Density Estimation using Quantile Variance and Quantile-Mean Covariance

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Abstract

Based on asymptotic properties of sample Quantile Distribution derived by Hall & Martin (1988) and Ferguson (1999), we propose a novel method which explodes Quantile Variance, and Quantile-Mean Covariance to estimate distributional density from samples. The process consists in firstly estimate sample Quantile Variance and sample Quantile-Mean Covariance using bootstrap techniques and after use them to compute distributional density. We conducted Montecarlo Simulations for different Data Generating Process, sample size and parameters and we discovered that for many cases Quantile Density Estimators perform better in terms of Mean Integrated Squared Error than standard Kernel Density Estimator. Finally, we propose some smoothing techniques in order to reduce estimators variance and increase their accuracy.

Keywords:Density Estimation, Quantile Variance, Quantile-Mean Covariance, Bootstrap

JEL Classification:C13, C14, C15, C46

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1 Introduction

If the continuous random variable X has density $f(x)$ and median $Q(0.5) = \theta$, then the sample variance of $\hat{Q}_n(0.5)$[1] is known to be approximately $\frac{1}{[4nf(\theta)^2]}$[2]. As $n \to \infty$ we also know that the asymptotic variance of the median converges to $\frac{1}{[4f(\theta)^2]}$. It can be shown that this result is true for any quantile τ being the asymptotic variance of $Q(\tau) = x_\tau$ equal to $\frac{\tau(1-\tau)}{f(x_\tau)^2}$. Ferguson (1999) extended this results and proved that sample Quantile-Mean joint distribution has asymptotic covariance equal to $\frac{\varpi(\tau)}{f(x_\tau)}$[3]. This results had been widely used to estimate order statistics variance and construct confidence intervals for robust inference. This usually requires to make assumptions about $f(x)$ functional form or instead firstly estimate this density function using non-parametric techniques. Based on properties of sample quantile variance bootstrap estimation derived by Hall & Martin (1988) we explore the revert problem of firstly estimate sample quantile variance $(\hat{Qv})$, and sample quantile-mean covariance $(\hat{Qc})$, and then use them to non-parametrically estimate density function $\hat{f(x)}$. We call them Qv and Qc density estimators.

2 Quantile Variance Density Estimator

Let $(X1, ..., Xn)$ be *i.i.d.* with distribution function $F(x)$, density $f(x)$, mean μ and finite variance σ^2. Let $0 < \tau < 1$ and let q_τ denote the τ quantile of F, so that $F(q_\tau) = \tau$. Assume that the density $f(x)$ is continuous and positive at q_τ. Let $Q(\tau)$ be the quantile function so that $Q(\tau) \equiv F^{-1}(\tau) = q_\tau$ and $Y_{\tau,n} = X_{n:n\tau}$ denote the sample τ quantile. Then:

$$\sqrt{n}(Y_{\tau,n} - q_\tau) \xrightarrow{L} N\left(0, \frac{\tau(1-\tau)}{f(x_\tau)^2}\right) \tag{1}$$

[1] i.e. the sample median of n observations

[2] According to Stigler (1973) this result, jointly with asymptotic properties of Median estimation, was firstly stated by Laplace in 1818

[3] Where $\varpi(\tau) = \text{argmin}_a\, E[\rho_\tau(X - a)]$ is the expected quantile loss function

Definition 1 *The quantile variance function, $V(\tau)$, is the asymptotic variance of the sample quantiles, i.e. $Y_{\tau,n}$, with index $\tau \in (0,1)$*

$$V(\tau) = \lim_{n\to\infty} \sqrt{n} V(Y_{\tau,n}). \tag{2}$$

From Equations 1 and 2 we know that at the limit $V(\tau) = \frac{\tau(1-\tau)}{f(x_\tau)^2}$ so density at quantile τ is:

$$f(Q(\tau)) = \sqrt{\frac{\tau(1-\tau)}{V(\tau)}}. \tag{3}$$

The Quantile Variance estimator of $f(Q(\tau))$ requires only a consistent estimator of $V(\tau)$. Following subsections sketch the steps we need to achieve that.

2.1 Consistent estimator of $V(\tau)$

We propose a naive non parametric Bootstrap estimator which tooks B random sub-samples of size n out of n observations, sort them into a order rank $j = 1, ..., n$ so $X_1^b < ... < X_n^b$ and:

1. Compute sub-sample quantile τ for sub-sample b as:
$$\hat{Q}_n^b(\tau) = inf \left\{ X_j^b | \left(\frac{\sum_{j=1}^n X_j^b 1\left(X_j^b \leq Q_n^b(\tau)\right)}{\sum_{j=1}^n X_j^b} \right) \geq \tau \right\} b = 1, 2, ..., B; j = 1, .., n$$
2. Compute bootstraped quantile mean as: $\overline{\hat{Q}}_n(\tau) = \frac{1}{B}\sum_{b=1}^B \hat{Q}_n^b(\tau), b = 1, 2, ..., B$
3. Finally Compute bootstraped quantile variance as: $\hat{V}(\tau) = \frac{1}{B}\sum_{b=1}^B (\hat{Q}_n^b(\tau) - \overline{\hat{Q}}_n(\tau))^2, b = 1, 2, ..., B$

Lemma 1 $\hat{V}_n^B(\tau) = V(\tau) + o_p(1)$ *as $n \to \infty$, $B \to \infty$.*

2.2 QV Density Estimator

Then our proposed density estimator is

$$\hat{f}(Q(\tau)) = \sqrt{\frac{\tau(1-\tau)}{n\hat{V}(\tau)}}. \tag{4}$$

Lemma 2 *$\hat{f}_n(Q(\tau)) = f(Q(\tau)) + O_p(n^{-\frac{1}{4}})$ as $n \to \infty$, $B \to \infty$.*

