entropy (4), all $I_f(\theta)$ exhibit roughly the same qualitative behavior as a function of θ, i.e., they all vanish at $\theta = 0$ and $\pi/2$

and exhibit an intermediate maximum. The slope discontinuity at the peak for the cases $q = 2$ and $q = 4$ indicates a sharp change in the minimizing measurement direction $\mathbf{k}$ (from $\theta = 0$ to $\theta = \pi/2$). In contrast, in the von Neumann case ($q = 1$), the transition is smoothed, while in the case of the original quantum discord the optimum measurement is always $\theta = \pi/2$ [20]. This result entails that while in the vicinity of the factorizing field the pair entanglement will be vanishingly small, the quantum discord as well as any of the present measures $I_f(\rho)$ will have nonzero and non-negligible values between any pair, irrespective of separation, confirming such fields as quantum critical points for the finite chain [20, 42].

Other different examples can be found in [26]. It is important to point out that these measures are, in general, not upper bounds of the corresponding entanglement monotones for mixed states, which are the convex roof extension of the pure state generalized entanglement entropy. In the von Neumann case, this monotone is the entanglement of formation, whereas in the case (3) it is just the square of the concurrence C_{AB} for the case of two qubits. Nonetheless, for some families of states, $I_f^B(\rho_{AB})$ can provide an upper bound to the associated entanglement monotone for some functions f, i.e., for some values of q for the case (4), which may exclude the von Neumann case $q = 1$ [26] but include, in contrast, the quadratic case $q = 2$ [26].

6. Conclusions

On the basis of generalized entropic forms and majorization concepts, we have introduced a general entropic measure of quantum correlations, $I_f^B(\rho_{AB})$, which represents the minimum loss of information, according to the selected entropy S_f, due to a local unread complete projective measurement. The positivity of this measure is strictly based on the majorization relations fulfilled between the post-measurement state and the original state and is applicable to general entropic forms based on arbitrary strictly concave functions. This quantity possesses similar properties to those of the original quantum discord, coinciding with the corresponding entanglement measure, the generalized entanglement entropy, in the case of pure states, and vanishing for the same type of partially classical states. As a particular case of the present formalism ($S_f = S_2$), a quadratic measure $I_2^B(\rho_{AB})$ can be derived, which represents the minimum Hilbert–Schmidt distance to a classical state and coincides with the so-called geometric discord. We have shown, on a specific example, that while these measures exhibit the same basic qualitative properties, details (such as minimizing measurement and order relation with the corresponding entanglement monotone) may differ. The explicit evaluation of these measures in general states of two or more qubits or spins is currently under investigation.

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