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DIFFERENTIAL FORMS ON DIFFERENTIABLE SPACES

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Introduction

In |5| we have defined the category of N-differentiable (not necessarily reduced) spaces and have described certain subcategories that seem to be of special interest. For one of these restricted categories we have, for example, established certain embedding theorems of the classical type for manifolds. At this point then one should describe certain constructions to "smooth" differentiable spaces, so that starting with a general space, one ends with a space in a more restricted, but nicer class. By this we are able, for example, to extend certain embedding results of |5| to a far more general class of differentiable spaces (see |7|).

Here we want to continue the description of differentiable spaces. First we define "products" for differentiable spaces. This can be done in several ways and depends on what one wishes to have. We describe here two products (the "product" in §1 and the "pseudoproduct" in §2). In the classical complex analytic case these both coincide. Also a fiberproduct can be introduced.

In connection with products we are interested in contractible differentiable spaces (in the reduced complex case see |2|, but see also |3| for some related results). We have two kinds of contractions (r-contractibility and weak r-contractibility) according to the different products. Contractions are described to some detail in §3.

However our main object here are "differential forms on differentiable spaces". They are defined in §4. The sheaves of these forms constitute a certain complex, but exactness in general fails. Under some contractibility assumption exactness from some degree on will be established (§5). In a special case then one gets of course a type of de Rham theorem for N-differentiable spaces, generalizing the corresponding theorem for differentiable manifolds. These results have first been described in |2| for reduced complex spaces. We have not only generalized |2| for the real analytic and moreover finite differentiable case, but also for the unreduced cases. Even in the reduced complex analytic case the result here is slightly more general than |2|, because our notion of an ω^*-differentiable space is more general than that of a complex space. When at La Plata, I learned that a generalization of Reiffen's theorem to the classical unreduced complex analytic case is also studied by R. Slutzki. We should mention that our C^∞-case can also be considered as a generalization of Reiffen's theorem to formal powerseries-algebras over the real or complex numbers. But in this direction there even exists a generalization to the theory of differential modules due to G. Scheja (not yet published). So one has the two possible generalizations either to N-differentiable spaces (analysis) or to differential modules (algebra) with powerseries-algebras as a non empty intersection.

Finally in §5 we describe some connections between differentiable and holomorphic differential forms on complex spaces. But this is yet not more than a beginning of what one should like to know.

After all we hoped to show by an example, how for certain questions in complex analysis the notion of an N-differentiable space just gives the appropiate set up. This task will be continued elsewhere.

§ 1. The product for N-differentiable spaces.

(A) We recall from [5] : An N-differentiable space X is a ringed space, which is locally of the form $D = (\mathbb{D}, \mathcal{D}^N/\mathcal{I}) \subset R^n$. That means:

$D \subset R^n$ is an arbitrary subset,

$\mathcal{D}^N$ is the sheaf of germs of complex valued functions of class C^N on R^n (: C^N- functions or N-differentiable functions. $1 \leq N \leq \infty$. $N = \omega$ means realanalytic, $N = \omega^*$ means complex analytic),

$\mathcal{I} \subset \mathcal{D}^N|D$ is any idealsubsheaf with $\mathcal{I}_x \neq \mathcal{D}_x^N$ for all $x \in D$,

$\mathcal{D}^N/\mathcal{I} := (\mathcal{D}^N|D)/\mathcal{I}$.

We say, that X satisfies A_3, if all local representations $(D, \mathcal{D}^N/\mathcal{I})$ of X satisfy $(\mathcal{I}_x \cdot \mathcal{D}_x^{N-1}) \cap \mathcal{D}_x^N = \mathcal{I}_x$ (where $N \pm 1 = N$ for $N = \infty, \omega, \omega^*$) for all $x \in D$.

In [7] we denote by $\mathcal{R}_{3N}$ the category of N-differentiable mappings and spaces, which satisfy A_3. Here we shorten: $\mathcal{R}_N = \mathcal{R}_{3N}$. Let $\mathcal{R}_N^2$ denote the category of all pairs (X_1, X_2), $X_i \in \mathcal{R}_N$ with differentiable mappings $h = (h_1, h_2)$: $(X_1, X_2) \longrightarrow (Y_1, Y_2)$ as morphisms, where $h_i : X_i \rightarrow Y_i \in \mathcal{R}_N$ are differentiable. If $g_i : Y_i \longrightarrow Z_i \in \mathcal{R}_N$, $i=1,2$, are differentiable and $g = (g_1, g_1)$, then $g \circ h = (g_1 \circ h_1, g_2 \circ h_2)$. Identifying $X = (X, X)$, $h = (h,h)$ for any $h: X \rightarrow Y$ we consider $\mathcal{R}_N$ as a subcategory of $\mathcal{R}_N^2$: $\mathcal{R}_N \subset \mathcal{R}_N^2$.

Let $\tilde{\mathcal{R}}_N \subset \mathcal{R}_N$ respectively $\tilde{\mathcal{R}}_N^2 \subset \mathcal{R}_N^2$ be the respective subcategories of N-differentiable spaces embeded in some R^m: $D \subset R^m$ (m not fixed).

(B) An N-differentiable space in this note will always mean an element of $\mathcal{R}_N$. The following spaces from $\tilde{\mathcal{R}}_N$ shall be given:

$$A_j = (A_j, \mathcal{D}_j^N/\mathcal{I}_j) \subset R^{m_j} \quad , \quad B_j = (B_j, \mathcal{D}_j^{*N}/\mathcal{I}_j^{*N}) \subset R^{m_j}, \; j=1,2.$$

For a while we shall drop the indices j. Then let $(x,y,) \in A \times B$ and

$$\| \quad (\mathcal{I} + \mathcal{I}^*)_{(x,y)} := \{ f;\; f \in \tilde{\mathcal{D}}^N_{(x,y)} \;,\; f = \sum_\nu c_\nu \cdot f_\nu + \sum_\mu d_\mu \cdot g_\mu \;,\; \text{where}$$
$$c_\nu, d_\mu \in \tilde{\mathcal{D}}^{N-1}_{(x,y)} \;,\; f_\nu \in \mathcal{I}_x \;,\; g_\mu \in \mathcal{I}^*_y \}.$$

Here of course $\tilde{\mathcal{D}}^\ell$ means the sheaf of germs of l-differentiable functions on $R^n \times R^m$ and the f_ν, g_μ are considered as elements of $\tilde{\mathcal{D}}^{N-1}$ in the natural way.

$\mathfrak{J} + \mathfrak{J}^*$:= the ideal subsheaf of $\tilde{\mathfrak{D}}^N | A \times B$, defined by the rings $(\mathfrak{J} + \mathfrak{J}^*)_{(x,y)}$ above.

$A \times B$:= $(A \times B, \tilde{\mathfrak{D}}^N / \mathfrak{J} + \mathfrak{J}^*)$.

We obviously have $A \times B \in \tilde{\mathcal{R}}_N$. The projections

$\pi^1 : R^n \times R^m \to R^n$, $\pi^2 : R^n \times R^m \longrightarrow R^m$

induce N-differentiable mappings, which we also denote by π^i, namely:

$\pi^1 : A \times B \to A$, $\pi^2 : A \times B \longrightarrow B$,

the projections on the first respectively second component. We write

$$\pi_j = (\pi_j^1, \pi_j^2) : A_j \times B_j \longrightarrow (A_j, B_j)$$

(with the identification $\tilde{\mathcal{R}}_N \subset \tilde{\mathcal{R}}_N^2$ from above). Now we have

Lemma 1,1. Let $D \in \tilde{\mathcal{R}}_N$, $f : D \longrightarrow (A, B)$ be differentiable. Then there exists exactly one differentiable mapping $g : D \to A \times B$ such that $f = \pi \circ g$. Hence π is left universal, the following diagram is commutative:

$$\begin{tikzcd} A \times B \arrow[rr, "\pi"] & & (A, B) \\ & D \arrow[ul, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proof. Let $D = (D, \mathfrak{D}^N / \tilde{\mathfrak{J}}) \subset R^\ell$. We show first the

Uniqueness: Let $g^1 = g$ and also $g^2 : D \longrightarrow (A, B)$ be differentiable with $f = \pi \circ g^2$. We have to schow: $g^1 = g^2$. The problem is of a local nature. Without any restriction we may therefore assume, that g^1 and g^2 are induced by some N-differentiable mappings

$$G^i : O \longrightarrow R^n \times R^m, \ i = 1,2,$$

where $D \subset O \subset_{\text{open}} R^\ell$. Let $G^i = (G_1^i, G_2^i)$ with $G_1^i : O \to R^n$, $G_2^i : O \to R^m$. Because of

$$\pi \circ g^2 = f = \pi \circ g^1 ,$$

we have for each $z \in D$: The components of $(G_i^1 - G_i^2)_z$ are in $\mathfrak{J}_z$. By [W], Lemma 3,1 therefore G^1 and G^2 generate the same differentiable mapping $D \longrightarrow A \times B$.

Existence. Because of the uniqueness it suffices to show: To each $z \in D$ there exists a neighborhood $U(z) \subset D$ of z and a differentiable mapping

$$g^z : D | U(z) \longrightarrow A_j \times B_j$$

such that $$f | U(z) = \pi \circ g^z.$$

So we even may assume, that $f = (f_1, f_2)$ is generated by some N-differentiable

mappings $F_1 : O \longrightarrow R^n$, $F_2 : O \longrightarrow R^m$, where $D \subset O \subset_{open} R^l$. We claim:
$F = (F_1, F_2) : O \longrightarrow R^n \times R^m$ generates our differentiable mapping

$$g : D \longrightarrow A \times B .$$

For this we have only to show : If $z \in D$, $y = F(z)$, $h_y \in (\mathfrak{J} + \mathfrak{J}^*)_y$, then $h_y \circ F_z \in \tilde{\mathfrak{J}}_z$. But this is obvious according to the definition of F_1, F_2 and $(\mathfrak{J} + \mathfrak{J}^*)_z$
From all that now follows 1,1. q. e. d.

