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FINITE ELEMENT VIBRATION ANALYSIS OF FLUID–SOLID SYSTEMS WITHOUT SPURIOUS MODES

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ABSTRACT. This paper deals with the finite element approximation of the vibration modes of a problem with fluid–structure interaction. Displacements variables are used for both the fluid and the solid. To avoid the typical spurious modes of this formulation we introduce a non conforming discretization. Error estimates for the approximation of eigenvalues and eigenvectors are given.

1. INTRODUCTION

Increasing attention has recently been paid to problems involving fluid–structure interactions. (For a survey of current results see [10] and references therein). In this paper, we will be concerned with a typical problem of this type: the elastoacoustic one; i.e.: the interaction between a compressible fluid and an elastic structure.

In this case, under the usual assumptions leading to linear problems, the evolution of the structure is governed by a second order in time linear equation. Its solution can be written in terms of the vibrations modes of the coupled system which are solutions of a linear eigenvalue problem; (see for instance [2]).

Different approaches have been proposed to analyze this problem. Displacement variables are generally chosen for the structure while the fluid is described by different variables (displacements, pressure, velocity potential). Some of these formulations have been analyzed from a theoretical point of view [2]. Others have been used for numerical experimentations [9,15,12].

We will consider the elastoacoustic problem consisting of a bounded domain completely filled by the fluid and limited by the solid. Displacement variables will be used for both the fluid and the solid. We will introduce and analize a finite element method to treat this problem in 2D.

Piecewise linear and bilinear finite elements, for both the fluid and the solid, have been numerically experimented for this formulation of the problem [9]. Spurious modes with almost zero frequencies arise in these discretizations. Several approaches have been tried to avoid this drawback. In [8] a penalty method is

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deviced to distinguish these modes. In [3] reduced integration in the stiffness matrix of the fluid combined with a projection on the element mass matrix is used. Numerical experiments seem to show that this last procedure is useful to avoid the spurious modes.

We are going to introduce an alternative approach. Our method consists of piecewise linear elements for the solid and Raviart–Thomas elements of lowest order [13] for the fluid. The coupling of both discretizations is of non–conforming type. We will prove the convergence of this scheme and an error estimate will be given. We will also prove that spurious modes do not arise when sufficiently refined meshes are used. Numerical experiments showing the efficiency of this method are reported in [1].

2. Modeling of the problem

2.1 The model problem.

We study the problem of determining the vibration modes of an ideal inviscid barotropic fluid contained in a linear elastic structure. We consider as a model problem the case of a vessel completely filled by the fluid. We restrict ourselves to the 2D case and assume polygonal boundaries and interfaces.

Let Ω_F and Ω_S be the domains occupied by the fluid and the solid respectively as in Fig. 1. We assume Ω_F to be simply connected. Γ_I denotes the interface between the solid and the fluid and $\boldsymbol{\nu}$ its unit normal vector pointing outwards Ω_F. The exterior boundary of the solid is the union of Γ_D and Γ_N: the structure is fixed along Γ_D and free of stress along Γ_N; let $\boldsymbol{\eta}$ denote the unit outward normal vector along Γ_N.

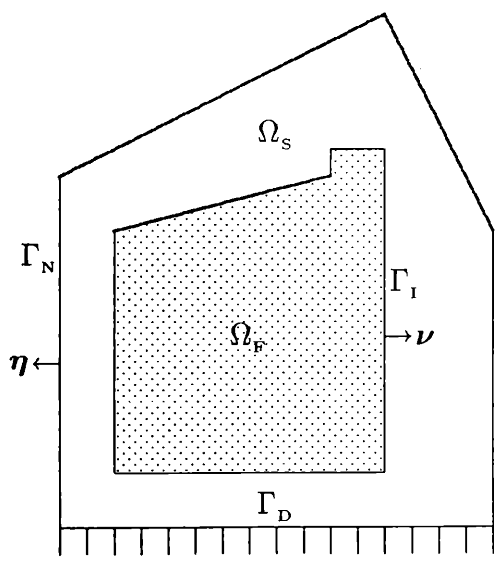

Fig. 1

Throughout this paper we use the standard notation for Sobolev spaces, norms and seminorms. We also denote

$$H^{\alpha,\beta}(\operatorname{div},\Omega_F) := \left\{ \mathbf{u} \in [H^\alpha(\Omega_F)]^2 \ : \ \operatorname{div}\mathbf{u} \in H^\beta(\Omega_F) \right\},$$

$\|\mathbf{u}\|^2_{H^{\alpha,\beta}(\operatorname{div},\Omega_F)} := \|\mathbf{u}\|^2_{H^\alpha(\Omega_F)} + \|\operatorname{div}\mathbf{u}\|^2_{H^\beta(\Omega_F)}$ and $H(\operatorname{div},\Omega_F) := H^{0,0}(\operatorname{div},\Omega_F)$.

The classical acoustic approximation for the small amplitude motions yields the following eigenvalue problem for the vibration modes of the coupled system (see, for instance, [2,8]).

SP. *Find* $\lambda \geq 0$ *and* $(\mathbf{u}, \mathbf{v}) \in H^{0,1}(\mathrm{div}, \Omega_{\mathrm{F}}) \times \left[H^1(\Omega_{\mathrm{S}})\right]^2$, $(\mathbf{u}, \mathbf{v}) \neq (\mathbf{0}, \mathbf{0})$ *such that:*

$$\begin{aligned}
&\lambda \mathbf{u} + c^2 \nabla(\,\mathrm{div}\,\mathbf{u}) = \mathbf{0}, && \text{in } \Omega_{\mathrm{F}} && (2.1)\\
&\lambda \rho_{\mathrm{S}} \mathbf{v} + \mathcal{L}(\mathbf{v}) = \mathbf{0}, && \text{in } \Omega_{\mathrm{S}} && (2.2)\\
&\boldsymbol{\sigma}(\mathbf{v})\boldsymbol{\nu} - (\rho_{\mathrm{F}} c^2 \,\mathrm{div}\,\mathbf{u})\boldsymbol{\nu} = \mathbf{0}, && \text{on } \Gamma_{\mathrm{I}} && (2.3)\\
&(\mathbf{u} - \mathbf{v}) \cdot \boldsymbol{\nu} = 0, && \text{on } \Gamma_{\mathrm{I}} && (2.4)\\
&\boldsymbol{\sigma}(\mathbf{v})\boldsymbol{\eta} = \mathbf{0}, && \text{on } \Gamma_{\mathrm{N}} && (2.5)\\
&\mathbf{v} = \mathbf{0}, && \text{on } \Gamma_{\mathrm{D}} && (2.6)
\end{aligned}$$