Hall & Martin(1988) derives convergence properties for Quantile variance bootstrap estimation and shows that relative error of the bootstrap quantile variance estimator and bootstrap sparsity function estimator is of precise order $n^{-1/4}$[4]. Given that:

$$n\sigma_{q_\tau}^2 = \frac{\tau(1-\tau)}{f(x_\tau)^2} + O(n^{-1})$$

$$\frac{\hat{\sigma_{q_\tau}}^2}{\sigma_{q_\tau}^2} = 1 + O(n^{-1/4})$$

Then, the rate of convergence of $\hat{f}_n(Q(\tau)$ is also of precise order $n^{-1/4}$[5]:

$$n^{\frac{1}{4}}(\hat{f}_n(Q(\tau)) - f(Q(\tau))) \xrightarrow{L} N\left(0, T^2\right) \text{ [6]} \tag{5}$$

Then:

$$\frac{\hat{f}_n(Q(\tau)}{f(Q(\tau))} = 1 + O(n^{-1/4})$$

3 Quantile-Mean Covariance Density Estimator

Suppose $\{X_1, ..., X_n\}$ is an i.i.d. (independent and identically distributed) sample with distribution function $F(.)$, density $f(.)$, quantile function $Q(\tau), \tau \in (0,1)$, and mean μ.

[4] Where n is *Sample Size*

[5] This rate is inferior to regular Kernel rate of converge $n^{-2/5}$. However, this rate can be improved by smoothing techniques discussed above or using a different technique to estimate variance such as m out n bootstrap as showed by Cheung & Lee (2005)

[6] Where $T^2 \equiv \frac{f(x_\tau)^2}{\sqrt{2\pi^{\frac{1}{2}}\tau(1-\tau)}}$. For a formal proof please refer to Hall $ Martin (1988)

Definition 2 *The quantile-mean covariance function, $C(\tau)$, is the asymptotic covariance between the sample quantiles, i.e. $\hat{Q}_n(\tau)$, with index $\tau \in (0,1)$ and the sample mean, i.e., $\bar{X}_n = \sum_{i=1}^n X_i$,*

$$C(\tau) = \lim_{n\to\infty} \sqrt{n} COV(\hat{Q}_n(\tau), \bar{X}_n). \tag{6}$$

Definition 3 *Define the expected quantile loss function (EQLF)*

$$\varpi(\tau) = \operatorname*{argmin}_a E[\rho_\tau(X-a)] = \tau(\mu - E[X|X < Q(\tau)]) = \tau\left(\mu - \frac{1}{\tau}E[1[X < Q(\tau)]X]\right), \tag{7}$$

where $\rho_\tau(u) = \{\tau - 1[u \leq 0]\}u$ is the quantile check function in Koenker and Bassett (1978).

By Ferguson (1999),

$$f(Q(\tau)) = \frac{\varpi(\tau)}{C(\tau)}. \tag{8}$$

The estimator of $f(Q(\tau))$ requires consistent estimators of $\varpi(\tau)$ and $C(\tau)$. The following subsections sketch the steps we need for each element.

3.1 Consistent estimator of $\varpi(\tau)$

Consider the following estimator:

$$\hat{\varpi}_n(\tau) = \tau(\bar{X}_n - \frac{1}{\tau}\frac{1}{n}\sum_{i=1}^n X_i 1[X_i < \hat{Q}_n(\tau)],$$

Lemma 3 *$\hat{\varpi}_n(\tau) = \varpi(\tau) + o_p(1)$ as $n \to \infty$.*

3.2 Consistent estimator of $C(\tau)$

The key point is how to estimate $C(\tau)$. Basically we just want to estimate a "covariance" between two random variables. Think about how we would estimate the covariance between $\bar{X}_n$ and $\hat{\sigma}_n^2$ or any other two "moments".

Proposal: The bootstrap

1. Consider B bootstrap samples of size n, $\{x_i^b\}_{i=1}^n, b = 1, 2, ..., B$;
2. Compute $(\bar{X}_n^b, \hat{Q}_n^b(\tau)), b = 1, 2, ..., B$;
3. Compute $\hat{C}_n^B(\tau) = \frac{1}{B}\sum_{b=1}^B \left(\bar{X}_n^b \times \hat{Q}_n^b(\tau)\right) - \left(\frac{1}{B}\sum_{b=1}^B \bar{X}_n^b\right)\left(\frac{1}{B}\sum_{b=1}^B \hat{Q}_n^b(\tau)\right)$ (i.e. sample covariance)

Lemma 4 *$\hat{C}_n^B(\tau) = C(\tau) + o_p(1)$ as $n \to \infty$, $B \to \infty$.*

3.3 QC Density Estimator

Then our proposed density estimator is

$$\hat{f}_n(Q(\tau)) = \frac{\hat{\varpi}_n(\tau)}{n\hat{C}_n^B(\tau)} \tag{9}$$

Lemma 5 *$\hat{f}_n(Q(\tau)) = f(Q(\tau)) + o_p(1)$ as $n \to \infty$, $B \to \infty$.*

4 Smoothing Methods

In practice, the density estimator may have several kinks in small samples because of the non continuous nature of quantile estimators. Then we can propose smoothing strategies to obtain smoother estimators.

4.1 Moving Average

Suppose a given grid of τ values $\mathcal{T} = \tau_1, \tau_2, ..., \tau_T$ such that $\tau_i < \tau_i + 1, i = 1, 2, ..., T - 1$.

The consider a moving average of $2m + 1$ (i.e. we would consider m quantiles to the left and m quantiles to the right to take an average), for τ_i, $i = m + 1, ..., m - 1$,

$$
\begin{aligned}
\hat{f}_n(Q(\tau_i))^{MA} &= \frac{1}{2m+1}\Big(\hat{f}_n(Q(\tau_{i-m})) + \hat{f}_n(Q(\tau_{i-m+1})) + ... \\
&\quad + \hat{f}_n(Q(\tau_{i-1})) + \hat{f}_n(Q(\tau_i)) + \hat{f}_n(Q(\tau_{i+1})) + ... + \hat{f}_n(Q(\tau_{i+m}))\Big)
\end{aligned}
$$

4.2 Weighted Moving Average

Alternatively, consider a smoothing Moving Average weighted by kernel function Ψ,

$$
\begin{aligned}
\hat{f}_n(Q(\tau_i))^{WMA} &= \frac{1}{2m+1}\Big(\Psi(\tau_{i-m} - \tau_i)\hat{f}_n(Q(\tau_{i-m})) + \Psi(\tau_{i-m+1} - \tau_i)\hat{f}_n(Q(\tau_{i-m+1})) + ... \\
&\quad + \Psi(\tau_{i-1} - \tau_i)\hat{f}_n(Q(\tau_{i-1})) + \Psi(0)\hat{f}_n(Q(\tau_i)) + \Psi(\tau_{i+1} - \tau_i)\hat{f}_n(Q(\tau_{i+1})) + ... \\
&\quad + \Psi(\tau_{i+m} - \tau_i)\hat{f}_n(Q(\tau_{i+m}))\Big)
\end{aligned}
$$

m is here a smoothing parameter and we can thus analyze the asymptotic properties with respect to n and m to get optimality properties.