Corollary 1,2. Let $h = (h_1, h_2) : (A_1, B_1) \longrightarrow (A_2, B_2)$ be differentiable. Then there exists a unique differentiable mapping $\tilde{h} : A_1 \times B_1 \longrightarrow A_2 \times B_2$ such that $\pi_2 \circ \tilde{h} = h \circ \pi_1$. Hence the following diagram is commutative:

$$\begin{array}{ccc} A_1 \times B_1 & \xrightarrow{\pi_1} & (A_1, B_1) \\ \tilde{h} \downarrow & & \downarrow h \\ A_2 \times B_2 & \xrightarrow{\pi_2} & (A_2, B_3). \end{array}$$

We write $\tilde{h} = h_1 \times h_2$.

Proof. Take 1,1 for $f := h \circ \pi_1$.

Corollary 1,3. The mapping

$$\mathcal{T} : (A, B) \longrightarrow A \times B \; , \; \mathcal{T} : h \longrightarrow \tilde{h}$$

described above is a covariant functor from $\tilde{\mathcal{R}}_N^2$ into $\tilde{\mathcal{R}}_N$. Especially:

$\mathcal{T}(g \circ h) = \mathcal{T}(g) \circ \mathcal{T}(h)$; if h is a diffeomorphism, so is $\mathcal{T}(h)$.

The statements 1,1 to 1,3 now also hold for $\mathcal{R}_N^2$ and $\mathcal{R}_N$:

Theorem 1,4. In $\mathcal{R}_N$ exists a product: To each $(X, Y) \in \mathcal{R}_N^2$ there exists a $Z \in \mathcal{R}_N$ with a differentiable mapping $\pi : Z \longrightarrow (X, Y)$ such that we have: To each $D \in \mathcal{R}_N$, $f : D \longrightarrow (X, Y)$ differentiable there exists exactly one differentiable mapping $g : D \longrightarrow Z$, such that $f = \pi \circ g$, it est :

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & (X, Y) \\ & {}_{g}\nwarrow \; D \; \nearrow_{f} & \end{array}$$

is commutative. Z is up to diffeomorphisms uniquely determined. We write $Z = X \times Y$.

Proof. The uniqueness is obvious. We show the

Existence. Let

$\{(U_i, \varphi_i); i \in I\}$ be an atlas of X

$\{(V_j, \psi_j); j \in J\}$ be an atlas of Y (see [#], p. 284).

Let $\varphi_i(X|U_i) = A_i$, $\psi_j(Y|V_j) = B_j$

$\varphi_{i_1}(X|U_{i_1} \cap U_{i_2}) = A_{i_1 i_2}$, $\psi_{j_1}(Y|V_{j_1} \cap V_{j_2}) = B_{j_1 j_2}$

For simplicity of notation we consider the sheaf $\underline{\underline{A_i \times B_j}}$ on $\underline{A_i \times B_j}$ also as a sheaf on $W_{ij} := U_i \times V_j \subset \underline{X} \times \underline{Y}$. Then we have

$$\underline{\underline{A_{i_1} \times B_{j_1}}} \mid W_{i_1 j_1} \cap W_{i_2 j_2} = \underline{\underline{A_{i_1 i_2} \times B_{j_1 j_2}}} .$$

The isomorphisms

$$\varphi_{i_2} \circ \varphi_{i_1}^{-1} : A_{i_1 i_2} \longrightarrow A_{i_2 i_1}, \qquad \psi_{j_2} \circ \psi_{j_1}^{-1} : B_{j_1 j_2} \longrightarrow B_{j_2 j_1}$$

induce isomorphisms

$$\tau_{i_2 j_2 i_1 j_1} := \varphi_{i_2} \circ \varphi_{i_1}^{-1} \times \psi_{j_2} \circ \psi_{j_1}^{-1} : A_{i_1 i_2} \times B_{j_1 j_2} \longrightarrow A_{i_2 i_1} \times B_{j_2 j_1} ,$$

hence

$$\underline{\underline{\tau_{i_2 j_2 i_1 j_1}}} : \underline{\underline{A_{i_1} \times B_{j_1}}} \mid W_{i_1 j_1} \cap W_{i_2 j_2} \longrightarrow \underline{\underline{A_{i_2} \times B_{j_2}}} \mid W_{i_1 j_1} \cap W_{i_2 j_2} .$$

The mappings $\tau_{i_2 j_2 i_1 j_1}$ satisfy the usual condicions of compatibility and therefore do define a sheaf $\mathcal{G}$ on $\underline{X} \times \underline{Y}$, which makes $Z := (\underline{X} \times \underline{Y} , \mathcal{G})$ to a ringed space in the sense of [5]. We have local isomorphisms

$$\sigma_{ij} : \underline{\underline{A_i \times B_j}} \longrightarrow \mathcal{G} \mid U_i \times V_j .$$

For each $i \in I$, $j \in J$ we have a bimorphism

$$\phi_{ij} := (\underline{\varphi}_i \times \underline{\psi}_j , \sigma_{ij}) : \quad (U_i \times V_j , \mathcal{G} \mid U_i \times V_j) \longrightarrow A_j \times B_j ,$$

such that

$$\phi_{i_2 j_2} \circ \phi_{i_1 j_1}^{-1} = \tau_{i_2 j_2 i_1 j_1} .$$

$\{ (U_i \times V_j , \phi_{ij}) ; i \in I, j \in J \}$ is a C^N- atlas on Z, which makes Z to an N-differentiable space.

Z with this structure of differentiability satisfies the conditions of the theorem, as follows from the constructions and 1,1 to 1,3. q.e.d.

1,2 and 1,3 now generalise for $\mathcal{R}_N^2$ and $\mathcal{R}_N$. Especially we have a functor $T : \mathcal{R}_N^2 \longrightarrow \mathcal{R}_N$ such that:

$$T (X , Y) \in \mathcal{R}_N \text{ is product of } X \text{ and } Y .$$

Up to equivalence, T is uniquely determined. We write

$$X \times Y := T (X , Y) , \quad h_1 \times h_2 := T (h_1 , h_2)$$

and denote by π the "projection" $\pi : X \times Y \longrightarrow (X , Y)$ from 1,4. We have

$$X \mid U \times Y \mid V \cong X \times Y \mid U \times V .$$

The identity mapping $i : X \longrightarrow X$ gives rise to a differentiable mapping i^*, such that the following diagram is commutative:

$$\begin{tikzcd} X \times X \arrow[rr, "\pi"] & & X \\ & X \arrow[ul, "i^*"] \arrow[ur, "i"'] & \end{tikzcd}$$

i^* is a closed embeding and hence a bimorphism onto an N-differentiable subspace $\Delta_X \subset X \times X$, the diagonal of $X \times X$. Obviously, in this special situation, the bimorphism is even a diffeomorphism. Any differentiable mapping $g: Y \to X$ then can be factored in a unique way, such that the following diagram is commutative:

$$\begin{tikzcd} \Delta_X \arrow[r] & X \times X \arrow[r, "\pi"] & X \\ & Y \arrow[ul] \arrow[ur, "g"'] & \end{tikzcd}$$

The construction of the fiberproduct now is again as usual:

Let $(X, Y) \xrightarrow{f} Z$ be a differentiable mapping. Then there exists an N-differentiable space $X \times_f Y$ with a differentiable mapping $\pi: X \times_f Y \longrightarrow (X, Y)$, which up to diffeomorphisms is uniquely determined by the following property: To each differentiable mapping $g: W \longrightarrow (X, Y)$, $g^*: W \longrightarrow Z$ such that $g^* = f \circ g$, there exists a uniquely determined mapping $h: W \longrightarrow X \times_f Y$ with $g = \pi \circ h$:

$$\begin{tikzcd} X \times_f Y \arrow[r, "\pi"] & (X, Y) \arrow[r, "f"] & Z \\ & W \arrow[ul, "h"] \arrow[u, "g"] \arrow[ur, "g^*"'] & \end{tikzcd}$$

Proof. Let $F: X \times Y \longrightarrow Z \times Z$ be the mapping induced by f. If $Z \times Z = (Z \times Z, \mathcal{C})$, then there exists an ideal subsheaf $\mathcal{C}^* \subset \mathcal{C}$ such that

$$\underline{\Delta_Z} = \text{support } \mathcal{C}/\mathcal{C}^*, \quad \underline{\underline{\Delta_Z}} = \mathcal{C}/\mathcal{C}^* \mid \underline{\Delta_Z}$$

If $X \times Y = (E, \mathcal{E})$, then let $\mathcal{E}^* \subset \mathcal{E}$ be the subsheaf, generated by the image of $\mathcal{C}^*$ under $\underline{\underline{F}}$ in $\mathcal{E}$. Then with $D :=$ support $\mathcal{E}/\mathcal{E}^*$ let

$$X \times_f Y := F_3(D, \mathcal{E}/\mathcal{E}^* | D),$$

where F_3 denotes the smoothing functor of [7], which carries arbitray differenttiable spaces to those of the class $\mathcal{R}_{3N} = \mathcal{R}_N$. For this space the unique factorisation of g by h exists, as follows from the commutative diagramm:

$$\begin{tikzcd} \Delta_Z \arrow[r, hook] & Z \times Z \arrow[rrd] & & \\ X \times_f Y \arrow[u] \arrow[r, shift left] & X \times Y \arrow[l, shift left] \arrow[u] \arrow[r] & (X, Y) \arrow[r] & Z \\ & W \arrow[ul, dashed] \arrow[u] \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

First there exists the mapping $W \longrightarrow X \times Y \longrightarrow Z \times Z$, which factors through $W \longrightarrow \Delta_Z$ and induces a mapping $W \longrightarrow X \times_f Y$, which factors $W \longrightarrow X \times Y$. Existence and uniqueness of h then give, that $X \times_f Y$ is uniquely determined.