where $\omega = \sqrt{\lambda}$ is the frequency of the eigenmode, $\mathbf{u}$ and $\mathbf{v}$ are the displacements in the fluid and the solid respectively. ρ_{F} and ρ_{S} are the respective densities; c is the acoustic speed in the fluid. $\boldsymbol{\sigma}$ is the 2×2 stress tensor; i.e.:

$$\sigma_{i,j}(\mathbf{v}) := \lambda_{\mathrm{S}} \sum_{k=1}^{2} \varepsilon_{kk}(\mathbf{v})\delta_{ij} + 2\mu_{\mathrm{S}}\varepsilon_{ij}(\mathbf{v}), \quad i,j = 1,2,$$

λ_{S} and μ_{S} being the Lamé coefficients of the structure and $\varepsilon_{ij}(\mathbf{v}) := \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ being the components of the strain tensor. Finally $\mathcal{L}(\mathbf{v}) := (\lambda_{\mathrm{S}} + \mu_{\mathrm{S}})\nabla(\,\mathrm{div}\,\mathbf{v}) + \mu_{\mathrm{S}}\Delta\mathbf{v}$ is the linear elasticity operator.

Remark 2.1. The second derivatives in (2.1) and (2.2) must be understood in the sense of distributions. Equations (2.3) and (2.4) are equalities in $H^{-1/2}(\Gamma_{\mathrm{I}})$, but, since $\mathrm{div}\,\mathbf{u} \in H^1(\Omega_{\mathrm{F}})$ (and hence $\mathrm{div}\,\mathbf{u}|_{\Gamma_{\mathrm{I}}} \in H^{1/2}(\Gamma_{\mathrm{I}})$) and $\mathbf{v} \cdot \boldsymbol{\nu} \in H^{1/2}(\Gamma_{\mathrm{I}})$, both can be considered as equalities in $L^2(\Gamma_{\mathrm{I}})$.

Problem **SP** is equivalent to a variational problem. To describe it we introduce the following notation. Let $\mathbf{H} := \left[L^2(\Omega_{\mathrm{F}})\right]^2 \times \left[L^2(\Omega_{\mathrm{S}})\right]^2$ and $|(\mathbf{u}, \mathbf{v})|$ be the standard $\boldsymbol{L}^2$–norm. Let $\mathbf{X} := H(\mathrm{div}, \Omega_{\mathrm{F}}) \times \left[H^1_{\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{S}})\right]^2$, $\left[H^1_{\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{S}})\right]^2$ being the subspace of functions in $\left[H^1(\Omega_{\mathrm{S}})\right]^2$ vanishing in Γ_{D}, and $\|(\mathbf{u}, \mathbf{v})\|$ be the $H(\mathrm{div}, \Omega_{\mathrm{F}}) \times \left[H^1(\Omega_{\mathrm{S}})\right]^2$ norm. Let $\mathbf{V}$ be the closed subspace of $\mathbf{X}$ defined by $\mathbf{V} := \{(\mathbf{u}, \mathbf{v}) \in \mathbf{X} \;:\; \mathbf{u} \cdot \boldsymbol{\nu} = \mathbf{v} \cdot \boldsymbol{\nu} \text{ on } \Gamma_{\mathrm{I}}\}$; (as in Remark 2.1, the equality of the normal traces $\mathbf{u} \cdot \boldsymbol{\nu}$ and $\mathbf{v} \cdot \boldsymbol{\nu}$ can be understood in the sense of $L^2(\Gamma_{\mathrm{I}})$).

Let a be the bilinear, symmetric, continuous, positive form defined on $\mathbf{X}$ by

$$a((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) := \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 (\,\mathrm{div}\,\mathbf{u})(\,\mathrm{div}\,\boldsymbol{\phi}) + \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}), \quad (\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{X},$$

where $\boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}) := \sum_{i,j=1,2} \sigma_{ij}(\mathbf{v})\varepsilon_{ij}(\boldsymbol{\psi})$. Let b be the bilinear, symmetric, continuous, coercive form defined on $\mathbf{H}$ by

$$b((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) := \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} \mathbf{u} \cdot \boldsymbol{\phi} + \int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} \mathbf{v} \cdot \boldsymbol{\psi}, \quad (\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{H}.$$

Integrating by parts, it is straightforward to see that, if the eigenpair $(\lambda, (\mathbf{u}, \mathbf{v}))$ is a solution of problem **SP** then it is also a solution of the following variational problem

VP. *Find* $\lambda \in \mathbf{R}$ *and* $(\mathbf{u}, \mathbf{v}) \in \mathbf{V}$, $(\mathbf{u}, \mathbf{v}) \neq (\mathbf{0}, \mathbf{0})$ *such that*

$$a((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) = \lambda b((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})), \quad \forall (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{V}. \tag{2.7}$$

Theorem 2.1. *Problems* **SP** *and* **VP** *are equivalent.*

Proof. We have already observed that every solution of **SP** is a solution of **VP**.

Conversely, let $(\lambda, (\mathbf{u}, \mathbf{v}))$ be an eigenpair of **VP**. Since $(\mathbf{u}, \mathbf{v}) \in \mathbf{V}$ then (2.4) and (2.6) are automatically satisfied. Equalities (2.1) and (2.2) can be easily proved by respectively taking $(\mathbf{0}, \boldsymbol{\psi})$, with $\boldsymbol{\psi} \in [\mathcal{D}(\Omega_{\mathrm{S}})]^2$, and $(\boldsymbol{\phi}, \mathbf{0})$, with $\boldsymbol{\phi} \in [\mathcal{D}(\Omega_{\mathrm{F}})]^2$, as test functions in (2.7).