4.3 HP Filter

Hodrick and Prescott (1997) proposed a very popular method for decomposing time-series into trend and cycle. Paige & Trindade (2010) proved that HP-Filter is a special case of *Penalized Spline Smoother*. Then, our HP smoothed estimator results from:

$$
\begin{aligned}
\hat{f}_n(Q(\tau_i))^{HP} &= \min_{\hat{f}_n(Q(\tau_i))^{HP}} \{\sum_{i=1}^{T}(\hat{f}(Q(\tau_i)) - \hat{f}_n(Q(\tau_i))^{HP})^2 + ... \\
&\quad + \lambda\sum_{i=1}^{T}[(\hat{f}_n(Q(\tau_i))^{HP} - \hat{f}_n(Q(\tau_{i-1}))^{HP}) - (\hat{f}_n(Q(\tau_{i-1}))^{HP} - \hat{f}_n(Q(\tau_{i-2}))^{HP})]^2\}
\end{aligned}
$$

Where λ is a free smoothing parameter.

5 Montecarlo Simulations

This section shows alternative scenarios where we firstly simulate known Data Generating Process (DGP), secondly we try to recover those DGP using techniques presented before (Quantile Variance Density Estimator, Quantile-Mean Covariance Density Estimator) and thirdly we evaluate the performance of estimators through their Mean Integrated Square Error ($MISE$). For a given simulation i, Integrated Square Eerror ISE_i is defined as:

$$ISE_i = \int (f(x) - \hat{f}_i(x))^2 dx \tag{10}$$

Where $f(x)$ is the known density function and $\hat{f}_i(x)$ is the estimated density for simulation i.

Then, Montecarlo estimated $M\hat{I}SE$ will be:

$$M\hat{I}SE = E[ISE] = \frac{1}{M}\sum_{i=1}^{M} ISE_i \tag{11}$$

Where M is the number of simulations.

As a benchmark for each DGP we also estimate its density with a kernel function:

$$K_i(x_0) = \frac{1}{nh}\sum_{i=j}^{n} K(\frac{x_j - x_0}{h}) \tag{12}$$

Where n is the sample size, h is the bandwidth, and $K(.)$ the kernel.[7] Quantil Variance (Qv), and Quantil-Mean Covariance (Qc) were estimated using nonparametric bootstrap as described in sections 3.2.

5.1 Standard Distributions

In order to test the accuracy of our estimators we generate $M = 1000$ random samples from:

1. Bell Shaped Gaussian $Normal(0, 1)$

[7] We use the default $K(.)$ and h set by Stata which are Epanechnikov Kernel and Optimal Gaussian Bandwidth.

2. Leptokurtic $Laplace(0,\sqrt{2})$[8]

3. Asymmetric $Gumbel(-\gamma\frac{\sqrt{6}}{\pi}, \frac{\sqrt{6}}{\pi})$[9]

4. $Gamma(1, 1)$[10].

Parameters of Distributions described before have been selected in order to assure 0 mean[11] and variance equal to 1. Figure 1 shows examples of one sample of each standard DGP describing Theoretical Distribution, Histogram, and *Qv Density Estimation* and *Qc Density Estimation* with their respective *Smoothing*.

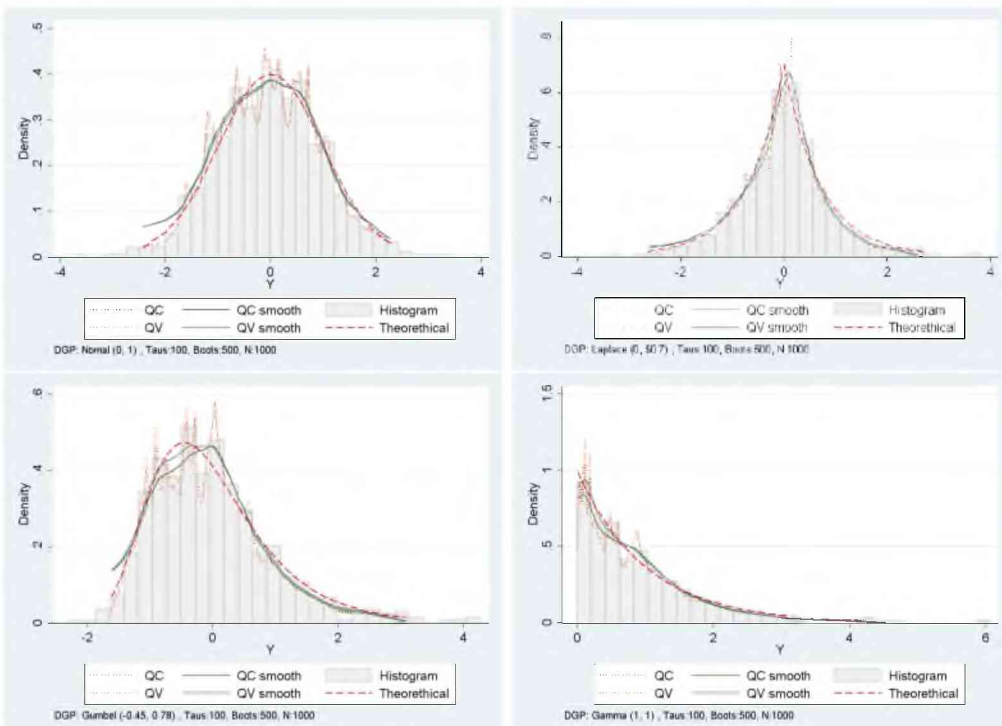

Figure 1: Normal, Laplace, Gumbel and Gamma Distributions. Graph shows Theoretical DGP, Histogram, and Empirical Density plotted from estimators: QC, QV, Smoothed QC and Smoothed QV

