Truncated γ-exponential models for multi-mass systems

Y.J. Gomez-Leyton[1] & L. Velazquez[1]

[1] *Universidad Católica del Norte, Antofagasta, Chile*

Contact / ygomez@ucn.cl

Resumen / Los modelos γ-exponenciales se propusieron previamente como un intento fenomenológico para caracterizar las propiedades de los sistemas estelares durante la evolución cuasi estacionaria bajo la incidencia de la evaporación: por ejemplo, cúmulos globulares. Estos modelos representan una familia paramétrica de distribuciones que unifican perfiles con núcleos isotérmicos y halos politrópicos, proporcionando así una generalización adecuada para varios modelos disponibles en la literatura. Comenzamos nuestra discusión revisando algunos resultados sobre el caso de sistemas de una sola masa. En particular, enfatizamos que estos modelos predicen la existencia de un nuevo tipo de fenómeno colectivo: el *colapso gravitacional asintótico*. Esta inestabilidad gravitacional difiere del *colapso gravotérmico normal* (por ejemplo, el asociado con el modelo isotérmico) porque su aparición requiere que el sistema libere una cantidad infinita de energía. Posteriormente analizamos cómo la presencia de un espectro de masas modifica la termodinámica de estos modelos, en particular, la aparición de fenómenos colectivos. Si bien la descripción teórica concierne a cualquier sistema de múltiples masas, nuestro estudio computacional aborda el caso más simple: el sistema de dos componentes. Este análisis permite una comprensión importante sobre la termodinámica de los sistemas estelares bajo la incidencia simultánea de la evaporación y los efectos de segregación de masa. Para los modelos actuales, el incremento de los efectos de segregación de masa no afecta la interrupción de la evaporación del sistema, pero favorece la aparición del colapso gravitacional, por ejemplo: tiende a reducir el intervalo de estabilidad de la energía al aumentar el límite inferior de energía crítica para la aparición de este fenómeno. Los casos extremos aparecen bajo ciertas condiciones, donde el colapso gravotérmico cambia su carácter de asintótico a normal.

Abstract / The γ-exponential models were previously proposed as a phenomenological attempt to characterize the properties of stellar systems with a quasi-stationary evolution under the incidence evaporation: e.g.: globular clusters. They represent a parametric family of distributions that unify profiles with isothermal cores and polytropic haloes, thus providing a suitable generalization for several models available in the literature. We start our discussion revisiting some results concerning the case of single-mass systems. In particular, we emphasized that these models predict the existence of a new type of collective phenomenon: the *asymptotic gravothermal collapse*. This gravitational instability differs from the *normal gravothermal collapse* (e.g.: the one associated with isothermal model) because its occurrence requires that the system releases an infinite amount of energy. Afterwards, we analyze how the presence of a mass spectrum modifies the thermodynamics of these models, in particular, the occurrence of collective phenomena. Although the theoretical description concerns to any multi-mass system, our computational study addresses the simplest case: the bi-component system. This analysis allows a major understanding about the thermodynamics of stellar systems under the simultaneous incidence of the evaporation and the mass-segregation effects. For the present models, the growth of mass-segregation effects does not affect the system evaporation disruption but favors the occurrence of gravothermal collapse, e.g.: it tends to reduce the energy interval of stability by increasing the lower bound critical energy for the occurrence of this phenomenon. Extreme cases appear under certain conditions, where gravothermal collapse changes its character from asymptotic to normal.

Keywords / cosmology: dark matter — galaxies: statistics — methods: numerical — methods: statistical

1. Truncated γ-exponential models

Let us consider a system composed of several species of identical particles, and let us denote by m_i and N_i the mass and number of particles of i-th specie. Considering $N = \sum_i N_i$ the total number of particles, one can introduce relative abundance $c_i = N_i/N$ and the average mass $m = \sum_i m_i N_i/N$. The average mass m can be employed to introduce the mass fraction ν_i for the i-th specie $\nu_i = m_i/m$. Accordingly, the mass fraction ν_i and the relative abundances c_i satisfy the normalization conditions, $\sum_i \nu_i c_i = 1$ and $\sum_i c_i = 1$. The one-body quasi-stationary distribution function (DF) of the i-th specie $f^{(i)}_{\gamma}(q,p|\beta) = A_i E\left\{\beta\left[\varepsilon^{(i)}_c - \varepsilon_i(q,p)\right];\gamma\right\}$, where β is the inverse temperature parameter and (q,p) are the canonical variables of the phase space, position and moment, respectively. The DF is based on simple thermo-statistical models (Gunn & Griffin, 1979; Meylan & Heggie, 1997; Zocchi et al., 2012), where $E(x;\gamma)$ is the truncated γ-exponential function that we have proposed (Gomez-Leyton & Velazquez, 2014) which considers a continuous deformation of the ordinary exponential function, given by:

$$E(x;\gamma) = \Theta(x)\sum_{k=0}^{\infty}\frac{x^{\gamma+k}}{\Gamma(\gamma+1+k)}, \tag{1}$$

whose introduction is based on fractional calculus (Miller & Ross, 1993), with $\Theta(x)$ and $\Gamma(x)$ being the Heaviside step and Gamma function, respectively. This proposed parametric family of models generalizes other models that include evaporations, such as Woolley models (Woolley, 1954) when $\gamma = 0$, King models (King, 1966a,b) for $\gamma = 1$ and Wilson models (Wilson, 1975) for $\gamma = 2$. A subset of polytropic models (Chandrasekhar, 1960) is obtained in the high energy limit, including the known model of (Plummer, 1911).

We have introduced here the individual mechanical energy for particles of the i-th specie

$$\varepsilon_i(q,p) = \frac{1}{2m_i}p^2 + m_i\phi(q) \tag{2}$$

(Binney & Tremaine, 1987) where the constant A_i must satisfy the normalization condition

$$\int f_\gamma^{(i)}(q,p|\beta)\, d^3q\, d^3p = 1. \tag{3}$$

The escape energy for the i-th specie $\varepsilon_c^{(i)} = m_i\phi_c$ is determined by its mass $m_i = \nu_i m$ and the tidal potential $\phi_c = -GM/R_t$, where M is the total mass of the system, and R_t the called tidal radius. This last quantity is assumed to be the same for all species because it is determined from the external gravitational influence of other systems. Introducing the dimensionless potential $\Phi(q) = \beta m[\phi_c - \phi(q)]$. The mathematical problem of this investigation is to solve the following second-order non-linear differential Poisson equation in terms of the dimensionless potential $\Phi(\xi)$ and assuming spherical symmetry:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d}{d\xi}\Phi(\xi)\right] = -4\pi\eta\sum_i\frac{\nu_i c_i}{Z_i}E\left[\nu_i\Phi(\xi);\gamma+\frac{3}{2}\right] \tag{4}$$

where $\xi = r/R_t$ is the dimensionless radius, $\eta = \beta GMm/R_t$ is the inverse dimensionless temperature and

$$Z_i = \int E\left[\nu_i\Phi(q);\gamma+\frac{3}{2}\right] d^3q/R_t^3 \tag{5}$$

is the partition function in terms of the tidal radius R_t. This problem can be numerically integrated by demanding the following conditions at the origin of the system, $\Phi(0) = \Phi_0$ and $\partial\Phi(0)/\partial\xi = 0$, and considering the boundary conditions $\xi = 1$, $\Phi(1) = 0$ and $\partial\Phi(1)/\partial\xi = -\eta$. Suitable values for partition functions $Z = Z_i$ and dimensionless inverse temperature η are required to perform the numerical integration of differential equation. Since these quantities are unknown, one needs to implement a scheme of successive iterations to obtain their values by self-consistence (Velazquez et al., 2009).

2. Results for the bi-component system

The mass spectrum of the bi-component system is defined from two parameters: the relative mass abundance of the first component $p = M_1/(M_1+M_2)$ and the mass ratio parameter $\theta = m_1/m_2$ of individual masses (m_1, m_2). According to the results shown in Fig. 1, the presence of a mass-spectrum and the incidence of evaporation are important realistic factors for understanding the thermodynamics of stellar systems. The incidence of evaporation manifests as a truncation of quasi-stationary distribution functions with isothermal cores; which also leads to the truncation of spatial distributions at the called *tidal radius*, as well as the existence of an upper bound for energy of quasi-stationary solutions where stellar systems undergo the evaporation disruption (Gomez-Leyton & Velazquez, 2014). Moreover, under a major incidence of evaporation, the system increases its stability against gravothermal collapse, which is translated into the existence of quasi-stationary configurations with lower energies, and cores more dense and hotter. However, the most significant consequence of evaporation is the distinction between the normal gravothermal collapse, where quasi-stationary configuration with minimal energy exhibits finite values for its energy, temperature and density, and the asymptotic gravothermal collapse, where the corresponding values of such a configuration turn infinite when deformation parameter γ of the present models falls inside a certain interval $\gamma_c < \gamma \leq \gamma_m = 7/2$, whose lower bound γ_c depends on the mass spectrum.