Now, $\forall (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{V}$, integration by parts in (2.7), together with (2,1), (2.2), (2.4) and (2.6), gives

$$\int_{\Gamma_{\mathrm{I}}} \left[(\rho_{\mathrm{F}} c^2 \operatorname{div} \mathbf{u}) \boldsymbol{\nu} - \boldsymbol{\sigma}(\mathbf{v}) \boldsymbol{\nu} \right] \cdot \boldsymbol{\psi} + \int_{\Gamma_{\mathrm{N}}} \left[\boldsymbol{\sigma}(\mathbf{v}) \boldsymbol{\nu} \right] \cdot \boldsymbol{\psi} = 0, \tag{2.8}$$

proving (2.3) and (2.5) in the sense of $H^{-1/2}(\Gamma_{\mathrm{I}})$. In fact, we may take any $\boldsymbol{\psi} \in \left[H^1_{\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{S}}) \right]^2$ as a test function in (2.8) since there always exists $\boldsymbol{\phi} \in H(\operatorname{div}, \Omega_{\mathrm{F}})$ such that $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{V}$; (for instance, $\boldsymbol{\phi} = \nabla q$, with q a harmonic function satisfying $\frac{\partial q}{\partial \nu} = \boldsymbol{\psi} \cdot \boldsymbol{\nu}$ on Γ_{I}). $\square$

2.2 The eigenspace for $\lambda = 0$.

There exist eigenmodes of problem (2.7) which do not induce vibrations into the solid. They are pure rotational motions of the fluid and correspond to the eigenvalue $\lambda = 0$. The following theorem shows that the eigenspace associated to this eigenvalue contains exclusively such rotational motions.

Theorem 2.2. *Let* $\mathbf{K}$ *be the eigenspace corresponding to* $\lambda = 0$ *in problem (2.7). Then* $\mathbf{K} = \left\{ (\operatorname{\mathbf{curl}} \xi, \mathbf{0}) \ : \ \xi \in H^1_0(\Omega_{\mathrm{F}}) \right\}$.

Proof. $\forall \xi \in H^1_0(\Omega_{\mathrm{F}})$, $\operatorname{\mathbf{curl}} \xi \cdot \boldsymbol{\nu} = \frac{\partial \xi}{\partial \boldsymbol{\tau}} = 0$, ($\boldsymbol{\tau}$ being the tangential unit vector to Γ_{I}). Hence $(\operatorname{\mathbf{curl}} \xi, \mathbf{0}) \in \mathbf{V}$ and $a((\operatorname{\mathbf{curl}} \xi, \mathbf{0}), (\boldsymbol{\phi}, \boldsymbol{\psi})) = 0$, $\forall (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{V}$.

Conversely, let $(\mathbf{u}, \mathbf{v}) \in \mathbf{K}$, then

$$a((\mathbf{u}, \mathbf{v}), (\mathbf{u}, \mathbf{v})) = \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 (\operatorname{div} \mathbf{u})^2 + \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) = 0.$$

Since $\int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) = 0$ and $\mathbf{v}|_{\Gamma_{\mathrm{D}}} = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$. On the other hand, $\operatorname{div} \mathbf{u} = 0$, then $\mathbf{u} = \operatorname{\mathbf{curl}} \xi$ for some $\xi \in H^1(\Omega_{\mathrm{F}})$. Furthermore, since $(\mathbf{u}, \mathbf{v}) \in \mathbf{V}$, the tangential derivative of ξ along Γ_{I} is $\operatorname{\mathbf{curl}} \xi \cdot \boldsymbol{\nu} = \mathbf{v} \cdot \boldsymbol{\nu} = 0$. Therefore, ξ can be chosen in $H^1_0(\Omega_{\mathrm{F}})$. $\square$

The following lemma gives simple characterizations of the orthogonal complements of $\mathbf{K}$ in $\mathbf{H}$ and $\mathbf{V}$ which will be used below.

Lemma 2.3. *Let* $\mathbf{G} := \{(\nabla q, \mathbf{v}) : q \in H^1(\Omega_{\mathrm{F}}),\ \mathbf{v} \in \boldsymbol{L}^2(\Omega_{\mathrm{S}})\}$ *and* $\mathbf{G}_{\mathbf{V}} := \mathbf{G} \cap \mathbf{V}$. *Then*

i) $\mathbf{H} = \mathbf{K} \oplus \mathbf{G}$ *is an orthogonal decomposition in* $|\cdot|$.

ii) $\mathbf{V} = \mathbf{K} \oplus \mathbf{G}_{\mathbf{V}}$ *is an orthogonal decomposition in both* $|\cdot|$ *and* $\|\cdot\|$.

Proof. It is well known that any function in $L^2(\Omega_{\mathrm{F}})$ can be orthogonally decomposed into a gradient plus the curl of a function vanishing on $\partial\Omega_{\mathrm{F}}$ (see [6]). So, (i) follows immediately.

On the other hand, for every $(\mathbf{u}, \mathbf{v}) \in \mathbf{V}$, the Neumann problem

$$\begin{cases} \Delta q = \operatorname{div} \mathbf{u}, & \text{in } \Omega_{\mathrm{F}} \\ \frac{\partial q}{\partial \nu} = \mathbf{v} \cdot \boldsymbol{\nu}, & \text{on } \Gamma_{\mathrm{I}} \end{cases} \tag{2.9}$$

is compatible and, for any solution $q \in H^1(\Omega_{\mathrm{F}})$, $(\nabla q, \mathbf{v}) \in \mathbf{V}$. Since $\operatorname{div}(\mathbf{u} - \nabla q) = 0$, then, proceeding as in Theorem 2.2, we have that $(\mathbf{u}, \mathbf{v}) - (\nabla q, \mathbf{v}) = (\mathbf{curl}\,\xi, \mathbf{0})$ with $\xi \in H_0^1(\Omega_{\mathrm{F}})$. This decomposition is orthogonal in $|\cdot|$ as well as in $\|\cdot\|$, so we conclude the theorem. □

2.3 Description of the spectrum and an a–priori estimate.

Let us recall that our goal is to determine those eigenmodes inducing vibrations into the solid. It can be easily seen from the previous lemma that these eigenmodes correspond to irrotational motions of the fluid and are associated to strictly positive eigenvalues.

The bilinear form a is not coercive on $\mathbf{V}$. However, $a^* := a + b$, can be used in our problem and it turns out to be coercive. In fact,

$$\begin{aligned} a^*((\mathbf{u},\mathbf{v}),(\mathbf{u},\mathbf{v})) &:= \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 (\operatorname{div} \mathbf{u})^2 + \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} |\mathbf{u}|^2 + \int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} |\mathbf{v}|^2 \\ &\geq C \left\{ \|\mathbf{u}\|^2_{H(\mathrm{div},\Omega_{\mathrm{F}})} + \|\mathbf{v}\|^2_{[H^1(\Omega_{\mathrm{S}})]^2} \right\} \\ &= C\|(\mathbf{u},\mathbf{v})\|^2, \quad \forall (\mathbf{u},\mathbf{v}) \in \mathbf{X}. \end{aligned} \tag{2.10}$$

(Throughout this paper C denotes a constant, not necessarily the same at each occurrence).

