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Duality and the DeRham Cohomology of Infinitesimal Neighborhoods

by D. Lieberman[1] and M. Herrera

Our principal application of the techniques developed in this note appears in §7 where we establish the following result. Let X be a nonsingular compact analytic space (resp. a complete algebraic variety over $\mathbb{C}$) of dimension m and let Y be an arbitrary closed subvariety of X, $U = X - Y$. Then the standard exact sequences

$$1) \ \ldots \longrightarrow H^p_c(U,\mathbb{C}) \longrightarrow H^p(X,\mathbb{C}) \longrightarrow H^p(Y,\mathbb{C}) \longrightarrow H^{p+1}_c(U,\mathbb{C}) \longrightarrow \ldots$$

$$2) \ \ldots \longleftarrow H^{2m-p}(U,\mathbb{C}) \longleftarrow H^{2m-p}(X,\mathbb{C}) \longleftarrow H^{2m-p}_Y(X,\mathbb{C}) \longleftarrow H^{2m-p-1}(U,\mathbb{C}) \longleftarrow \ldots$$

can be computed purely in terms of the analytic (resp. algebraic) deRham cohomology. Explicitly if $Y^{(r)}$ denotes the r^{th} order infinitesimal neighborhood of Y in X then sequence 1) is obtained as the inverse limit of sequences

$$1)_r \ \ldots \longrightarrow H^p_{DR}(X,Y^{(r)}) \longrightarrow H^p_{DR}(X) \longrightarrow H^p_{DR}(Y^{(r)}) \longrightarrow \ldots$$

where H_{DR} denotes the hypercohomology of the deRham complex $\Omega^\cdot$ of holomorphic (resp. algebraic) differential forms. The second sequence is obtained as the direct limit of hyperext sequences

$$2)_r \ \ldots \longleftarrow \underline{\underline{\mathrm{Ext}}}^{2m-p}(\Sigma^\cdot_r, \Omega^\cdot_X) \longleftarrow \underline{\underline{H}}^{2m-p}_{DR}(X) \longleftarrow \underline{\underline{\mathrm{Ext}}}^{2m-p}(\Omega^\cdot_{Y^{(r)}}, \Omega^\cdot_X) \longleftarrow \ldots$$

where $\Omega^\cdot_{Y^{(r)}}$ is the deRham complex on $Y^{(r)}$ and Σ_r is the kernel of the map $\Omega_X \longrightarrow \Omega_{Y^{(r)}}$. The standard Poincaré duality between 1) and 2) is exhibited as the limit of dualities for each finite r (essentially Serre duality.) In the case that Y is nonsingular the limiting procedure is irrelevant, since for every finite r the sequences $1)_r$ and $2)_r$ already calculate the classical cohomology.

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In the presence of singularities it is well known that $H_{DR}(Y)$ may fail to calculate the correct result, cf. [14] or [12]. An example of R. Slutzki (§7) shows that no finite r need calculate cohomology correctly. In the event that Y has codimension one the representation of $H^{m-p}(U,C)$ as $\varinjlim \underline{\underline{\mathrm{Ext}}}^{m-p}(\Sigma_r, \Omega_X^{\cdot})$ is the result of Grothendieck [8] that the cohomology of U may be calculated using forms on X with only polar singularities on Y, and we obtain an expression for the residue for a closed meromorphic form with pole order r as defining an element of $H_Y(X,C) = (H^{\cdot}(Y,C))^*$.

Our work on this result began jointly in the summer of 1969 and had established the results only for Y of codimension one when the first author heard Deligne announce the result $\varprojlim \underline{\underline{H}}(\Omega_{Y(r)}) = H(Y,C)$, for Y of arbitrary codimension, in his lectures at Harvard (Fall, 1969). Our method of proof, employing duality, is quite distinct from the argument of Deligne which is based on the appendix in [10] written by Deligne. The duality theory developed here for use in the proof is new, as is the formalism for complexes with differential operators of order one.

The duality theorem referred to is the following. Let X be nonsingular and compact and let $\mathcal{F}^{\cdot}$ be a complex of coherent $\mathcal{O}_X$ modules with the maps $\mathcal{F}^i \longrightarrow \mathcal{F}^{i+1}$ being differential operators of order one (in particular they may be $\mathcal{O}_X$ linear.) The spectral sequence of hypercohomology

$$E_1^{p,q} = H^q(X,\mathcal{F}^p) \Longrightarrow \underline{\underline{H}}^{p+q}(\mathcal{F}^{\cdot})$$

is dual to a natural spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^q(\mathcal{F}^{m-p},\Omega^m) \Longrightarrow \underline{\underline{\mathrm{Ext}}}^{p+q}(\mathcal{F}^{\cdot},\Omega^{\cdot})$$

where this second spectral sequence is obtained as follows. Let $0 \longrightarrow \Omega^{\cdot} \longrightarrow Q^{\cdot\cdot}$ be a resolution of $\Omega^{\cdot}$ by injective complexes (cf. §2), and form the double complex

$$K^{i,j} = \mathrm{Hom}^{i}(\mathcal{F}^{\cdot}, Q^{\cdot j}) \qquad \text{(cf. §2)}$$

Then the first spectral sequence of this double complex is the spectral sequence above. The duality is established by a Yoneda pairing, §4, of the spectral sequences, which is perfect at level E_1, (Serre duality.)

§1. Jet sheaves.

We work throughout in the category of "spaces" which is either the category of complex analytic spaces with holomorphic maps or the category of complex algebraic varieties with regular maps. By a compact space, we mean either an analytic space or a complete algebraic variety.

Given a space X let J_n denote the $\mathcal{O}_X$ algebra of n jets on X, with the standard differential operator of order n: $d_n : \mathcal{O} \longrightarrow J_n$. The sheaf J_n is simply the restriction to the diagonal in $X \times X$ of the sheaf $\mathcal{O}_{X\times X}/I^{n+1}$ where I is the ideal sheaf of the diagonal, see for example [16]. The $\mathcal{O}_X$ algebra structure of J_n is provided by the morphism $\pi_1^* : \mathcal{O}_X \longrightarrow \mathcal{O}_{X\times X}/I^{n+1}$, while the operator d_n is then given by π_2^*.

The sheaves J_n could also be viewed as $\mathcal{O}_X$ modules via π_2^*, and adopting this point of view momentarily, one may define for any $\mathcal{O}_X$ module E, the J_n module of E-jets $J_n(E) = J_n \otimes_{\pi_2^*\mathcal{O}} E$ which may then be viewed as a sheaf of $\mathcal{O}_X$ modules via the morphism $\pi_1^* : \mathcal{O} \longrightarrow J_n$. The natural π_2 linear map $d_n : E \longrightarrow J_n(E)$ is not a morphism of $\mathcal{O}_X$ modules (via π_1).

When the space X is nonsingular the sheaves $J_n = J_n(\mathcal{O})$ are locally free $\mathcal{O}$ modules (via either π_1^* or π_2^*) as can be seen readily from the standard exact sequences

$$0 \longrightarrow S^n\Omega^1 \longrightarrow J_n \longrightarrow J_{n-1} \longrightarrow 0$$

$$J_0 = \mathcal{O}$$

where $S^n\Omega^1$ is the n^{th} symmetric product of the differential 1 forms.

Consequently for any $\mathcal{O}_X$ module E the sequences of $\pi_2^*\mathcal{O}$ modules

$$0 \longrightarrow S^n\Omega^1 \otimes E \longrightarrow J_n(E) \longrightarrow J_{n-1}(E) \longrightarrow 0$$

$$0 \longrightarrow J_0(E) \longrightarrow E \longrightarrow 0$$

remain exact, $(\mathrm{Tor}(J_{n-1},E) = 0)$. Viewing the sheaves as $\mathcal{O}$ modules via π_1 does not alter the exactness, (the maps above are all clearly $\pi_1^*\mathcal{O}$ linear.) We note that the standard splitting

$$J_1 = \mathcal{O} \oplus \Omega^1$$

of J_1 as a $\pi_1^*(\mathcal{O})$ module is not a π_2 linear splitting so that $J_1(E) \neq E \oplus E \otimes \Omega^1$ as an $\mathcal{O}$ module. (In fact if E is locally free the class of the extension $J_1(E)$, in $H^1(X,\mathrm{Hom}(E, \otimes\,\Omega^1))$, is the "Chern class" of E.)

Note the mapping associating to an $\mathcal{O}$ module E the $\mathcal{O}$ module $J_n(E)$ is clearly a functor, the composite of extension of scalars from $\pi_2^*(\mathcal{O})$ to J_n and restriction of scalars from J_n to $\pi_1^*\mathcal{O}$. Furthermore in the nonsingular case $J_n(\ )$ is an <u>exact</u> functor, being essentially tensor product with a locally free sheaf.

<u>Remark</u>: Define $\tilde{J}_n$ by

$$0 \longrightarrow \tilde{J}_n \longrightarrow J_n \longrightarrow \mathcal{O} \longrightarrow 0 \quad ,$$

and $\tilde{J}_n(E) = \tilde{J}_n \otimes E$. The nonlinearity of the differential operator $d_n : E \longrightarrow J_n(E)$ may then be exhibited explicitly by

1.1 $$d_n(f\alpha) = f\, d_n(\alpha) + \tilde{d}_n(f) \otimes \alpha.$$

where α a local section of E, f a section of $\mathcal{O}$ and $d_n(f) = f + \tilde{d}_n(f)$ is the decomposition of $d_n(f)$ in the direct sum $J_n = \mathcal{O} \oplus \tilde{J}_n(\mathcal{O})$.

We shall employ the standard definition of differential operator of order n, namely a morphism of sheaves, $E \xrightarrow{D} \mathcal{F}$ is such an object, if and only if there is an $\mathcal{O}$ linear map $\tilde{D}: J_n(E) \longrightarrow \mathcal{F}$ making the diagram

$$\begin{tikzcd} J_n(E) \arrow[dr, "\tilde{D}"] & \\ E \arrow[u, "d_n"] \arrow[r, "D"'] & \mathcal{F} \end{tikzcd}$$

commutative. Note that the mapping $\tilde{D}$ is completely determined by D, i.e. $J_n(E)$ is generated as an $\mathcal{O}$ module by $d_n(E)$, as is clear from 1.1. Associated to the operator D we have the "symbol" $\tilde{\sigma}(D)$: $\tilde{J}_n \otimes E \longrightarrow \mathcal{F}$, given by restricting

Given a function f and a section α of E we employ the notation $\tilde{d}_n(f)\cdot\alpha$ to denote the section $\tilde{\sigma}(D)(\tilde{d}_n(f) \otimes \alpha)$ thereby obtaining the formula:

1.2 $$D(f\cdot\alpha) = fD(\alpha) + \tilde{d}_n(f)\cdot\alpha \qquad \text{cf. 1.1.}$$

§2. Complexes of differential operators of order 1.

Given a space X we denote by $C(X)$ the category of complexes of $\mathcal{O}_X$ modules with differential operators of order 1. An object $\{\mathcal{F}^\cdot\}$ of $C(X)$ is a collection of $\mathcal{O}_X$ modules, $\mathcal{F}^i$, $i \in \mathbb{Z}$, with differential operators of order 1 $D_i : \mathcal{F}^i \longrightarrow \mathcal{F}^{i+1}$ satisfying $D_{i+1} \circ D_i = 0$. A canonical example of such an object is the deRham complex $\Omega^\cdot_X$. By a morphism of complexes $f: \mathcal{F}^\cdot \longrightarrow \mathcal{G}^\cdot$ we mean a collection of $\mathcal{O}$-linear maps $f_i : \mathcal{F}^i \longrightarrow \mathcal{G}^i$ such that $f_i \circ D_i = D_i \circ f_i$. The set of morphisms is denoted $\mathrm{Hom}(\mathcal{F}^\cdot, \mathcal{G}^\cdot)$.

Given a complex $\mathcal{F}^\cdot$ we employ the standard notation $\mathcal{F}^\cdot[i]$ to denote the complex $\mathcal{F}^j[i] = \mathcal{F}^{i+j}$ with $d: \mathcal{F}^j[i] \longrightarrow \mathcal{F}^{j+i}[i]$ defined to be $(-1)^i$ times the differential $\mathcal{F}^{i+j} \longrightarrow \mathcal{F}^{i+j+1}$.

$C(X)$ is readily seen to be an abelian category which satisfies axioms AB 5) and AB 3* of [6].

Proposition 2.1: If $\mathcal{F}^\cdot \in C$ then $\mathcal{F}^\cdot$ has a natural structure of graded module over the graded ring $\Omega^\cdot$, with the property

$$D(\varphi\cdot\alpha) = d\varphi\cdot\alpha + (-1)^i\varphi\cdot D\alpha \tag{2.2}$$

where φ is a local section of Ω^i and α is a local section of $\mathcal{F}^j$.

Proof: For $\varphi \in \Omega^0 = \mathcal{O}$ and $\alpha \in \mathcal{F}^j$, $\varphi\cdot\alpha$ is well defined. The natural pairing $\Omega^1 \otimes \mathcal{F}^j \longrightarrow \mathcal{F}^{j+1}$ is given by the symbol of D_j. By iterating the symbol one obtains the $\mathcal{O}$ multilinear map $(\Omega^1)^{\otimes i} \otimes \mathcal{F}^j \longrightarrow \mathcal{F}^{i+j}$ denoted $\varphi_1 \otimes \ldots \otimes \varphi_i \otimes \alpha \longrightarrow \varphi_1\cdot(\varphi_2\cdot\ldots(\cdot(\varphi_i\cdot\alpha)))$. To see that this induces an $\mathcal{O}$ linear map $\Omega^i \otimes \mathcal{J}^j \longrightarrow \mathcal{J}^{i+j}$ it suffices to check that for any function f and α a section of an $\mathcal{J}^j$ that

$$df\cdot(df\cdot\alpha) = 0$$

But note that

2.3 $$0 = D^2(f\alpha) = D(f\cdot D\alpha + df\cdot\alpha) = df\cdot D\alpha + D(df\cdot\alpha)$$

or

$$D(df\cdot\alpha) = -df\cdot D(\alpha)$$

and

$$0 = D^2(f^2\alpha) = 2fdf\cdot D\alpha + 2fD(df\cdot\alpha) + 2df\cdot(df\cdot\alpha) = 2df\cdot(df\cdot\alpha)$$

The formula 2.2, follows immediately from 2.3, and 1.2.