On the other hand, a direct consequence of mass spectrum is the existence of mass-segregation effects, that is, the tendency of heavier members of a gravitationally bound system to concentrate toward the center, while lighter members are displaced to the outer regions. By itself, mass-segregation phenomenon introduces important quantitative and qualitative modifications in most of thermodynamic dependencies. The most important of all them is the growth of the critical energy associated with the occurrence of gravothermal collapse, which implies that mass-segregation favors the occurrence of the gravothermal collapse. Under certain conditions, the presence of a mass spectrum can even modify the nature of gravothermal collapse from asymptotic to normal. Our computational study of the bi-component system evidences that the effects of evaporation and mass-segregation on the thermodynamics of the present stellar models follow opposite tendencies or exhibit opposite consequences, which enable us to talk about a certain competition among evaporation and mass-segregation effects.

Acknowledgements: Authors thank the partial financial support of VRIDT-UCN research programme, and CONICYT ACT1204, FONDECYT REGULAR 1170834 and CONICYT grant 22140036 (Chilean agencies). The authors thank Professor A. Zarate to allow us the access to computational facilities of Research Group on Science Materials and Nanotechnology-UCN.

References

Binney J., Tremaine S., 1987, *Galactic dynamics*, Princeton University Press, Princeton, NJ

Chandrasekhar S., 1960, *Principles of stellar dynamics*, Dover, New York

Gomez-Leyton Y.J., Velazquez L., 2014, J. Stat. Mech.: Theory Exp., 4, 04006

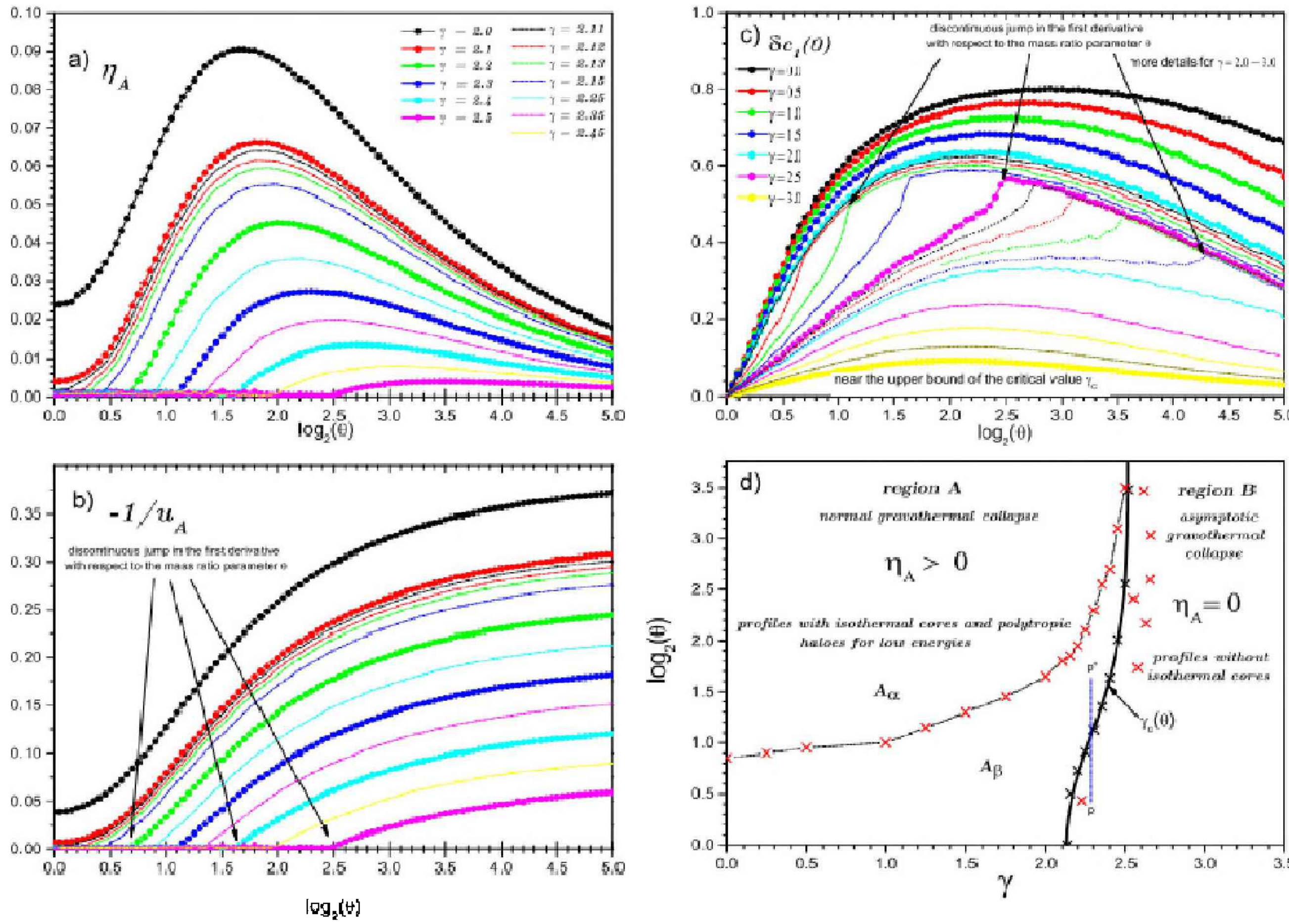

Figure 1: Panel a) and b) Dependencies of the dimensionless inverse temperature $\eta = \beta GMm/R_t$ and the negative of the inverse dimensionless energy $-1/u$ evaluated at the critical points of gravothermal collapses vs. the mass ratio parameter $\theta = m_1/m_2$. The calculations were performed with relative mass abundance of the first component fixed at the value $p = M_1/(M_1 + M_2) = 0.25$ and some values of deformation parameter γ near the critical value $\gamma_c \simeq 2.13$ (single-mass case) in order to show the