The eigenvalue problem associated to a^* is

VP*. *Find* $\lambda \in \mathbf{R}$ *and* $(\mathbf{u}, \mathbf{v}) \in \mathbf{V}$, $(\mathbf{u}, \mathbf{v}) \neq (\mathbf{0}, \mathbf{0})$ *such that*

$$a^*((\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})) = \lambda b((\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})), \quad \forall (\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathbf{V}.$$

It is clear that $(\lambda, (\mathbf{u}, \mathbf{v}))$ is an eigenpair of **VP*** if and only if $(\lambda - 1, (\mathbf{u}, \mathbf{v}))$ is an eigenpair of **VP**. Since a^* is $\mathbf{V}$–elliptic, we can define the bounded linear operator $A : \mathbf{H} \longrightarrow \mathbf{V}$ as given by $A(\mathbf{f}, \mathbf{g}) = (\mathbf{u}, \mathbf{v}) \in \mathbf{V}$ such that

$$a^*((\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})) = b((\mathbf{f},\mathbf{g}),(\boldsymbol{\phi},\boldsymbol{\psi})), \quad \forall (\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathbf{V}.$$

For different A–invariant subspaces $\mathbf{E} \subset \mathbf{H}$, we denote $A_{\mathbf{E}}$ the restriction $A|_{\mathbf{E}} : \mathbf{E} \longrightarrow \mathbf{E}$. In particular, let $A_{\mathbf{V}} := A|_{\mathbf{V}} : \mathbf{V} \longrightarrow \mathbf{V}$; $A_{\mathbf{V}}$ is selfadjoint and, clearly, $(\lambda, (\mathbf{u}, \mathbf{v}))$ is an eigenpair of $A_{\mathbf{V}}$ if and only if $(\frac{1}{\lambda}, (\mathbf{u}, \mathbf{v}))$ is a solution of **VP*** and then $(\frac{1}{\lambda} - 1, (\mathbf{u}, \mathbf{v}))$ is a solution of **VP**. Since the eigenvalues of **VP** are positive, then those of $A_{\mathbf{V}}$ satisfy $0 < \lambda \leq 1$.

$A_{\mathbf{V}}$ is not compact; in fact, $A_{\mathbf{K}}$ is the identity on the infinite dimensional subspace $\mathbf{K} \subset \mathbf{V}$. Nevertheless, the following results show that $\mathbf{G}_{\mathbf{V}}$ is A–invariant and $A_{\mathbf{G}_{\mathbf{V}}}$ is compact. This will be used below to show that the spectrum of $A_{\mathbf{V}}$ consists of $\lambda = 1$ and a sequence of eigenvalues $\lambda_n \underset{n}{\longrightarrow} 0$.

Lemma 2.4. $A(\mathbf{G}) \subset \mathbf{G}_{\mathbf{V}}$.

Proof. Decompositions in lemma 2.3 are also orthogonal for the inner products $b(\cdot,\cdot)$ and $a^*(\cdot,\cdot)$. Therefore, for $(\mathbf{f}, \mathbf{g}) \in \mathbf{G} = \mathbf{K}^{\perp}$, $(\mathbf{u}, \mathbf{v}) = A(\mathbf{f}, \mathbf{g}) \in \mathbf{V}$ satisfies

$$a^*((\mathbf{u}, \mathbf{v}), (\mathbf{curl}\,\xi, \mathbf{0})) = b((\mathbf{f}, \mathbf{g}), (\mathbf{curl}\,\xi, \mathbf{0})) = 0, \quad \forall \xi \in H_0^1(\Omega_{\mathrm{F}}).$$

Hence, according to Theorem 2.2, $(\mathbf{u}, \mathbf{v})$ is orthogonal to $\mathbf{K}$ in $a^*(\cdot,\cdot)$, so $(\mathbf{u}, \mathbf{v}) \in \mathbf{G}_{\mathbf{V}}$. □

The following theorem gives an a–priori estimate for the eigenvectors of **VP*** not corresponding to $\lambda = 1$. This result is basic for the error estimates of the numerical method that we introduce in Section 3. As a by–product it yields the compactness of $A_{\mathbf{G}_{\mathbf{V}}}$.

Theorem 2.5. *There exist* $\alpha \in \left(\frac{1}{2}, 1\right]$, $\beta \in (0, 1]$ *and* $C > 0$ *such that if* $(\mathbf{f}, \mathbf{g}) \in \mathbf{G}$, *then* $(\mathbf{u}, \mathbf{v}) := A(\mathbf{f}, \mathbf{g}) \in H^{\alpha,1}(\mathrm{div}, \Omega_{\mathrm{F}}) \times \left[H^{1+\beta}(\Omega_{\mathrm{S}})\right]^2$ *and*

$$\left\{ \|\mathbf{u}\|_{H^{\alpha,1}(\mathrm{div},\Omega_{\mathrm{F}})} + \|\mathbf{v}\|_{[H^{1+\beta}(\Omega_{\mathrm{S}})]^2} \right\} \leq C|(\mathbf{f}, \mathbf{g})|.$$

Proof. Let $(\mathbf{f}, \mathbf{g}) \in \mathbf{G}$. Because of Lemma 2.4, $(\mathbf{u}, \mathbf{v}) \in \mathbf{G}_{\mathbf{V}}$. Hence, there exists $q \in H^1(\Omega_{\mathrm{F}})$ such that $\mathbf{u} = \nabla q$. q is a solution of problem (2.9), (i.e.: $\Delta q = \mathrm{div}\,\mathbf{u}$ in Ω_{F} and $\frac{\partial q}{\partial \nu} = \mathbf{v} \cdot \boldsymbol{\nu}$ on Γ_{I}). Since $A : \mathbf{H} \longrightarrow \mathbf{V}$ is continuous, then

$$\|\mathbf{v} \cdot \boldsymbol{\nu}\|_{H^{1/2}(\Gamma_{\mathrm{I}})} \leq C \|\mathbf{v}\|_{[H^1(\Omega_{\mathrm{S}})]^2} \leq \|A\| \, |(\mathbf{f}, \mathbf{g})|$$

and

$$\|\,\mathrm{div}\,\mathbf{u}\|_{[L^2(\Omega_{\mathrm{F}})]^2} \leq \|A\| \, |(\mathbf{f}, \mathbf{g})|.$$