Remark 2.4: If $\Phi: \mathcal{J}^\cdot \longrightarrow \mathcal{S}^\cdot$ is a morphism of complexes then Φ is automatically an $\Omega^\cdot$ module map.

Thus the category $C(X)$ admits another description, namely as the category of graded $\Omega^\cdot$ modules $\mathcal{J}^\cdot$ with differential $d: \mathcal{J}^\cdot \longrightarrow \mathcal{J}^{\cdot+1}$ satisfying

$$d(\varphi\cdot\alpha) = d\varphi\cdot\alpha + (-1)^p\,\varphi.d\alpha$$

for φ, (resp. α) a local section of Ω^p, (resp. $\mathcal{J}^q$). (The map $\mathcal{J}^i \longrightarrow \mathcal{J}^{i+1}$ factors through $J^1(\mathcal{J}^i)$ $\mathcal{O}$ linearly in an obvious manner.) Many constructions are more evident from this point of view.

Viewing the elements of C as $\Omega^\cdot$ modules, one may introduce a natural "internal" tensor product and an "internal" Hom into the category. Namely $(\mathcal{J}^\cdot \otimes \mathcal{S}^\cdot)^\cdot = \mathcal{J}^\cdot \otimes_{\Omega^\cdot} \mathcal{S}^\cdot$ is the complex defined by

$$(\mathcal{J}^\cdot \otimes \mathcal{S}^\cdot)^k = \bigoplus_{i+j=k} \mathcal{J}^i \otimes_{\mathcal{O}} \mathcal{S}^j / R$$

where R is the subsheaf generated by elements of the form $\varphi\cdot\alpha \otimes \beta + (-1)^{a(i-a)+1} \alpha \otimes \varphi\cdot\beta$, with $\varphi \in \Omega^a$, $\alpha \in \mathcal{J}^{i-a}$ and $\beta \in \mathcal{S}^j$. The differential operator is the standard $D = D \otimes 1 + (-1)^i\, 1 \otimes D$. (One must check that $D(\varphi\cdot\alpha \otimes \beta) = (-1)^{a(i-a)} D(\alpha \otimes \varphi\cdot\beta)$, a straightforward calculation.)

Given a space X we may view X as a ringed space $(X, \Omega^\cdot_X)$, a point of view which is most convenient for the study of the category $C(X)$. Given a morphism of spaces $f\colon Y \longrightarrow X$, the pair (f, df), $(df\colon \Omega^\cdot_X \longrightarrow \Omega^\cdot_Y)$ defines a homomorphism of ringed spaces. Viewing spaces as ringed spaces in this manner one has associated to a morphism $f\colon Y \longrightarrow X$ the natural functors $f_*\colon C(Y) \longrightarrow C(X)$ and $f^*\colon C(X) \longrightarrow C(Y)$. Namely, given $\mathcal{J} \in C(Y)$, $\mathcal{J}$ is a sheaf of $\Omega^\cdot_Y$ modules, so that the sheaf theoretic direct image $f_*(\mathcal{J})$ carries a natural $\Omega^\cdot_X$ graded structure via $df\colon \Omega^\cdot_X \longrightarrow \Omega^\cdot_Y$. Further the morphisms $d\colon f_*(\mathcal{J}^p) \longrightarrow f_*(\mathcal{J}^{p+1})$ induced by the differentials in $\mathcal{J}^\cdot$, make $f_*(\mathcal{J})$ a differential $\Omega^\cdot_X$ module, i.e. an element of $C(X)$. Similarly given $\mathcal{S} \in C(X)$, we note that $f^{-1}(\mathcal{S})$, (the sheaf theoretic inverse image) is a sheaf of graded $f^{-1}(\Omega_X)$ modules, while Ω_Y is a sheaf of graded $f^{-1}(\Omega_X)$ algebras, and one obtains a sheaf of graded Ω_Y modules, $f^*(\mathcal{S}) = f^{-1}(\mathcal{S}) \otimes_{f^{-1}(\Omega_X)} \Omega_Y$. $f^*(\mathcal{S})$ equipped with the differential

$d(\alpha \otimes \beta) = d\alpha \otimes \beta + (-1)^p \alpha \otimes d\beta$ for $\alpha \in f^{-1}(\mathcal{S}^p)$ and $\beta \in \Omega_Y$ defines an element of $C(Y)$. One has the formula $\mathrm{Hom}(f^*\mathcal{S}, \mathcal{J}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{S}, f_*\mathcal{J})$. Note that

$f^*(\Omega_X) = \Omega_Y$, and that if $\mathcal{O}_X$ denotes the complex which is $\mathcal{O}_X$ in degree zero and zero in other degrees then $f^*(\mathcal{O}_X) = \Omega^{\cdot}_{Y|X}$ is the complex of <u>relative</u> differentials, this may be seen by noting that Ω^1_X acts trivially on $\mathcal{O}_X$ so that $f^*(\mathcal{O}_X) = f^{-1}(\mathcal{O}_X) \otimes_{f^{-1}(\Omega^{\cdot}_X)} \Omega^{\cdot}_Y = \Omega^{\cdot}_Y/\Omega^1_X \cdot \Omega^{\cdot -1}_Y$. In the special case that f is an immersion $f^*(\mathcal{O}_X) = \mathcal{O}_Y$.

Similarly one may define $Hom^k(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$ to be the sheaf of morphisms of degree k of graded $\Omega^{\cdot}$ modules, i.e, the sheaf obtained from the presheaf which associates to an open set U , the collection of $\Phi \in \prod_{j \in \mathbb{Z}} Hom(\mathcal{F}^j_U , \mathcal{G}^{k+j}_U)$ such that

$$\Phi(\varphi\cdot\alpha) = (-1)^{ik}\varphi\,\Phi(\alpha)$$

for all $\varphi \in \Omega^i$, $\alpha \in \mathcal{F}^j$. (The global sections of the sheaf Hom^k are denoted by Hom^k .) The differential operator $D : Hom^k(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot}) \longrightarrow Hom^{k+1}(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$ is given by the standard formula

$$D\ (\Phi) = D_G \circ \Phi \ + \ (-1)^{k+1}\ \Phi \circ D_F \ .$$

The fact that for $\varphi \in \Omega^i$ and $\alpha \in \mathcal{F}^j$ one has

$$D(\Phi)\ (\varphi\cdot\alpha) = (-1)^{i(k+1)}\varphi \cdot D(\alpha)$$

is obtained by a straightforward calculation.

<u>Remark</u>: We note that the cycles in $Hom^k(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$ are precisely the morphisms of complexes $\mathcal{F}^{\cdot} \longrightarrow \mathcal{G}^{\cdot}[k]$. The boundaries are of course the homotopies $\mathcal{F}^{\cdot} \longrightarrow \mathcal{G}^{\cdot}[k]$, so that the homology group $H^k(Hom(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot}))$ is simply the homotopy classes of morphisms.

It is especially easy to calculate $\mathcal{T}^{\cdot} \otimes \mathcal{S}^{\cdot}$ and $\mathrm{Hom}^{\cdot}(\mathcal{T}^{\cdot}, \mathcal{S}^{\cdot})$ when $\mathcal{T}^{\cdot}$ or $\mathcal{S}^{\cdot}$ is simply the complex $\Omega^{\cdot}$:

Proposition 2.5: 1) $\Omega^{\cdot} \otimes \mathcal{S}^{\cdot} \xrightarrow{\sim} \mathcal{S}^{\cdot}$

2) $\mathrm{Hom}^{\cdot}(\Omega^{\cdot}, \mathcal{S}^{\cdot}) \xrightarrow{\sim} \mathcal{S}^{\cdot}$

3) If X is nonsingular, $\mathrm{Hom}^{k}(\mathcal{T}^{\cdot}, \Omega^{\cdot}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\mathcal{T}^{m-k}, \Omega^{m})$.

Proof: 1) The natural morphisms $\Omega^{\cdot} \otimes \mathcal{S}^{\cdot} \longrightarrow \mathcal{S}^{\cdot}$, $\varphi \otimes \alpha \longrightarrow \varphi \cdot \alpha$, and $\mathcal{S}^{\cdot} \longrightarrow \Omega^{\cdot} \otimes \mathcal{S}^{\cdot}$, $\alpha \longrightarrow 1 \otimes \alpha$ are inverse to one another.

2) The standard map $\mathrm{Hom}^{\cdot}(\Omega^{\cdot}, \mathcal{S}^{\cdot}) \longrightarrow \mathcal{S}^{\cdot}$, $\Phi \longrightarrow \Phi(1)$ is an isomorphism.

3) Instead of proving this result directly, we shall prove a more useful generalization in the following lemma.

Definition 2.6: Let $\mathcal{M}$ be any $\mathcal{O}$ module, and denote by $(\Omega^{m-\cdot})^{*} \otimes \mathcal{M}$ the graded $\Omega^{\cdot}$ module with structure

$$(\Omega^{j} \otimes ([\Omega^{m-i})^{*}) \otimes \mathcal{M}] \xrightarrow{c\otimes 1} (\Omega^{m-i-j})^{*} \otimes \mathcal{M}$$

where $c\colon \Omega^{j} \otimes (\Omega^{m-i})^{*} \longrightarrow (\Omega^{m-i-j})^{*}$ is contraction. (If α, φ, β are local sections of Ω^{j}, $(\Omega^{m-i})^{*}$ and Ω^{m-i-j} then c is defined by $c(\alpha \otimes \varphi)\ (\beta) = \varphi\ (\beta \wedge \alpha)$.)

Note that $(\Omega^{m-i})^{*} \otimes \mathcal{M} = 0$ unless $0 \leq i \leq m$, and that the m^{th} level is $\mathcal{M}$, (X nonsingular.) Further, if $\mathcal{M} = \Omega^{m}$, and X is nonsingular, then $(\Omega^{m-\cdot})^{*} \otimes \Omega^{m} = \Omega^{\cdot}$, [2]. In view of this fact, the following lemma generalizes 3) of the preceding proposition.

<u>Lemma</u> 2.7: Let X be nonsingular. Let $\mathcal{J}^{\cdot}$ be a graded $\Omega^{\cdot}_X$ module, and $\mathcal{M}$ an $\mathcal{O}_X$ module, then

$$\mathrm{Hom}^k_{\Omega}\,(\mathcal{J}^{\cdot},\ (\Omega^{m-\bullet})^* \otimes \mathcal{M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}\,(\mathcal{J}^{m-k},\ \mathcal{M})\ .$$

<u>Proof</u>: One has a natural map $\mathrm{Hom}^k(\mathcal{J}^{\cdot},\ (\Omega^{m-\bullet})^* \otimes \mathcal{M}) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(\mathcal{J}^{m-k},\ \mathcal{M})$ obtained by restricting $\Phi \in \mathrm{Hom}^k$ to yield $\Phi_{m-k}\colon \mathcal{J}^{m-k} \longrightarrow \mathcal{M}$. The map is clearly injective; indeed, if $\Phi_{m-k} = 0$, then for any $\alpha \in \mathcal{J}^{m-k-i}$ and $\varphi \in \Omega^i$ we have $0 = \Phi(\varphi\cdot\alpha) = (-1)^{ik}\,\varphi\cdot\Phi(\alpha)$. Thus the section $\Phi(\alpha)$ of $(\Omega^i)^* \otimes \mathcal{M}$ is annihilated by every section φ of Ω^i and is hence 0. To obtain surjectivity, assume $\lambda\colon \mathcal{J}^{m-k} \longrightarrow \mathcal{M}$ is given. Define $(-1)^{kj}\cdot\Phi_{m-k-j}\colon \mathcal{J}^{m-k-j} \longrightarrow (\Omega^j)^* \otimes \mathcal{M}$ by

2.8 $$\mathcal{J}^{m-k-j} \longrightarrow (\Omega^j)^* \otimes \Omega^j \otimes \mathcal{J}^{m-k-j} \longrightarrow (\Omega^j)^* \otimes \mathcal{J}^{m-k} \xrightarrow{1\otimes\lambda} (\Omega^j)^* \otimes \mathcal{M}\ .$$

Clearly $\Phi_{m-k} = \lambda$, and the $\Omega^{\cdot}$ linearity of Φ is checked easily in coordinates, as follows. Let $e_1,\ldots,e_m$ be a local basis for Ω^1 then the map 2.8 is described by

$$\alpha \longrightarrow \sum_{|J|=j} e_J^* \otimes e_J \otimes \alpha \longrightarrow \Sigma\, e_J^* \otimes e_J\cdot\alpha \longrightarrow \Sigma\, e_J^* \otimes \lambda(e_J\alpha)$$

where α is a local section of $\mathcal{J}^{m-k-j}$ and we employ the multi-index notation $e_J = e_{n_1} \wedge \ldots \wedge e_{n_j}$ where $J = \{n_1 \le n_2 \le \ldots \le n_j\}$. Now let e_I be an arbitrary basal element for Ω^i then

$$e_I\cdot\alpha \longrightarrow \sum_{|K|=j-i} e_K^* \otimes \lambda((e_K \wedge e_I)\cdot\alpha) = \Sigma\, \epsilon_{KI}\, e_K^* \otimes \lambda(e_{K\cup I}\cdot\alpha)$$

where the sum extends over $K \cap I = \emptyset$ and ϵ_{KI} is the sign of the permutation required to put KI in ascending order.

Comparing this with

$$e_I \sum_J e_J^* \otimes \lambda(e_J\alpha) = \sum_J c(e_I \otimes e_J^*) \otimes \lambda(e_J\alpha) = \sum_{J\geq I} \varepsilon_{J-I,I}\, e_{J-I}^* \lambda(e_J\cdot\alpha)$$

we obtain the asserted $\Omega^\cdot$ linearity.

Corollary 2.9: Let Q be an injective $\mathcal{O}_X$ module then $(\Omega^{m-\cdot})^* \otimes Q^\cdot$ is an injective $\Omega^\cdot$ module, i.e. if $0 \longrightarrow \mathcal{M}^\cdot \longrightarrow N^\cdot$ is an exact sequence of graded $\Omega^\cdot$ modules then $\mathrm{Hom}^\cdot(N^\cdot,(\Omega^{m-\bullet})^* \otimes Q) \longrightarrow \mathrm{Hom}^\cdot(\mathcal{M}^\cdot,(\Omega^{m-\bullet})^* \otimes Q) \longrightarrow 0$ is exact.