In many cases a somewhat weaker notion for a product of differentiable spaces is sufficient. This pseudoproduct, which we are going to describe now, is solution of a weaker universal problem than that in §1. In the classical complex analytic case bothe coincide, in general they certainly do not. So the question arises, in what cases they do coincide.

We stick to the notations of §1 and start with the local description: Let $(x^\circ, y^\circ) \in A_j \times B_j$. Again, let us drop for a while the indices j. Define

|| $(\mathfrak{J} \mp \mathfrak{J}^*)_{(x^\circ,y^\circ)} := \{ f: f \in \widetilde{\mathfrak{D}}^N_{(x^\circ,y^\circ)}$ has in a neighborhood $U(x^\circ) \times V(y^\circ)$ of (x°, y°) a representation $F: U \times V \longrightarrow R$ such that $F^y_x \in \mathfrak{J}_x$, $F^x_y \in \mathfrak{J}^*_y$ for each $x \in U \cap A$, $y \in V \cap B \}$.

Here we used $F^y(x) := F(x,y)$ for fixed y, $F^x(y) := F(x,y,)$ for each fixed x, and $F^y_x \in \mathfrak{D}^N_x$, $F^x_y \in \mathfrak{D}^{*N}_y$ are the corresponding germs.

|| $\mathfrak{J} \mp \mathfrak{J}^*$:= ideal subsheaf of $\widetilde{\mathfrak{D}}^N | A \times B$, defined by the rings $(\mathfrak{J} \mp \mathfrak{J}^*)_{(x,y)}$ described above.

|| $A \mathbin{\bar{\times}} B$:= $(A \times B, \widetilde{\mathfrak{D}}^N / (\mathfrak{J} \mp \mathfrak{J}^*))$.

We obviously have $A \mathbin{\bar{\times}} B \in \widetilde{\mathcal{R}}_N$. Again we have natural induced differentiable projections, also denoted by π:

|| $\pi^1: A \mathbin{\bar{\times}} B \longrightarrow A$, $\pi^2: A \mathbin{\bar{\times}} B \longrightarrow B$, $\pi = (\pi^1, \pi^2)$.

The following definition will be usefull:

|| A differentiable mapping $\tilde{h}: A_1 \mathbin{\bar{\times}} B_1 \longrightarrow A_2 \mathbin{\bar{\times}} B_2$ is called of product type, if there exists a differentiable mapping $h: (A_1, B_1) \longrightarrow (A_2, B_2)$ such that $h \circ \pi_1 = \pi_2 \circ \tilde{h}$. Hence the following diagramm commutes:

$$(*) \qquad \begin{array}{ccc} A_1 \mathbin{\bar{\times}} B_1 & \xrightarrow{\tilde{h}} & A_2 \mathbin{\bar{\times}} B_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ (A_1, B_1) & \xrightarrow{h} & (A_2, B_2). \end{array}$$

The class of N-differentiable spaces $A \mathbin{\bar{\times}} B$ with $A, B \in \widetilde{\mathcal{R}}_N$ together with the differentiable mappings of product type constitutes a subcategory $\widetilde{\mathcal{R}}^*_N \subset \widetilde{\mathcal{R}}_N$. The following corresponds to corrollary 1,2:

Lemma 2,1. Let $h = (h_1, h_2): (A_1, B_1) \longrightarrow (A_2, B_2)$ be differentiable. Then there exists exactly one differentiable mapping $\tilde{h}: A_1 \mathbin{\bar{\times}} B_1 \longrightarrow A_2 \mathbin{\bar{\times}} B_2$, such that the corresponding diagram (*) is commutative. We write $\widetilde{h} := h_1 \bar{\times} h_2$.

Proof. Uniqueness: Let h and h^* be given mappings $A_1 \bar{\times} B_1 \longrightarrow A_2 \bar{\times} B_2$ such that $\pi_2 \circ h^* = h \circ \pi_1 = \pi_2 \circ \widetilde{h}$. We have to show: $\widetilde{h} = h^*$. Let $z = (x,y) \in A_1 \times B_1$. In z our $\widetilde{h}$ and h^* are generated by N-differentiable mappings $(\widetilde{H}_1, \widetilde{H}_2)$ and (H_1^*, H_2^*) with $\widetilde{H}_1, H_1^* : (R^{m_1} \times R^{m_1})_z \longrightarrow R_u^{n_2}$, $\widetilde{H}_2, H_2^* : (R^{m_1} \times R^{m_1})_z \longrightarrow R_v^{m_2}$. Using $\pi_2 \circ h^* = \pi_2 \circ \widetilde{h}$ we find, that the components of $\widetilde{H}_1 - H_1^*$ and of $\widetilde{H}_2 - H_2^*$ are in $(\mathfrak{I}_1 \mp \mathfrak{I}_1^*)$. By [5], Lemma 3,1 we therefore have $h^* = \widetilde{h}$, because $z \in A_1 \times B_1$ was arbitrary.

Existence. Let again $z = (x,y) \in A_1 \times B_1$. If h_1 is in x generated by an N-differentiable mapping $H_1 : R_x^{m_1} \longrightarrow R_u^{n_2}$, h_2 in y by $H_2 : R_y^{m_1} \longrightarrow R_v^{m_2}$, then, as is immediately verified, (H_1, H_2) generate for some neighbourhoods $U(x) \subset A$, $V(y) \subset B$ a differentiable mapping

$$\widetilde{h}^{(x,y)} : A_1 \bar{\times} B_1 \mid U(x) \times V(y) \longrightarrow A_2 \bar{\times} B_2$$

with $\pi_2 \circ \widetilde{h}^{(x,y)} = h \circ \pi_1 \mid U(x) \times V(y)$. Because of uniqueness the different mappings $\widetilde{h}^{(x,y)}$ define a global differentiable mapping of the required type.

Corollary 2,2. (compare with 1,3). By

$$\Gamma : (A, B) \longrightarrow A \bar{\times} B, \qquad \Gamma : h \longrightarrow \widetilde{h}$$

there is defined a covariant functor of $\widetilde{\mathcal{R}}_N^2$ into $\widetilde{\mathcal{R}}_N^*$. Especially we have:

$$\Gamma(g \circ h) = \Gamma(g) \circ \Gamma(h) \ ; \text{ is h a diffeomorphism, so is } \Gamma(h).$$

To describe the universal property, which is the basis for the pseudoproduct define:

A differentiable mapping $f : A_1 \bar{\times} B_1 \longrightarrow (A_2, B_2)$ is called of product type, if there exists a differentiable mapping $h : (A_1, B_1) \longrightarrow (A_2, B_2)$ such that $f = h \circ \pi_1$.

Corollary 2,3. Let $D \in \widetilde{\mathcal{R}}_N^*$, $f : D \longrightarrow (A_2, B_2)$ be of product type. Then there exists exactly one differentiable mapping

$$g : D \longrightarrow A_2 \bar{\times} B_2 \quad \text{with} \quad f = \pi_2 \circ g.$$

Proof. Let $D = A_1 \bar{\times} B_1$, $h : (A_1, B_1) \longrightarrow (A_2, B_2)$ with $f = h \circ \pi_1$, then 2,3 follows from 2,1:

$$\begin{tikzcd}
A_2 \bar{\times} B_2 \arrow[r, "\pi_2"] & (A_2, B_2) \\
A_1 \bar{\times} B_1 \arrow[u, "g"] \arrow[ur, "f"] \arrow[r, "\pi_1"'] & (A_1, B_1) \arrow[u, "h"']
\end{tikzcd}$$

The following obvious remark however should be noted:

Remark 2,4. α) Let $D \subset \tilde{\mathcal{R}}_N$ be reduced, $f: D \longrightarrow (A_2, B_2)$ differentiable. Then there exists exactly one differentiable mapping

$$g: D \longrightarrow A_2 \tilde{\times} B_2 \quad \text{with} \quad f = \pi_2 \circ g.$$

β) If A_2, B_2 are reduced, so is $A_2 \tilde{\times} B_2$.

γ) $\tilde{\times}$ is in the category of reduced differentiable spaces a product.

Due to lemma 2,1 one can now construct to each $X, Y \in \mathcal{R}_N$ a pseudoproduct $X \tilde{\times} Y$ and to each differentiable mapping $h: (X, Y) \longrightarrow (Z, W)$ a differentiable mapping $\tilde{h}: X \tilde{\times} Y \longrightarrow Z \tilde{\times} W$ in a similar way as was described in §1 for the product $\times$. Again we have a natural projection, also denoted by π:

$$\pi: \quad X \tilde{\times} Y \longrightarrow (X, Y)$$

2,1, 2,2 and 2,3 now generalise with the corresponding definitions. Let $\mathcal{R}_N^*$ denote the category of N-differentiable spaces of product type with differentiable mappings of product type, then:

We have a natural covariant functor $T: \mathcal{R}_N^2 \longrightarrow \mathcal{R}_N^*$ defined by

$$T(X, Y) = X \tilde{\times} Y, \qquad T(h) = \tilde{h}.$$

Remark 2,5. Let $X, Y \in \mathcal{R}_N$, $y^\circ \in \underline{Y}$. y° has a unique reduced N-differentiable structure for all N, namely (y°, C), short: y°. The natural embedding $y^\circ \longrightarrow Y$ together with the identity $X \longrightarrow X$ therefore induce differentiable mappings:

$$X \times y^\circ \longrightarrow X \times Y, \quad X \tilde{\times} y^\circ \longrightarrow X \tilde{\times} Y.$$

One finds, that the projections

$$X \times y^\circ \longrightarrow X \quad \text{and} \quad X \tilde{\times} y^\circ \longrightarrow X$$

are diffeomorphisms and $X \times y^\circ = X \tilde{\times} y^\circ$.

§ 3 Contractible differentiable spaces.