From the usual a–priori estimate for the Neumann problem, (see [7]), we have $q \in \mathbf{H}^{1+\alpha}(\Omega_{\mathrm{F}})$ and

$$\begin{aligned}\|\mathbf{u}\|_{H^{\alpha,1}(\mathrm{div},\Omega_{\mathrm{F}})} &= \|\nabla q\|_{H^{\alpha,1}(\mathrm{div},\Omega_{\mathrm{F}})} \\ &\leq C\left[\|\,\mathrm{div}\,\mathbf{u}\|_{[L^2(\Omega_{\mathrm{F}})]^2} + \|\mathbf{v} \cdot \boldsymbol{\nu}\|_{H^{1/2}(\Gamma_{\mathrm{I}})}\right] \leq C|(\mathbf{f}, \mathbf{g})|,\end{aligned}$$

where $\alpha = 1$ if Ω_{F} is convex and $\alpha = \frac{\pi}{\theta}$, with θ the biggest reentrant corner, otherwise.

Proceeding as in Theorem 2.1, it can be shown that $-c^2\nabla(\,\mathrm{div}\,\mathbf{u}) + \mathbf{u} = \mathbf{f}$ in Ω_{F} and then $\mathrm{div}\,\mathbf{u}|_{\Gamma_{\mathrm{I}}} \in H^{1/2}(\Gamma_{\mathrm{I}})$ with $\|\,\mathrm{div}\,\mathbf{u}\|_{H^{1/2}(\Gamma_{\mathrm{I}})} \leq C \,\|\,\mathrm{div}\,\mathbf{u}\|_{H^1(\Omega_{\mathrm{F}})} \leq C|(\mathbf{f}, \mathbf{g})|$.

Arguing again as in Theorem 2.1, $\mathbf{v}$ is shown to be the solution of the following linear elasticity problem:

$$\begin{aligned}
-\mathcal{L}(\mathbf{v}) + \rho_{\mathrm{F}}\mathbf{v} &= \rho_{\mathrm{F}}\mathbf{g}, && \text{in } \Omega_{\mathrm{S}}\\
\boldsymbol{\sigma}(\mathbf{v})\boldsymbol{\nu} &= (\rho_{\mathrm{F}} c^2 \operatorname{div}\mathbf{u})\boldsymbol{\nu}, && \text{on } \Gamma_{\mathrm{I}}\\
\boldsymbol{\sigma}(\mathbf{v})\boldsymbol{\eta} &= \mathbf{0}, && \text{on } \Gamma_{\mathrm{N}}\\
\mathbf{v} &= \mathbf{0}, && \text{on } \Gamma_{\mathrm{D}}
\end{aligned}$$

From the a–priori estimate for this problem ([7]), it turns out that $\mathbf{v} \in \left[H^{1+\beta}(\Omega_{\mathrm{S}})\right]^2$ with $\|\mathbf{v}\|_{[H^{1+\beta}(\Omega_{\mathrm{S}})]^2} \le C\left[\|\mathbf{g}\|_{[L^2(\Omega_{\mathrm{S}})]^2} + \|\operatorname{div}\mathbf{u}\|_{H^{1/2}(\Gamma_{\mathrm{I}})}\right] \le C|(\mathbf{f},\mathbf{g})|$, where $\beta \in (0,1]$ depends on the reentrant corners of $\partial\Omega_{\mathrm{S}}$, on the angles between Γ_{N} and Γ_{D} and on the Lamé coefficients λ_{S} and μ_{S}. □

This result yields the regularity of the irrotational eigenmodes.

Theorem 2.6. *The eigenfunctions* $(\mathbf{u},\mathbf{v})$ *of* $\mathbf{VP}^*$ *corresponding to eigenvalues* $\lambda \neq 1$ *belong to* $\left\{H^{\alpha,1}(\operatorname{div},\Omega_{\mathrm{F}}) \times \left[H^{1+\beta}(\Omega_{\mathrm{S}})\right]^2\right\} \cap \mathbf{G}_{\mathbf{V}}$.

Proof. Since $A_{\mathbf{V}}$ is selfadjoint and $\mathbf{K}$ is the eigenspace associated to the eigenvalue 1, for $\lambda \neq 1$, $(\mathbf{u},\mathbf{v}) \in \mathbf{K}^{\perp} = \mathbf{G}_{\mathbf{V}}$ because of Lemma 2.3. So, this theorem is an immediate consequence of applying Theorem 2.5 to $(\mathbf{f},\mathbf{g}) = \frac{1}{\lambda}(\mathbf{u},\mathbf{v})$. □

Now, we can give a characterization of the spectrum of $A_{\mathbf{V}}$.

Theorem 2.7. *The spectrum of* $A_{\mathbf{V}}$ *consists of the eigenvalue* $\lambda = 1$ *and a sequence of finite multiplicity eigenvalues* $\{\lambda_n : n \in \mathrm{N}\} \subset (0,1)$ *converging to* 0. $\mathbf{K}$ *is the eigenspace of* $\lambda = 1$ *and each eigenvector* $(\mathbf{u}_n,\mathbf{v}_n)$ *associated to* λ_n *satisfies* $\operatorname{curl}\mathbf{u}_n = 0$.

Proof. It only remains to prove that $A_{\mathbf{G}_{\mathbf{V}}}$ is compact. Now, because of Theorem 2.5, $A_{\mathbf{G}_{\mathbf{V}}} : \mathbf{G}_{\mathbf{V}} \longrightarrow \left\{H^{\alpha,1}(\operatorname{div},\Omega_{\mathrm{F}}) \times \left[H^{1+\beta}(\Omega_{\mathrm{S}})\right]^2\right\} \cap \mathbf{G}_{\mathbf{V}}$ is continuous. Then, our claim is a simple consequence of the compactness of the inclusion

$$\left\{H^{\alpha,1}(\operatorname{div},\Omega_{\mathrm{F}}) \times \left[H^{1+\beta}(\Omega_{\mathrm{S}})\right]^2\right\} \cap \mathbf{G}_{\mathbf{V}} \hookrightarrow \mathbf{G}_{\mathbf{V}}.$$

□

We conclude this section giving an analogous of Theorem 2.5 for $(\mathbf{f},\mathbf{g}) \notin \mathbf{G}$. This result will be used for the proof of Lemma 5.7 below.

