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Sampling Formulae and Optimal Factorizations of Projections

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Abstract

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{W}$ a closed subspace of $\mathcal{H}$, and Q a (linear bounded) projection from $\mathcal{H}$ onto $\mathcal{W}$ with null space $\mathcal{M}^{\perp}$. We study decompositions like $Qf = \sum_{n\in\mathbb{N}} \langle f, h_n \rangle f_n$, where $\{f_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ are frames for the subspaces $\mathcal{W}$ and $\mathcal{M}$, respectively. This type of decompositions corresponds to sampling formulae. By considering the synthesis operator F (resp. H) of the sequence $\{f_n\}_{n\in\mathbb{N}}$ (resp. $\{h_n\}_{n\in\mathbb{N}}$), the formula above can be expressed as the factorization $Q = FH^*$. We study different properties of these factorizations and decompositions of oblique and orthogonal projections. Several characterizations of these decompositions are presented. By means of an operator inequality for positive operators, we get a result which minimizes the norm of $F - H$.

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1 Introduction

The classical sampling formula of Whittaker-Kotelnikov-Shannon says that for every $x \in \mathbb{R}$ it holds

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinc}(x-n), \tag{1}$$

for every function $f \in L^2(\mathbb{R})$ whose Fourier transform is supported in the interval $[-1,1]$; here, $\mathrm{sinc}\,x = \dfrac{\sin \pi x}{\pi x}$. The convergence of this series is uniform on $\mathbb{R}$, a fortiori it converges in the L^2 sense. The set of those functions with Fourier transform supported in $[-1,1]$ is one of the well-known Paley-Wiener spaces and it will be denoted PW. This point of view has been the basis for many generalizations of the classical sampling theory, mainly for nonuniform sampling (see Aldroubi-Gröchenig [1], Benedetto [5], Jerri [24], and Kramer [25]). A unified view is obtained from representations in Reproducing Kernel Hilbert Spaces (Nashed and Walter [32,33], and Yao [41]). The reader will also find excellent accounts of the WKS formula and its applications and generalizations in the surveys by Jerri [23], Higgins [22], and Unser [38], and in the books by Higgins [20] and Higgins and Stens [21], among many other sources. PW is a closed subspace of $L^2(\mathbb{R})$ and formula (1) provides the orthogonal projection P from $L^2(\mathbb{R})$ onto PW; in fact, if $s_n = \mathrm{sinc}(\cdot - n)$, then we note that $f(n) = \langle\, f,\, s_n \,\rangle$ for every $f \in PW$ and that $\{s_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of PW. As Unser observes in [38], these facts were first proven by G. H. Hardy [18] who wrote: " It is odd that, although these functions occur repeatedly in analysis, especially in the theory of interpolation, it does not seem to have been remarked explicitly that they form an orthogonal system." Therefore, we get

$$Pf = \sum_{n\in\mathbb{Z}} \langle\, f,\, s_n \,\rangle\, s_n \quad \forall\, f \in L^2(\mathbb{R})\,. \tag{2}$$

On the other hand, if T is the operator defined on some fixed orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ of $L^2(R)$ by $Te_n = s_n$, then (2) is the explicit form of the factorization $P = TT^*$. In this factorization the fact that P is a self-adjoint projection is not independent of the fact that the vectors $s_n = Te_n$ constitute an orthonormal basis of PW. Indeed, this is the well-known method in linear algebra used to construct the orthogonal projection onto a fixed subspace explicitly. This remark shows that any orthonormal basis $\{e'_n\}$ of $L^2(\mathbb{R})$ and any orthonormal basis $\{s'_n\}$ of PW induces a factorization $P = \widetilde{T}\widetilde{T}^*$, or, which is the same, a formula $Pf = \sum_n \langle\, f,\, s'_n \,\rangle\, s'_n$ similar to formula (2).

Unser's observation is the starting point of our paper, in which we will consider more general factorizations of projections, in a sense that we describe now. It is useful to notice that for every bounded linear projection $Q \in L(\mathcal{H})$ its range and null space induce a direct sum decomposition of $\mathcal{H}$ and, conversely, every direct sum decomposition $\mathcal{H} = \mathcal{W} + \mathcal{M}^\perp$ induces a unique projection Q with range $\mathcal{W}$ and null space $\mathcal{M}^\perp$. These correspondences preserve orthogonality in the sense that Q is orthogonal if and only if $\mathcal{W}$ is orthogonal to $\mathcal{M}^\perp$, i.e., $\mathcal{W} = \mathcal{M}$. Let $\mathcal{W}$ be a closed subspace of $\mathcal{H}$ and Q a (bounded linear) projection from $\mathcal{H}$ onto $\mathcal{W}$ with null space $\mathcal{M}^\perp$. We study factorizations, such as

$$Q = FH^*, \tag{3}$$

where $F, H \in L(\ell^2, \mathcal{H})$, $R(F) = \mathcal{W}$ and $R(H) = \mathcal{M}$; throughout, $R(T)$ denotes the range of the operator T. This leads to the study of sampling formulae like

$$f = \sum_{n\in\mathbb{N}} \langle f, h_n \rangle f_n, \quad f \in \mathcal{W} \tag{4}$$

where $\{f_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ are not necessarily bases of their respective subspaces: they are frames for these subspaces, which means that they respectively span $\mathcal{W}$ and $\mathcal{M}$ in a controlled way (see definitions below). Christensen and Eldar [8,14], describe this situation by saying that $\{h_n\}_{n\in\mathbb{N}}$ is an oblique dual frame of $\{f_n\}_{n\in\mathbb{N}}$ on $\mathcal{M}$. The reader should observe that the most general sampling formula,

$$f = \sum_{n\in\mathbb{N}} \langle f, h_n \rangle f_n, \quad f \in \mathcal{W},$$