Proof: The diagrams

$$\begin{array}{ccc} \mathrm{Hom}^k(N^\cdot, (\Omega^{m-\bullet})^* \otimes Q) & \longrightarrow & \mathrm{Hom}^k(\mathcal{M}^\cdot,(\Omega^{m-\bullet})^* \otimes Q) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Hom}_{\mathcal{O}}(N^{m-k}, Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(\mathcal{M}^{m-k}, Q) \longrightarrow 0 \end{array}$$

are commutative.

2.10 One further fundamental construction deserves mention. One may associate to every $\mathcal{O}_X$ module E a canonical complex $C^\cdot(E)$ with

$$C^j(E) = 0 \qquad j < 0$$

$$C^0(E) = E\ , \quad C^1(E) = J_1(E)$$

which is universal for $\mathcal{O}$ linear maps of E into the zero'th level of a complex of differential operators of order 1. Our complex $C^\cdot(E)$ is the same as the complex $C_1^\cdot$ constructed in Spencer [16]. Once $C^\cdot(E)$ is constructed, one will clearly have that $C^\cdot(E)[-i]$ will be a complex which is universal for maps of E to the i^{th} level of a complex with differential operators of order 1. To visualize the form which $C^\cdot(E)$ must take, we note that for $E = \mathcal{O}_X$ the complex $C^\cdot = C^\cdot(\mathcal{O})$ has the following description: $C^j = \Omega^{j-1} \oplus \Omega^j$ and the differential $C^j \longrightarrow C^{j+1}$ is given by $(\varphi,\psi) \longrightarrow (d\varphi + (-1)^j \psi, d\psi)$. Given any complex $\mathcal{J}^\cdot$ and morphism $\mathcal{O} \xrightarrow{\lambda} \mathcal{J}^0$ defined by $1 \longrightarrow \alpha$, λ extends uniquely to $C^j \longrightarrow \mathcal{J}^j$ via $\lambda(\varphi,\psi) = \varphi.d\alpha + \psi\cdot\alpha$. N.B. The complex $\Omega^\cdot$ is not universal for maps $\mathcal{O} \xrightarrow{\lambda} \mathcal{J}^\cdot$, although an extension can be made to give a unique element of $\mathrm{Hom}^0(\Omega^\cdot,\mathcal{J}^\cdot)$ but this element will be a cycle, i.e. a morphism of complexes, if and only if $\lambda(1)$ is a cycle.

To obtain the construction of $C^\cdot(E)$ we proceed as follows. Let $J^0(E) = E$, define $J^n(E) = J_1(J^{n-1}(E))$, and note the natural sequence

$$J^\cdot(E): \quad 0 \longrightarrow J^0(E) \longrightarrow J^1(E) \longrightarrow J^2(E) \longrightarrow \cdots$$

which is not a complex but which clearly has the property that given any $\mathcal{J}^\cdot \in C$ and any $\mathcal{O}$ linear map $\lambda: E \longrightarrow \mathcal{J}^0$ there is a unique morphism of sequences $\tilde{\lambda}: J^\cdot(E) \longrightarrow \mathcal{J}$; extending λ. If one sets $K^0(E) = 0$ and defines K^n to be the smallest $\mathcal{O}$ submodule of $J^n(E)$ containing $d(K^{n-1})$ and $dd(J^{n-2})$, then clearly $C^\cdot(E) = J^\cdot(E)/K^\cdot$ is a complex. Given any morphism of sequences $\tilde{\lambda}: J^\cdot(E) \longrightarrow \mathcal{J}^\cdot$ where $\mathcal{J}^\cdot$ is a complex, one must have $\tilde{\lambda}(K^\cdot) = 0$, so that there is a unique induced morphism of complexes, $\tilde{\lambda}: C^\cdot(E) \longrightarrow \mathcal{J}^\cdot$. To obtain a more manageable expression for $C^\cdot(E)$, we note that $C^\cdot(\mathcal{O})$ carries two $\mathcal{O}$ module structures: the standard "or left" structure, $\mathcal{O} \otimes \Omega^{j-1} \oplus \Omega^j \longrightarrow \Omega^{j-1} \oplus \Omega^j$ given by $f(\varphi,\psi) = (f\varphi, f\psi)$ and the "π_2" or "right" structure given by $(\varphi,\psi)f = (f\varphi, f\psi + \varphi df)$. Given any $\mathcal{O}$ module E , $C^\cdot(E)$ has an expression analogous to that for $J_1(E)$, namely, viewing $C^\cdot(\mathcal{O})$ as a "π_2" module

$$C^\cdot(E) = C^\cdot(\mathcal{O}) \otimes_{\pi_2^*\mathcal{O}} E$$

with the $\mathcal{O}$ module structure on $C^\cdot(E)$ given by the "π_1" structure on $C^\cdot(\mathcal{O})$. The differential on $C^\cdot(E)$ is given by $(\varphi,\psi)\otimes\sigma \longrightarrow (d\varphi + (-1)^j\psi, d\psi)\otimes\sigma$.

Remark 2.11: In the event that X is nonsingular the functor $E \longrightarrow C^\cdot(E)$ is exact, being essentially tensor product with the locally free $\mathcal{O}$ module $C^\cdot(\mathcal{O})$.

Proposition 2.12: The category $C(X)$ of complexes of $\mathcal{O}_X$ modules with differential operators of order 1 has enough injectives.

Proof: Since $C(X)$ is an abelian category which satisfies AB 5) and AB 3), of [6] it suffices to show that $C(X)$ has generators, ([6], §1.10). As U runs through the open sets of X the sheaves $\mathcal{O}_U$ (extended by zero) define a system of generators for the category of $\mathcal{O}_X$ modules, and the complexes $C^\cdot(\mathcal{O}_U)[i]$ clearly form a system of generators for $C(X)$, as i ranges over the integers, and U ranges over the open sets.

While the appearance of injective complexes is not at all evident from the preceding proposition, one can easily check that if $Q^\cdot$ is injective then each Q^i is an injective $\mathcal{O}$ module provided X is nonsingular. We show below that $Q^\cdot$ is injective as a graded $\Omega^\cdot$ module whether or not X has singularities. We must first generalize the construction 2.10.

<u>Proposition</u> 2.13: Let $\mathcal{M}^\cdot$ be any graded $\Omega^\cdot$ module. There exists a complex $C^\cdot(\mathcal{M}^\cdot)$ and a morphism $\Psi: \mathcal{M}^\cdot \longrightarrow C^\cdot(\mathcal{M}^\cdot)$ of graded $\Omega^\cdot$ modules such that given any graded module map $\Phi: \mathcal{M}^\cdot \longrightarrow K^\cdot$ with $K^\cdot$ a complex, Φ factors uniquely:

$$\begin{tikzcd} C^\cdot(\mathcal{M}^\cdot) \arrow[dr, "\tilde{\Phi}"] & \\ \mathcal{M}^\cdot \arrow[u, "\Psi"] \arrow[r, "\Phi"'] & K^\cdot \end{tikzcd}$$

where $\tilde{\Phi}$ is a morphism of complexes.

<u>Proof</u>: $C^\cdot(\mathcal{M}^\cdot)$ is clearly the quotient of the complex $\bigoplus_a C^\cdot(\mathcal{M}^a)[-a]$ by a suitable subcomplex R. More instructively, we may obtain $C^\cdot(\mathcal{M}^\cdot)$ by noting first that $C^\cdot(\Omega^\cdot)$ is clearly just $C^\cdot(\mathcal{O})$. Indeed, a map $\Omega^\cdot \longrightarrow K^\cdot$ of graded $\Omega^\cdot$ modules is determined by an $\mathcal{O}$ linear map $\mathcal{O} \longrightarrow K^0$, which uniquely defines a morphism $C^\cdot(\mathcal{O}) \longrightarrow K^\cdot$.

Now, note that $C^\cdot(\mathcal{O})$ has again two graded $\Omega^\cdot$ module structures given explicitly by

π_1 or left: $\alpha(\varphi,\psi) = (\alpha\wedge\varphi,\ \alpha\wedge\psi)$

π_2 or right: $(\varphi,\psi)\alpha = (\varphi\wedge\alpha,\ (-1)^i\ (\psi\wedge\alpha + \varphi\wedge d\alpha))$

where α and (φ,ψ) are respectively local sections of Ω^i and $C^j(\mathcal{O}) = \Omega^{j-1} \oplus \Omega^j$ The differential in $C^\cdot(\mathcal{O})$ is, of course, linear in the π_2 structure. Note that $C^\cdot(\mathcal{O})$ is a free $\Omega^\cdot$ module of rank 2 in either structure, with the global sections $(0,1) \in \Gamma(C^0(\mathcal{O})) = \Gamma\ (\Omega^{-1}) \oplus \Gamma(\Omega^0)$ and $(1,0) \in \Gamma(C^1(\mathcal{O})) = \Gamma(\Omega^0) \oplus \Gamma(\Omega^1)$ as a basis. Given any $\mathcal{M}^\cdot$, a graded $\Omega^\cdot$ module, the complex

$$C^{\cdot}(\mathcal{M}^{\cdot}) = C^{\cdot}(\mathcal{O}) \otimes_{\pi_2\Omega} \mathcal{M}^{\cdot}$$

is the universal object we seek.

Remark 2.14: In view of the freeness of $C^{\cdot}(\mathcal{O})$ as an $\Omega^{\cdot}$ module, the functor $\mathcal{M}^{\cdot} \longrightarrow C^{\cdot}(\mathcal{M}^{\cdot})$ is clearly exact. In fact the complex $C^{\cdot}(\mathcal{M}^{\cdot})$ has the disarmingly simple description $C^j(\mathcal{M}^{\cdot}) \xrightarrow{\sim} \mathcal{M}^{j-1} \oplus \mathcal{M}^j$ (as $\pi_2^*\mathcal{O}$ modules) where we identify $(\varphi,\psi) = (1,0) \otimes \varphi + (0,1) \otimes \psi$ and the differentiation is given by $d(\varphi,\psi) = (\psi,0)$. This splitting is, of course, not a $\pi_1^*\mathcal{O}$ splitting. In particular the π_1 splitting $C^j(\Omega^{\cdot}) = \Omega^{j-1} \oplus \Omega^j$ is quite distinct from the π_2 splitting $C^j(\Omega^{\cdot}) \xrightarrow{\sim} \Omega^{j-1} \oplus \Omega^j$ the relationship being given by

$$(\varphi,\psi)_2 = (\varphi,(-1)^j\,(d\varphi - \psi))_1 .$$

Proposition 2.15: Let $Q^{\cdot}$ be an injective complex with differential operators, then $Q^{\cdot}$ is an injective graded $\Omega^{\cdot}$ module.

Proof: Given $0 \longrightarrow E^{\cdot} \longrightarrow \mathcal{J}^{\cdot}$ a morphism of graded $\Omega^{\cdot}$ modules and an $\Omega^{\cdot}$ morphism $E^{\cdot} \xrightarrow{\lambda} Q^{\cdot}$ we obtain the exact (cf. 2.14) commutative diagram

$$\begin{tikzcd}
\mathcal{J}^{\cdot} \arrow[r] & C^{\cdot}(\mathcal{J}^{\cdot}) & \\
E^{\cdot} \arrow[u] \arrow[r, "i"] & C^{\cdot}(E^{\cdot}) \arrow[u] \arrow[r, "\tilde{\lambda}"] & Q^{\cdot} \\
0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where the morphism $\tilde{\lambda}$ is a morphism of complexes and $\lambda = \tilde{\lambda} \circ i$. Since $Q^{\cdot}$ is an injective complex the map $\tilde{\lambda}$ extends to $C^\cdot(\mathcal{J}^\cdot)$, yielding an extension of λ to $\mathcal{T}$

One further remark on the structure of injective complexes is useful.

<u>Proposition</u> 2.16: Let $Q^\cdot$ be an injective complex, then $Q^\cdot$ is acyclic, and the complex $\Gamma(Q^\cdot)$ is also acyclic.

<u>Proof</u>: We show that the complex $\Gamma(Q^\cdot)$ has no cohomology, then given any open set U, $\Gamma(U,Q^\cdot)$ is acyclic since the restriction of $Q^\cdot$ to U is injective on U, and the acyclicity of $Q^\cdot$ follows. Suppose $\alpha \in \Gamma(Q^i)$ is a cycle, and consider the morphism of complexes $\Omega^\cdot[-i] \longrightarrow Q^\cdot$ $\lambda(\varphi) = \varphi\cdot\alpha$. But we have a natural injection of complexes, $\Omega^\cdot[-i] \longrightarrow C^\cdot(\Omega^\cdot)[-i+1]$, $\Omega^j \longrightarrow \Omega^j \oplus \Omega^{j+1}$, defined by $\varphi \longrightarrow (\varphi,0)$. Therefore one can extend λ to give $\tilde{\lambda}: C^\cdot(\Omega^\cdot)[-i+1] \longrightarrow Q^\cdot$ a morphism of complexes, since $Q^\cdot$ is injective. Noting that $\alpha = \tilde{\lambda}((1,0)) = \tilde{\lambda}(d(0,1)) = d(\tilde{\lambda}((0,1)))$, we see that α must bound in $\Gamma^\cdot(Q^\cdot)$, as asserted.

§3 Hyperext

We construct the functor hyperext which is to provide the dual to the hypercohomology of a complex. The construction is quite analogous to the hypercohomology construction (cf. [9], 0 , 11.4) which we briefly review. Given any complex $E^{\cdot} \in C(X)$ one may find a resolution

$$0 \longrightarrow E^{\cdot} \longrightarrow Q^{\cdot,0} \longrightarrow Q^{\cdot,1} \longrightarrow \ldots .$$

where the $Q^{\cdot,i}$ are injectives (cf.2.12). Given $0 \longrightarrow E^{\cdot} \longrightarrow Q^{\cdot\cdot}$ and $0 \longrightarrow \mathcal{J}^{\cdot} \longrightarrow P^{\cdot\cdot}$ injective resolutions, a morphism $f: E^{\cdot} \longrightarrow \mathcal{J}^{\cdot}$ lifts to a morphism $\hat{f}: Q^{\cdot\cdot} \longrightarrow P^{\cdot\cdot}$ of double complexes, and the lifting $\hat{f}$ is determined up to a homotopy in the following strong sense: If $\hat{f}$ and $\hat{f}'$ both lift f then there exist morphism of complexes $\Phi_i: Q^{\cdot i} \longrightarrow P^{\cdot i-1}$ such that

$$d''\Phi + \Phi d'' = \hat{f} - \hat{f}' .$$

Note that if d denotes the total differential in the double complexes

$$d\Phi + \Phi d = \hat{f} - \hat{f}'$$

since $d'\Phi = \Phi d'$. Thus Φ is a homotopy of degree 0 with respect to the first filtration. Given any complex $E^{\cdot}$, the first spectral sequence of hypercohomology of $E^{\cdot}$ is the first spectral sequence of the double complex $\Gamma(Q^{\cdot\cdot})$ where $Q^{\cdot\cdot}$ is an arbitrary injective resolution of $E^{\cdot}$. The hypercohomology sequence

$$E_1^{p,q} = H_{II}^q\ (\Gamma(Q^{p\cdot})) = H^q(X, E^p) \Longrightarrow H^{p+q}(E^\cdot) = H^{p+q}(\Gamma Q^\cdot))$$

is functorial in $E^\cdot$, and independent of the choice of injective resolution, in view of the special form of homotopies, (cf. [3], p. 321.)