Theorem 2.8. *The image of* A *is contained in* $H^{0,1}(\operatorname{div},\Omega_{\mathrm{F}}) \times \left[H^1(\Omega_{\mathrm{S}})\right]^2$ *and* $A : \mathbf{H} \longrightarrow H^{0,1}(\operatorname{div},\Omega_{\mathrm{F}}) \times \left[H^1(\Omega_{\mathrm{S}})\right]^2$ *is continuous.*

Proof. It is similar to that of Theorem 2.5 since the assumption $(\mathbf{f},\mathbf{g}) \in \mathbf{G}$ in that theorem was only used to ensure further smoothness of the solution. □

3. Finite element discretization

In spite of the fact that $A_{\mathbf{G_V}}$ is compact, it would not be convenient to reduce our analysis to $\mathbf{G_V}$ because of the difficulty of finding finite element spaces consisting of irrotational functions. So, we will deal with the non compact operator $A_{\mathbf{V}}$ through the variational problem $\mathbf{VP^*}$.

In this problem, the infinite dimensional eigenspace $\mathbf{K}$, associated to $\lambda = 1$, consists of pure rotational motions which are not physically relevant since they do not induce vibrations into the structure. However, a suitable numerical approximation should take care of them. Otherwise, spurious modes may appear. This is the case, for instance, when continuous piecewise linear finite elements are used for both, the fluid and the solid (see [8]).

Such spurious modes are eigenvalues of the discrete problem which do not approximate any eigenvalue of the continuous one. They arise as a consequence of the fact that, in this discretization, the eigenspace associated to $\lambda = 1$ is very small. Because of it, in the discretized problem, this eigenvalue splits into several spurious eigenvalues which are placed among the physical ones. A procedure to distinguish them was deviced in [8].

To avoid this drawback, we use the well known lowest order Raviart–Thomas elements for the fluid. In our approach, the eigenspaces associated to $\lambda = 1$ in the discretizations of $\mathbf{VP^*}$ have increasing dimension and yield good approximations of $\mathbf{K}$ as the meshsize becomes smaller. This fact will turn out to be highly relevant in the proofs of Section 5.

Let $\{\mathcal{T}_h\}$ be a family of regular triangulations of $\Omega_{\mathrm{F}} \cup \Omega_{\mathrm{S}}$ such that every triangle is completely contained either in Ω_{F} or in Ω_{S}. For each component of the displacements in the solid we use the standard linear finite element space

$$L_h(\Omega_{\mathrm{S}}) := \left\{ v \in H^1(\Omega_{\mathrm{S}}) \, : \, v|_T \in \mathcal{P}_1(T), \, \forall T \in \mathcal{T}_h, \, T \subset \Omega_{\mathrm{S}} \right\}$$

and, for the fluid, the Raviart–Thomas space [13]

$$\mathbf{R}_h(\Omega_{\mathrm{F}}) := \{ \mathbf{u} \in H(\mathrm{div}, \Omega_{\mathrm{F}}) \, : \, \mathbf{u}|_T \in \mathcal{R}_0(T), \, \forall T \in \mathcal{T}_h, \, T \subset \Omega_{\mathrm{F}} \},$$

where

$$\mathcal{R}_0(T) := \left\{ \mathbf{u} \in \mathcal{P}_1(T)^2 \, : \, \mathbf{u}(x, y) = (a + bx, c + by), \, a, b, c \in \mathbf{R} \right\}.$$

The degrees of freedom in $\mathbf{R}_h(\Omega_{\mathrm{F}})$ are the (constant) values of the normal component of $\mathbf{u}$ along each edge of the triangulation. Therefore, the discrete analogous of $\mathbf{X}$ is

$$\mathbf{X}_h := \left\{ (\mathbf{u}, \mathbf{v}) \in \mathbf{R}_h(\Omega_{\mathrm{F}}) \times [L_h(\Omega_{\mathrm{S}})]^2 \, : \, \mathbf{v}|_{\Gamma_{\mathrm{D}}} = \mathbf{0} \right\}.$$

As we shall see in Theorem 4.3, $\lambda = 1$ is an eigenvalue of our discrete problem with eigenspace $\mathbf{K} \cap \mathbf{X}_h$. This space contains good approximants for any function of $\mathbf{K}$ since, as it is straightforward to prove,

$$\mathbf{K} \cap \mathbf{X}_h = \left\{ (\mathbf{curl}\, \xi, \mathbf{0}) \, : \, \xi \in L_h(\Omega_{\mathrm{F}}), \, \xi|_{\Gamma_{\mathrm{I}}} = 0 \right\}. \tag{3.1}$$

The naive choice of finite element spaces to approximate $\mathbf{V}$ would be to choose the conforming ones $\mathbf{V} \cap \mathbf{X}_h$. However, this is not a suitable choice. In fact, any function of these spaces has constant normal components along each segment of Γ_{I} and, hence, only functions with this same property could be well approximated. Nevertheless, the vibration modes of the physical problem are far from having constant normal components along these segments. So, we are led to impose a weaker condition than (2.4) to define our discrete spaces. In fact, we use the following finite element spaces:

$$\mathbf{V}_h := \left\{ (\mathbf{u}, \mathbf{v}) \in \mathbf{X}_h \, : \, \int_\ell (\mathbf{u} - \mathbf{v}) \cdot \boldsymbol{\nu} = 0, \, \forall \ell \subset \Gamma_{\mathrm{I}} \right\}.$$

Let us remark that for $(\mathbf{u}, \mathbf{v}) \in \mathbf{V}_h$, $\mathbf{u} \cdot \boldsymbol{\nu}$ and $\mathbf{v} \cdot \boldsymbol{\nu}$ coincide at the middle point of each edge $\ell \subset \Gamma_{\mathrm{I}}$ but, in general, they do not coincide on the whole edge. Hence, $\mathbf{V}_h \not\subset \mathbf{V}$; that is, our method turns out to be non conforming.