can be written as $FH^*Q = Q$, where F and H are defined by $Fe_n = f_n$, $He_n = h_n$ and Q is any (bounded linear) projection onto $\mathcal{W}$. Here f_n (resp. h_n) is not supposed to belong to $\mathcal{W}$ (resp. $\mathcal{M}$). This type of factorization, studied by Li and Ogawa (and also by Ogawa and Berrached [34,35] for finite dimensional spaces) comes from the notion of pseudoframes of subspaces extensively studied in [30] (see also [9]). Li and Ogawa refer to $\{h_n\}_{n\in\mathbb{N}}$ as an interpolating sequence and $\{f_n\}_{n\in\mathbb{N}}$ as an approximating sequence. Since the analysis operator of a pseudoframe is in general an unbounded operator, we prefer to restrict our study to more specific factorizations which allow us to work with bounded linear operators. We intend to study the general case elsewhere. A main difference between our approach and those of Li-Ogawa and Christensen-Eldar is that we study pairs (F, H) as described in (3), while those authors fix a certain F (resp. H) and then study the fiber $pr_1^{-1}(F)$ (resp. $pr_2^{-1}(H)$), where $pr_1(F, H) = F$ (resp. $pr_2(F, H) = H$). We should mention here that Unser and Aldroubi [39] were among the first to use oblique projections in sampling theory.

In order to describe the results of this work, it is convenient to introduce the subset $\mathfrak{X}_Q$ of $L(\ell^2, \mathcal{H}) \times L(\ell^2, \mathcal{H})$ defined by

$$\mathfrak{X}_Q := \{(F, H) : \; FH^* = Q, \; R(F) = R(Q) \text{ and } R(H) = N(Q)^\perp\}.$$

Each pair $(F,H) \in \mathfrak{X}_Q$ corresponds to a sampling formula like (4) with an additional condition, namely that the right-hand side of (4) vanishes for $f \in \mathcal{M}^\perp$. More precisely,

$$Qf = \sum_{n\in\mathbb{N}} \langle\, f,\, h_n \,\rangle\, f_n = FH^* f\,, \quad f \in \mathcal{H} \tag{5}$$

where $Fe_n = f_n$ and $He_n = h_n$. The operator version of the sampling formulae by Li and Ogawa is given by $\mathcal{F}_Q = \{(F,H) : FH^*Q = Q\}$. Since F and H may be unbounded and $R(F)$ (resp. $R(H)$) may be strictly bigger than $\mathcal{W}$ (resp. $\mathcal{M}$), it follows that $\mathfrak{X}_Q$ is significantly smaller than $\mathcal{F}_Q$. In Proposition 3.2 we show how to identify, inside $\mathfrak{X}_Q$, those pairs that produce biorthogonal sampling formulae, i.e., those that satisfy $\langle\, f_n, h_m \,\rangle = \delta_{m,n}$. By using techniques of generalized inverses, we show that, given a projection Q with range $\mathcal{W}$ and null space $\mathcal{M}^\perp$ and an injective operator $F \in L(\ell^2,\mathcal{H})$ with $R(F) = \mathcal{W}$, there is a unique $H \in L(\ell^2,\mathcal{H})$ such that $Q = FH^*$. This corresponds to the fact that every Riesz basis of $\mathcal{W}$ determines a unique Riesz basis of $\mathcal{M}$ such that $(\{f_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}})$ is biorthogonal. The first main result identifies the set $\mathfrak{X}_Q$ with a set of pairs of operators previously studied by Gramsch [17] and Corach, Porta and Recht [11]. This set is a homogeneous space with a very rich differential structure that is studied in [11]. The second main result gives a criteria of metric optimality, in the sense that we solve, without uniqueness, the problem

$$\arg\min\{\|F-H\|^2 :\ (F,H) \in \mathfrak{X}_Q\}\,.$$

More explicitly, we prove that

$$\min\{\|F-H\|^2 :\ (F,H) \in \mathfrak{X}_Q\} = 2\big(\|Q\| - 1\big),$$

and we show a set of pairs $(F,H) \in \mathfrak{X}_Q$ that realize the minimum. As a corollary, we obtain that only for orthogonal projections this minimum is equal to zero. The key part of the proof of the minimization theorem is an operator inequality, which may have interest by itself: for every positive invertible operators C, B on $\mathcal{H}$ such that $B \geq 1$, it holds that

$$\|C + C^{-1/2}BC^{-1/2}\| \geq 2\|B^{1/2}\|;$$

this means that

$$\inf\big\{\|C + C^{-1/2}BC^{-1/2}\| :\ C \in GL(\mathcal{H})^+\big\}$$

is attained in $C = B^{1/2}$.

We show then that our methods give a short proof of a result by Christensen and Eldar [8, Cor. 4.4] on frames of translates for shift invariant spaces.

2 Preliminaries

Given two separable Hilbert spaces, $\mathcal{K}$ and $\mathcal{H}$, the set of bounded linear operators from $\mathcal{K}$ to $\mathcal{H}$ is denoted by $L(\mathcal{K},\mathcal{H})$. For an operator, $A \in L(\mathcal{K},\mathcal{H})$, we denote by $R(A)$ the range or image of A; $N(A)$, the null space of A; A^*, the adjoint of A; $\|A\|$, the usual norm of A; and, if $R(A)$ is closed, $A^\dagger$, the Moore-Penrose pseudoinverse of A. The set $L(\mathcal{H},\mathcal{H})$ is denoted $L(\mathcal{H})$, $GL(\mathcal{H})$ denotes the group of invertible operators on $\mathcal{H}$, and $L(\mathcal{H})^+$ denotes the cone of positive (semi-definite) operators of $L(\mathcal{H})$. If $\mathcal{H} = \mathcal{W} \oplus \mathcal{M}^\perp$, then the projection Q onto $\mathcal{W}$ defined by this decomposition is denoted by $P_{\mathcal{W}||\mathcal{M}^\perp}$. Observe that $P^*_{\mathcal{W}||\mathcal{M}^\perp} = P_{\mathcal{M}||\mathcal{W}^\perp}$.