The second spectral sequence of this double complex

$$E_2^{p,q} = H_{II}^p\ (H_I^q\ (\Gamma(Q^{\cdot\cdot}))) \Longrightarrow \underline{\underline{H}}^{p+q}(E^\cdot)$$

is also functorial and independent of the choice of Q, but is not interesting since $H_I^q\ (\Gamma(Q^{\cdot p})) = 0$ for all p and q in view of 2.16. The label "second spectral sequence of hypercohomology" is therefore reserved for the local-global spectral sequence

$$E_2^{p,q} = H^p(X, H^q(E^\cdot)) \Longrightarrow \underline{\underline{H}}^{p+q}(E^\cdot)$$

where H^q are the cohomology sheaves of $E^\cdot$.

Now, given two complexes $E^\cdot, \mathcal{J}^\cdot \in C(X)$ the spectral sequences for hyperext are obtained as follows. Taking an injective resolution $0 \longrightarrow \mathcal{J}^\cdot \longrightarrow Q^{\cdot\cdot}$, one forms the double complex $K^{p,q} = \mathrm{Hom}^p(E^\cdot, Q^{\cdot q})$ where the differential $d'\colon \mathrm{Hom}^p(E^\cdot, Q^{\cdot q}) \longrightarrow \mathrm{Hom}^{p+1}(E^\cdot, Q^{\cdot q})$ is defined by

$$d'\Phi = d_Q \circ \Phi + (-1)^{p+q+1}\ \Phi \circ d_E$$

while the differential $d''\colon \mathrm{Hom}(E^a, Q^{a+p,q}) \longrightarrow \mathrm{Hom}(E^a, Q^{a+p,q+1})$ is given by

$$d''\Phi = (-1)^a\ j_q \circ \Phi$$

where j_q is the map $Q^{\cdot,q} \longrightarrow Q^{\cdot,q+1}$. Note that $d'd'' = d''d'$, so that the total differential d of the double complex is defined by $d = d' + (-1)^p d''$ on the (p,q) level. Recall that elements of $\mathrm{Hom}^\cdot$ are required to possess $\Omega^\cdot$ linearity, a property which is preserved by d' and d''. The two spectral sequence of the double complex $K^{\cdot\cdot}$

$$3.1 \qquad {}'E_1^{p,q} = H_{II}^{q}\,(K^{p\cdot}) \Longrightarrow H^{p+q}(K^{\cdot})$$

and

$$''E_2^{p,q} = H_{II}^{p}\, H_{I}^{q}\,(K^{\cdot\cdot}) \Longrightarrow H^{p+q}(K^{\cdot})$$

are clearly bifunctorial in $E^{\cdot}$ (contravariant) and $\mathcal{J}^{\cdot}$ (covariant) and are independent of the particular injective resolution of $\mathcal{J}^{\cdot}$. We define the bifunctor $\underline{\underline{\mathrm{Ext}}}^p(E^{\cdot},\mathcal{J}^{\cdot})$ to be the abuttment $H^p((K^{\cdot})$ of these spectral sequences. Given $E_1^{\cdot} \xrightarrow{e} E_2^{\cdot}$ we denote by $e^*: \underline{\underline{\mathrm{Ext}}}\,(E_2^{\cdot},\ \ ) \longrightarrow \underline{\underline{\mathrm{Ext}}}\,(E_1^{\cdot},\ \ )$ and $e_*: \underline{\underline{\mathrm{Ext}}}\,(\ \ , E_1^{\cdot}) \longrightarrow \underline{\underline{\mathrm{Ext}}}\,(\ \ , E_2^{\cdot})$ the natural transformations of functors.

The first spectral sequence above is called the first spectral sequence of hyperext, while the term "second spectral sequence" is reserved for the local global sequence obtained as follows.

Define the sheaf $\underline{\underline{\mathrm{Ext}}}^{\cdot}(\mathcal{J}^{\cdot},\mathcal{S}^{\cdot})$ by taking an injective resolution $0 \longrightarrow \mathcal{S}^{\cdot} \longrightarrow Q^{\cdot\cdot}$, forming the double complex of sheaves $K^{p,q} = \mathrm{Hom}^p(\mathcal{J}^{\cdot},Q^{\cdot q})$ and letting $\underline{\underline{\mathrm{Ext}}}^{\cdot}$ to be the cohomology sheaves of the single complex $K^{\cdot}$. Local $\underline{\underline{\mathrm{Ext}}}$ is related to $\underline{\underline{\mathrm{Ext}}}$ by a spectral sequence

$$3.2 \qquad E_2^{p,q} = H^p(X,\underline{\underline{\mathrm{Ext}}}^q(\mathcal{J}^{\cdot},\mathcal{S}^{\cdot})) \Longrightarrow \underline{\underline{\mathrm{Ext}}}^{p+q}(\mathcal{J}^{\cdot},\mathcal{S}^{\cdot})$$

In fact 3.2 is the second spectral sequence of hypercohomology for the complex $K^{\cdot}$:

3.2' $$E_2^{p,q} = H^p(X, \underline{\underline{\mathrm{Ext}}}^q(\mathcal{J}^\cdot, \mathcal{B}^\cdot)) \Longrightarrow \underline{\underline{H}}^{p+q}(\mathbf{K}^\cdot)$$

as may be seen from the first spectral sequence $H^q(X, K^p) \Longrightarrow \underline{\underline{H}}^{p+q}(K^\cdot)$. The sheaves K^p are flasque (standard argument cf. ([4], II, 7.3.2) so that $H^q(K^p) = 0$ $q > 0$ and $\underline{\underline{H}}^{p+q}(K^\cdot)$ is the $p+q$'th cohomology of the complex $\Gamma(K^\cdot)$, i.e. $\underline{\underline{\mathrm{Ext}}}^{p+q}(\mathcal{J}^\cdot, \mathcal{B}^\cdot)$.

<u>Remark</u> 3.3: Notice that for $E^\cdot = \Omega^\cdot$, one has

$$\underline{\underline{\mathrm{Ext}}}^\cdot\,(\Omega^\cdot, \mathcal{J}^\cdot) = \underline{\underline{H}}^\cdot(\mathcal{J}^\cdot)$$
$$\underline{\underline{\mathrm{Ext}}}^\cdot\,(\Omega^\cdot, \mathcal{J}^\cdot) = \mathcal{H}^\cdot(\mathcal{J}^\cdot)$$

In fact if $Q^{\cdot\cdot}$ is any injective resolution of $\mathcal{J}^\cdot$ then $\mathrm{Hom}^\cdot(\Omega^\cdot, Q^{\cdot j}) \xrightarrow{\sim} \Gamma(Q^{\cdot j})$ and $\mathrm{Hom}^\cdot(\Omega^\cdot, Q^{\cdot j}) \xrightarrow{\sim} Q^{\cdot j}$ in view of 2.4, (2), and the cohomology of these complexes is $\underline{\underline{H}}(\mathcal{J}^\cdot)$ and $\mathcal{H}(\mathcal{J}^\cdot)$.

<u>Remark</u> 3.4: The bifunctors $\underline{\underline{\mathrm{Ext}}}$ and $\underline{\underline{\mathrm{Ext}}}$ are cohomological, i.e. given $E^\cdot$ and an exact sequence $0 \to \mathcal{J}_1^\cdot \xrightarrow{\Phi} \mathcal{J}_2^\cdot \xrightarrow{\Psi} \mathcal{J}_3^\cdot \to 0$ one obtains a long exact sequence

$$\cdots \to \underline{\underline{\mathrm{Ext}}}^{i-1}(E^\cdot, \mathcal{J}_3^\cdot) \xrightarrow{\delta_*} \underline{\underline{\mathrm{Ext}}}^i(E^\cdot, \mathcal{J}_1^\cdot) \xrightarrow{\Phi_*} \underline{\underline{\mathrm{Ext}}}^i(E^\cdot, \mathcal{J}_2^\cdot) \xrightarrow{\Psi_*} \underline{\underline{\mathrm{Ext}}}^i(E^\cdot, \mathcal{J}_3^\cdot) \xrightarrow{\delta_*} \cdots$$

by appropriate choice of injective resolutions. Similarly, given $\mathcal{J}^\cdot$ and $0 \to E_1^\cdot \xrightarrow{\Phi} E_2^\cdot \xrightarrow{\Psi} E_3^\cdot \to 0$, there is a long exact sequence

$$\cdots \to \underline{\underline{\mathrm{Ext}}}^{i-1}(E_1^\cdot, \mathcal{J}^\cdot) \xrightarrow{\delta^*} \underline{\underline{\mathrm{Ext}}}^i(E_3^\cdot, \mathcal{J}^\cdot) \xrightarrow{\Psi^*} \underline{\underline{\mathrm{Ext}}}^i(E_2^\cdot, \mathcal{J}^\cdot) \xrightarrow{\Phi^*} \underline{\underline{\mathrm{Ext}}}^i(E_1^\cdot, \mathcal{J}^\cdot) \xrightarrow{\delta^*} \cdots$$

This is obtained by taking $0 \to \mathcal{J}^\cdot \to Q^{\cdot\cdot}$ any injective resolution and forming the sequence of double complexes

3.6 $$0 \to \mathrm{Hom}^\cdot(E_3^\cdot, Q^{\cdot\cdot}) \to \mathrm{Hom}^\cdot(E_2^\cdot, Q^{\cdot\cdot}) \to \mathrm{Hom}^\cdot(E_1^\cdot, Q^{\cdot\cdot}) \to 0$$

which is <u>exact</u> in view of 2.15. The exact sequence of single complexes formed from 3.6 has 3.5 as its cohomology sequence.

§4. The Yoneda Pairing

Let R be a ring.

Definition 4.1: Let A, B, C be spectral sequences of R-modules. A pairing $A \times B \longrightarrow C$ of spectral sequences is

1) For each r, p, q, p', q' there is a bilinear pairing

$$A_r^{p,q} \times B_r^{p',q'} \longrightarrow C_r^{p+p',q+q'}$$

denoted $a \times b \longrightarrow a \cdot b$, which satisfies

$$d_r^{p+p',q+q'}(a \cdot b) = d_r^{p,q}(a) \cdot b + (-1^{p+q}\, a \cdot d_r^{p',q'}(b)$$

2) The pairing on level $r+1$ coincides with the pairing induced on the cohomology of level r.

3) There is a pairing on the abutments $A^n \times B^m \longrightarrow C^{n+m}$ compatible with the filtrations, i.e.

$$F_p(A^n) \times F_q(B^m) \longrightarrow F_{p+q}\,(C^{n+m})$$

4) The induced pairing on the associated graded objects

$$gr_p(A^n) \times gr_q(B^m) \longrightarrow gr_{p+q}(C^{n+m})$$

is compatible with the pairings on the E_∞ terms, (i.e. the pairing of the E_2 terms is assumed to satisfy $Z_\infty(A) \times Z_\infty(B) \longrightarrow Z_\infty(C)$, $B_\infty(A) \times Z_\infty(B) \longrightarrow B_\infty(C)$

and $Z_\infty(A) \times B_\infty(B) \longrightarrow B_\infty(C)$ thereby inducing a pairing of the E_∞ terms.)

The construction of the Yoneda pairing which follows is essentially standard. See Hartshorne [10], for example.

Proposition 4.2: Let $E^\cdot, \mathcal{F}^\cdot, \mathcal{G}^\cdot \in C(X)$, then there is a natural pairing of the first spectral sequences for hyperext

$$\underline{\underline{\mathrm{Ext}}}\,(\mathcal{F}^\cdot,\mathcal{G}^\cdot) \times \underline{\underline{\mathrm{Ext}}}\,(E^\cdot,\mathcal{F}^\cdot) \longrightarrow \underline{\underline{\mathrm{Ext}}}\,(E^\cdot,\mathcal{G}^\cdot)\ .$$

Proof: Let $0 \longrightarrow \mathcal{F}^\cdot \longrightarrow Q^{\cdot\cdot}$ and $0 \longrightarrow \mathcal{G}^\cdot \longrightarrow P^{\cdot\cdot}$ be injective resolutions. Denote by $\mathrm{Hom}^{r,s}(Q^{\cdot\cdot},P^{\cdot\cdot})$ the sheaf maps $\Phi\colon Q^{\cdot,\cdot} \longrightarrow P^{\cdot+r,\cdot+s}$ which are $\Omega^\cdot$ linear in the sense that

$$\Phi(\varphi.\alpha) = (-1)^{(r+s)j}\,\varphi\cdot\Phi(\alpha)$$

for $\varphi \in \Gamma(\Omega^j)$. Then $\mathrm{Hom}^{\cdot\cdot}(Q^{\cdot\cdot},P^{\cdot\cdot}) = \bigoplus_r \bigoplus_{s\geq 0} \mathrm{Hom}^{r,s}\,(Q^{\cdot\cdot},P^{\cdot\cdot})$ has a natural structure of double complex by defining the total differential d by the formula

$$d\Phi = d\circ\Phi + (-1)^{r+s+1}\,\Phi\circ d \quad , \quad \Phi \in \mathrm{Hom}^{r,s}$$

One can check that $d\Phi$ is again $\Omega^\cdot$ linear. Note that the differentials in our double complexes commute and the total derivative d in a double complex is $d = d' + (-1)^p d''$ on the (p,q) elements. This leads to the formulae

$$d'\Phi = d'\circ\Phi + (-1)^{r+s+1}\,\Phi\circ d'$$

and $$(d''\Phi)(\alpha) = (-1)^a(d''(\Phi(\alpha)) + (-1)^{s+1}\,\Phi(d''(\alpha)) \quad , \quad \alpha \in \Gamma(Q^{a,\cdot})$$

Note that one has a natural pairing

$$\mathrm{Hom}^{r,s}(Q^{\cdot\cdot},P^{\cdot\cdot}) \times \mathrm{Hom}^{r'}(E^{\cdot},Q^{\cdot s'}) \longrightarrow \mathrm{Hom}^{r+r'}(E^{\cdot},P^{\cdot,s+s'})$$

by composition of morphisms, that is $\varphi \times \psi \longrightarrow \varphi\circ\psi$. This is a differential pairing in the sense that

$$d(\varphi\circ\psi) = d\varphi\circ\psi + (-1)^{r+s}\varphi\circ d\psi$$

indeed

$$d(\varphi\circ\psi) = d\circ\varphi\circ\psi + (-1)^{r+r'+s+s'+1}\varphi\circ\psi\circ d$$

while

$$d\varphi\circ\psi = d\circ\varphi\circ\psi + (-1)^{r+s+1}\varphi\circ d\circ\psi$$

$$\varphi\circ d\psi = \varphi\circ d\circ\psi + (-1)^{r'+s'+1}\varphi\circ\psi\circ d\ .$$

Such a differential pairing of double complexes gives rise to a pairing of the first (and second) spectral sequences of the double complexes.