influence of the mass spectrum on the character of gravothermal collapse. In particular, the qualitative changes observed for dependencies with $\gamma = 2.5$ between the values $k = 2$ and 3, which are associated with the suppression of divergence of the dimensionless energy at the gravothermal collapse u_A (minimum energy that the system can reach), actually occurs when $k = \log_2(\theta) \simeq 2.5$.
Panel c) the central value of deviation of local relative abundance of the heavy specie $\delta c_1(0)$ vs. the mass ratio parameter $\theta = m_1/m_2$ at the point of gravothermal collapse for different values of deformation parameter γ, we have highlighted more details for $\gamma = 2.0 - 3.0$ with a step of 0.1 and near of the upper bound of the critical value γ_c with a step of 0.02 since $2.52 - 2.58$.
Panel d) Phase diagram in the semi-plane $(\gamma, \log_2 \theta)$ with different qualitative behaviors of the order parameter η_A (the dimensionless inverse temperature $\eta = \beta GM^2/R_t$ evaluated at the critical point u_A of gravothermal collapse) for the bi-component system with relative mass abundance of the heavy component fixed at the value $p = M_1/(M_1 + M_2) = 0.25$. This scheme summarizes the nature of gravothermal collapse under the incidence of evaporation effects and the presence of a mass spectrum. Crosses are points estimated from our computational study, which were employed to conform this scheme. The dash-dot segment $p - p'$ (blue) represents a change of mass ratio parameter $\theta = m_1/m_2$ that keeps fixed the deformation parameter γ. If this segment crosses the boundary between the regions A and B [the curve $\gamma_c(\theta)$], one can observe the discontinuous jump in the first derivative of some observables with respect to the mass ratio parameter θ.

Gunn J.E., Griffin R.F., 1979, AJ, 84, 752
King I.R., 1966a, AJ, 71, 64
King I.R., 1966b, AJ, 71, 276
Meylan G., Heggie D.C., 1997, A&A Rv, 8, 1
Miller K., Ross B., 1993, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley
Plummer H.C., 1911, MNRAS, 71, 460
Velazquez L., García S.G., Guzmán F., 2009, PhRvE, 79, 011120
Wilson C.P., 1975, AJ, 80, 175
Woolley R.V.D.R., 1954, MNRAS, 114, 191
Zocchi A., Bertin G., Varri A.L., 2012, A&A, 539, A65