Generalized inverses

In this subsection we mention the definition and basic facts on generalized inverses. The reader is referred to the books by Nashed [31], by Ben-Israel and Greville [4], and Campbell and Meyer Jr. [6] for more information. Throughout this section, $\mathcal{K}$ and $\mathcal{H}$ are Hilbert spaces.

Definition 2.1. Let $A \in L(\mathcal{K},\mathcal{H})$. A *generalized inverse (or pseudoinverse)* of A is an operator $B \in L(\mathcal{H},\mathcal{K})$ such that $ABA = A$ and $BAB = B$.

It is a well-known fact that $A \in L(\mathcal{K},\mathcal{H})$ has a generalized inverse if and only if $R(A)$ is closed. Also recall that A has a closed range if and only if A^* has a closed range. The next proposition relates generalized inverses with oblique projections.

Proposition 2.2. *Let $A \in L(\mathcal{K},\mathcal{H})$.*

1. *If $B \in L(\mathcal{H},\mathcal{K})$ is a generalized inverse of A, then*
   i. *AB is an oblique projection onto $R(A)$.*
   ii. *BA is an oblique projection whose null space is $N(A)$.*
2. *Given a pair of projections $Q \in L(\mathcal{H})$ and $P \in L(\mathcal{K})$ such that $R(Q) = R(A)$ and $N(P) = N(A)$, there is a unique generalized inverse B of A such that $AB = Q$ and $BA = P$.*

Definition 2.3. Given $A \in L(\mathcal{K},\mathcal{H})$ with a closed range, the *Moore-Penrose's generalized inverse* of A, denoted by $A^\dagger$, is the unique generalized inverse associated to the orthogonal projections onto $R(A)$ and $N(A)^\perp$, respectively. In other words, $A^\dagger$ is the unique generalized inverse of A such that $A^\dagger A$ and $AA^\dagger$ are self-adjoint projections.

In terms of the Moore-Penrose generalized inverse, the set of generalized inverses can be parametrized in the following way.

Proposition 2.4. If $A \in L(\mathcal{K}, \mathcal{H})$ has a closed range and projections Q and $\widetilde{Q}$ satisfy $R(Q) = R(A)$ and $N(\widetilde{Q}) = N(A)$, then the unique generalized inverse B of A such that $AB = Q$ and $BA = \widetilde{Q}$ is given by

$$B = \widetilde{Q} A^{\dagger} Q.$$

Reduced solutions of the equation $AX = B$

Along this note, the following result of Douglas [12] (see also [16]) will be used several times.

Theorem 2.5. Given $A, B \in L(\mathcal{K}, \mathcal{H})$, the following conditions are equivalent:

1. $R(B) \subseteq R(A)$.
2. There exists a positive number λ such that $BB^* \leq \lambda\ AA^*$.
3. There exists $D \in L(\mathcal{K})$ such that $B = AD$.

Moreover, given a complement $\mathcal{S}$ of $N(A)$ in $\mathcal{K}$ there is a unique operator $D \in L(\mathcal{K})$ that satisfies one of the conditions above and also $R(D) \subseteq \mathcal{S}$. It also holds that $N(D) = N(B)$. This D is called a *reduced solution* of the equation $B = AX$. Among the reduced solutions, there is only one corresponding to $N(A)^{\perp}$. This D is called the *Douglas solution* of the equation $AX = B$ and it satisfies

$$\|D\|^2 = \inf \Big\{ \lambda \in \mathbb{R} \ : \ BB^* \leq \lambda\ AA^* \Big\} . \tag{6}$$

■

Remark 2.6. Note that if D is a solution of $AX = B$, then $BB^* = ADD^*A^* \leq \|D\|^2 AA^*$. So, if D_0 is the Douglas solution of $AX = B$, by the characterization of $\|D_0\|$ given in (6), we get $\|D_0\| \leq \|D\|$. ▲

Remark 2.7. If $A \in L(\mathcal{K}, \mathcal{H})$ has closed range, then the Douglas solution of $AX = B$ is $A^{\dagger}B$: in fact, $A(A^{\dagger}B) = B$ because $AA^{\dagger} = P_{R(A)}$ and $P_{R(A)}B = B$ because $R(B) \subseteq R(A)$. On the other side, $R(A^{\dagger}B) \subseteq R(A^{\dagger}) = N(A)^{\perp}$. ▲

Frames

We introduce some basic facts about frames in Hilbert spaces. For complete descriptions of frame theory and applications, the reader is referred to the review by Heil and Walnut [19] or the books by Young [42] and Christensen [7].

Consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements of a Hilbert space $\mathcal{H}$. $\{f_n\}_{n\in\mathbb{N}}$ is a *Bessel sequence* if there exists a positive number B such that

$$\sum_{n\in\mathbb{N}} |\left\langle f, f_n \right\rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}.$$

A *frame* for a closed subspace $\mathcal{W}$ of $\mathcal{H}$ is a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that each f_n belongs to $\mathcal{W}$ and there exist numbers $\alpha, \beta > 0$ such that, for every $f \in \mathcal{W}$,

$$\alpha\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n \rangle|^2 \leq \beta\|f\|^2. \tag{7}$$

The optimal constants α, β for equation (7) are called the *frame bounds* for $\mathcal{F}$. $\mathcal{F}$ is a *Parseval frame* if $\alpha = \beta = 1$. Note that, as each $f_n \in \mathcal{W}$, for every $f \in \mathcal{W}^\perp$ it holds that $\langle f, f_n \rangle = 0$. This shows that every frame for $\mathcal{W}$ is in particular a Bessel sequence in $\mathcal{H}$.