Moreover, one has a natural morphism of double complexes $\mathrm{Hom}^{r,s}(Q^{\cdot\cdot},P^{\cdot\cdot}) \longrightarrow \mathrm{Hom}^{r}(\mathcal{F}^{\cdot},P^{\cdot,s})$ obtained by composing a map $Q^{\cdot\cdot} \longrightarrow P^{\cdot+r,\cdot+s}$ with the inclusion $\mathcal{F}^{\cdot} \longrightarrow Q^{\cdot,0}$. This morphism induces an isomorphism of first spectral sequences. Indeed the fact that $0 \longrightarrow \mathcal{G}^{\cdot} \longrightarrow P^{\cdot\cdot}$ is a resolution of $\mathcal{G}^{\cdot}$ by injective $\Omega^{\cdot}$ modules, while $Q^{\cdot\cdot}$ is a resolution of $\mathcal{F}^{\cdot}$ implies that the d'' cohomology of the two complexes is identical, (standard argument.)

Moreover, since both double complexes are bounded below by $s \geq 0$, the first spectral sequences of these double complexes are regular [9], $\underline{0}$, 11.3.3), and the spectral sequences are therefore isomorphic [9], $\underline{0}$, 11.1.5).

Remark 4.3: 1) The double complex $\bigoplus_{\substack{r \\ s\geq 0}} (\mathrm{Hom}^{r,s})$ used in the preceding proof can be replaced by $\bigoplus_{r,s} (\mathrm{Hom}^{r,s})$, which also has the same first spectral sequence, although the proof becomes more arduous since this latter complex is not regular.

4.4 The Yoneda pairing exhibits the following functoriality properties. Given morphisms $E_1^{\cdot} \xrightarrow{e} E_2^{\cdot}$, $\mathcal{J}_1^{\cdot} \xrightarrow{f} \mathcal{J}_2^{\cdot}$, $\mathcal{S}_1 \xrightarrow{g} \mathcal{S}_2$ one has the basic formulae

1) $\alpha\cdot e^*(\beta) = e^*(\alpha\cdot\beta)$ $\qquad \alpha \in \underline{\underline{\mathrm{Ext}}}\,(\mathcal{J},\mathcal{S})$, $\beta \in \underline{\underline{\mathrm{Ext}}}\,(E_2,\mathcal{J})$

2) $g_*(\alpha)\cdot\beta = g_*((\alpha\cdot\beta)$ $\qquad \alpha \in \underline{\mathrm{Ext}}\,(\mathcal{J},\mathcal{S}_1)$, $\beta \in \underline{\underline{\mathrm{Ext}}}\,(E,\mathcal{J})$

3) $f^*(\alpha)\cdot\beta = \alpha . f_*(\beta)$ $\qquad \alpha \in \underline{\underline{\mathrm{Ext}}}\,(\mathcal{J}_2,\mathcal{S})$, $\beta \in \underline{\mathrm{Ext}}\,(E,\mathcal{J}_1)$.

which express the fact that composition of morphisms is associative. We check 3). Fix injective resolutions $0 \longrightarrow \mathcal{J}_i^{\bullet} \longrightarrow Q_i^{\cdot\cdot}$ and $0 \longrightarrow \mathcal{S} \longrightarrow P^{\cdot}$, and let $\tilde{f}\colon Q_1^{\cdot\cdot} \longrightarrow Q_2^{\cdot\cdot}$ denote a lift of $f\colon \mathcal{J}_1 \longrightarrow \mathcal{J}_2$, so that $\tilde{f}$ is a morphism of double complexes. Then given $\hat{\alpha} \in \mathrm{Hom}^{\cdot\cdot}(Q_2^{\cdot\cdot}, P^{\cdot\cdot})$ and $\hat{\beta} \in \mathrm{Hom}^{\cdot}(E^{\cdot},Q_1^{\cdot\cdot})$ we have $(\hat{\alpha}\circ\hat{f})\circ\hat{\beta} = \hat{\alpha}\circ(\hat{f}\circ\hat{\beta})$, and 3) follows.

4.5 The relationship of the Yoneda pairing with the connecting morphisms δ_*, δ^* is also readily calculated. Assume $0 \longrightarrow \mathcal{J}_1^{\cdot} \xrightarrow{f_1} \mathcal{J}_2^{\cdot} \xrightarrow{f_2} \mathcal{J}_3 \longrightarrow 0$ is exact then

$$\delta^*(\alpha)\cdot\beta = (-1)^{r+s+1}\, \alpha\cdot\delta_*(\beta)$$

where $\alpha \in \underline{\underline{\mathrm{Ext}}}^{r+s}\,(\mathcal{J}_1,\mathcal{S})$ and $\beta \in \underline{\mathrm{Ext}}\,(E,\mathcal{J}_3)$.

§5. Duality Theorem

Throughout this section X denotes a compact nonsingular space of dimension m . Given $E^{\cdot} \in C(X)$, we have the natural Yoneda pairing of first spectral sequences

$$\underline{\underline{\mathrm{Ext}}}(E^{\cdot}, \Omega^{\cdot}) \times \underline{\underline{\mathrm{Ext}}}(\Omega^{\cdot}, E^{\cdot}) \longrightarrow \underline{\underline{\mathrm{Ext}}}(\Omega^{\cdot}, \Omega^{\cdot})$$

which may be rewritten as a pairing

$$\underline{\underline{\mathrm{Ext}}}(E^{\cdot},\Omega^{\cdot}) \times \underline{\underline{H}}(E^{\cdot}) \longrightarrow \underline{\underline{H}}(\Omega^{\cdot})$$

in view of 3.3 . In particular we have for each r, p, q,

5.1 $$E_r^{p,q} \times E_r^{m-p,m-q} \longrightarrow E_r^{m,m}$$

and on the abuttments

5.2 $$F_a\underline{\underline{\mathrm{Ext}}}^{p+q} \times F_b\, \underline{\underline{H}}^{2m-p-q} \longrightarrow F_{a+b}\, H^{2m}(\Omega^{\cdot}) .$$

We note the following facts about the spectral sequence for $\underline{\underline{H}}(\Omega^{\cdot})$:

$$(5.3)\quad E_1^{m,m} \xrightarrow{\sim} E_2^{m,m} \xrightarrow{\sim} \cdots \xrightarrow{\sim} E_\infty^{m,m} \xrightarrow{\sim} \underline{\underline{H}}^{2m}(\Omega^{\cdot}) = \mathbb{C}$$

and $$(5.4)\quad F_p\, \underline{\underline{H}}^{2m}(\Omega^{\cdot}) = \begin{cases} 0 & p > m \\ \underline{\underline{H}}^{2m}(\Omega) & p \le m \end{cases}$$

Indeed, $H^q(\Omega^p) = 0$ if $q > m$, and 5.3 and 5.4 follow immediately except for the observation $E_1^{m,m} \xrightarrow{\sim} E_2^{m,m}$ which is evident from the exact sequence

$$H^m(\Omega^{m-1}) \xrightarrow{d_1^{m-1,m}} H^m(\Omega^m) \longrightarrow E_2^{m,m} \longrightarrow 0$$

and the fact that $H^m(\Omega^m) = (H^0(\Omega^0))^* = C$.

Thus the pairings 5.1, 5.2 , are into C , and we note for further reference that 5.4 implies that

5.5 $F_a \underline{\underline{\mathrm{Ext}}}^{p+q}(E^\cdot,\Omega^\cdot)$ and $F_b(\underline{\underline{H}}^{2m-p-q}(E^\cdot))$ are orthogonal if $a+b>m$.

Note, moreover, that the vanishing of the differentials $d_r^{m-r,m+r-1}$ in the sequence for $\underline{\underline{H}}(\Omega^\cdot)$ implies that the differentials in the $\underline{\underline{\mathrm{Ext}}}$ sequence are <u>dual</u> to the differentials in the sequence $\underline{\underline{H}}(E^\cdot)$, i.e.

5.6
$$(d_r\alpha)\cdot\beta = (-1)^{p+q+1}\,\alpha . d_r\beta$$

if $\alpha\in E_r^{p,q}$, $\beta\in E_r^{m-p-r,m-q+r-1}$ since

$$0 = d_r(\alpha\cdot\beta) = (d_r\alpha)\cdot\beta + (-1)^{p+q}\,\alpha\cdot d_r\beta$$

<u>Theorem</u> 5.7: Let X be a compact nonsingular space. Let $E^\cdot\in C(X)$ be a quasicoherent complex, (each E^i is a quasicoherent $\mathcal{O}_X$ module) then the pairings 5.1 and 5.2 are perfect. In particular

$$\underline{\underline{\mathrm{Ext}}}^k(E^\cdot,\Omega^\cdot) \xrightarrow{\sim} (\underline{\underline{H}}^{2m-k}(E^\cdot))^* ,$$

is an isomorphism of filtered vector spaces, i.e.

$$F_p\,\underline{\underline{\mathrm{Ext}}}^k\,(E^\cdot,\Omega^\cdot) \xrightarrow{\sim} (\underline{\underline{H}}^{2m-k}(E^\cdot)/F_{m-p+1}\,\underline{\underline{H}}^{2m-k}(E^\cdot))^*$$

<u>Proof:</u> Let $0 \longrightarrow \Omega^{\cdot} \longrightarrow Q^{\cdot\cdot}$ be a resolution of $\Omega^{\cdot}$ by injective complexes. The pairing at level E_1 may then be described as

5.8 $$H^q_{II}\ (\mathrm{Hom}^p(E^{\cdot},Q^{\cdot\cdot})) \times H^{2m-q}(E^{2m-p}) \longrightarrow H^m(\Omega^m) = C\ .$$

We claim that this pairing is perfect, being in fact standard Serre duality. To check this fact we compute H^q_{II} .

Consider the left exact functor F from graded $\Omega^{\cdot}$ modules to groups defined by $F(\mathcal{J}^{\cdot}) = \mathrm{Hom}^p(E^{\cdot},\mathcal{J}^{\cdot})$. We note that $R^qF(\Omega^{\cdot}) = H^q_{II}\ (\mathrm{Hom}^p(E^{\cdot},Q^{\cdot\cdot}))$ since $Q^{\cdot\cdot}$ is an injective resolution of $\Omega^{\cdot}$ in the category of $\Omega^{\cdot}$ modules (2.15). To to calculate these groups we may replace $Q^{\cdot\cdot}$ by any resolution of $\Omega^{\cdot}$ by $\Omega^{\cdot}$ injectives. Let $0 \longrightarrow \Omega^m \longrightarrow P^0 \longrightarrow P^1 \ldots$ be any $\mathcal{O}_X$ injective resolution of Ω^m , and consider the resolution

$$\begin{array}{c} 0 \longrightarrow (\Omega^{m-\bullet})^* \otimes \Omega^m \longrightarrow (\Omega^{m-\bullet})^* \otimes P^0 \longrightarrow (\Omega^{m-\bullet})^* \otimes P^1 \longrightarrow \ldots \\ \uparrow \wr \\ \Omega^{\cdot} \end{array}$$

of $\Omega^{\cdot}$ by $\Omega^{\cdot}$ injectives, (cf. 2.9).

Now $\mathrm{Hom}^p(E^{\cdot}, (\Omega^{m-\bullet})^* \otimes P^q) \xrightarrow{\sim} \mathrm{Hom}(E^{m-p}, P^q)$ by 2.7 and hence

5.9 $$H^q_{II}\ (\mathrm{Hom}^p(E^{\cdot},Q^{\cdot\cdot})) \xrightarrow{\sim} H^q(\mathrm{Hom}(E^{m-p},P^{\cdot})) \xrightarrow{\sim} \mathrm{Ext}^q_{\mathcal{O}_X}(E^{m-p},\Omega^m)$$

making this identification, the pairing 5.8 is simply the standard Yoneda pairing and is well known to be perfect, cf. Hartshorne [10] for X algebraic, and Suominen for X analytic, [17][1]. Thus the pairing of spectral sequences at level E_1 is perfect. But both of the spectral sequences in question are biregular.

In fact both spectral sequences are defined by double complexes $K^{r,s}$ with $s \geq 0$ so that $E_r^{p,q} = 0$ if $q < 0$. By the perfectness of the pairings $E_r^{p,q} = 0$ if $q > m$ (this is simply the fact that $H^q(X,\mathcal{J}) = 0$ and $\operatorname{Ext}^q(\mathcal{J},\Omega^m) = 0$ for $q > m$ and $\mathcal{J}$ quasi-coherent.) Moreover in both sequences $Z_\infty^{p,q} = Z_r^{p,q}$ for $r \geq q+1$ and $B_\infty^{p,q} = B_r^{p,q}$ for $r \geq m-q+1$. Hence

$$5.10 \qquad E_r^{p,q} = E_\infty^{p,q} \qquad r \geq \max(q+1,\ m-q+1)$$

and the pairing is perfect on E_∞. Moreover, on the abuttment we clearly have $F_{p+1} E^p = 0$ and since $E_\infty^{p,q} = 0$ for $q > m$, $F_{p-m} E^p = E^p$ so that the filtration is finite, (and in fact of length $\leq m+1$.) The perfection of the pairing on the abuttment then follows from the result on the associated graded.