[8]For a Laplace (μ, β) $Var = 2\beta^2$ and $Mean = \mu$

[9]For a Gumbel (μ, β) $Var = \pi^2\frac{\beta^2}{6}$ and $Mean = \mu + \gamma\beta$ ($\gamma \approx 0.5772...$ is Euler-Marechoni Constant)

[10]For a Gamma (α, β) $Var = \alpha\beta^2$ and $Mean = \alpha\beta$

[11]Gamma distribution is the exception just to let it start from 0, then its mean and variance are both equal to 1

Given that quantile variance and quantile-mean covariance are very volatile at extremes of distributions, we also explore Qc and Qv performance after trimming distribution support for left and right at $\pm 1\%$, $\pm 2.5\%$ and $\pm 5\%$.

In order to avoid upper-extreme values of Qv and Qc coming from $near - zero$ quantile variance and quantile-mean covariance estimations, we also explore replacing extreme estimated values at the top 1%, 2.5% and 5% for an interpolation between nearest neighbors.

6 Results

6.1 Alternative DGPs

Tables 1 to 4 in Appendix A present *Bias* and $MISE$ for each DGP and *estimator*. As a benchmark for performance we also show the *Ratio* between each *estimator* $MISE$ and the traditional $Kernel$ density $MISE$.[12]

Before smoothing and trimming or interpolating, and relative to Kernel Density Estimation, the best performance for Qv *estimator* was obtained for Gamma distribution. With a $R_{Qv} = 0.40$ Qv estimator best performs indicates that it is 2.5 times more accurate than Kernel. This is true only for Gamma distribution while the traditional Kernel has smaller $MISE$ for Normal, Laplace and Gumbel distributions. Best performance for Qc was obtained for *Laplace* distribution. Even so, that is not enough to reach Kernel overall performance which is 3.2 times more accurate than Qc estimation.

As was described in section 4, is inherent to discrete nature of quantiles to produce dis-

[12] If the *Ratio* of estimator e takes value one ($R_e = 1$) we can conclude that *Estimator* e has in average the same performance than $Kernel$. If $R_e > 1$ we can say that e perform R_e times worse than Kernel. If $R_e < 1$ we can conclude that e perform $\frac{1}{R_e}$ times better than $Kernel$.

perse estimations. In order to reduce variability and improve estimators performance[13] we also computed two smoothed versions (MA, WMA, and HP) of each estimator and evaluate their $MISE$. Smoothed Qv and Qc best performance was obtained both for Gamma distribution but respectively using HP and MA smoothing. This time we reach a relative accuracy of $R_{HP_{Qv}} = 0.08$ and $R_{MA_{Qc}} = 0.94$ reflecting that for this DGP MA_{Qc} performs slightly better than $Kernel$ and HP_{Qv} performs more than 10 times better. We also find that HP_{Qv} overrides Kernel for Laplace distribution, but smoothed Qv and Qc perform worse for remaining distributions.

Due to bootstrap sub-sampling nature, Qv and Qc $estimators$ are very unstable at the extremes of distribution support. A mayor treat for stability arises from sporadic sub-sample $quantile\ covariace$ or $quantile - mean\ covariance$ near to 0 ($\hat{V(\tau)} \approx 0$ or $\hat{C(\tau)} \approx 0$). Remembering from Equations 8 and 2 that $\hat{C(\tau)}$ and $\hat{V(\tau)}$ enters into the denominator of density estimation, having for example only one simulation with estimated variance or covariance ≈ 0 will lead to a $f(\hat{Q(\tau)})$ near to infinite and consequently to a huge $MISE$.That is why we also computed density estimators after trimming DGP support. We find that Trimming first and last percentiles ($\pm 1\%$)[14] of density estimation improves performance notoriously. For Normal Distribution, HP_{Qc} performs 2 times better than Kernel, while HP_{Qv} performs 4 times better. Given that after $\pm 1\%$ trimming smoothed estimators perform better than Kernel for the four distributions we can conclude that this is a general result. Even before smoothing we find that after trimming Qc and Qv estimators override Kernel for most of Distributions.

For last, given that near 0 variance mostly produces density estimations going to ∞ we

[13]Remembering that $MISE$ is a combination of $Bias$ a $Variance$ in the form $MISE = B^2 - V$. Given that estimation $Bias$ in absolute value of Qv and k are quite similar, we can attribute most of the lack of performance to relative $Variance$ rather than $Bias$.

[14]For Density estimation we proceed as before using the whole sample to compute density. After doing that, we trimmed support before computing MISE

explore estimators performance after replacing upper-extreme values. Firstly, we estimate density as before using all the information available. Secondly, we detected upper-extreme values of point estimation, and finally we replace this extreme values for a lineal interpolation between nearest point estimation. This process has the advantage that does not affect support extent and produce a complete density estimation. We find that replacing 1% of upper-extreme values for a lineal interpolation has slightly better results in terms of $MISE$ than cutting extremes of support in $\pm 1\%$. A notable exception is Qc estimation of Gamma distribution that performs much better after support trimming.

6.2 Alternative Parameters

Figures 2 to 4 in Appendix B show graphics of Montecarlo Simulation results for a Normal DGP. In section B.1 we show how $MISE$ of Qv and Qc estimation decreases as *Sample Size* grows. We can also see that estimators are dominated by Kernel estimation confirming Hall & Martin(1988) theoretical results which states that kernel has greater rate of convergence. However, we can also see that HP_{Qv} after a n-size around 1200 performs almost identical than Kernel.

Section B.2 evaluate Qc and Qv performance when increasing *Number of Bootstraps* leaving *Sample Size* and remaining parameters fixed. From this graphic we can conclude that Qv estimator requires less bootstraps than Qc to become stable, and that increasing the number of bootstraps leads to convergence. However, after reaching sample size there are few gains in terms of $MISE$.