Any Bessel sequence $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ defines a bounded linear operator $T : \ell^2 \to \mathcal{H}$ by $Te_n = f_n$, where $\{e_n\}_{n\in\mathbb{N}}$ denotes the "canonical" basis of ℓ^2. This operator is called the *synthesis operator* of $\mathcal{F}$, $T^* \in L(\mathcal{H}, \ell^2)$, is called the *analysis operator* of $\mathcal{F}$, and $S = TT^*$ is called *frame operator* of $\mathcal{F}$. It is easy to see that $T^*f = \sum_{n\in\mathbb{N}} \langle f, f_n\rangle e_n$ and, therefore,

$$Sf = \sum_{n\in\mathbb{N}} \langle f, f_n \rangle f_n \qquad \forall f \in \mathcal{H}. \tag{8}$$

Observe that, in the case of a frame for a (closed) subspace $\mathcal{W}$, from (7) we obtain the operator inequality $\alpha \cdot P_{\mathcal{W}} \leq S \leq \beta \cdot P_{\mathcal{W}}$. Hence, $S|_{\mathcal{W}}$ is invertible in $L(\mathcal{W})$ and $R(T) = \mathcal{W}$. The dimension of $N(T)$ is sometimes called *excess* of $\mathcal{F}$. A Riesz basis for a closed subspace $\mathcal{W}$ is a frame for this subspace with excess equal to zero (see Balan et al. [3]).

3 Biorthogonal decompositions of oblique projections

In this section we will study decompositions of oblique projections by using biorthogonal systems. First of all, we obtain some basic results about biorthogonal systems from the point of view of generalized inverses.

Definition 3.1. *Given Bessel sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$, then the pair $\{\{f_n\}, \{h_n\}\}$ is called biorthogonal system if for every $m, n \in \mathbb{N}$ it holds that*

$$\langle f_m, h_n \rangle = \delta_{m,n}.$$

Proposition 3.2. *Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ be Bessel sequences with synthesis operators denoted by F and H, respectively, and let $\mathcal{W} = \overline{R(F)}$ and $\mathcal{M} = \overline{R(H)}$. The following statements are equivalent:*

1. *$\{\{f_n\}, \{h_n\}\}$ is a biorthogonal system;*
2. *$H^*F = I$.*

3. $\mathcal{H} = \mathcal{W} \oplus \mathcal{M}^\perp$, $\{f_n\}_{n\in\mathbb{N}}$ *and* $\{h_n\}_{n\in\mathbb{N}}$ *are Riesz bases of* $\mathcal{W}$ *and* $\mathcal{M}$*, respectively, and for every* $f \in \mathcal{H}$ *it holds*

$$Qf = \sum_{n\in\mathbb{N}} \langle\, f, h_n \,\rangle\, f_n\,, \tag{9}$$

where Q *is the oblique projection onto* $\mathcal{W}$ *parallel to* $\mathcal{M}^\perp$*.*

Proof.

$1 \Leftrightarrow 2)$ Note that the matrix of H^*F with respect to the canonical basis of ℓ^2 is

$$\begin{pmatrix} \langle\, f_1,\, h_1 \,\rangle & \langle\, f_2,\, h_1 \,\rangle & \langle\, f_3,\, h_1 \,\rangle & \dots \\ \langle\, f_1,\, h_2 \,\rangle & \langle\, f_2,\, h_2 \,\rangle & \langle\, f_3,\, h_2 \,\rangle & \dots \\ \langle\, f_1,\, h_3 \,\rangle & \langle\, f_2,\, h_3 \,\rangle & \langle\, f_3,\, h_3 \,\rangle & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

So, the equivalence between 1. and 2. follows from the definition of biorthogonal systems.

$2 \Rightarrow 3)$ As $H^*F = I$, H^* is surjective and F is injective. Analogously, taking the adjoint, we get that $F^*H = I$, which shows that F^* is surjective and H is injective. On the other hand, as F^* and H^* have closed ranges, F and H also have closed ranges. In particular, $\{f_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ are Riesz bases for $\mathcal{W}$ and $\mathcal{M}$, respectively. Finally, as $H^*F = I$, it holds that FH^* is a projection onto $R(F)(=\mathcal{W})$ with null space $N(H^*)(=\mathcal{M}^\perp)$. So $Q = FH^*$, which is equivalent to (9).

$3 \Rightarrow 1)$ As $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis and $f_m = \sum_{n\in\mathbb{N}} \langle\, f_m, h_n \,\rangle\, f_n$, we get that for every $m, n \in \mathbb{N}$ it holds that $\langle\, f_m,\, h_n \,\rangle = \delta_{mn}$.

∎

The following proposition states that any oblique projection with infinite dimensional range can be decomposed by using biorthogonal systems.

Proposition 3.3. *Let* $\mathcal{W}$ *and* $\mathcal{M}$ *be two infinite dimensional closed subspaces of* $\mathcal{H}$ *such that* $\mathcal{H} = \mathcal{W} \oplus \mathcal{M}^\perp$*, and let* Q *be the oblique projection onto* $\mathcal{W}$ *parallel to* $\mathcal{M}^\perp$*, i.e.,* $Q = P_{\mathcal{W}||\mathcal{M}^\perp}$*. For every Riesz basis* $\{f_n\}_{n\in\mathbb{N}}$ *of* $\mathcal{W}$ *there is a unique Riesz basis* $\{h_n\}_{n\in\mathbb{N}}$ *of* $\mathcal{M}$ *such that:*

$$Qf = \sum_{n\in\mathbb{N}} \langle\, f, h_n \,\rangle\, f_n\,. \tag{10}$$

Remark 3.4. Note that, in order to have a biorthogonal decomposition of a projection Q, it is necessary that $\dim R(Q) = \infty$. In the above proposition the hypothesis $\dim \mathcal{W} = \infty$ is used to guarantee this necessary condition.

Proof of Proposition 3.3. Let F be the synthesis operator of $\{f_n\}_{n\in\mathbb{N}}$ and let $H : \ell^2 \to \mathcal{H}$ be the adjoint of the unique generalized inverse of F such that $FH^* = Q$ and $H^*F = I$ (see Prop. 2.4). If we define $h_n = H(e_n)$, then we get, for every $f \in \mathcal{H}$,

$$\sum_{n\in\mathbb{N}} \langle f, h_n \rangle f_n = FH^*f = Qf\,.$$

As $H^*F = I$, by Proposition 3.2 $\{\{f_n\}, \{h_n\}\}$ is a biorthogonal system. So, $\{h_n\}_{n\in\mathbb{N}}$ is a Riesz basis for $\mathcal{M}$.