<u>Problem</u>: The duality theorem, at least for coherent complexes should ideally not require compactness of X. The statement should of course be modified from the pairing of $\underline{\underline{\operatorname{Ext}}}$ and $\underline{\underline{H}}$ to $\underline{\underline{H}}$, to a pairing of $\underline{\underline{\operatorname{Ext}}}_c$ and $\underline{\underline{H}}$ to $\underline{\underline{H}}_c$,

[1] The analytic duality theorem $\operatorname{Ext}^q(E,\Omega^m) \xrightarrow{\sim} (H^{m-q}(E))^*$ is stated in Suominen only for the case of coherent E. If E is quasi-coherent and $E = \varinjlim E_\alpha$ where E_α is coherent then $\varinjlim H(X,E_\alpha) = H(X, \varinjlim E_\alpha)$, since X is compact, while $\varprojlim \operatorname{Ext}(E_\alpha,\Omega^m) = \operatorname{Ext}(\varinjlim E_\alpha,\Omega^m)$. Thus duality extends to the quasi-coherent case.

(c denoting compact supports.) If X is Stein, for example, the pairing is still "perfect" at level E_1 by standard duality, but the duality is between topological vector spaces. Moreover the differentials $d_1^{p,q}$ need not have closed image so that the duality for E_2 is no longer automatic. An example is furnished by $X = \mathbb{C} - \mathbb{Z}$ and $E^\cdot = \Omega_X^\cdot$, which still possesses duality, topological duality on E_1 and vector space duality on E_2 and the abuttment.

The duality theory for locally free complexes is considerably simpler than the result for quasi-coherent complexes and deserves special mention. First of all if $\mathcal{F}^\cdot$ is a locally free complex (i.e. each $\mathcal{F}^i$ is locally free) then the $\underline{\underline{\mathrm{Ext}}}\,(\mathcal{F}^\cdot,\Omega^\cdot)$ have a simpler description. Note first that for any $\mathcal{F}^\cdot,\mathcal{G}^\cdot$ one has a natural homomorphism

5.11 $$\underline{\underline{H}}^\cdot(\mathrm{Hom}^\cdot(\mathcal{F}^\cdot,\mathcal{G}^\cdot)) \longrightarrow \underline{\underline{\mathrm{Ext}}}^\cdot(\mathcal{F}^\cdot,\mathcal{G}^\cdot)$$

defined by taking $0 \longrightarrow \mathcal{G}^\cdot \longrightarrow Q^{\cdot\cdot}$ an injective resolution, setting $K^\cdot = \bigoplus_{r+s=\cdot} \mathrm{Hom}^r(\mathcal{F}^\cdot,Q^{\cdot s})$, so that the homomorphism 5.11 is the map

$$\underline{\underline{H}}^\cdot(\mathrm{Hom}^\cdot(\mathcal{F}^\cdot,\mathcal{G}^\cdot)) \longrightarrow \underline{H}^\cdot(K^\cdot) = \underline{\underline{\mathrm{Ext}}}^\cdot(\mathcal{F}^\cdot,\mathcal{G}^\cdot)$$

(cf. 3.2') arising from the natural map of complexes

$$\mathrm{Hom}^\cdot(\mathcal{F}^\cdot,\mathcal{G}^\cdot) \longrightarrow K^\cdot$$

The mapping 5.11 is in general neither injective nor surjective.

<u>Proposition</u> 5.13: Let $\mathcal{F}^\cdot \in C(X)$ be a locally free complex, then the natural map

$$\underline{H}^\cdot(\mathrm{Hom}^\cdot(\mathcal{F}^\cdot,\Omega^\cdot)) \longrightarrow \underline{\mathrm{Ext}}^\cdot(\mathcal{F}^\cdot,\Omega^\cdot)$$

is an isomorphism.

Proof: Let $0 \longrightarrow \Omega^{\cdot} \longrightarrow Q^{\cdot\cdot}$ be a resolution by injective complexes. The sequence of complexes

5.14 $$0 \longrightarrow \mathrm{Hom}^{\cdot}(\mathcal{J}^{\cdot},\Omega^{\cdot}) \longrightarrow \mathrm{Hom}^{\cdot}(\mathcal{J}^{\cdot},Q^{\cdot,0}) \longrightarrow \mathrm{Hom}^{\cdot}(\mathcal{J}^{\cdot},Q^{\cdot,1}) \longrightarrow \cdots$$

is exact. Indeed, the cohomology groups in the sequence are the derived functors of $\mathrm{Hom}^{\cdot}(\mathcal{J}^{\cdot},\ )$ evaluated at $\Omega^{\cdot}$, and may be calculated by replacing $Q^{\cdot\cdot}$ by an arbitrary resolution of $\Omega^{\cdot}$ by injective $\Omega^{\cdot}$ modules $P^{\cdot\cdot}$, for example $P^{r,s} = (\Omega^{m-r})^{*} \otimes P^{s}$ where $0 \longrightarrow \Omega^{m} \longrightarrow P^{\cdot}$ is an injective resolution. The cohomology of 5.14 is then the cohomology of

$$0 \longrightarrow \mathrm{Hom}(\mathcal{J}^{m-\cdot},\Omega^{m}) \longrightarrow \mathrm{Hom}(\mathcal{J}^{m-\cdot},P^{0}) \longrightarrow \mathrm{Hom}(\mathcal{J}^{m-\cdot},P^{1}) \longrightarrow \cdots$$

which is clearly acyclic since the $\mathcal{J}^{m-i}$ are locally free.

Thus 5.14 defines a resolution of $\mathrm{Hom}^{\cdot}(\mathcal{J}^{\cdot},\Omega^{\cdot})$ and consequently the map

$$\mathrm{Hom}^{\cdot}(\mathcal{J}^{\cdot},\Omega^{\cdot}) \longrightarrow \bigoplus_{r+s=\cdot} \mathrm{Hom}^{r}(\mathcal{J}^{\cdot},Q^{\cdot,s})$$

induces an isomorphism of cohomology sheaves. Therefore $\underline{\underline{H}}(\mathrm{Hom}^{\cdot}(\mathcal{J}^{\cdot},\Omega^{\cdot})) \xrightarrow{\sim} \underline{\underline{\mathrm{Ext}}}^{\cdot}(\mathcal{J}^{\cdot},\Omega^{\cdot})$ as asserted.

Remark 5.15: We know that $\mathrm{Hom}^{p}(\mathcal{J}^{\cdot},\Omega^{\cdot}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{J}^{m-p},\Omega^{m})$ by 2.5. Thus for $\mathcal{J}$ locally free $\underline{\underline{\mathrm{Ext}}}(\mathcal{J}^{\cdot},\Omega^{\cdot})$ is simply calculated as the hypercohomology of the complex $(\mathcal{J}^{m-\cdot})^{*} \otimes \Omega^{m}$. The differential operator in this complex is just the usual "adjoint" to the differential in $\mathcal{J}^{\cdot}$, cf. ([1], §5 and §11).

Moreover, the Yoneda pairing $\underline{\underline{\mathrm{Ext}}}(\mathcal{J}^{\cdot},\Omega^{\cdot}) \times \underline{\underline{H}}(\mathcal{J}^{\cdot}) \longrightarrow \underline{\underline{H}}(\Omega^{\cdot})$ is simply the cup product

$$\underline{\underline{H}}(\mathrm{Hom}^\cdot(\mathcal{J}^\cdot,\Omega^\cdot)) \times \underline{\underline{H}}(\mathcal{J}^\cdot) \longrightarrow \underline{\underline{H}}(\Omega^\cdot)$$

and the perfection of the pairing at level E_1 is standard Serre duality, $H^q(X,\mathrm{Hom}^p(\mathcal{J}^\cdot,\Omega^\cdot)) = H^q(X,(\mathcal{J}^{m-p})^* \otimes \Omega^m)$ being paired with $H^{m-q}(X,\mathcal{J}^{m-p})$ into $H^m(X,\Omega^m)$ by cup product.

5.16 Finally, we note that in the locally free case the C^∞ differential forms, $E_X^{\cdot\cdot}$, may be used to calculate all the pairings in question. Namely, if $\mathcal{J}^\cdot \in C(X)$ define $(\mathcal{J}^\cdot \otimes_{\Omega^\cdot} E^{\cdot,s})^r = E^{r,s}(\mathcal{J}^\cdot)$. If $\mathcal{J}^\cdot$ is locally free, then $E^{\cdot,s}(\mathcal{J}^\cdot)$ is a fine resolution of $\mathcal{J}^\cdot$ and the global double complex $\Gamma(E^{rs}(\mathcal{J}^\cdot))$ has for total cohomology $\underline{\underline{H}}(\mathcal{J}^\cdot)$. The Yoneda pairing may be directly calculated as the cup pairing

$$\mathrm{Hom}^\cdot(\mathcal{J}^\cdot,E^{\cdot\cdot}) \times \Gamma(\mathcal{J}^\cdot \otimes_{\Omega^\cdot} E^{\cdot\cdot}) \longrightarrow \Gamma(E^{\cdot\cdot}) .$$

$$\phi \times \alpha \otimes \beta \longrightarrow \phi(\alpha) \wedge \beta$$

§6. Poincare duality

The Yoneda pairing may well be perfect even though the complexes are not quasi-coherent. A typical example of this is classical Poincare duality ...

Let X be a complex manifold and $Y \xrightarrow{f} X$ a closed subvariety of X, $U = X - Y$. Consider the exact sequence of complexes

$$6.1 \qquad 0 \longrightarrow \Omega_U^{\cdot} \longrightarrow \Omega_X^{\cdot} \longrightarrow f^{-1}\Omega_X^{\cdot} \longrightarrow 0$$

where Ω_U (resp., $f^{-1}\Omega_X$) is the (sheaf theoretic) restriction of Ω_X to U, (resp. to Y), extended by zero. Note that $\underline{H}(X,\Omega_U^{\cdot}) = \underline{H}_c(U,\Omega_U)$ so that the hypercohomology sequence of the above sequence may be written

$$6.2 \qquad \ldots \longrightarrow \underline{\underline{H}}^{i-1}(f^{-1}\Omega_X) \longrightarrow \underline{\underline{H}}_c^{i}(U,\Omega_U) \longrightarrow \underline{H}^{i}(\Omega_X) \longrightarrow \underline{H}^{i}(f^{-1}(\Omega_X^{\cdot})) \longrightarrow \ldots$$

Since X is nonsingular the sequence 6.1 is a resolution of the sequence $0 \longrightarrow \mathbb{C}_U \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}_Y \longrightarrow 0$ where $\mathbb{C}$ denotes the sheaf of constants. Consequently the hypercohomology sequence above is simply

$$\ldots \longrightarrow H_c^i(U,\mathbb{C}) \longrightarrow H^i(X,\mathbb{C}) \longrightarrow H^i(Y,\mathbb{C}) \longrightarrow \ldots .$$

On the other hand we have the $\underline{\underline{\mathrm{Ext}}}$ sequence

$$6.3 \ldots \leftarrow \underline{\underline{\mathrm{Ext}}}^{2m-i+1}(f^{-1}\Omega_X,\Omega_X) \leftarrow \underline{\underline{\mathrm{Ext}}}^{2m-i}(\Omega_U^{\cdot},\Omega_X) \leftarrow \underline{\underline{\mathrm{Ext}}}^{2m-i}(\Omega_X,\Omega_X) \leftarrow \underline{\underline{\mathrm{Ext}}}^{2m-i}(f^{-1}\Omega_X,\Omega_X) \leftarrow$$

which pairs with the sequence 6.2 into $\mathbb{C}$, (cf. 4.4,4.5) and this pairing is perfect even though Ω_U and $f^{-1}\Omega_X$ are not quasi-coherent. Fixing an injective resolution $0 \longrightarrow \Omega^{\cdot} \longrightarrow Q^{\cdot\cdot}$, we recall that $\underline{\underline{\mathrm{Ext}}}(\Omega_U^{\cdot}\ \Omega^{\cdot})$ is the cohomology of $\mathrm{Hom}^{\cdot}(\Omega_U^{\cdot},Q^{\cdot\cdot})$.

But $\mathrm{Hom}^{\cdot}(\Omega_U^{\cdot}, Q^{\cdot\cdot}) \xrightarrow{\sim} \Gamma(U, Q^{\cdot\cdot})$ by identifying a homomorphism with the image of the constant section 1 over U. The restriction of the sheaves $Q^{\cdot\cdot}$ to U gives an injective resolution on U of Ω_U, so that $\underline{\underline{\mathrm{Ext}}}(\Omega_U^{\cdot}, \Omega^{\cdot}) \xrightarrow{\sim} \underline{\underline{H}}(U, \Omega^{\cdot})$. Similarly, $\mathrm{Hom}(f^{-1}\Omega_X^{\cdot}, Q^{\cdot\cdot}) = \Gamma_Y(Q^{\cdot\cdot})$ is the sections of $Q^{\cdot\cdot}$ with support in Y and $\underline{\underline{\mathrm{Ext}}}(f^{-1}\Omega_X, \Omega^{\cdot}) = \underline{\underline{H}}_Y(\Omega_X)$. Thus the sequence 6.3 is the local hypercohomology sequence:

$$\ldots \longleftarrow \underline{\underline{H}}_Y^{2m-i+1}(\Omega_X) \longleftarrow \underline{\underline{H}}^{2m-i}(U, \Omega_U) \longleftarrow \underline{\underline{H}}^{2m-i}(\Omega_X) \longleftarrow \underline{\underline{H}}_Y^{2m-i}(\Omega_X) \longleftarrow \ldots .$$

Since X is nonsingular $0 \longrightarrow C \longrightarrow \Omega_X^{\cdot}$ is a resolution and hence the above sequence is simply the local cohomology sequence

$$\ldots \longleftarrow H_Y^{2m-i+1}(X, \mathbb{C}) \longleftarrow H^{2m-i}(U, \mathbb{C}) \longleftarrow H^{2m-i}(X, \mathbb{C}) \longleftarrow H_Y^{2m-i}(X, \mathbb{C}) \longleftarrow \ldots .$$