Let $D \subset R^n$, $x^\circ \in D$ and $D^N \in \mathcal{R}_N$ be the reduced space defined by D. Then D^N is called r-contractible at x°, if to each neighbourhood $U(x^\circ) \subset D$ there exists a neighbourhood $V(x^\circ) \subset U(x^\circ)$ of x° and a N-differentiable mapping

$$\phi : D \cap V \times I \longrightarrow D \cap U \quad \text{with} \quad I = \{t;\ 0 \le t \le 1\}, \text{ such that}$$

$$\phi(x, 1) = x \qquad \text{for all } x \in D \cap V$$

$$\phi(x, 0) \subset B \subset D \cap U \quad \text{for all } x \in D \cap V, \text{ where embdim } B \le r.$$

This generalises for arbitrary N-differentiable spaces as follows:

Def.3,1. $X \in \mathcal{R}_N$ is called strongly (weakly) r-contractible at $x^\circ \in X$, if to each neighbourhood $U(x^\circ) \subset X$ there exists a neighbourhood $V(x^\circ) \subset U(x^\circ)$ of x°, a subspace $Y \in \mathcal{R}_N$ of X : $Y \hookrightarrow X \mid U(x^\circ)$ with embdim $Y \le r$, satisfying A_4 of [5], p.275, and a differentiable mapping

$$\phi : \quad (X \mid V(x^\circ)) \times I^N \longrightarrow X \mid U(x^\circ)$$

$$\text{(resp.} \quad \phi : \quad (X \mid V(x^\circ)) \bar{\times} I^N \longrightarrow X \mid U(x^\circ) \quad),$$

such that for the composed mappings

$$\phi_i : (X \mid V(x^\circ)) \times i \longrightarrow (X \mid V(x^\circ)) \times I^N \longrightarrow X \mid U(x^\circ), \quad i \in I,$$

(resp. the same for $\bar{\times}$) we have :

ϕ_1 is the natural projection $X \mid V(x^\circ) \times 1 \longrightarrow X \mid V(x^\circ)$

ϕ_0 maps $X \mid V(x^\circ) \times 0$ into Y.

ϕ is called r-contraction. We say contractible instead of 0-contractible.

If $r = 0$, then Y above is reduced and consists of isolated points. If X is reduced and in $x^\circ \in X$ weakly r-contractible, then we may assume without any restriction, that also Y above is reduced.

Let $X = (D, \mathcal{D}^N/\mathcal{J}) \subset R^n$. We want to compare the following properties for $x^\circ \in X$:

a) X is in x° strongly r-contractible.

b) X is in x° weakly r-contractible.

c) To each neighbourhood $U(x^\circ) \subset R^n$ of x° there exists a neighbourhood $V(x^\circ) \subset U(x^\circ)$ and an N-differentiable mapping $\varphi : V \times I \longrightarrow U$ such that:

1) φ is an r-contraction,

2) φ induces an r-contraction $(X \mid V) \times I^N \longrightarrow X \mid U$.

d) The same as c), with $(X \mid V) \overline{\times} I^N$ instead of $(X \mid V) \times I^N$ in c), 2).

Lemma 3,2. For $X = (D, \mathcal{D}^N/\mathcal{J})$ we have

1) a) $\rightsquigarrow$ b), c) $\rightsquigarrow$ d). 2) a) $\leftrightsquigarrow$ c), b) $\leftrightsquigarrow$ d).

3) b) $\rightsquigarrow$ a), d) $\rightsquigarrow$ c), if $\infty \leq N$ and D satisfies properties A_{13} and A_2 of [4], p. 275.

Proof. 1) is obvious, because one hase the natural mapping $X \overline{\times} Y \longrightarrow X \times Y$.

2): c) $\rightsquigarrow$ a), d) $\rightsquigarrow$ b) follow by definition. We show a) $\rightsquigarrow$ c), b) $\rightsquigarrow$ d):

Let $U(x^\circ) \subset R^m$ be given. Then there exists a neighbourhood $V(x^\circ) \subset U(x^\circ)$ and an r-contraction

$$\phi : (X \mid D \cap V) \times I^N \longrightarrow X \mid D \cap U .$$

Let $Y = (B, \mathcal{D}^N/\tilde{\mathcal{J}}) \subset X \mid D \cap V$ be an N-differentiable space with embdim Y $\leq r$ and $\phi_0((X \mid D \cap V) \times 0) \subset Y$. I is compact. Then there exist finitely many relatively open intervals $I^\nu \subset I$ and a neighbourhood $W(x^\circ) \subset D$, such that

α) $\bigcup_{\nu=1}^{\lambda} I^\nu = I$,

β) To each ν there exists a $\varphi^\nu \in H^0(W \times I^\nu, \tilde{\mathcal{D}}^N)^n$, which generates $\phi \mid W \times I^\nu$. Here $\tilde{\mathcal{D}}^N$ is the sheaf of germs of functions in $R^m \times R^1$ (respectively in $C^m \times C^1$ in case $N = \omega^*$) of class C^N. For each component j we therefore have:

(*) $(\varphi^\nu - \varphi^\mu)_j \mid \{x^\circ\} \times (I^\nu \cap I^\mu) \quad \in \quad H^0(\{x^\circ\} \times (I^\nu \cap I^\mu), \mathcal{J} + 0)$.

(Here 0 is the zero-idealsheaf of the structure sheaf $\mathcal{D}^{*N}$ on R^1 (resp. on C^1). So the elements in (*) define an element in $H^1(\{x^\circ\} \times I, \mathcal{J} + 0) =: H^1$. But we have $H^1 = 0$ (In case $N \leq \infty$, because $\mathcal{J} + 0$ is fine; in case $N = \omega$ or $N = \omega^*$ this follows essentially from theorem B of the analytic sheaftheory). Hence there exists a $\varphi^* \in H^0(\{x^\circ\} \times I, \tilde{\mathcal{D}}^N)^m$ such that:

$$(\varphi^* - \varphi^\nu)_j \mid \{x^\circ\} \times I^\nu \quad \in \quad H^0(\{x^\circ\} \times I^\nu, \mathcal{J} + 0).$$

Hence we may suppose $\varphi^* \in H^0(V(x^\circ) \times I, \tilde{\mathcal{D}}^N)^m$ and

$$(\varphi^* - \varphi^\nu)_j \mid W(x^\circ) \times I^\nu \quad \in \quad H^0(W(x^\circ) \times I^\nu, \mathcal{J} + 0)$$

for all j and a sufficiently small neighborhood $W(x^\circ) \subset D$ of x°(eventually we make the I^ν a little bit smaller). It follows, that φ^* generates the differentiable mapping $\phi \mid W \times I$. Making $V(x^\circ) \subset R^m$ smaller, if necessary, as well as $W(x^\circ)$, we may assume, that φ^* is induced by an element

$$\varphi' \quad \in \quad H^0(V(x^\circ) \times U(I), \tilde{\mathcal{D}}^N)^m,$$

with $U(I) \subset R^1$ some neighbourhood of I. Let

$$\varphi_0(x) := \varphi'(x,0), \quad \varphi_1(x) := \varphi'(x,1), \quad y^o := \varphi_0(x^o).$$

Because of embdim $Y \leq r$ there exists in a neighbourhood $V(y^o) \subset U(x^o)$ an r-dimensional submanifold $M \subset V(y^o)$ such that

α) $M \supset B \cap V(y^o)$,

β) If $f_z \in \mathcal{D}_z^N$, $f_z | M_z = 0$, then $f_z \in \tilde{\mathcal{J}}_z$ for all $z \in B \cap V(y^o)$.

If $V(y^o)$ is properly chosen, then M is an N-differentiable retract of $V(y^o)$, that means

γ) There exists an N-differentiable mapping

$\pi : V(y^o) \longrightarrow M$ such that $\pi | M = \mathrm{id}$.

Hence: $(\mathrm{id} - \pi)_{jz} \in \tilde{\mathcal{J}}_z$ for each component j and each $z \in B \cap V(y^o)$.

Now let $V(x^o)$ be so small, such that $\varphi_0(V(x^o)) \subset V(y^o)$. For each $x \in D \cap V(x^o)$ we have than $\varphi_0(x) \in B$ and

$$((\mathrm{id} - \pi) \circ \varphi_0)_{jx} = (\varphi_0 - \pi \circ \varphi_0)_{jx} \in \mathcal{J}_x ,$$

$$(\varphi_1 - \mathrm{id})_{jx} \in \mathcal{J}_x .$$

If

$$\varphi'' := (\varphi_0 - \pi \circ \varphi_0) + t \cdot (\varphi_1 - \mathrm{id} - (\varphi_0 - \pi \circ \varphi_0)),$$

we have $\varphi''_{j(x,t)} \in \mathcal{J}_x + 0_t$ for all $(x,t) \in D \cap V(x^o) \times I$ and each j. With

$$\varphi := \varphi' - \varphi''$$

finally we have :

α) $\varphi_{j(x,t)} - \varphi'_{j(x,t)} \in \mathcal{J}_x + 0_t$ for all $(x,t) \in D \cap V(x^o) \times I$. That means: φ generates the same morphism as φ', hence it generates $\phi | D \cap V(x^o) \times I$.

β) $\varphi(x,0) = \pi \circ \varphi_0(x)$, hence $\varphi(x,0) \subset M$ for all $x \in V(x^o)$,

$\varphi(x,1) = \mathrm{id}$.

Hence $\varphi : V(x^o) \times I \longrightarrow U(x^o)$ is a required mapping for V small.q.e.d. Finally b) $\rightsquigarrow$ d) follows in the same way by substituting $\bar{+}$ for $+$ and $\bar{\times}$ for $\times$.

3) In this case one can show, that $\bar{\times} = \times$.

Remark 3.3. D^N is r-contractible at $x^o \in D$, iff D^N is weakly r-contractible at $x^o \in D$.

Proof. Let D^N be r-contractible at $x^o \in D$, $U(x^o) \subset D$ a neighbourhood of x^o. then there exists a neighbourhood $V(x^o) \subset U(x^o)$ and an r-contraction

$$\phi : V(x^o) \times I \longrightarrow U(x^o).$$

φ obviously defines also an r-contraction $(D^n \mid V(x^o)) \times \dot{I}^n \longrightarrow D^N \mid U(x^o)$.
This proves one direction, the other direction follows from 3.2 , b) $\to$ d).