As it was shown in (2.10), a^* is $\mathbf{X}$–elliptic and hence $\mathbf{V}_h$–elliptic. Then, we can define a linear operator $A_h : \mathbf{V}_h \longrightarrow \mathbf{V}_h$ such that, for $(\mathbf{u}, \mathbf{v}) \in \mathbf{V}_h$,

$$a^*(A_h(\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) = b((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})), \quad \forall (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{V}_h.$$

The spectrum of A_h furnishes the approximation of the spectrum of $A_{\mathbf{V}}$ that we are going to analize.

4. Spectral approximation

We are going to make use of the theory developed in [4] for non compact operators. This theory does not cover our case since it assumes a conforming discretization. However, a simple trick allows us to set our problem within this framework.

Since $\mathbf{V}_h \subset \mathbf{X}$, A_h can be considered as a conforming discretization of the operator $A_{\mathbf{X}} : \mathbf{X} \longrightarrow \mathbf{X}$. On the other hand, the knowledge of the spectrum of $A_{\mathbf{X}}$ gives complete information about the spectrum of $A_{\mathbf{V}}$. More precisely, we have the following lemma.

Lemma 4.1. *The spectra of $A_{\mathbf{X}}$ and $A_{\mathbf{V}}$ satisfy*

$$\sigma(A_{\mathbf{X}}) = \sigma(A_{\mathbf{V}}) \cup \{0\}.$$

Proof. Let $z \notin \sigma(A_{\mathbf{V}})$, $z \neq 0$. As we show now, $(z - A_{\mathbf{X}}) : \mathbf{X} \longrightarrow \mathbf{X}$ is one to one and onto. It is onto since, given $\mathbf{y} \in \mathbf{X}$, taking $\mathbf{x} := \frac{1}{z}\left[\mathbf{y} + (z - A_{\mathbf{V}})^{-1} A_{\mathbf{X}} \mathbf{y}\right]$ we have $(z - A_{\mathbf{X}})\mathbf{x} = \mathbf{y}$. On the other hand, if $(z - A_{\mathbf{X}})\mathbf{x} = 0$, then $\mathbf{x} = \frac{1}{z} A_{\mathbf{X}} \mathbf{x} \in \mathbf{V}$ and so $(z - A_{\mathbf{V}})\mathbf{x} = (z - A_{\mathbf{X}})\mathbf{x} = 0$. Since $z \notin \sigma(A_{\mathbf{V}})$, then $\mathbf{x} = \mathbf{0}$. Hence $(z - A_{\mathbf{X}})$ is one to one. Therefore, because of the open mapping theorem, $z \notin \sigma(A_{\mathbf{X}})$.

Conversely, let $z \notin \sigma(A_{\mathbf{X}})$. Firstly, $z \neq 0$ since $A_{\mathbf{X}}(\mathbf{X}) \subset \mathbf{V}$ and so $A_{\mathbf{X}}$ is not onto. Secondly, given $\mathbf{y} \in \mathbf{V}$, there exists a unique $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{y} = (z - A_{\mathbf{X}})\mathbf{x}$. Moreover, $\mathbf{x} = \frac{1}{z}(\mathbf{y} + A_{\mathbf{X}}\mathbf{x}) \in \mathbf{V}$. Hence, this is the unique $\mathbf{x} \in \mathbf{V}$ such that $(z - A_{\mathbf{V}})\mathbf{x} = (z - A_{\mathbf{X}})\mathbf{x} = \mathbf{y}$. Therefore, $(z - A_{\mathbf{V}}) : \mathbf{V} \longrightarrow \mathbf{V}$ is invertible and, as before, $z \notin \sigma(A_{\mathbf{V}})$. $\square$

From Theorem 2.7 and this lemma we know that the spectrum of $A_{\mathbf{X}}$ consists of a sequence of eigenvalues $\lambda_n \subset (0, 1)$, $\lambda = 1$ and $\lambda = 0$ (the latter being no relevant in our problem). As we claimed above, the eigenspace associated to $\lambda = 1$ is well represented in our discretization. In fact, we have the following theorem.

Theorem 4.2. *$\lambda = 1$ is an eigenvalue of A_h and its eigenspace is $\mathbf{K}\cap\mathbf{X}_h = \mathbf{K}\cap\mathbf{V}_h$.*

Proof. From (3.1) it is immediate to see that $\mathbf{K}\cap\mathbf{X}_h$ is contained in $\mathbf{V}_h$. Now, let $(\mathbf{u},\mathbf{v}) \in \mathbf{V}_h$ such that $A_h(\mathbf{u},\mathbf{v}) = (\mathbf{u},\mathbf{v})$. It satisfies $a((\mathbf{u},\mathbf{v}),(\boldsymbol{\phi},\boldsymbol{\psi})) = 0$ for all $(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathbf{V}_h$. In particular $a((\mathbf{u},\mathbf{v}),(\mathbf{u},\mathbf{v})) = 0$ and hence $\operatorname{div}\mathbf{u} = 0$ and $\mathbf{v} = \mathbf{0}$. So, $\mathbf{u} = \mathbf{curl}\,\xi$ with $\xi \in H^1(\Omega_{\mathrm{F}})$. Since $\mathbf{u} \in \mathbf{R}_h$, then $\mathbf{curl}\,\xi\cdot\boldsymbol{\nu}|_\ell$ is constant for every $\ell \subset \Gamma_{\mathrm{I}}$ and, since $(\mathbf{u},\mathbf{v}) \in \mathbf{V}_h$ and $\mathbf{v} = \mathbf{0}$, then $\left.\frac{\partial\xi}{\partial\tau}\right|_\ell = \mathbf{curl}\,\xi\cdot\boldsymbol{\nu}|_\ell = 0$ and so ξ can be chosen in $H_0^1(\Omega_{\mathrm{F}})$. On the other hand, $\operatorname{div}\mathbf{u} = 0$ implies that, for each $T \subset \Omega_{\mathrm{F}}$, $\mathbf{u}|_T \in [\mathcal{P}_0(T)]^2$ and then $\xi \in L_h(\Omega_{\mathrm{F}})$. Therefore $(\mathbf{u},\mathbf{v}) = (\mathbf{curl}\,\xi,\mathbf{0}) \in \mathbf{K}\cap\mathbf{X}_h$ according to (3.1).