The pairings

$$H^{2m-i}(X, \mathbb{C}) \times H^i(X, \mathbb{C}) \longrightarrow \mathbb{C}$$
$$H^{2m-i}(U, \mathbb{C}) \times H_c^i(U, \mathbb{C}) \longrightarrow \mathbb{C}$$

are well known to be perfect and it follows that the pairing $H_Y^{2m-i}(X, \mathbb{C}) \times H(Y, \mathbb{C}) \longrightarrow \mathbb{C}$ is perfect by the five lemma. (One should check that the pairings above are in fact the classical DeRham-Poincare duality pairings.) This is easy to verify. Let $E_U^{\cdot\cdot}$ denote the sheaf of $\mathbb{C}^{\infty}$ forms on U extended by zero to X. Then $0 \longrightarrow \Omega_U \longrightarrow E_U^{\cdot\cdot}$ is a fine resolution of Ω_U and $\Gamma(X, E_U^{\cdot\cdot}) = \Gamma_c(U, E_U^{\cdot\cdot})$ can be used to calculate $\underline{\underline{H}}_c(U, \Omega_U)$. The Yoneda pairing is calculated by

$$\mathrm{Hom}(E_U^{\cdot\cdot}, Q^{\cdot\cdot}) \times \Gamma_c(E_U^{\cdot\cdot}) \longrightarrow \Gamma(Q^{\cdot\cdot})$$

and the fact that $\mathrm{Hom}(E_U, Q^{\cdot\cdot}) \longrightarrow \mathrm{Hom}(\Omega_U, Q^{\cdot\cdot})$ gives an isomorphism of spectral sequences. Let $E^{\cdot\cdot}$ denote the sheaf of $\mathbb{C}^{\infty}$-forms on X and let $E^{\cdot\cdot} \xrightarrow{\lambda} Q^{\cdot\cdot}$

lift the identity map $\Omega_X \xrightarrow{\sim} \Omega_X$. Note that one has a natural map of double complexes $\Gamma(U,E^{\cdot\cdot}) \longrightarrow \mathrm{Hom}(E_U,Q^{\cdot\cdot})$ given by "wedge product", i.e. $\alpha \in \Gamma(U,E^{\cdot\cdot})$ defines the map $E_U \longrightarrow Q^{\cdot\cdot}$, $\beta \longrightarrow \lambda(\alpha\wedge\beta)$. Observe that $\Gamma(U,E^{\cdot\cdot}) \longrightarrow \mathrm{Hom}(E_U,Q^{\cdot\cdot}) \longrightarrow \mathrm{Hom}(\Omega_U,Q^{\cdot\cdot}) = \Gamma(U,Q^{\cdot\cdot})$ is simply the map induced by λ and yields an isomorphism of spectral sequences. Thus the Yoneda pairing of spectral sequences is that given by wedge product. The pairing to $\mathbb{C}$ is then achieved by integrating over X the wedge products of forms on U with forms compactly supported on U , and is well known to be perfect. Analogous remarks apply to the pairing $H(X,\mathbb{C}) \times H(X,\mathbb{C}) \longrightarrow \mathbb{C}$.)

Remark 6.4: Note that when X is a complex manifold one has for any subvariety Y of dimension d that

$$H_p(Y,\mathbb{C}) \xrightarrow{\sim} H_Y^{2m-p}(X,\mathbb{C})$$

as is evident from the duality between $H_Y^p(X,\mathbb{C})$ and $H^{2m-p}(Y,\mathbb{C})$ noted above. In fact the local cohomology sequence is simply the homology sequence of the pair (X,Y):

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_p(Y,\mathbb{C}) & \longrightarrow & H_p(X,\mathbb{C}) & \longrightarrow & H_p(X,Y;\mathbb{C}) & \longrightarrow \cdots\cdot \\
& \uparrow\wr & & \uparrow\wr & & \uparrow\wr & \\
\cdots \longrightarrow & H_Y^{2m-p}(X,\mathbb{C}) & \longrightarrow & H^{2m-p}(X,\mathbb{C}) & \longrightarrow & H^{2m-p}(U,\mathbb{C}) & \longrightarrow \cdots\cdot
\end{array}$$

§7. DeRham cohomology of infinitesimal neighborhoods.

Let $Y \xrightarrow{f} X$ be a closed subvariety of X defined by the ideal sheaf I. Let $U = X - Y$. Consider the closed subvariety $Y^{(n)}$ defined by the ideal sheaf I^{n+1}, (the n^{th} order neighborhood of Y.) Let $\Omega_{Y(n)}$ be the sheaf of holomorphic differentials on $Y^{(n)}$. Define Σ_n by the exact sequence

$$0 \longrightarrow \Sigma_n \longrightarrow \Omega_X \longrightarrow \Omega_{Y(n)} \longrightarrow 0$$

or more explicitly by $\Sigma_n^{\cdot} = I^{n+1}\Omega_X^{\cdot} + d(I^{n+1}\Omega^{\cdot -1})$. It will be convenient to define $Y^{(\infty)}$ to be the "subvariety" defined by the ideal sheaf $\varprojlim I^{n+1} = \mathcal{O}_U$ and to employ the notation $\Sigma_\infty = \varprojlim \Sigma_n = \Omega_U$ and $\Omega_{Y(\infty)} = \Omega_X/\Sigma_\infty = f^{-1}\Omega_X$. (Of course, $\Omega_{Y(\infty)} \neq \varprojlim \Omega_{Y(n)}$). Then for $n = 0,1,\ldots,\infty$ one has the hypercohomology sequence

$$\cdots \longrightarrow \underline{\underline{H}}^p(\Sigma_n) \longrightarrow \underline{\underline{H}}^p(\Omega_X) \longrightarrow \underline{\underline{H}}^p(\Omega_{Y(n)}) \longrightarrow \underline{\underline{H}}^{p+1}(\Sigma_n) \longrightarrow \cdots$$

and for $\infty \geq m \geq n$ a homomorphism from the sequence for m to that for n induced by the inclusion $Y^{(n)} \longrightarrow Y^{(m)}$.

<u>Theorem</u> 7.1: Let X be a compact complex manifold. In the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \varprojlim \underline{\underline{H}}^p(\Sigma_n) \arrow[r] & \varprojlim \underline{\underline{H}}^p(\Omega_X) \arrow[r] & \varprojlim \underline{\underline{H}}^p(\Omega_{Y(n)}) \arrow[r] & \cdots \\
\cdots \arrow[r] & \underline{\underline{H}}^p(\Sigma_\infty) \arrow[r] \arrow[u] \arrow[d, equal] & \underline{\underline{H}}^p(\Omega_X) \arrow[r] \arrow[u] \arrow[d, equal] & \underline{\underline{H}}^p(\Omega_{Y(\infty)}) \arrow[r] \arrow[u] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H_c^p(U,\mathbb{C}) \arrow[r] & H^p(X,\mathbb{C}) \arrow[r] & H^p(Y,\mathbb{C}) \arrow[r] & \cdots
\end{tikzcd}$$

the vertical arrows are isomorphisms, and consequently the top row is exact.

The equality signs in the above diagram result from the nonsingularity of X (cf. §6.).

Remark 7.2: The above theorem asserts that "in the limit" the deRham cohomology of $Y^{(n)}$ calculates the classical cohomology of Y. The limiting process need not be "essentially constant," that is no finite value of n need give the "right" cohomology[2] of Y. An example of this phenomenon was discovered by R. Slutzki based upon an example of Reiffen, [14] . Taking $X = \mathbb{P}^2$ and Y a reduced singular curve, the deRham complex

$$0 \longrightarrow \Omega^0_{Y(n)} \longrightarrow \Omega^1_{Y(n)} \longrightarrow 0$$

has $H^0 = C_Y$ and will calculate the "correct" cohomology of Y if and only if $H^1 = 0$ as can be seen from the exact sequence

$$0 \longrightarrow H^1(C_Y) \longrightarrow \underline{\underline{H}}^1(\Omega_{Y(n)}) \longrightarrow H^0(H^1) \longrightarrow H^2(C_Y) \longrightarrow \underline{\underline{H}}^2(\Omega_{Y(n)}) \longrightarrow 0$$

coming from the spectral sequence $H^p(H^q(\Omega_{Y(n)}) \Longrightarrow \underline{\underline{H}}^{p+q}(\Omega_{Y(n)})$. (Note that H^1

2) In the event that Y is nonsingular, the hypercohomology $\underline{\underline{H}}(\Omega_{Y(n)})$ calculates the classical cohomology of Y for all n . More generally, if every point $p \in Y$ has a neighborhood V in X with a homotopy $\varphi: V \times [0,1] \longrightarrow V$ with $\varphi(x,t)$ holomorphic in x for fixed t , $\varphi(x,0) = p$, $\varphi(x,1) = x$ and, moreover, $f \circ \varphi(x,t) \in I \cdot E^{\cdot}_{V \times [0,1]}$ for all $f \in I$, then $\underline{\underline{H}}(\Omega_{y(n)})$ calculates classical cohomology. This is seen by using φ to construct a homotopy for $\Omega^{\cdot}_X$ which preserves $\Sigma^{\cdot}_n$.

is supported on the singular locus of Y and hence $H^i(H^1) = 0$ for $i > 0$.) The nonvanishing of H^1 is a local question, and is equivalent to the nonvanishing of H^2 for the complex $0 \to \Sigma_n^0 \to \Sigma_n^1 \to \Sigma_n^2 \to 0$, i.e. one must show that $I^{n+1}\Omega_X^2 \not\subset d(I^{n+1}\Omega_X^1)$ to obtain an example. Letting Y be defined by the affine equation $f = x^4 + y^5 + y^3x^2$, one may easily check that $f^n dx\, dy$ is not of the form $d(f^n g\, dx + f^n h\, dy)$ for any g,h. (Note that $f^n = x^{4n} + y^{5n} + ny^{5(n-1)+3}x^2 \pmod{x^4y^3}$ so that if one seeks $g = \Sigma a_{ij}x^iy^j$, $h = \Sigma b_{ij}x^iy^j$ by recursively solving for a_{ij} and b_{ij} one obtains three linear equations in a_{01} and b_{10} which have no simultaneous solution.)

To prove theorem 7.1 we need only establish the <u>dual</u> assertion, namely that in

7.3
$$\begin{tikzcd}
\cdots \arrow[r] & \varinjlim \underline{\underline{\mathrm{Ext}}}^p(\Omega_{Y(n)},\Omega) \arrow[r] \arrow[d] & \varinjlim \underline{\underline{\mathrm{Ext}}}^p(\Omega,\Omega) \arrow[r] \arrow[d] & \varinjlim \underline{\underline{\mathrm{Ext}}}^p(\Sigma_n,\Omega) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \underline{\underline{\mathrm{Ext}}}^p(\Omega_{Y(\infty)},\Omega) \arrow[r] \arrow[d, equal] & \underline{\underline{\mathrm{Ext}}}^p(\Omega,\Omega) \arrow[r] \arrow[d, equal] & \underline{\underline{\mathrm{Ext}}}^p(\Sigma_\infty),\Omega) \arrow[r] \arrow[d, equal] & \cdots \\
\cdot \arrow[r] & H_Y^p(X,\mathbb{C}) \arrow[r] & H^p(X,\mathbb{C}) \arrow[r] & H^p(U,\mathbb{C}) \arrow[r] & \cdots
\end{tikzcd}$$

The vertical arrows are isomorphisms. (Again the equality signs come from §6.) From this dual assertion we also see how to calculate the homology $H.(Y,C)$ as a suitable limit of analytic calculations. Moreover, since $\varinjlim$ is exact, we know that the top row in 7.3 is exact and to show that the maps are isomorphisms we need only check that

7.4
$$\varinjlim \underline{\underline{\mathrm{Ext}}}^p(\Sigma_n,\Omega) \xrightarrow{\sim} \underline{\underline{\mathrm{Ext}}}^p(\Sigma_\infty,\Omega) = H^p(U,\mathbb{C})$$

This result is in fact true in a more general setting, namely let X be a compact complex reduced analytic space and let Y be a closed subvariety containing the singularities of X , $U = X - Y$.

To prove this we replace the question by a local question, by a procedure analogous to the local reduction in [8].

Theorem 7.5: Let X be a reduced analytic space and Y a subvariety containing the singularities of X , $U = X - Y$. Then the sheaf homomorphisms

$$\varinjlim \underline{\underline{\mathrm{Ext}}}^q(\Sigma_n,\Omega) \longrightarrow \underline{\underline{\mathrm{Ext}}}^q(\Sigma_\infty,\Omega) = R^q i_*(\mathbb{C}_U)$$

analogous to 7.4 , are isomorphisms.

Assuming this theorem one obtains 7.4 immediately. Indeed for X compact we have

7.6 $$\varinjlim H^p(X,\underline{\underline{\mathrm{Ext}}}^q(\Sigma_n,\Omega)) \xleftarrow{\sim} H^p(X, \varinjlim \underline{\underline{\mathrm{Ext}}}^q(\Sigma_n,\Omega)$$

Thus the morphism of biregular spectral sequences (cf. 3.2)

$$\begin{array}{ccc} \varinjlim \underline{\underline{\mathrm{Ext}}}^{p-q}(\Sigma_n,\Omega) & \longrightarrow & \underline{\underline{\mathrm{Ext}}}(\Sigma_\infty,\Omega) \\ \Uparrow & & \Uparrow \\ \varinjlim H^p(X,\underline{\underline{\mathrm{Ext}}}^q(\Sigma_n,\Omega)) & \longrightarrow & H^p(X,\underline{\underline{\mathrm{Ext}}}^q(\Sigma_\infty,\Omega)) \end{array}$$

is an isomorphism, being an isomorphism at level E_2 in view of 7.5 and 7.6.

Theorem 7.5 is local, and we may therefore assume that $r = \sup_{x \in X} \{$(minimum) number of generators of $I_x \subseteq \mathcal{O}_x\}$ is a finite number, and we proceed by induction on r .

Remark 7.7: To prove theorem 7.5 we may clearly replace the sequence of complexes $\Sigma_n^\cdot$ by any cofinal sequence of complexes. In particular let $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_k \supseteq \ldots$ be a sequence of ideals, cofinal with the powers of I, which satisfy $d(I_i\,\Omega_X^\cdot) \subseteq I_{i-1}\,\Omega_X^\cdot$, (eg. $I_i = I^i$) and consider the subcomplexes of $\Omega_X^\cdot$ defined for $n \geq 1$ by

$I_{n+m-\cdot}\,\Omega^\cdot$: $0 \longrightarrow I_{n+m}\,\Omega^0 \longrightarrow I_{n+m-1}\,\Omega^1 \longrightarrow \ldots \longrightarrow I_n\Omega^m \longrightarrow 0$.