Now, suppose that $\{g_n'\}_{n\in\mathbb{N}}$ is another Riesz basis of $\mathcal{M}$ with synthesis operator H' that also satisfies equation (10). Then $F(H')^* = Q$ and, by Proposition 3.2, $(H')^*F = I$. Therefore, H^* and $(H')^*$ are pseudoinverses of F associated to the same pair of projections. As a consequence, by Proposition 2.2 $H^* = (H')^*$, which proves the uniqueness of the Riesz basis $\{h_n\}_{n\in\mathbb{N}}$. ■

If we identify a biorthogonal system $\{\{f_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}}\}$ with the pair of operator (F, H) consisting of the synthesis operators of $\{f_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$, respectively, the set of biorthogonal systems that satisfy (10) can be identified with the set

$$\mathcal{B}_Q = \{(F,\, H) \in L(\ell^2, \mathcal{H}) \times L(\ell^2, \mathcal{H}) :\; FH^* = Q, \;\text{ and }\; H^*F = I\}.$$

In terms of this set, Proposition 3.3 can be rewritten in the following way.

Corollary 3.5. *For every $F \in L(\ell^2, \mathcal{H})$ such that $R(F) = R(Q)$ and $N(F) = \{0\}$ there is a unique operator $H \in L(\ell^2, \mathcal{H})$ such that $(F, H) \in \mathcal{B}_Q$. Moreover, if Ω denote the set of synthesis operators of all the Riesz bases for $\mathcal{W}$, then $\Lambda : \Omega \to \mathcal{B}_Q$ given by $\Lambda(F) = \big(F, \big((F^*F)^{-1}F^*Q\big)^*\big)$ is an homeomorphism from Ω onto $\mathcal{B}_Q$.*

4 Frame decompositions of oblique projections

In the previous section we used biorthogonal systems to decompose oblique projections. In some applications, more general decompositions are required. Let $\mathcal{W}$ and $\mathcal{M}$ be closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{M}^\perp$ and let Q be the oblique projection onto $\mathcal{W}$ parallel to $\mathcal{M}^\perp$. If we use a biorthogonal system $\{\{f_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}}\}$ to decompose Q, then $\{f_n\}_{n\in\mathbb{N}}$ as well as $\{h_n\}_{n\in\mathbb{N}}$ have to be Riesz bases, as shown in Proposition 3.2. Suppose that this decomposition of Q is associated with a sampling-reconstruction procedure, where $\{h_n\}_{n\in\mathbb{N}}$ is used to sample a signal $f \in \mathcal{W}$ and $\{f_n\}_{n\in\mathbb{N}}$ is used to reconstruct the signal. As $\{h_n\}_{n\in\mathbb{N}}$ is a Riesz basis, if some part of the information contained in the sampling data $\{\langle f, h_n \rangle\}_{n\in\mathbb{N}}$ is lost, then it is impossible to get a perfect reconstruction of f. This is the reason why, in signal processing, frames instead of Riesz bases are used to sample signals. As frames are overcomplete systems of vectors, they permit the design of reconstruction methods that can identify errors in the received data. There are many ways to do this, and the linear reconstruction methods are closely related with generalized inverses.

Throughout this section we study decompositions of Q by means of frames. As in Corollary 3.5, we study such decompositions by means of the synthesis operators of the corresponding frames.
If $\{f_n\}_{n\in\mathbb{N}}$ is a frame of $\mathcal{W}$ and $\{h_n\}_{n\in\mathbb{N}}$ is a frame of $\mathcal{M}$ with synthesis operators F and H, respectively, then the frame decomposition of Q given by

$$Qf = \sum_{n\in\mathbb{N}} \langle f, h_n \rangle f_n$$

is equivalent to the factorization $Q = FH^*$. Observe that this factorization is not unique because different frames of $\mathcal{W}$, $\mathcal{M}$ produce different factorizations of Q. This leads us to define and study the following subset of $L(\ell^2,\mathcal{H}) \times L(\ell^2,\mathcal{H})$:

$$\mathfrak{X}_Q := \{(F,\, H) : \; FH^* = Q,\; R(F) = R(Q) \;\text{ and }\; R(H) = N(Q)^{\perp}\}.$$

In the following proposition we list some alternative characterizations of $\mathfrak{X}_Q$. On one side, these presentations of $\mathfrak{X}_Q$ permit us to observe that $\mathfrak{X}_Q$ has been studied before, under a quite different aspect; on the other side, later computations relay on some of these characterizations.

Theorem 4.1. *Given $F, H \in L(\ell^2,\mathcal{H})$, the following statements are equivalent:*

1. *$(F,H) \in \mathfrak{X}_Q$;*
2. *$FH^*F = F$, $H^*FH^* = H^*$, and $FH^* = Q$;*
3. *$R(F) = R(Q)$, $H = (\widetilde{Q}F^\dagger Q)^*$ where $\widetilde{Q}$ is an oblique projection such that $N(\widetilde{Q}) = N(F)$;*
4. *$R(F) = R(Q)$ and H^* is a reduced solution of $FX = Q$;*
5. *$FH^* = Q$, $QF = F$, and $H^*Q = H^*$*

Proof.

$1 \Rightarrow 2)$ Given $(F,H) \in \mathfrak{X}_Q$,

$$FH^* = Q\,, FH^*F = QF = F \quad \text{and} \quad H^*FH^* = H^*Q = (Q^*H)^* = H^*.$$

$2 \Rightarrow 3)$ It follows by Proposition 2.4 .

$3 \Rightarrow 4)$ As $F\widetilde{Q} = F$ and $R(FF^\dagger) = R(F) = R(Q)$,

$$FH^* = F\widetilde{Q}F^\dagger Q = FF^\dagger Q = Q,$$

which shows that H^* is a solution of $FX = Q$. On the other hand, $R(H^*) = R(\widetilde{Q})$, which is a complement of $N(F)$. Hence, H^* is a reduced solution.