§4 Differential forms on N-differentiable spaces ($N \geq 3$!).

Here we describe the analogon of differential forms in algebraic and analytic geometry for our differentiable geometry. We encounter some difficulties, which however can be overcome. It is usefull, to introduce three types of differential forms, which in general are different. In certain cases some of them coincide, for example if $N = \omega^*$ or in some cases, when $N = \infty$. We start with the local description. So let $D = (D , \mathcal{D}^N/\mathcal{J}) \subset R^n$ be an arbitrary N- differentiable space, always with $N \geq 3$ of course.

A) Sheaves of p-forms on D.

Let ${}^{\nu}\underset{\sim}{\Omega}{}^{p}$ denote the sheaf of germs of p-forms of class C^{ν} in R^n . Let ${}^{N-1}\Omega^{p} \subset {}^{N-1}\underset{\sim}{\Omega}{}^{p}$ denote the $\mathcal{D}^N$ - subsheaf of elements $\omega \in {}^{N-1}\underset{\sim}{\Omega}{}^{p}$, such that $d\omega \in {}^{N-1}\Omega^{p+1}$. Let ${}^{\nu}\underset{\sim}{\Omega}{}^{-1} = {}^{\nu}\Omega^{-1} = C$, ${}^{\nu}\underset{\sim}{\Omega}{}^{-2} = {}^{\nu}\Omega^{-2} = 0$. Let

$$\mathcal{J}_1 := \mathcal{J}$$

$\mathcal{J}_2 := \overline{\mathcal{J}\cdot\mathcal{D}^{N-2}}$ (i.e. $\mathcal{J}\cdot\mathcal{D}^{N-2}$ is the ($\mathcal{D}^{N-2}|D$) - subsheaf of $\mathcal{D}^{N-2}|D$, generated by $\mathcal{J}$, and $\overline{\mathcal{J}\cdot\mathcal{D}^{N-2}}$ its topological closure in the sense of [5], p. 276.

$\mathcal{J}_3 := {}^{N-2}\mathcal{J}(D) :=$ ideal sheaf of elements of $\mathcal{D}^{N-2}|D$, vanishing on D.

With these and $d: {}^{\nu}\underset{\sim}{\Omega}{}^{p} \longrightarrow {}^{\nu-1}\underset{\sim}{\Omega}{}^{p+1}$ the differential operator, define

$$\mathcal{J}_i^p := (\mathcal{J}_i \cdot {}^{N-2}\underset{\sim}{\Omega}{}^{p} + d(\mathcal{J}_i \cdot {}^{N-2}\underset{\sim}{\Omega}{}^{p-1})) \cap {}^{N-1}\Omega^{p}$$

$$\mathcal{J}_i^{o} := (\mathcal{J}_i \cdot {}^{N-2}\underset{\sim}{\Omega}{}^{o}) \cap {}^{N-1}\Omega^{o} \subset (\mathcal{J}\cdot\mathcal{D}^{N-2}) \cap \mathcal{D}^N \quad , \mathcal{J}_i^{-1} := 0.$$

We have: $\mathcal{J}_1 \subset \mathcal{J}_2 \subset \mathcal{J}_3$, then $\mathcal{J}_1^p \subset \mathcal{J}_2^p \subset \mathcal{J}_3^p$, and each $\mathcal{J}_i^p \subset {}^{N-1}\Omega^{p}|D$ is a $(\mathcal{D}^N|D)$ - subsheaf with $\mathcal{J}\cdot{}^{N-1}\Omega^{p} \subset \mathcal{J}_i^p$ for $i = 1,2,3$.

|| ${}^{N-1}\Omega^{p}/\mathcal{J}_i^p$ $(:= ({}^{N-1}\Omega^{p}|D)/\mathcal{J}_i^p)$ is a $\mathcal{D}^N/\mathcal{J}$ - sheaf, called

1) sheaf of p-forms on D , if $i = 1$,
2) sheaf of weak p-forms on D , if $i = 2$,
3) sheaf of reduced p-forms on D, if $i = 3$.

Remark 4,1 α). We have $\mathcal{D}^N = {}^N\Omega^0$, $\mathcal{J}_1^0 = \mathcal{J}$, if $N = \infty, \omega, \omega^*$;
p-form = weak p-form, if $N = \omega, \omega^*$, or if $N = \infty$ and $\mathcal{J}$ is closed;
p-form = weak p-form = reduced p-form, if D is reduced and $N = \infty, \omega, \omega^*$.

β) $\mathcal{J}_1$ may be substituted by $\mathcal{J}\cdot\mathcal{D}^{N-2}$ in the definition of $\mathcal{J}_1^p$. We have $d(\mathcal{J}\cdot\mathcal{D}^{N-1})\wedge {}^{N-1}\underset{\sim}{\Omega}^{p-1} \subset \mathcal{J}\cdot{}^{N-2}\underset{\sim}{\Omega}^p + d(\mathcal{J}\cdot{}^{N-2}\underset{\sim}{\Omega}^{p-1})$.

γ) We have $\mathcal{D}^N = {}^{N-1}\Omega^0$, hence $\mathcal{J}_1^0 = (\mathcal{J}\cdot\mathcal{D}^{N-2})\cap\mathcal{D}^N$, hence if $\mathcal{J}_1^0 = \mathcal{J}$ then $\mathcal{D}^N/\mathcal{J} = {}^{N-1}\Omega^0/\mathcal{J}_1^0$. This holds for example if D is reduced.

B) Differential operater.

We have the exact sequence (see the proofs on pages 18 - 19)

$$(*)\qquad {}^{N-1}\Omega^p \xrightarrow{d} {}^{N-1}\Omega^{p+1} \longrightarrow {}^{N-1}\Omega^{p+2} \longrightarrow {}^{N-1}\Omega^{p+3} \longrightarrow \cdots ,$$

and we have obviously $d(\mathcal{J}_i^p) \subset \mathcal{J}_i^{p+1}$ for i=1,2,3. Hence (*) induces the following complex

$$(**)\qquad \mathcal{J}_i^p \xrightarrow{d} \mathcal{J}_i^{p+1} \longrightarrow \mathcal{J}_i^{p+2} \longrightarrow \mathcal{J}_i^{p+3} \longrightarrow \cdots$$

and therefore the complex

$$(***)\qquad {}^{N-1}\Omega^p/\mathcal{J}_i^p \xrightarrow{d} {}^{N-1}\Omega^{p+1}/\mathcal{J}_i^{p+1} \longrightarrow {}^{N-1}\Omega^{p+2}/\mathcal{J}_i^{p+2} \longrightarrow \cdots$$

Remark 4,2. (***) is exact at $({}^{N-1}\Omega^{p+1}/\mathcal{J}_i^{p+1})_{x^0}$ iff (**) is exact at $\mathcal{J}_{i\,x^0}^{p+2}$.

The proof is obvious. Hence exactnes of (***) depends on exactnes of (**). In general exactnes fails (see [2]). We give condicions for exactnes of (**) in §5, which in the reduced complex case are due to Reiffen [2], but see also a related statement in [3] . But even in the case $N = \omega^*$ our §5 is slightly more general, because we do not assume neither that $(D , \mathcal{D}^{\omega^*}/\mathcal{J})$ is reduced, or $\mathcal{J}$ is coherent, or $\mathcal{D}$ is analytic.

C) Change under differentiable mappings.

Let $D' = (D', \mathcal{D}'^N/\mathcal{J}') \subset R^{n'}$, $D^* = (D^*, \mathcal{D}^{*N}/\mathcal{J}^*) \subset R^{n^*}$ be two other spaces from $\widetilde{\mathcal{R}}_N$ with differentiable mappings

$$\phi : D \longrightarrow D', \qquad \psi : D' \longrightarrow D^*.$$

In what follows, entities with a stipe " ' " refer to D', those with a star " * " refer to D^* .

Let $x^\circ \in D$, $y^\circ = \phi(x^\circ)$. There exists aneighbourhood $U(x^\circ) \subset R^n$ and an N-differentiable mapping $\psi : U(x^\circ) \longrightarrow R^m$, which generates $\phi \,|\, U(x^\circ) \cap D$. For each $x \in D \cap U$, $y = \psi(x)$, $1 \leq \nu < N$, ψ induces by $\omega \longrightarrow \psi^*(\omega)$ ($= \omega \circ \psi$) an homomorphism (because $d(\omega \circ \psi) = (d\omega) \circ \psi$):

$$\psi^* : {}^{\nu}\Omega'^{p}_{y} \longrightarrow {}^{\nu}\Omega^{p}_{x} .$$

One verifies, that $\psi^*(\mathcal{J}'^{p}_{iy}) \subset \mathcal{J}^{p}_{ix}$ for $i=1,2,3$. The following diagram is commutative:

$$\begin{tikzcd}
 & {}^{N-1}\Omega'^{p+1}_{y} \arrow[rr, "\psi^*"] & & {}^{N-1}\Omega^{p+1}_{x} \\
{}^{N-1}\Omega'^{p}_{y} \arrow[ur, "d"] \arrow[rr] & & {}^{N-1}\Omega^{p}_{x} \arrow[ur] & \\
 & \mathcal{J}'^{p+1}_{iy} \arrow[uu, hook] \arrow[rr, dashed] & & \mathcal{J}^{p+1}_{ix} \arrow[uu, hook] \\
\mathcal{J}'^{p}_{iy} \arrow[uu, hook] \arrow[ur, dashed] \arrow[rr] & & \mathcal{J}^{p}_{ix} \arrow[uu, hook] \arrow[ur] &
\end{tikzcd}$$

ψ now induces homomorphisms ${}^{N-1}\Omega'^{p}_{y}/\mathcal{J}'^{p}_{iy} \longrightarrow {}^{N}\Omega^{p}_{x}/\mathcal{J}^{p}_{iy}$ and therefore a sheafhomomorphimm

$$D \oplus_{\phi} {}^{N-1}\Omega'^{p}/\mathcal{J}'^{p}_{i} \Big| D \cap U \longrightarrow {}^{N}\Omega^{p}/\mathcal{J}^{p}_{i} .$$