Finally, we evaluate how the *Number of Quantiles* we chose to perform point estimation affects $MISE$ and we find that there is not trade-off between the number of quantiles and

MISE. Although, there is a trade-off between quantiles and MISE for *Smoothing* estimators. Figure 4 show that for Normal DGP, after some point, increasing the number of quantiles only increases the $MISE$. The reason for this finding is that smoothing techniques improves density estimation performance due to variance reduction. However, as the number of point estimation converges to sample size, variance of smoothed estimators also converges to raw estimator variance. That is why, estimator smoothed MISE converges to raw estimator MISE as the number of quantiles converges to sample size.

7 Summary

- Before trimming and Smoothing, in terms of $MISE$ quantile estimators Qv and Qc usually perform worse than kernel for standard distributions.

- Given the discrete nature of quantile estimations, differences in $MISE$ comes mostly from highest relative *variance* rather than higher $Bias$.

- $Variance$ can be reduced using smoothing methods like MA, WMA and HP.

- Given that great part of $MISE$ comes from instability of estimators at the extreme of the support, Trimming support by $\pm 1\%$ leads to a general over-performance of Quantile Density Estimators Qv and Qc relative to the traditional Kernel Density Estimation.

- Given that *near-zero* estimated variance leads to huge density point estimators, replacing 1% of the extreme values at the top for a lineal interpolation leads to a general over-performance of Quantile Density Estimators relative to Kernel.

- Qv estimator performed better in average than Qc estimator, but as *Sample Size* and *Number of Bootstraps* increase they converge to the same results.

- As the *Number of Quantiles* used for point estimation converges to *Sample Size*, the gains from smoothing disappears whereas the performance of Qv and Qc density estimators remains the same.

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Appendix

A Simulation Tables

Normal

Table 1: Normal Distribution - Bias and Mean Integrated Square Error

	Normal	Support Trim			Range Upper-Trim		
	($\sigma^2 = 1$)	± 0.01	± 0.025	± 0.05	-0.01	-0.025	-0.05
Bias							
Qc	-0.0080	-0.0081	-0.0082	-0.0084	-0.0077	-0.0067	-0.0055
MAqc	-0.0075	-0.0076	-0.0077	-0.0080	-0.0072	-0.0061	-0.0048
WMAqc	-0.0064	-0.0066	-0.0070	-0.0075	-0.0061	-0.0051	-0.0039
HPqc	-0.0080	-0.0081	-0.0082	-0.0084	-0.0077	-0.0067	-0.0055
Qv	-0.0039	-0.0039	-0.0040	-0.0041	-0.0036	-0.0027	-0.0015
MAqv	-0.0032	-0.0033	-0.0034	-0.0035	-0.0030	-0.0019	-0.0007
WMAqv	-0.0023	-0.0025	-0.0028	-0.0034	-0.0020	-0.0011	0.0001
HPqv	-0.0039	-0.0039	-0.0040	-0.0041	-0.0036	-0.0027	-0.0015
Kernel	0.0024	0.0079	0.0087	0.0098	0.0075	0.0074	0.0073
MISE							
Qc	0.0026	0.0026	0.0027	0.0028	0.0025	0.0023	0.0021
MAqc	0.0017	0.0017	0.0017	0.0018	0.0016	0.0015	0.0014
WMAqc	0.0012	0.0013	0.0013	0.0018	0.0012	0.0011	0.0010
HPqc	0.0008	0.0008	0.0009	0.0009	0.0008	0.0008	0.0007
Qv	0.0020	0.0020	0.0021	0.0022	0.0019	0.0018	0.0016
MAqv	0.0008	0.0012	0.0012	0.0012	0.0011	0.0010	0.0009
WMAqv	0.0011	0.0008	0.0009	0.0013	0.0008	0.0007	0.0007
HPqv	0.0004	0.0004	0.0004	0.0005	0.0004	0.0004	0.0004
Kernel	0.0003	0.0017	0.0017	0.0019	0.0016	0.0016	0.0016
Ratio MISE (Kernel=1)							
Qc	9.1981	1.5768	1.5579	1.5000	1.5568	1.4225	1.2984
MAqc	5.9198	1.0149	1.0032	0.9669	1.0065	0.9258	0.8462
WMAqc	4.4111	0.7491	0.7582	0.9794	0.7515	0.6960	0.6405
HPqc	2.8892	0.4956	0.4922	0.4807	0.4938	0.4633	0.4324
Qv	7.0496	1.2088	1.1946	1.1506	1.1925	1.0864	0.9919
MAqv	2.7991	0.6940	0.6865	0.6626	0.6875	0.6299	0.5758
WMAqv	4.0469	0.4760	0.4905	0.7041	0.4765	0.4401	0.4061
HPqv	1.5128	0.2584	0.2561	0.2502	0.2584	0.2424	0.2284

Simulations:1000 Boots:500 ; Taus:100 ; N:1000 Qc:Quantil-Mean Covariance Density, Qv:Quantil Variance Density. Kernel:Kernel density estimation MAqc and MAqv are Qc and Qv smoothed by Moving Average (3). WMAqc and WMAqv are Qc and Qv smoothed by a Kernel-Weighted Moving Average ($bw = 3$). HPqc and HPqv are Qc and Qv smoothed by HP filter ($\lambda = 1600$). Ratio MISE is each density estimation MISE in terms of Kernel MISE (MISE.f/MISE.k