$4 \Rightarrow 5)$ Clearly $FH^* = Q$ and $QF = Q$. As H^* is a reduced solution, $N(H^*) = N(Q)$. So, $H^*Q = H^*$.

$5 \Rightarrow 1)$ The equation $FH^* = Q$ implies that $R(Q) \subseteq R(F)$, and equation $QF = F$ implies the other inclusion. So, $R(F) = R(Q)$. By taking the adjoints, we obtain that $R(H) = N(Q)^\perp = R(Q^*)$.

■

Remark 4.2. Note that the projection onto the first coordinate, $pr_1((F,H)) = F$, defines a map from $\mathfrak{X}_Q$ onto the space of epimorphisms of $L(\ell^2, \mathcal{W})$ (and something analogous for $\mathcal{M}$). Item 3 of the theorem above gives a parameterization of $pr_1^{-1}(F)$ in terms of all oblique projections $\widetilde{Q}$ with the same null space as F. A more general result valid for pseudoframes is from Li and Ogawa [30, Thm. 2 and 5]; a similar parameterization is the following one by Christensen and Eldar [8, Thm. 3.2;13, Thm. 1]: they prove that $H \in L(\ell^2, \mathcal{H})$ belongs to $pr^{-1}(F)$ if and only if there exists $K \in L(\ell^2, \mathcal{H})$ such that $R(K) \subseteq \mathcal{M}$ and $H = P_{\mathcal{M}//\mathcal{W}^\perp}(FF^*)^\dagger F + K - KF^*(FF^*)^\dagger F$. Our conditions look more manageable.

Topological remarks. The trivial projection $Q = I$ produces the non-trivial space $\mathfrak{X}_I$. According to item 3 of Theorem 4.1,

$$\begin{aligned}\mathfrak{X}_I &= \big\{(F,H) : \ F, H \in \mathcal{E}\,,\ FH^* = I\big\} \\ &= \big\{(F, (\widetilde{Q}F^\dagger)^*) : \ F \in \mathcal{E},\ \widetilde{Q} \in L(\ell^2),\ Q = Q^2,\ N(\widetilde{Q}) = N(F)\big\},\end{aligned}$$

where $\mathcal{E}$ denotes the set of epimorphisms of $L(\ell^2, \mathcal{H})$, i.e., $\mathcal{E} = \{F \in L(\ell^2, \mathcal{H}) : R(F) = \mathcal{H}\}$. This is a set of continuity of the Moore-Penrose operation (see [28]). In fact, $F \in \mathcal{E}$ if and only if $FF^* \in GL(\mathcal{H})$; therefore, it is easy to check that $F^\dagger = F^*(FF^*)^{-1}$, and this shows that, on $\mathcal{E}$, $F \to F^\dagger$ is continuous (moreover, real analytic). About the topological properties of $\mathcal{E}$, the reader is referred to [10]. Recall from [10] that $\mathcal{E} = \{T \in L(\mathcal{H}) : \ R(T) = \mathcal{H}\}$ is an open subset of $L(\mathcal{H})$ with a natural action

$$GL(\mathcal{H}) \times \mathcal{E} \longrightarrow \mathcal{E}$$

defined by $(G,T) \to TG^{-1}$. For each $T \in \mathcal{E}$ its orbit $\mathcal{O}_T := \{TG^{-1} : \ G \in GL(\mathcal{H})\}$ is the connected components of T in $\mathcal{E}$. Moreover, the component is determined by the nullity of T: $T' \in \mathcal{E}$ belong to $\mathcal{O}_T$ if and only if $\dim N(T) = \dim N(T')$. Fix an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ and define the unilateral shift $S \in L(\mathcal{H})$ by $Se_n = e_{n+1}$ ($n \in \mathbb{N}$). Then $S^* \in \mathcal{E}$ and all "finite" connected components of $\mathcal{E}$ have the form $GL(\mathcal{H}) \cdot S^{*n}$. The map $\mathcal{E} \to \mathcal{E}$ defined by $T \to (TT^*)^{-1/2}T$ is a retraction from $\mathcal{E}$ onto the subset $\widetilde{\mathcal{E}} = \{T \in L(\mathcal{H}) : TT^* = I\}$. It is well known that $\mathcal{E}$ corresponds naturally to the set of all frames on $\mathcal{H}$ and, under this correspondence, $\widetilde{\mathcal{E}}$ is mapped onto the set of all Parseval frames. By means of the projection onto the first coordinate $pr_1 : \mathfrak{X}_I \to \mathcal{E}$, we can completely describe the topological and geometrical structure of $\mathfrak{X}_I$. An analogous statement holds for the set $\mathfrak{X}_Q$ and the map $pr_1 : \mathfrak{X}_Q \to \mathcal{CR}_{\mathcal{W}}$, where $\mathcal{CR}_{\mathcal{W}}$ is the set of all operators from ℓ^2 to $\mathcal{H}$ whose range is $\mathcal{W}$. These results will be described elsewhere. We only mention here that the connected components of $(F, H) \in \mathfrak{X}_Q$ can be characterized in terms of $\dim N(F)$: in fact, it can be proved that a pair $(S, T) \in \mathfrak{X}_Q$ belongs to the connected component of (F, H) if and only if $\dim N(S) = \dim N(F)$.

Optimal factorizations

Since we are generalizing factorizations as $P = TT^*$, where T is a partial isometry with final space $R(P)$, it seems natural to search for a way of minimizing, given $Q = FH^*$, the difference $F - H$. To minimize the norm of this difference may be one. Intuitively, $\|F - H\|$ measures how well distributed is one frame with respect to the other. In finite dimensional spaces a similar notion of optimality is defined to classify the different methods of orthogonalization (see [37]).