One shows (using 4,1 β)), that this homomorphism does not depend on the representation ψ of $\phi \,|\, D \cap U$. So we have an induced sheaf homomorphism

$$\tilde{\phi}_i : D \oplus_{\phi} {}^{N-1}\Omega'^{p}/\mathcal{J}'^{p}_{i} \longrightarrow {}^{N-1}\Omega^{p}/\mathcal{J}^{p}_{i}$$

(short: ${}^{N-1}\Omega'^{p}/\mathcal{J}'^{p}_{i} \longrightarrow {}^{N-1}\Omega^{p}/\mathcal{J}^{p}_{i}$). One has in a natural way

a) $\tilde{\phi}_i \circ \tilde{\psi}_i = \widetilde{\psi_i \circ \phi_i}$ b) $\tilde{\phi}_i = \mathrm{id}$, if $\phi = \mathrm{id}$.

c) $\tilde{\psi}_i$ injective, $\tilde{\phi}_i$ surjective, if $\psi \circ \phi = \mathrm{id}$

d) $\tilde{\phi}_i$ is an isomorphism, if ϕ is. e) $\tilde{\phi}_i \circ d = d \circ \tilde{\phi}_i$,

if d denotes each of the following natural homomorphisms:

$$d : {}^{N-1}\Omega^{p}/\mathcal{J}^{p}_{i} \longrightarrow {}^{N-1}\Omega^{p+1}/\mathcal{J}^{p+1}_{i} ,$$

$$d : {}^{N-1}\Omega'^{p}/\mathcal{J}'^{p}_{i} \longrightarrow {}^{N-1}\Omega'^{p+1}/\mathcal{J}'^{p+1}_{i}$$

$$d : D \oplus_{\phi} {}^{N-1}\Omega'^{p}/\mathcal{J}'^{p} \longrightarrow D \oplus_{\phi} {}^{N-1}\Omega'^{p+1}/\mathcal{J}'^{p+1} .$$

Especially there exists on each space $X = (X, \mathcal{A}) \in \mathfrak{R}_N$, $N \geq 3$, certain $\mathcal{A}$- sheaves Ω^{p}_{i}, $i=1,2,3$, called sheaf of p-forms, if $i=1$, sheaf of weak p-forms, if $i=2$; sheaf of reduced p-forms, if $i=3$; and homomorphisms $d : \Omega^{p}_{i} \longrightarrow \Omega^{p+1}_{i}$, such that

1) $\Omega^{p}_{i} \xrightarrow{d} \Omega^{p+1}_{i} \xrightarrow{d} \Omega^{p+2}_{i} \xrightarrow{d}$ is a complex .

2) If $\{(U_j, \phi_j);\ j \in J\}$ is an atlas of X, $\phi_j(X|U_j) = (D_j, \mathcal{D}_j^N/\mathcal{J}_j)$, then one hase isomorphisms for all p

$$\tilde{\phi}_{ji} : {}^{N-1}\Omega^p/\mathcal{J}_{ji}^p \;\; (= U_j \oplus_{\phi_j} {}^N\Omega^p/\mathcal{J}_{ji}^p) \longrightarrow \Omega_i^p|U_j$$

with $\tilde{\phi}_{\ell i}^{-1} \circ \tilde{\phi}_{ji} = \widetilde{(\phi_j \circ \phi_\ell^{-1})_i}$, $\tilde{\phi}_{ji} \circ d = d \circ \tilde{\phi}_{ji}$. a)-e) generalise for the sheaves Ω_i^p on X.

Finally: The product

$$\wedge : ({}^{N-1}\Omega^p, {}^{N-1}\Omega^q) \longrightarrow {}^{N-1}\Omega^{p+q}$$

induces a product

$$\wedge : ({}^{N-1}\Omega^p/\mathcal{J}_i^p, {}^{N-1}\Omega^q/\mathcal{J}_i^q) \longrightarrow {}^{N-1}\Omega^{p+q}/\mathcal{J}_i^{p+q}$$

with $d(\omega^p \wedge \omega^q) = d\omega^p \wedge (-1)^p \cdot d\omega^q$, $\quad \omega^p \wedge \omega^q = (-1)^{p \cdot q}\omega^q \wedge \omega^p$.

One sees then, that $\wedge$ induces for arbitrary $X \in \mathcal{R}_N$ a product on the p-forms on X :

$$\wedge : (\Omega_i^p, \Omega_i^q) \longrightarrow \Omega_i^{p+q}$$

with $\omega^p \wedge \omega^q = (-1)^{p \cdot q}\omega^q \wedge \omega^p$, $d(\omega^p \wedge \omega^q) = d\omega^p \wedge \omega^q + (-1)^p \cdot \omega^p \wedge d\omega^q$.

Theorem 5.1. Let $X \in \mathcal{R}_N$, $x^\circ \in X$ and for $i=1,2,3$

$$(i) \qquad \Omega^{r-1}_{i\,x^\circ} \xrightarrow{d} \Omega^{r}_{i\,x^\circ} \xrightarrow{d} \Omega^{r+1}_{i\,x^\circ} \longrightarrow \Omega^{r+2}_{i\,x^\circ} \longrightarrow \cdots$$

be the complexes of the three different types of differential forms on X at x°.

a) If X is strongly r-contractible at x°, then (1), (2), (3) are exact.

b) If X is weakly r-contractible at x°, then (2) and (3) are exact.

Proof. We may assume $X = (D, \mathcal{D}^N/\mathcal{J}) \subset R^n$. Then we have to show the exactness of

$$\mathcal{J}^{r}_{i\,x^\circ} \longrightarrow \mathcal{J}^{r+1}_{i\,x^\circ} \longrightarrow \mathcal{J}^{r+2}_{i\,x^\circ} \longrightarrow \cdots$$

Let $r < p$ and $\omega^p_{x^\circ} \in \mathcal{J}^p_{i\,x^\circ}$ for a fixed i. ω^p_x has in a neighbourhood $W(x^\circ) \subset \subset R^n$ some representation

$$\omega^p = \sum_j f_j \cdot {}^j\omega^p + d\Big(\sum_j g_j \cdot {}^j\omega^{p-1}\Big),$$

where $f_j, g_j \in H^\circ(W, \mathcal{D}^N) \cap H^\circ(W \cap D, \mathcal{J}_i)$, ${}^j\omega^\pi \in H^\circ(W, {}^{N-2}\Omega^\pi)$ for $\pi = p, p-1$.

There exists a neighbourhood $V(x^\circ) \subset W(x^\circ)$ and an N-differentiable r-contraction $\varphi : V \times I \longrightarrow W$, such that the properties c), 1), 2) of § 3, p. 11 hold. Write

$$\omega^p \circ \varphi = \omega^p_1 + \omega^{p-1}_2 \wedge dt,$$

$${}^j\omega^p \circ \varphi = {}^j\omega^p_1 + {}^j\omega^{p-1}_2 \wedge dt,$$

$${}^j\omega^{p-1} \circ \varphi = {}^j\omega^{p-1}_1 + {}^j\omega^{p-2}_2 \wedge dt.$$

Here all differential forms are defined on $V \times I$. The forms with index 1 do not contain dt, and ω^{p-1}_2 is of class C^{N-1}. Let d_x denote differentiation with respect to $x \in U(x^\circ)$, d_t differentiation with respect to $t \in I$, then we have:

$$(*) \qquad \omega^{p-1}_2 \wedge dt = \Big(\sum_j f_j\circ\varphi \cdot {}^j\omega^{p-1}_2\Big)\wedge dt + d_t\Big(\sum_j g_j\circ\varphi \cdot {}^j\omega^{p-1}_1\Big) + d_x\Big(\sum_j g_j\circ\varphi \cdot {}^j\omega^{p-2}_2\Big)\wedge dt.$$

If $d\,\omega^p_{x^\circ} = 0$, we may assume $d\,\omega^p = 0$. Then

$$0 = d(\omega^p \circ \varphi) = d_x\omega^p_1 + d_t\omega^p_1 + d_x\omega^{p-1}_2 \wedge dt$$

$$0 = d_x\omega^p_1, \quad 0 = d_t\omega^p_1 + d_x\omega^{p-1}_2 \wedge dt$$

$$\omega^p(x) = \omega^p_1(x,1) = \omega^p_1(x,1) - \omega^p_1(x,0)$$

$$= \int_0^1 d_t\omega^p_1 = -d_x\int_0^1 \omega^{p-1}_2 \wedge dt$$

(integration with respect to t). Here we have used: $\varphi(x,1) \equiv x$, hence $\omega^p(x) = \omega^p_1(x,1)$, and $\omega^p_1(x,0) = 0$, because $\varphi(V(x^\circ)\times\{0\}) \subset W$ is contained locally in a not more than r-dimensional n-differentiable manifold with $r < p$. If

$$\omega^{p-1} := -\int_0^1 \omega^{p-1}_2 \wedge dt, \text{ we have in } V(x^\circ):$$

$$\omega^p = d\,\omega^{p-1}.$$

Obviously, ω^{p-1} is of class C^{N-1}. [illegible] ω^{p}. Hence $d\omega^{p-1}$ is of class C^{N-1}. Hence $\omega^{p-1} \in H^{o}(V, {}^{N-1}\Omega^{p-1})$. Finally we even have

$$\omega^{p-1}_{x^o} \in (\mathfrak{I}_i \cdot {}^{N-2}\underset{\sim}{\Omega}^{p} + d(\mathfrak{I}_i \cdot {}^{N-2}\underset{\sim}{\Omega}^{p-1}))_{x^o},$$

because integrating (*) gives ω^{p-1} as a sum of elements of the form

$$F \cdot \tilde{\omega}^{p-1} \qquad \text{(integrating the first sum in (*))},$$

$$G \cdot \tilde{\omega}^{p-1} = g_j \circ \varphi(x,t_o) \cdot {}^{1}\omega^{p-1}_{1}(x,t_o) \quad \text{(integrating the second sum in (*))},$$

$$d(H \cdot \tilde{\omega}^{p-2}) \qquad \text{(integrating the third sum in (*))}.$$

Here $\tilde{\omega}^{\pi} \in H^{o}(V(x^o), {}^{N-2}\underset{\sim}{\Omega}^{\pi})$ for $\pi = p-1, p-2$ and

$$(**) \qquad F = \int_0^1 f_j \circ \varphi(x,t) \cdot k(x,t)\, dt,$$

$k \in H^{o}(V \times I, \tilde{\mathcal{D}}^{N-2})$ and H being of the same form as F. We now specialize.