Laplace

Table 2: Laplace Distribution - Bias and Mean Integrated Square Error

	Laplace	Support Trim			Range Upper-Trim		
	($\beta = \sqrt{2}$)	± 0.01	± 0.025	± 0.05	-0.01	-0.025	-0.05
Bias							
Qc	-0.0089	-0.0091	-0.0093	-0.0097	-0.0085	-0.0069	-0.0048
MAqc	-0.0088	-0.0089	-0.0091	-0.0094	-0.0083	-0.0066	-0.0044
WMAqc	-0.0076	-0.0076	-0.0076	-0.0075	-0.0072	-0.0055	-0.0035
HPqc	-0.0089	-0.0091	-0.0093	-0.0097	-0.0085	-0.0069	-0.0048
Qv	-0.0040	-0.0041	-0.0042	-0.0043	-0.0037	-0.0022	-0.0004
MAqv	-0.0036	-0.0037	-0.0037	-0.0039	-0.0033	-0.0017	0.0003
WMAqv	-0.0027	-0.0027	-0.0027	-0.0026	-0.0024	-0.0009	0.0010
HPqv	-0.0040	-0.0041	-0.0042	-0.0043	-0.0037	-0.0022	-0.0004
Kernel	0.0060	0.0151	0.0164	0.0173	0.0141	0.0121	0.0100
MISE							
Qc	0.0045	0.0046	0.0048	0.0050	0.0044	0.0040	0.0037
MAqc	0.0030	0.0031	0.0032	0.0034	0.0030	0.0028	0.0027
WMAqc	0.0023	0.0023	0.0023	0.0026	0.0022	0.0021	0.0021
HPqc	0.0016	0.0016	0.0017	0.0018	0.0016	0.0016	0.0016
Qv	0.0034	0.0035	0.0036	0.0038	0.0033	0.0030	0.0028
MAqv	0.0014	0.0021	0.0021	0.0023	0.0020	0.0019	0.0018
WMAqv	0.0020	0.0014	0.0015	0.0017	0.0014	0.0014	0.0014
HPqv	0.0009	0.0009	0.0009	0.0010	0.0009	0.0009	0.0010
Kernel	0.0014	0.0057	0.0061	0.0065	0.0055	0.0053	0.0050
Ratio MISE (Kernel=1)							
Qc	3.1905	0.8024	0.7820	0.7655	0.8002	0.7656	0.7468
MAqc	2.1379	0.5381	0.5252	0.5152	0.5412	0.5312	0.5298
WMAqc	1.6086	0.3971	0.3801	0.3958	0.4090	0.4085	0.4156
HPqc	1.1286	0.2844	0.2784	0.2750	0.2890	0.2981	0.3133
Qv	2.3960	0.6025	0.5872	0.5747	0.6018	0.5765	0.5678
MAqv	1.0043	0.3607	0.3521	0.3455	0.3634	0.3585	0.3643
WMAqv	1.4330	0.2481	0.2387	0.2604	0.2560	0.2589	0.2711
HPqv	0.6284	0.1583	0.1550	0.1532	0.1619	0.1723	0.1901

Simulations:1000 Boots:500 ; Taus:100 ; N:1000 Qc:Quantil-Mean Covariance Density, Qv:Quantil Variance Density. Kernel:Kernel density estimation MAqc and MAqv are Qc and Qv smoothed by Moving Average (3). WMAqc and WMAqv are Qc and Qv smoothed by a Kernel-Weighted Moving Average ($bw = 3$). HPqc and HPqv are Qc and Qv smoothed by HP filter ($\lambda = 1600$). Ratio MISE is each density estimation MISE in terms of Kernel MISE (MISE.f/MISE.k

Gumbel

Table 3: Gumbel Distribution - Bias and Mean Integrated Square Error

	Gumbel	Support Trim			Range Upper-Trim		
	$(\beta = \frac{\sqrt{6}}{\pi})$	± 0.01	± 0.025	± 0.05	-0.01	-0.025	-0.05
Bias							
Qv	-0.0043	-0.0044	-0.0044	-0.0045	-0.0040	-0.0028	-0.0016
MAqv	-0.0035	-0.0036	-0.0037	-0.0039	-0.0032	-0.0020	-0.0006
WMAqv	-0.0024	-0.0028	-0.0031	-0.0039	-0.0021	-0.0010	0.0003
HPqv	-0.0043	-0.0044	-0.0044	-0.0045	-0.0040	-0.0028	-0.0016
Qc	-0.0092	-0.0093	-0.0094	-0.0096	-0.0089	-0.0076	-0.0062
MAqc	-0.0085	-0.0086	-0.0088	-0.0090	-0.0081	-0.0068	-0.0053
WMAqc	-0.0073	-0.0076	-0.0080	-0.0088	-0.0070	-0.0057	-0.0043
HPqc	-0.0092	-0.0093	-0.0094	-0.0096	-0.0089	-0.0076	-0.0062
Kernel	0.0042	0.0119	0.0131	0.0145	0.0113	0.0110	0.0107
MISE							
Qv	0.0027	0.0027	0.0028	0.0029	0.0026	0.0023	0.0022
MAqv	0.0011	0.0015	0.0016	0.0016	0.0015	0.0013	0.0012
WMAqv	0.0015	0.0011	0.0011	0.0020	0.0010	0.0009	0.0009
HPqv	0.0006	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005
Qc	0.0036	0.0037	0.0038	0.0039	0.0035	0.0032	0.0030
MAqc	0.0024	0.0024	0.0025	0.0025	0.0023	0.0021	0.0019
WMAqc	0.0018	0.0018	0.0019	0.0030	0.0018	0.0016	0.0015
HPqc	0.0012	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011
Kernel	0.0004	0.0024	0.0024	0.0026	0.0023	0.0023	0.0023
Ratio MISE (k=1)							
Qv	6.7457	1.1506	1.1447	1.0959	1.1360	1.0252	0.9383
MAqv	2.6628	0.6535	0.6508	0.6236	0.6486	0.5891	0.5407
WMAqv	3.8317	0.4533	0.4627	0.7634	0.4518	0.4142	0.3842
HPqv	1.5397	0.2573	0.2537	0.2442	0.2628	0.2474	0.2357
Qc	9.2464	1.5737	1.5641	1.4913	1.5560	1.4073	1.2871
MAqc	5.9742	1.0173	1.0112	0.9626	1.0110	0.9221	0.8461
WMAqc	4.5464	0.7726	0.7796	1.1312	0.7713	0.7097	0.6567
HPqc	3.1642	0.5349	0.5313	0.5110	0.5395	0.5066	0.4777