In terms of sampling theory the operators F and H can be identified with the frames $\{f_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ used to reconstruct and to sample, respectively. Then $\|F - H\|$ measures the distance between both frames. If the sampling space $\mathcal{M}$ and the reconstruction space $\mathcal{W}$ coincide, i.e., the decomposition of the space and the corresponding projections are orthogonal, then we can use the same frame to sample and to rescontruct a given signal. If those spaces are different, i.e., the decomposition and the corresponding projections are oblique, this is, of course, impossible. The next result gives a minimum for $\|F - H\|$. Notice that the minimum is related with the angle between the spaces $\mathcal{W}$ and $\mathcal{M}^\perp$ by means of the norm of the oblique projection $P_{\mathcal{W}\|\mathcal{M}^\perp}$.

Theorem 4.3. *Let $Q = P_{\mathcal{W}\|\mathcal{M}^\perp}$. Then the problem*

$$\min_{(F,H)\in\mathfrak{X}_Q} \|F - H\|^2$$

has a solution. More precisely, $\|F - H\|^2 \geq 2\big(\|Q\| - 1\big)$ for all $(F, H) \in \mathfrak{X}_Q$, and the equality holds for every pair (F, H) such that $FF^* = (QQ^*)^{1/2}$ and H^* is the Douglas solution of the equation $FX = Q$.

Proof of Theorem 4.3. Given $(F,H) \in \mathfrak{X}_Q$, it holds

$$\begin{aligned}\|F-H\|^2 &= \|(F-H)(F-H)^*\| = \|FF^* + HH^* - FH^* - H^*F\| \\ &= \|FF^* + HH^* - (Q+Q^*)\|. \end{aligned} \tag{11}$$

Claim: it is enough to minimize over pairs (F,H_0) where H_0^* is the Douglas solution of $FX = Q$. Indeed, given an operator F such that $R(F) = R(Q)$, then for every H such that $(F,H) \in \mathfrak{X}_Q$ it holds that

$$F(F^* - H^*) = FF^* - Q.$$

If H_0^* is the Douglas solution of $FX = Q$, then $F^* - H_0^*$ is the Douglas solution of $FX = FF^* - Q$. So, by remark 2.6, we get

$$\|F - H_0\| = \|F^* - H_0^*\| \le \inf_{H\colon (F,H)\in\mathfrak{X}_Q} \|F^* - H^*\| = \inf_{H\colon (F,H)\in\mathfrak{X}_Q} \|F - H\|,$$

which proves our claim. So, it is enough to consider pairs $(F,H_0) \in \mathfrak{X}_Q$ such that H_0^* is the Douglas solution of $FX = Q$. As F has a closed range, H_0 has an explicit formula in terms of the Moore-Penrose inverse of F, namely $H_0 = Q^*(F^\dagger)^*$ (see Remark 2.7). Using this expression of H_0 in equation (11), we get

$$\|F-H_0\|^2 = \|FF^*+Q^*(F^\dagger)^*F^\dagger Q-(Q+Q^*)\| = \|FF^*+Q^*(FF^*)^\dagger Q-(Q+Q^*)\|. \tag{12}$$

Then, the decomposition $\mathcal{H} = \mathcal{W} \oplus \mathcal{W}^\perp$ induces the following 2×2 matrix representation of an operator $A \in L(\mathcal{H})$:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{11} := P_{\mathcal{W}} A P_{\mathcal{W}}|_{\mathcal{W}}$, $A_{12} := P_{\mathcal{W}} A (I - P_{\mathcal{W}})|_{\mathcal{W}^\perp}$, etc. With respect to this decomposition Q and FF^* have the next form:

$$Q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \qquad FF^* = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},$$

where $a : \mathcal{W} \to \mathcal{W}$ is invertible, 1 denotes the identity of $L(\mathcal{W})$, and the zeros denote the corresponding null operators. Using these matrix representations we obtain

$$\begin{aligned}
\|F - H_0\|^2 &= \left\| \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ x^* & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2 & x \\ x^* & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a^{-1} & a^{-1}x \\ x^*a^{-1} & x^*a^{-1}x \end{pmatrix} - \begin{pmatrix} 2 & x \\ x^* & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} a + a^{-1} - 2 & (a^{-1} - 1)x \\ x^*(a^{-1} - 1) & x^*a^{-1}x \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} a^{-1/2} - a^{1/2} & 0 \\ x^*a^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} a^{-1/2} - a^{1/2} & a^{-1/2}x \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} a^{-1/2} - a^{1/2} & a^{-1/2}x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^{-1/2} - a^{1/2} & 0 \\ x^*a^{-1/2} & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} a^{-1} + a - 2 + a^{-1/2}xx^*a^{-1/2} & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \|a + a^{-1/2}(1 + xx^*)a^{-1/2}\| - 2
\end{aligned}$$

where the last equality holds because $a + a^{-1/2}(1 + xx^*)a^{-1/2} \geq 2$. Therefore, our problem has been reduced to the next minimization problem: given a positive operator B such that $1 \leq B$, find positive invertible operators A such that

$$\|A + A^{-1/2}BA^{-1/2}\| = \inf_{C>0} \|C + C^{-1/2}BC^{-1/2}\|.$$

Fix $C > 0$ (this means that C is a positive invertible operator). Given $h \in \mathcal{H}$ with $\|h\| = 1$, let $f = \dfrac{C^{1/2}h}{\|C^{1/2}h\|}$. Then

$$\begin{aligned}
\|C + C^{-1/2}BC^{-1/2}\| &\geq \langle Cf, f \rangle + \left\langle BC^{-1/2}f, C^{-1/2}f \right\rangle \\
&= \frac{\langle C^2h, h \rangle}{\langle Ch, h \rangle} + \frac{\langle Bh, h \rangle}{\langle Ch, h \rangle} \\
&\geq \frac{\langle Ch, h \rangle^2}{\langle Ch, h \rangle} + \frac{\langle Bh, h \rangle}{\langle Ch, h \rangle} \\
&= \langle Ch, h \rangle + \frac{\langle Bh, h \rangle}{\langle Ch, h \rangle}
\end{aligned}$$

where the second inequality is a consequence of Jensen's inequality. Now, using the fact that the function $f(t) = t + \frac{\alpha}{t}$ attains its minimum in $(0, +\infty)$ at $t = \sqrt{\alpha}$, we get

$$\|C + C^{-1/2}BC^{-1/2}\| \geq 2 \langle Bh, h \rangle^{1/2} > 0.$$

As h is any unitary vector of $\mathcal{H}$, we obtain that

$$\|C + C^{-1/2}BC^{-1/2}\| \geq 2\|B\|^{1/2}, \quad C > 0\,.$$

On the other hand, if we take $C = B^{1/2}$, clearly we obtain that $C+C^{-1/2}BC^{-1/2} = 2B^{1/2}$. So, the minimum is attained in $C = B^{1/2}$.