Case a) for i=1 : If $V(x^o)$ is small enough, we have

$$f_j \circ \varphi(x,t) = \sum_i h_{ji}(x,t) \cdot k_{ji}(x)$$

with $k_{ji} \in H^{o}(V, \mathcal{D}^{N}) \cap H^{o}(V \cap D, \mathfrak{I})$, h_{ji} are of class C^{N-1}. Then of course (**) yields an element $F_{x^o} \in \mathfrak{I}_{x^o} \cdot \mathcal{D}^{N-2}_{x^o}$, similarly $H_{x^o} \in \mathfrak{I}_{x^o} \cdot \mathcal{D}^{N-2}_{x^o}$. So we get $F_{x^o} \cdot \tilde{\omega}^{p-1}_{x^o} \in \mathfrak{I}_{x^o} \cdot {}^{N-2}\underset{\sim}{\Omega}^{p-1}_{x^o}$, $H_{x^o} \cdot \tilde{\omega}^{p-2}_{x^o} \in \mathfrak{I}_{x^o} \cdot {}^{N-2}\underset{\sim}{\Omega}^{p-2}_{x^o}$, $G_{x^o} \cdot \tilde{\omega}^{p-1}_{x^o} \in \mathfrak{I}_{x^o} \cdot {}^{N-2}\Omega^{p-1}_{x^o}$.

$$\longrightarrow \qquad \omega^{p-1}_{x^o} \in \mathfrak{I}^{p-1}_{1x^o} \quad \text{q.e.d.}$$

Case b), i=2 : Here we have for each $t_o \in I$:

$$f_j \circ \varphi(\cdots, t_o) \cdot k(\cdots, t_o) \in H^{o}(V, \mathcal{D}^{N-2}) \cap H^{o}(V \cap D, \overline{\mathfrak{I} \cdot \mathcal{D}^{N-2}}).$$

But then of course: $F_{x^o} \in \overline{\mathfrak{I} \cdot \mathcal{D}^{N-2}_{x^o}}$, hence $F_{x^o} \cdot \tilde{\omega}^{p-1}_{x^o} \in \mathfrak{I}_{2x^o} \cdot {}^{N-2}\underset{\sim}{\Omega}^{p-1}_{x^o}$. Similarly $H_{x^o} \cdot \tilde{\omega}^{p-2}_{x^o} \in \mathfrak{I}_{2x^o} \cdot {}^{N-2}\underset{\sim}{\Omega}^{p-2}_{x^o}$, and of course $G_{x^o} \cdot \tilde{\omega}^{p-1}_{x^o} \in \mathfrak{I}_{2x^o} \cdot {}^{N-2}\Omega^{p-1}_{x^o}$ $\longrightarrow$

$$\omega^{p-1}_{x^o} \in \mathfrak{I}^{p-1}_{2x^o} \quad \text{q.e.d.}$$

Case b), i=3 : Similarly.

Case a), i=2,3 : because r-contractible implies weakly r-contractible. q.e.d

Theorem 5,2. Let $X \in \mathcal{R}_N$. Then the sequences of differential forms

$$0 \longrightarrow C \longrightarrow \Omega^{0}_{i} \longrightarrow \Omega^{1}_{i} \longrightarrow \Omega^{2}_{i} \longrightarrow \Omega^{3}_{i} \longrightarrow \cdots$$

are exact for

$i = 1,2,3$, if X is strongly contractible at each $x^o \in \underline{X}$.

$i = 2,3$, if X is weakly contractible at each $x^o \in \underline{X}$.

Hence, because the Ω^{j}_{i} are soft, we have for paracompact $\underline{X}$:

$$H^{q}(\underline{X}, C) \cong \text{kernel}(H^{o}(\underline{X}, \Omega^{q}_{i}) \longrightarrow H^{o}(\underline{X}, \Omega^{q+1}_{i})) / \text{im}(H^{o}(\underline{X}, \Omega^{q-1}_{i}) \longrightarrow H^{o}(\underline{X}, \Omega^{q}_{i}))$$

in the corresponding cases.

Remark 5,3. The proof of 5,1 gives in certain nice cases a homotopy operator. For this let our $\phi: X|V \times I^N \longrightarrow X|U$ be an r-contraction with $\phi(x^\circ \times I) = x^\circ$. Let $N = \infty, \omega, \omega^*$. Then if φ above is a ϕ inducing mapping, decompose each $\omega^p_{x^\circ} \in \Omega^p_{x^\circ}$ as above:

$$\omega^p \circ \varphi = \omega^p_1 + \omega^{p-1}_2 \wedge dt.$$

$$\sigma(\omega^p) = -\int_0^1 \omega^{p-1}_2 \wedge dt$$

then defines a mapping $\sigma: \Omega^p_{x^\circ} \longrightarrow \Omega^{p-1}_{x^\circ}$ satisfying

$$\sigma(\mathcal{I}^p_{ix^\circ}) \subset \mathcal{I}^{p-1}_{ix^\circ}$$

for $i=1,2,3$, if ϕ is a strong contraction, for $i=2,3$, if ϕ is a weak contraction (by the proof above). Hence we have induced mappings

$$\sigma: \Omega^p_{x^\circ}/\mathcal{I}^p_{ix^\circ} \longrightarrow \Omega^{p-1}_{x^\circ}/\mathcal{I}^{p-1}_{ix^\circ}, \quad \Omega^p_{ix^\circ} \longrightarrow \Omega^{p-1}_{ix^\circ}.$$

We have

$$(d \circ \sigma - \sigma \circ d)(\omega^p) = \omega^p_1(x,1) - \omega^p_1(x,0) = \omega^p_1(x,1) = \omega^p(x)$$

for $p > r$. Hence the induced mapping

$$(d \circ \sigma - \sigma \circ d): \Omega^p_{ix^\circ} \longrightarrow \Omega^p_{ix^\circ}$$

is the identity, if $p > r$, for

$i = 1,2,3$, if ϕ is a strong r-contraction, as above,

$i = 2,3$, if ϕ is a weak r-contraction, as above.

§ 6. Some connections to analytic spaces.

Let $A \subset C^m = \{z;\ z=(z_1,\ldots z_m) \text{ complex}\}$ be locally complex analytic, ${}^N\mathcal{J} \subset \mathcal{D}^N|A$ the ideal sheaf of germs of functions vanishing on A, N = 1,2,3,..... We are interested in some conntions between holomorphic and differentiable differential forms on A. With the notations from §4 and §5 we have:

Theorem 6,1. There exists a locally bounded function $N: A \longrightarrow N$ into the integers, such that for each $z^\circ \in A$, $N \geq N(z^\circ)$ we have:

$$ {}^{\omega^*}\mathcal{J}^p_{1z^\circ} = {}^N\mathcal{J}^p_{3z^\circ} \cap {}^{\omega^*}\Omega^p_{z^\circ} = {}^N\mathcal{J}^p_{1z^\circ} \cap {}^{\omega^*}\Omega^p_{z^\circ}. $$

Proof. Obviously: ${}^{\omega^*}\mathcal{J}^p_{1z^\circ} \subset {}^N\mathcal{J}^p_{1z^\circ} \cap {}^{\omega^*}\Omega^p_{z^\circ} \subset {}^N\mathcal{J}^p_{3z^\circ} \cap {}^{\omega^*}\Omega^p_{z^\circ}$. To show the other inclusion let $g_{z^\circ} \in {}^N\mathcal{J}^p_{3z^\circ} \cap {}^{\omega^*}\Omega^p_{z^\circ}$. Hence we have

$$ (1) \qquad g_{z^\circ} = \omega^p_{z^\circ} \in {}^{\omega^*}\Omega^p_{z^\circ} $$

$$ (2) \qquad = \sum_i f_{iz^\circ} \cdot {}^i\omega^p_{z^\circ} + \sum_j d(g_{jz^\circ} \cdot {}^j\omega^{p-1}_{z^\circ}), $$

where $f_{iz^\circ}, g_{jz^\circ} \in {}^{N-2}\mathcal{J}_{z^\circ}$, ${}^i\omega^\pi_{z^\circ} \in {}^{N-2}\Omega^\pi_{z^\circ}$ for $\pi = p, p-1$.

First let A be pure dimensional and f_i, g_j representations of the f_{iz°, g_{jz° in some neighbourhood $U(z^\circ) \subset C^m$. For each $z \in U(z^\circ)$ we have a decomposition

$$ f_{iz} = h_{iz} + a_{iz} + k_{iz}, $$

where $h_{iz} + a_{iz}$ is the taylorpolynomial of f_{iz} of degree N-3, h_{iz} its holomorphic part, and k_{iz} the taylor rest of f_{iz}. For each $z \in A \cap U$ we have

$$ h_{iz} + a_{iz} = -k_{iz} \mod {}^{N-2}\mathcal{J}_z $$

and therefore

$$ \limsup_{A \ni \tilde{z} \to z} |h_i(\tilde{z}) + a_i(\tilde{z})| / |z - z|^{N-2} < \infty . $$

As in [4] p. 149 we get

$$ (3) \qquad \limsup_{A \ni \tilde{z} \to z} |h_i(\tilde{z})| / |\tilde{z} - z|^{N-2} < \infty . $$

If $U(z^0)$ is sufficiently small, then there exists in $U(z^\circ)$ a holomorphic function u, which does not depend on the h_i and satisfies:

α) $u_z|A_z$ is not a zero divisor of $\mathcal{O}_z / {}^{\omega^*}\mathcal{J}_z$ for any $z \in A \cap U$.