These complexes are cofinal with the $\Sigma_n^\cdot$, since

$$I^{n+1}\Omega^\cdot \subseteq \Sigma_n^\cdot = I^{n+1}\Omega^\cdot + d(I^{n+1}\Omega^\cdot) \subseteq I^n\Omega^\cdot$$

We proceed with the inductive proof. Assume that $r = 1$, i.e. that $I = (f)$. We show that $\varinjlim \underline{\underline{\mathrm{Ext}}}^q(\Sigma_n,\Omega_X) \xrightarrow{\sim} H^q\,(\varinjlim I^{-n}\Omega_X)$ and the isomorphism with $R^q i_*(C_U)$ follows by Grothendieck's theorem 2 in [8] . Note first that $\varinjlim \underline{\underline{\mathrm{Ext}}}^q(\Sigma_n,\Omega_X) = \varinjlim \underline{\underline{\mathrm{Ext}}}^q(I^{n+m-\cdot}\Omega_X,\Omega_X) = \varinjlim \underline{\underline{\mathrm{Ext}}}^q(I^{m-\cdot}\Omega_X\ ,\ I^{-n}\Omega_X)$.

Fixing an injective resolution $0 \longrightarrow \Omega^\cdot \longrightarrow Q^{\cdot\cdot}$, we note that $I^{-n} \otimes Q^{\cdot\cdot}$ is a resolution of $I^{-n}\Omega^\cdot$ by injective $\Omega^\cdot$ modules and we seek to compute the total cohomology of double complex $\varinjlim \mathrm{Hom}^r(I^{m-\cdot}\Omega^\cdot,\ I^{-n} \otimes Q^{\cdot s})$. From the inclusion $0 \longrightarrow I^{m-\cdot}\Omega^\cdot \longrightarrow \Omega^\cdot$ we have

$$0 \longleftarrow \varinjlim \mathrm{Hom}^r(I^{m-\cdot}\Omega\ ,\ I^{-n} \otimes Q^{\cdot s}) \longleftarrow \varinjlim \mathrm{Hom}^r(\Omega^\cdot, I^{-n}Q^{\cdot s}) = \varinjlim I^{-n} \otimes Q^{rs}$$

in view of the injectivity of $I^{-n}Q^{\cdot s}$ and 2.5, (2) . The kernel of this map is zero. Indeed, a section α of $\varinjlim I^{-n} \otimes Q^{r,s}$ represented by a section $\hat{\alpha}$ of

$I^{-n} \otimes Q^{r,s}$ yields the element $f^{m-p}\varphi \longrightarrow f^{m-p} \otimes (\varphi \wedge \hat{\alpha})$ of $\varinjlim \mathrm{Hom}^r(I^{m-\cdot}\Omega, I^{-n} \otimes Q^{\cdot s})$ which is zero only if $f^m\alpha = 0$. But the mapping $f^m\colon \varinjlim I^{-n}Q^{r,s} \longrightarrow \varinjlim I^{-n}Q^{r,s}$ is the limit of the isomorphisms $f^m\colon I^{-n} \xrightarrow{\sim} I^{m-n}Q^{r,s}$, so that $f^m\alpha = 0$ implies $\alpha = 0$.

Thus $\varinjlim \underline{\underline{\mathrm{Ext}}}^q(\Sigma_n,\Omega_X) \xrightarrow{\sim} H^q(\varinjlim I^{-n}Q^\cdot)$. But the injections $0 \longrightarrow I^{-n}\Omega^\cdot \longrightarrow I^{-n} \otimes Q^\cdot$ induce isomorphisms in cohomology so that $\varinjlim \underline{\underline{\mathrm{Ext}}}^q(\Sigma_n,\Omega_X) \xrightarrow{\sim} H^q(\varinjlim I^{-n}\Omega^\cdot)$, as asserted.

Proceeding to the case of general r, we note that since the theorem is local, one may assume $I = J_1 \oplus J_2$ where $J_1 = (f)$ and $J_2 = (f_1,\dots,f_{r-1})$. The sequence of ideals $J_1^n + J_2^n$ is cofinal with the powers of I. Let Y_j be the subvariety defined by J_j and let $U_j = X - Y_j$. Let $A_n^\cdot = (J_1^{n+m-\cdot} \cap J_2^{n+m-\cdot})\,\Omega^\cdot$; $B_{j,n}^\cdot = J_j^{n+m-\cdot}\,\Omega^\cdot$; $C_n^\cdot = (J_1^{n+m-\cdot} + J_2^{n+m-\cdot})\,\Omega^\cdot$. Note the exact sequences

$$0 \longrightarrow A_n^\cdot \longrightarrow B_{1,n} \oplus B_{2,n} \longrightarrow C_n^\cdot \longrightarrow 0$$

and the hyperext sequences:

$$\text{7.8} \qquad \dots \longrightarrow \underline{\underline{\mathrm{Ext}}}^p(C_n,\Omega) \longrightarrow \underline{\underline{\mathrm{Ext}}}^p(B_{1,n},\Omega) \oplus \mathrm{Ext}^p(B_{2,n},\Omega) \longrightarrow \underline{\underline{\mathrm{Ext}}}^p(A_n,\Omega) \longrightarrow \dots$$

We know by induction that $\varinjlim \underline{\underline{\mathrm{Ext}}}^p(B_{j,n},\Omega) \xrightarrow{\sim} R^p i_{j*}(\mathbb{C}_{U_j})$. Moreover, by the Artin-Rees lemma (cf. [13]) we know that the sequence of ideals $J_1^{n+n-\cdot} \cap J_2^{n+m-\cdot}$ is cofinal with the powers of the ideal $J_1 \cdot J_2$. This ideal has $r-1$ generators $\{f \cdot f_i\}$, and defines $Y_1 \cup Y_2$. By induction we therefore have $\varinjlim \mathrm{Ext}^p(A_n^\cdot, \Omega^\cdot) \xrightarrow{\sim} R^p i_{12*}(\mathbb{C}_{U_1 \cap U_2})$. But considering the diagram

7.9
$$\begin{tikzcd}
\cdots \arrow[r] & \varinjlim \underline{\underline{\mathrm{Ext}}}^p(C_n,\Omega) \arrow[r] \arrow[d] & \varinjlim \underline{\underline{\mathrm{Ext}}}^p(B_{1,n},\Omega) \oplus \varinjlim \underline{\underline{\mathrm{Ext}}}^p(B_{2,n},\Omega) \arrow[r] \arrow[d] & \varinjlim \underline{\underline{\mathrm{Ext}}}^p(A_n,\Omega) \arrow[r] \arrow[d] & {} \\
\cdots \arrow[r] & R^p i_*(\mathbb{C}_U) \arrow[r] & R^p i_{1*}(\mathbb{C}_{U_1}) \oplus R^p i_{2*}(\mathbb{C}_{U_2}) \arrow[r] & R^p i_{12*}(\mathbb{C}_{U_1 \cap U_2}) \arrow[r] & {}.
\end{tikzcd}$$

in which the bottom row is the Mayer-Vietoris sequence, we find that $\varinjlim \underline{\underline{\mathrm{Ext}}}^p(C_n,\Omega) \xrightarrow{\sim} R^p i_*(\mathbb{C}_U)$ by the five lemma. Since $\varinjlim \underline{\underline{\mathrm{Ext}}}^p(\Sigma_n,\Omega) \xrightarrow{\sim} \varinjlim \underline{\underline{\mathrm{Ext}}}^p(C_n,\Omega)$, by cofinality, our proof is complete.

Theorem 7.10: Let X be a nonsingular complete algebraic variety over $\mathbb{C}$, Y a closed subvariety and $U = X - Y$. Let X' , Y' and U' denote the corresponding analytic spaces. In the commutative diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & \varprojlim \underline{\underline{H}}^p(\Sigma_n) \arrow[r] & \underline{\underline{H}}^p(\Omega_X) \arrow[r] & \varprojlim \underline{\underline{H}}^p(\Omega_{Y(n)}) \arrow[r] & \cdots \\
{} \arrow[r] & H^p_c(U',\mathbb{C}) \arrow[r] \arrow[u] & H^p(X',\mathbb{C}) \arrow[r] \arrow[u] & H^p(Y',\mathbb{C}) \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\cdots \arrow[r] & \varinjlim \underline{\underline{\mathrm{Ext}}}^p(\Omega_{Y(n)},\Omega_X) \arrow[r] \arrow[d] & \underline{\underline{H}}^p(\Omega_X) \arrow[r] \arrow[d] & \varinjlim \underline{\underline{\mathrm{Ext}}}^p(\Sigma_n,\Omega_X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^p_{Y'}(X',\mathbb{C}) \arrow[r] & H^p(X',\mathbb{C}) \arrow[r] & H^p(U',\mathbb{C}) \arrow[r] & \cdots
\end{tikzcd}$$

the vertical arrows are isomorphisms.

Proof: This may be seen most easily by noting that for each finite n the groups $\underline{\underline{H}}^p(\Sigma_n)$, $\underline{\underline{\mathrm{Ext}}}^p(\Sigma_n,\Omega_X)$, etc. are isomorphic to the corresponding groups calculated with analytic forms, [GAGA] , and one reduces to 7.1, 7.3. In fact a direct proof in the algebraic case is "easier" than the analytic case. As in the analytic case the two assertions are dual to one another and it suffices to prove that $\varinjlim \underline{\underline{\mathrm{Ext}}}^p(\Sigma_n,\Omega_X) \xrightarrow{\sim} H^p(U',\mathbb{C})$ or that $\varinjlim \underline{\underline{\mathrm{Ext}}}^p(I^{n+m-\cdot}\Omega_X^\cdot,\Omega_X^\cdot) \xrightarrow{\sim} \underline{\underline{H}}^p(U,\Omega_U)$ since this latter group calculates classical cohomology [8].

But note the hyperext spectral sequence, 3.2,

$$\begin{array}{ccc} E_1^{a,b} = \mathrm{Ext}^b(I^{n+a}\Omega^{m-a},\Omega^m) & \Longrightarrow & \underline{\underline{\mathrm{Ext}}}^{a+b}(I^{n+m-\cdot}\,\Omega_X^\cdot,\Omega_X^\cdot) \\ \downarrow\wr & & \\ \mathrm{Ext}^b(I^{n+a},\Omega^a) & & \end{array}$$

and the morphism of spectral sequences

$$\begin{array}{ccc} \varinjlim \underline{\underline{\mathrm{Ext}}}^{a+b}(I^{n+m-\cdot}\,\Omega^\cdot,\Omega^\cdot) & \longrightarrow & \underline{\underline{H}}^{a+b}(U,\Omega_U) \\ \Uparrow & & \Uparrow \\ \varinjlim \mathrm{Ext}^b(I^{n+a},\Omega^a) & \longrightarrow & H^b(U,\Omega^a) \end{array}$$

which in the <u>algebraic</u> case is already an isomorphism at level E_1, (cf. [7], Theorem 2.8.) This E_1 isomorphism fails, in the analytic case.

<u>Remark</u> 7.11: One should note the other description of $\varprojlim \underline{\underline{H}}^q(\Omega^\cdot_{Y(n)})$ which has been conscientiously ignored in this note. Namely

7.12 $$\varprojlim \underline{\underline{H}}^q(\Omega^\cdot_{Y(n)}) \xleftarrow{\sim} \underline{\underline{H}}^q(\varprojlim \Omega^\cdot_{Y(n)})$$

where the morphism is induced by the maps $\varprojlim \Omega^\cdot_{Y(n)} \longrightarrow \Omega^\cdot_{Y(n)}$. This follows from two observations, first that $\varprojlim \underline{\underline{H}}^q(\Omega^\cdot_{Y(n)})$ is the abuttment of a spectral sequence with $E_1^{p,q} = \varprojlim H^q(X,\Omega^p_{Y(n)})$, that is, the "dual" spectral sequence to that for $\varinjlim \underline{\underline{\mathrm{Ext}}}(\Omega_{Y(n)},\Omega)$. To show that 7.12 is an isomorphism it suffices to establish that the map $H^q(X,\varprojlim \Omega^p_{Y(n)}) \longrightarrow \varprojlim H^q(X,\Omega^p_{Y(n)})$ at level E_1 is an isomorphism, which a standard Mittag-Leffler argument ([9], $\underline{0}$, 13.3).

Bibliography

1. M. Atiyah and R. Bott, Notes on the Lefschetz Fixed Point Theorem for Elliptic Complexes, Harvard, 1964.

2. N. Bourbaki, Algèbre multilinéaire, Hermann, Paris, 1948.

3. H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, 1956.

4. R. Godemont, Théorie des faisceaux, Hermann, Paris, 1964.

5. P. Griffiths, Periods of Integrals on Algebraic Manifolds, III, (to appear).

6. A. Grothendieck, Sur quelques points d'Algèbre Homologique, Tohoku Mathematical Journal, vol. IX (1957), pp. 119-221.

7. A. Grothendieck, Local Cohomology, Lecture Notes in Mathematics, 41, Springer Verlag, Berlin, 1967.

8. A. Grothendieck, On the deRham Cohomology of Algebraic Varieties, Publ. Math., IHES, 29, (1966).

9. A. Grothendieck, Eléments de Géometrie Algébrique, Chapter III, Publ. Math., IHES, 11, (1961).

10. R. Hartshorne, Residues and Duality, Lecture Notes in Mathematics, 20, Springer Verlag, Berlin, 1966.

11. R. Hartshorne, Cohomological Dimension of Algebraic Varieties, Ann. of Math., vol. 88 (1968), pp. 403-450.

12. M. Herrera and T. Bloom, DeRham cohomology of an analytic space, Inventiones Math., 7 (1969), pp. 275-296.

13. M. Nagata, Local Rings, Interscience, New York, 1962.

14. H. J. Reiffen, Das Lemma von Poincare für holomorphe Differentialformen auf Komplexen Raumen, Math. Zeitsch. 101 (1964), pp. 269-284.

15. J. P. Serre, Géometrie algébrique et géometrie analytique, Ann. Institute Fourier, 6 (1955-56), pp. 1-42.

16. D. C. Spencer, Overdetermined Systems of Linear Partial Differential Equations, BAMS, vol. 75 (1969), pp. 179-239.

17. K. Suominen, Duality for coherent sheaves on an analytic manifold, Ann. Acad. Sci. Fenn. 424 (1968).