Simulations:1000 Boots:500 ; Taus:100 ; N:1000 Qc:Quantil-Mean Covariance Density, Qv:Quantil Variance Density. Kernel:Kernel density estimation MAqc and MAqv are Qc and Qv smoothed by Moving Average (3). WMAqc and WMAqv are Qc and Qv smoothed by a Kernel-Weighted Moving Average ($bw = 3$). HPqc and HPqv are Qc and Qv smoothed by HP filter ($\lambda = 1600$). Ratio MISE is each density estimation MISE in terms of Kernel MISE (MISE.f/MISE.k

Gamma

Table 4: Gamma Distribution - Bias and Mean Integrated Square Error

	Gamma	Support Trim			Range Upper-Trim		
	($\alpha = \beta = 1$)	± 0.01	± 0.025	± 0.05	-0.01	-0.025	-0.05
Bias							
Qv	-0.0076	-0.0071	-0.0068	-0.0066	-0.0054	-0.0018	0.0014
MAqv	-0.0068	-0.0067	-0.0067	-0.0065	-0.0059	-0.0034	-0.0009
WMAqv	-0.0079	-0.0075	-0.0072	-0.0056	-0.0053	-0.0010	0.0026
HPqv	-0.0076	-0.0071	-0.0068	-0.0066	-0.0054	-0.0018	0.0014
Qc	-0.0209	-0.0205	-0.0186	-0.0163	-0.0088	0.0010	0.0070
MAqc	-0.0181	-0.0173	-0.0167	-0.0154	-0.0146	-0.0092	-0.0050
WMAqc	-0.0216	-0.0236	-0.0214	-0.0184	-0.0061	0.0052	0.0116
HPqc	-0.0209	-0.0205	-0.0186	-0.0163	-0.0088	0.0010	0.0070
Kernel	0.0658	0.0938	0.0923	0.0792	0.0944	0.0870	0.0805
MISE							
Qv	0.0103	0.0093	0.0088	0.0078	0.0089	0.0076	0.0070
MAqv	0.0039	0.0039	0.0039	0.0037	0.0039	0.0036	0.0034
WMAqv	0.0040	0.0044	0.0054	0.0175	0.0037	0.0037	0.0040
HPqv	0.0020	0.0019	0.0019	0.0018	0.0018	0.0018	0.0019
Qc	0.6270	0.0382	0.0198	0.0135	0.5456	0.7360	0.7576
MAqc	0.0242	0.0103	0.0090	0.0075	0.0304	0.0737	0.0777
WMAqc	0.3119	0.0404	0.0231	0.0329	0.4029	0.7122	0.7529
HPqc	0.1034	0.0137	0.0085	0.0060	0.1381	0.2940	0.3123
Kernel	0.0258	0.0412	0.0397	0.0324	0.0418	0.0381	0.0351
Ratio MISE (Kernel=1)							
Qv	0.4014	0.2261	0.2216	0.2403	0.2136	0.2006	0.1990
MAqv	0.1526	0.0957	0.0989	0.1145	0.0924	0.0941	0.0976
WMAqv	0.1556	0.1074	0.1358	0.5407	0.0886	0.0961	0.1131
HPqv	0.0768	0.0461	0.0473	0.0565	0.0434	0.0464	0.0530
Qc	24.3396	0.9267	0.4981	0.4174	13.0656	19.3351	21.5978
MAqc	0.9394	0.2501	0.2273	0.2333	0.7279	1.9367	2.2148
WMAqc	12.1093	0.9813	0.5811	1.0152	9.6493	18.7112	21.4627
HPqc	4.0142	0.3329	0.2151	0.1839	3.3071	7.7230	8.9031

Simulations:1000 Boots:500 ; Taus:100 ; N:1000 Qc:Quantil-Mean Covariance Density, Qv:Quantil Variance Density. Kernel:Kernel density estimation MAqc and MAqv are Qc and Qv smoothed by Moving Average (3). WMAqc and WMAqv are Qc and Qv smoothed by a Kernel-Weighted Moving Average ($bw = 3$). HPqc and HPqv are Qc and Qv smoothed by HP filter ($\lambda = 1600$). Ratio MISE is each density estimation MISE in terms of Kernel MISE (MISE.f/MISE.k

B Simulation Graphics (Normal Distribution)