β) For some $l > 3$ and all $z \in A \cap U$ statement (3) implies:

$$ u_z \cdot h_z \in m_z^{N-l} \mod {}^{\omega^*}\mathcal{J}_z \quad (\text{see [4], p. 149-150}) $$

($\mathcal{O}_z = \mathcal{D}_z^{\omega^*}$, $m_z \subset \mathcal{O}_z$ maximal ideal). Hence for all $z \in U(z^\circ) \cap A$:

$$ (4) \qquad u_z \cdot h_{iz} \in u_z \cdot \mathcal{O}_z \cap (m_z^{N-l} \mod {}^{\omega^*}\mathcal{J}_z). $$

Following the theorem of Artin - Rees, there exists to each function $\mu: A \longrightarrow N$ a function $\nu: A \longrightarrow N$ such that: $\nu \geq \mu$ and

(5a) $(u_z \cdot \mathcal{O}_z) \cap (m_z^{\nu(z)} \mod {}^{\omega^*}\mathcal{J}_z) \subset u_z \cdot (\mathcal{O}_z \cdot m_z^{\mu(z)}) \mod {}^{\omega^*}\mathcal{J}_z$.

Again, if $U(z^\circ)$ is sufficiently small, μ = constant, one also may choose ν = constant $\geq \mu$ in (5a) (see [6], p. 83). Hence

(5b) $(u_z \cdot \mathcal{O}_z) \cap (m_z^\nu \mod {}^{\omega^*}\mathcal{J}_z) \subset u_z \cdot (\mathcal{O}_z \cdot m_z^\mu) \mod {}^{\omega^*}\mathcal{J}_z$.

But u_z is not a zero divisor in $\mathcal{O}_z / {}^{\omega^*}\mathcal{J}_z$, therefore (4) and (5b) imply for $N - 1 \geq \nu$ and all $z \in A \cap U$:

$$(6) \qquad \begin{aligned} h_{iz} &\in m_z^\mu + {}^{\omega^*}\mathcal{J}_z \\ f_{iz} &= h'_{iz} + k'_{iz} , \text{ with } h'_{iz} \in {}^{\omega^*}\mathcal{J}_z, \end{aligned}$$

and where the taylor polynomial of k'_{iz} up to the degree μ does not contain any holomorphic part. A similar decomposition exists for g_{jz} . Again for $U(z^\circ)$ being small, there exist $c_1, \ldots c_s \in H^\circ(U, {}^{\omega^*}\mathcal{J})$, generating ${}^{\omega^*}\mathcal{J}|U$. By (6) we may write for (2) :

$$(7) \quad \omega_z^p = \sum_i c_{iz} \cdot {}^i\omega_z^p + \sum_j dc_{jz} \wedge {}^j\omega_z^{p-1} + \omega_z'^p ,$$

with $z \in U(z^\circ)$, ${}^i\omega_z^p , \omega_z'^p \in {}^{N-3}\Omega_z^p$, ${}^j\omega_z^{p-1} \in {}^{N-3}\Omega_z^{p-1}$, and the taylor polonomials of the coefficients of $\omega_z'^p$ up to degree μ do not contain any holomorphic part. Multiplying out in (7) and seperating corresponding coeffi= cients, we find (7) to be aquivalent to a system of equations

$$(8) \qquad A_z = C_z \cdot d_z + r_z ,$$

where A is a column vector and C is a matrix with in $U(z^\circ)$ holomorphic coefficients. d_z and r_z are row vectors with germs of functions of class C^{N-3} as entities. Moreover, the taylor polynomials of the coefficients of r_z do not contain a holomorphic part up to the order μ . Let $\mathcal{C}$ denote the $\mathcal{O}$ - submodule generated by the column vectors of C, then (8) implies

$$(9) \qquad A_z \in \mathcal{C}_z \mod \sigma \cdot (m_z)^{\mu-1} \quad (\sigma\text{-times direct sum})$$

for all $z \in U(z^\circ)$ and some $\sigma \in \mathbb{N}$. Hence by [7] , for sufficiently large μ:

$$A_{z^\circ} \in \mathcal{C}_{z^\circ} \quad , \quad \Rightarrow$$

$$g_{z^\circ} = \sum_i c_{iz^\circ} \cdot {}^i\omega_{z^\circ}^p + \sum_i dc_{iz^\circ} \wedge {}^i\omega_{z^\circ}^{p-1}$$

for some ${}^i\omega_{z^\circ}^\pi \in {}^{\omega^*}\Omega_{z^\circ}^\pi$, $\pi = p, p-1$. It follows $g_{z^\circ} \in {}^{\omega^*}\mathcal{J}_{z^\circ}^p$. The existence of the locally boundet function $N(z)$ is now obvious.

If A is not puredimensional, then combine the proof given here with some additional considerations described in [4], p.151.

Theorem 6,2. If $A \subset R^n$ is only locally real analytic, $z^o \in A$, then

$$ {}^{\omega}\mathfrak{J}^p_{1z^o} = {}^{\infty}\mathfrak{J}^p_{3z^o} \cap {}^{\omega}\Omega^p_{z^o} $$

Proof. similar as the proof for 6,1 , but instead using results of [4] and [6] use [1], Theorem 2 . We find a decomposition similar to that of (6):

$$ f_{iz^o} = h'_{iz^o} + k'_{iz^o} \quad , \quad h_{iz^o} \in {}^{\omega}\mathfrak{J}_{1z^o}, $$

where the Taylor series of k'_{iz^o} vanishes.

Corollary to 6,1. $\quad {}^{\omega^*}\Omega^p_{z^o}/{}^{\omega^*}\mathfrak{J}^p_{1z^o} \subset {}^{N}\Omega^p_{z^o}/{}^{N}\mathfrak{J}^p_{3z^o}$, if $N \geq N(z^o)$.

Corollary to 6,2. $\quad {}^{\omega}\Omega^p_{z^o}/{}^{\omega}\mathfrak{J}^p_{1z^o} \subset {}^{\infty}\Omega^p_{z^o}/{}^{\infty}\mathfrak{J}^p_{3z^o}$.

Hence holomorphic p-forms can be identified with certain n-differentiable p-forms, analytic p-forms can be identified with certain ∞-differentiable p-forms. Let us call a holomorphic (analytic) p-form on A_{z^o}

N-differentiably closed ,

if this form, when considered as a N-differentiable form, is closed. Then we have:

Corollary to 6,1. Each N-differentiably closed holomorphic form on A_{z^o} is closed, $N \geq N(z^o)$.

Corollary to 6,2. Each ∞-differentiably closed analytic p-form on A_{z^o} is closed.

The problem arises of course, whether similar results hold for boundedness. This will probably be true. However here we can only show for $A \subset C^n$ locally complex analytic:

Theorem 6,3. For some locally bounded function $N: A \longrightarrow N$ one has : Each holomorphic 1-form on A_{z^o}, which is $N(z^o)$-differentiably bounded, is bounded.

Proof. Let $N \geq N(z^o)$, $\omega^p \in {}^{\omega^*}\Omega^p_{z^o}$, $\omega^{p-1} \in {}^{N-2}\Omega^{p-1}_{z^o}$ and

$$ d\,\omega^{p-1} = \omega^p \mod {}^{N}\mathfrak{J}^p_{3z^o} . $$

We may assume

$$ (1) \qquad d\,\omega^{p-1} = \omega^p + \sum_i h_i \cdot {}^i\omega^p , $$

with $h_i \in \mathcal{D}^{N-2}_{z^o}$, $h_i | A_{z^o} = 0$, ${}^i\omega^p \in {}^{N-2}\Omega^p_{z^o}$. Write $\omega^{p-1} = \omega_1^{p-1} + \omega_2^{p-1}$, where ω_1^{p-1} contains exactly all forms of ω^{p-1} of the sort $a \cdot dz_{i_1} \wedge \dots \wedge dz_{i_{p-1}}$. Let $d = \partial + \bar\partial$, then (1) implies

$$ \bar\partial\omega_1^{p-1} + \partial\omega_2^{p-1} + \bar\partial\omega_2^{p-1} = \sum h_i \cdot {}^i\omega''^p $$

with ${}^i\omega''^p \in {}^{N-2}\Omega^p_{z^o}$ and $\partial\omega_1^{p-1} = \omega^p + \sum h_i \cdot {}^i\omega'^p$. If p=1, we have

$\omega^{p-1} = \omega^{o} = g_{z^o} \in \mathcal{D}^{N-2}_{z^o}$ and

$$\bar\partial g_{z^o} = \Sigma h_i \cdot {}^{1}\omega^{*1} , \partial g_{z^o}/\partial \bar z_\nu | A_{z^o} = 0.$$

$g_{z^o}|A_{z^o}$ is weakly holomorphic and of class C^{N-2}. By [6] then g_{z^o} is holomorphic for a suitable locally boundet function N: A ⟶ N . q.e.d.

Remark 6,4. With the notations from above: If A^N is contractible at $z^o \in A$, then the following complexes are exact:

1) $0 \to C \to {}^{N-1}\Omega^{0}_{z^o}/{}^{N-1}\mathcal{J}^{0}_{3z^o} \longrightarrow {}^{N-1}\Omega^{1}_{z^o}/{}^{N-1}\mathcal{J}^{1}_{3z^o} \longrightarrow \cdots$

2) $0 \to C \to {}^{\omega^*}\Omega^{0}_{z^o}/{}^{\omega^*}\mathcal{J}^{0}_{3z^o} \longrightarrow {}^{\omega^*}\Omega^{1}_{z^o}/{}^{\omega^*}\mathcal{J}^{1}_{3z^o}$,

if A is complex analytic and N sufficiently large, or if A is real analytic and $N = \infty$.

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