With respect to our original problem, this says that if $a_0 = (1 + xx^*)^{1/2}$ or, equivalently, if $F_0F_0^* = (QQ^*)^{1/2}$, then

$$\|F_0F_0^*+Q^*(F_0F_0^*)^\dagger Q-(Q+Q^*)\| = \min_{F:R(F)=R(Q)} \|FF^*+Q^*(FF^*)^\dagger Q-(Q+Q^*)\|,$$

and $\|F_0F_0^* + Q^*(F_0F_0^*)^\dagger Q - (Q + Q^*)\| = 2\|QQ^*\|^{1/2} - 2 = 2\|Q\| - 2$. This implies that the pair (F_0, H_0) with $H_0 = Q^*(F_0^\dagger)^*$ satisfies

$$\|F_0 - H_0\|^2 = 2\left(\|Q\| - 1\right) = \min_{(F,H)\in\mathfrak{X}_Q} \|F - H\|^2,$$

which concludes the proof. ∎

Remark 4.4. Note that the trivial decompositions $Q = QQ$ and $Q = P_{\mathcal{W}}Q$ induce pairs of the form (QU, Q^*U) and $(P_{\mathcal{W}}U, Q^*U)$ of $\mathfrak{X}_Q$, where U is any isometric isomorphism between ℓ^2 and $\mathcal{H}$. In both cases it holds that

$$\|QU-Q^*U\|^2 = \|P_{\mathcal{W}}U-Q^*U\|^2 = \|Q\|^2-1 = (\|Q\|+1)(\|Q\|-1) \geq 2(\|Q\|-1).$$

Moreover, the more acute the angle between $\mathcal{W}$ and $\mathcal{M}^\perp$ is, the greater $\|Q\|+1$ is, so that these decompositions are far from being optimal. It should also be mentioned that in the theorem we describe only some of the minimizers, but we do not know the general form of all of them.

Application to frames of translates

Given $k \in \mathbb{Z}$, let $T_k \in L(L^2(\mathbb{R}))$ be the unitary operator defined by

$$T_k(\phi)(x) = \phi(x - k).$$

A subspace $\mathcal{W}$ of $L^2(\mathbb{R})$ is called *shift invariant* if $T_k(\mathcal{W}) \subseteq \mathcal{W}$ for every $k \in \mathbb{Z}$. On the other hand, a frame of (integer) translates (or *shift invariant frame*) for a subspace $\mathcal{W}$ is a frame for $\mathcal{W}$ that has the form $\{T_k\phi\}_{k\in\mathbb{Z}}$ for some $\phi \in L^2(\mathbb{R})$. Throughout this subsection, we suppose that the domain of the synthesis operator is $\ell^2(\mathbb{Z})$ instead of $\ell^2(\mathbb{N})$. So, given a shift invariant subspace $\mathcal{W}$ of $L^2(\mathbb{R})$ and a frame $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ for $\mathcal{W}$ with synthesis operator F, $\mathcal{F}$ is a shift invariant frame if and only if the following identity holds:

$$T_1\, F = F\, S, \tag{13}$$

where $S \in L(\ell^2(\mathbb{Z}))$ is the shift operator, defined on the canonical basis of $\ell^2(\mathbb{Z})$ as $Se_n = e_{n+1}$.

The next result is of Christensen and Eldar ([8, Corol. 4.4]), but their proof is much longer and uses completely different techniques.

Corollary 4.5. *Let $\mathcal{W}$ and $\mathcal{M}$ be shift invariant subspaces of $\mathcal{H} = L^2(\mathbb{R})$ and suppose that there is a frame of translates $\{T_k\phi\}_{k\in\mathbb{Z}}$ for $\mathcal{W}$. If F is the synthesis operator of $\{T_k\phi\}$, then $\mathfrak{X}_Q$ contains only one pair (F, H) such that H is the synthesis operator of a shift invariant frame.*

Proof. Observe, first, that $T_1F = FS$. As $\mathcal{M}$ is also shift invariant, $T_1^*QT_1 = Q$ or equivalently $T_1^*Q = Q\,T_1^*$. So, $H = (F^\dagger Q)^*$ is not only a frame for $\mathcal{M}$ such that $(F, H) \in \mathfrak{X}_Q$ (Thm. 4.1) but it is also a shift invariant frame. In fact, as $F^\dagger T_1 = SF^\dagger$, $HS = (F^\dagger Q)^*S = (S^{-1}F^\dagger Q)^* = (F^\dagger T_1^*Q)^* = (F^\dagger QT_1^*)^* = T_1H$, which implies that the frame associated to H is also shift invariant.

Conversely, suppose that (F, H) is a pair of $\mathfrak{X}_Q$ such that are synthesis operators of shift invariant frames. Then, by Theorem 4.1, we get that $H = (PT^\dagger Q)^*$. Moreover, $P = H^*F$. As F and H are synthesis operators of shift invariant frames, $FS = TF$ and $HS = TS$. Therefore, $S^*(H^*F)S = H^*F$. In other words, the projection P commutes with the bilateral shift operator S, which implies that P is self-adjoint. As the null space of P is fixed, the operator H is unique. ■

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