B.1 Sample Size

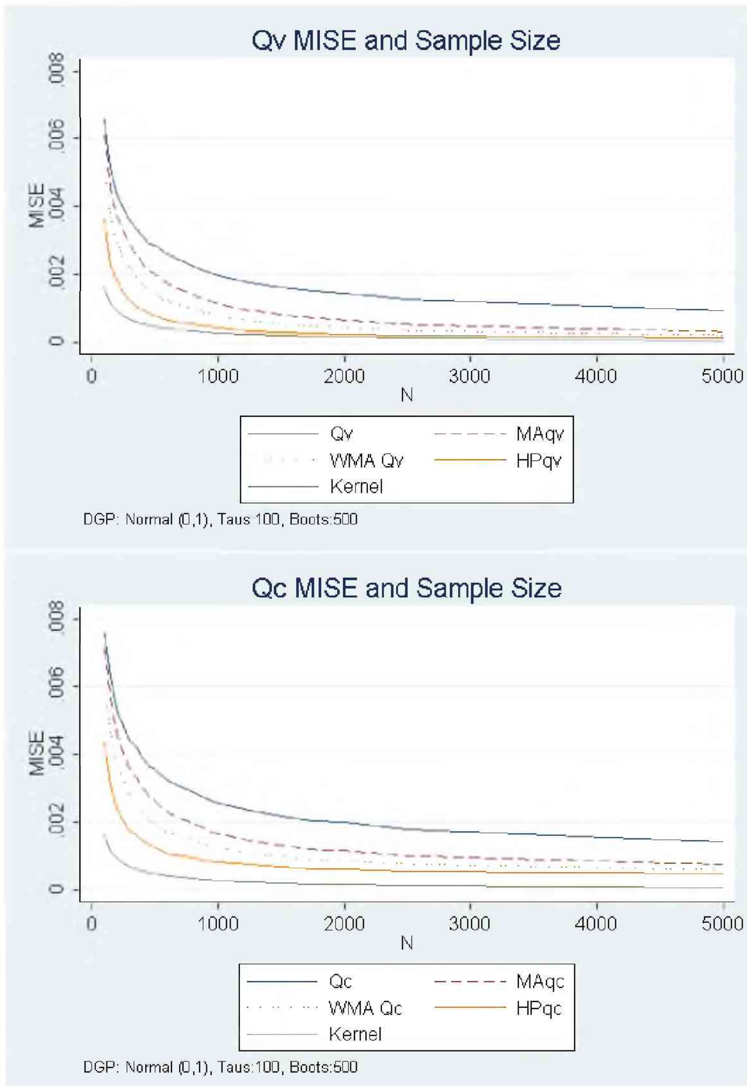

Figure 2: MSE and Sample Size

B.2 Number of Bootstraps

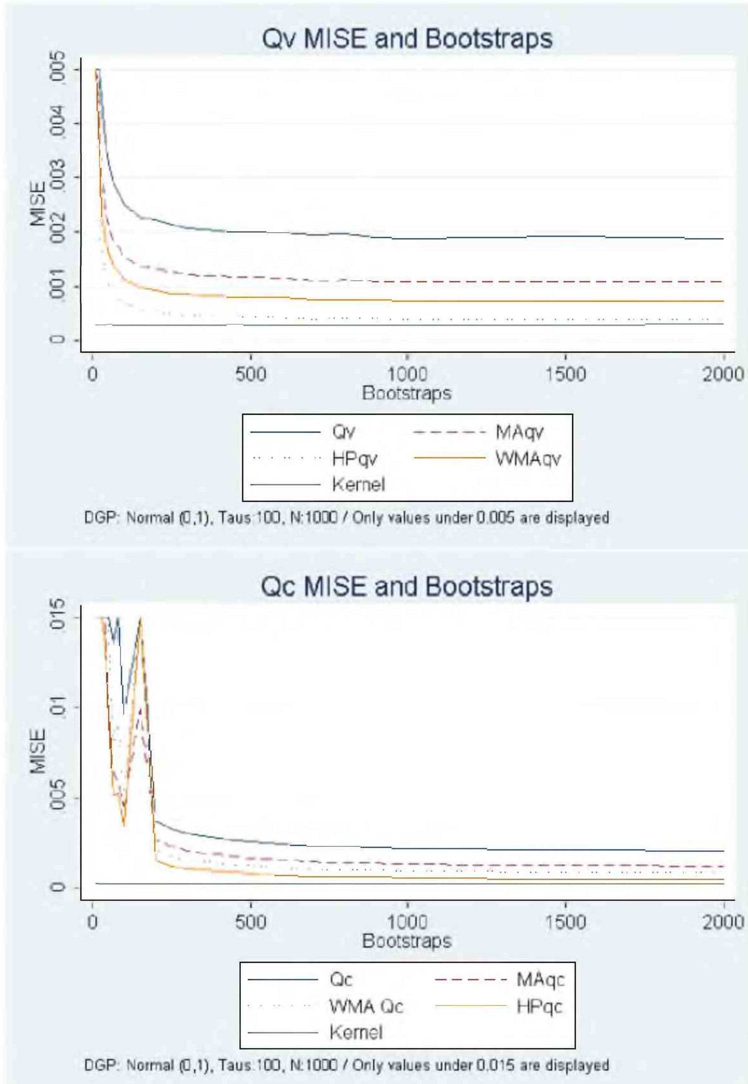

Figure 3: MSE and Number of Bootstraps

B.3 Number of Quantiles

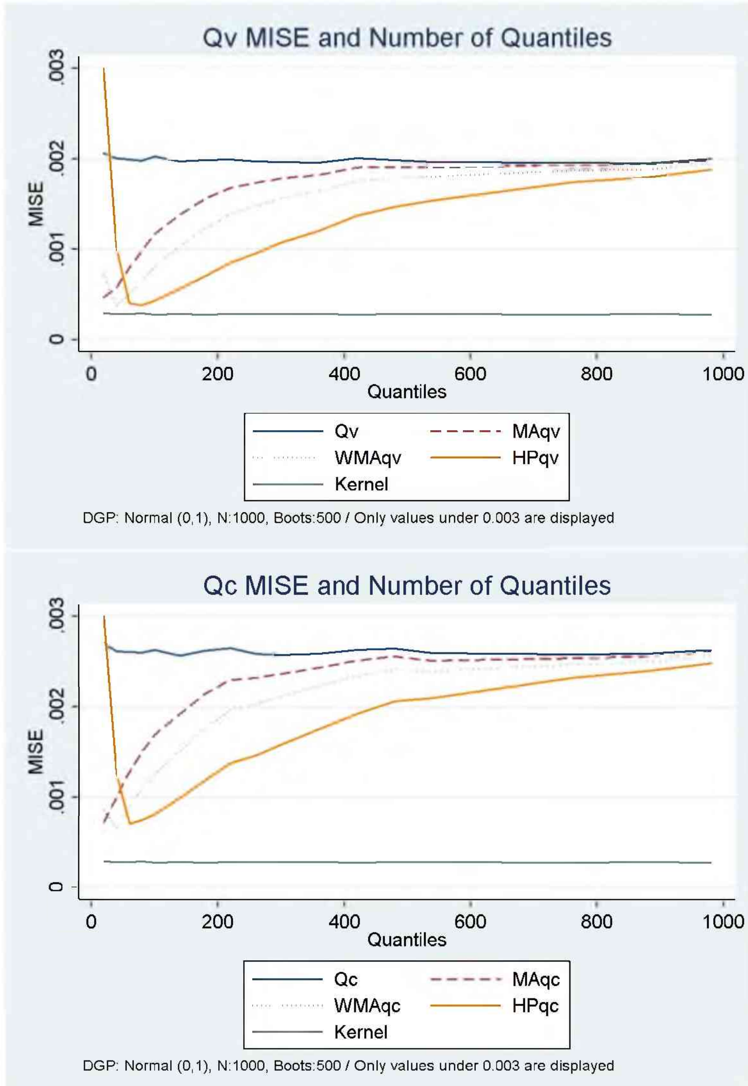

Figure 4: MSE and Number of Quantiles