Conversely, because of (3.1) again, every $(\mathbf{u},\mathbf{v}) \in \mathbf{K}\cap\mathbf{X}_h$ is clearly an eigenvector of A_h associated to $\lambda = 1$. □

Let us denote $\|A_{\mathbf{X}} - A_h\|_h := \sup_{\{\mathbf{x}\in\mathbf{V}_h\,:\,\|\mathbf{x}\|=1\}} \|(A_{\mathbf{X}} - A_h)\mathbf{x}\|$ and, for $\mathbf{x} \in \mathbf{X}$ and $\mathbf{E}$ a subspace of $\mathbf{X}$, $\operatorname{dist}(\mathbf{x},\mathbf{E}) := \inf_{\{\mathbf{y}\in\mathbf{E}\}} \|\mathbf{x}-\mathbf{y}\|$. Let $\alpha \in \left(\frac{1}{2},1\right]$ and $\beta \in (0,1]$ be the constants arising in Theorem 2.5 and let $\gamma := \min\{\alpha,\beta\}$. In the remainder of this section, we assume the two following properties which will be proved below:

P1. *For each eigenfunction $\mathbf{x}$ of $A_{\mathbf{X}}$ associated to $\lambda \in (0,1)$ with $\|\mathbf{x}\| = 1$, there exist strictly positive constants C and h_0 such that, if $h \le h_0$, then*

$$\operatorname{dist}(\mathbf{x},\mathbf{V}_h) \le Ch^\gamma.$$

P2. *There exist strictly positive constants C and h_0 such that, if $h \le h_0$, then*

$$\|A_{\mathbf{X}} - A_h\|_h \le Ch^\gamma.$$

Theorems 4.3 and 4.4 below are specializations of the theory in [4] to our case, where the spectrum $\sigma(A_{\mathbf{X}})$ is discrete, real and positive, with $\lambda = 0$ as the only accumulation point. Let us remark that, since A_h is selfadjoint, $\sigma(A_h)$ also consists of real positive eigenvalues.

The next theorem shows that there are not spurious eigenvalues for h small enough.

Theorem 4.3. *Let J be a closed interval such that $J\cap\sigma(A_{\mathbf{X}}) = \varnothing$. There exists $h_J > 0$ such that, if $h \le h_J$, then $J\cap\sigma(A_h) = \varnothing$.*

Proof. See [4]. □

Now, we are going to prove that the eigenvalues in $(0,1)$ and their eigenvectors are well approximated in our discretization. For any open interval $I := (a,b) \subset (0,1)$ with $a,b \notin \sigma(A_{\mathbf{X}})$, we denote $\mathbf{E}_I$ the direct sum of the eigenspaces of $A_{\mathbf{X}}$ associated to its eigenvalues $\lambda \in I$. Let $\mathbf{E}_I^h$ denote the analogous for A_h. The theorem below gives bounds for the distance between $\mathbf{E}_I$ and $\mathbf{E}_I^h$.

Theorem 4.4. *There exist strictly positive constants C and h_I such that, if $h \le h_I$, then:*

i) for each $\mathbf{x} \in \mathbf{E}_I^h$ with $\|\mathbf{x}\| = 1$,

$$\operatorname{dist}(\mathbf{x},\mathbf{E}_I) \le C\,\|A_{\mathbf{X}} - A_h\|_h \le Ch^\gamma; \tag{4.1}$$

ii) for each $\mathbf{x} \in \mathbf{E}_I$ *with* $\|\mathbf{x}\| = 1$,

$$\operatorname{dist}(\mathbf{x}, \mathbf{E}_I^h) \leq C\,[\operatorname{dist}(\mathbf{x}, \mathbf{V}_h) + \|A_{\mathbf{X}} - A_h\|_h] \leq Ch^\gamma. \tag{4.2}$$

Proof. It follows by combining the proofs in [4] with **P1** and **P2**. □

As a consequence of Theorem 4.4, we may assert that if $I \cap \sigma(A_{\mathbf{X}}) = \{\lambda\}$, then $\dim \mathbf{E}_I^h = \dim \mathbf{E}_I (=: n)$ for h small enough. This last property implies the convergence to λ of the (not necessarily different) eigenvalues of the discrete problem $\lambda_1^h, \ldots, \lambda_n^h$. We cannot use directly the theory in [4] to obtain error estimates because $A_{\mathbf{X}}$ is not selfadjoint. However, we are going to show that a slight variation of this theory works in our case. Let us remark that the theory for non conforming methods in [11] does not either apply to our case since it assumes the operator to be compact.

Theorem 4.5. *There exist strictly positive constants* C *and* h_I *such that, if* $h \leq h_I$, *then*

i)
$$\left| \lambda - \frac{1}{n} \sum_{i=1}^{n} \lambda_i^h \right| \leq Ch^\gamma;$$

ii)
$$\left| \frac{1}{\lambda} - \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda_i^h} \right| \leq Ch^\gamma;$$

iii)
$$\max_{1 \leq i \leq n} \left| \lambda - \lambda_i^h \right| \leq Ch^\gamma.$$

Proof. Let $\Pi_h : \mathbf{X} \longrightarrow \mathbf{E}_I^h$ be the $a^*(\cdot,\cdot)$ orthogonal projection and let Λ_h be its restriction to $\mathbf{E}_I$. Since a^* defines a norm equivalent to $\|\cdot\|$, by using (4.2) we have

$$\|\mathbf{x} - \Lambda_h \mathbf{x}\| \leq C \operatorname{dist}(\mathbf{x}, \mathbf{E}_i^h) \leq Ch^\gamma \|\mathbf{x}\| \tag{4.3}$$

for $\mathbf{x} \in \mathbf{E}_I$ and h small enough. Furthermore, for h such that $Ch^\gamma \leq \frac{1}{2}$, $\|\mathbf{x} - \Lambda_h \mathbf{x}\| \leq \frac{1}{2}\|\mathbf{x}\|$ and hence

$$\|\Lambda_h \mathbf{x}\| \geq \|\mathbf{x}\| - \|\mathbf{x} - \Lambda_h \mathbf{x}\| \geq \frac{1}{2}\|\mathbf{x}\|.$$

So, since $\dim \mathbf{E}_I = \dim \mathbf{E}_I^h$, Λ_h is a bijection and its inverse satisfies $\|\Lambda_h^{-1}\| \leq 2$.

Let $A_{\mathbf{E}_I} := A_{\mathbf{X}}|_{\mathbf{E}_I} : \mathbf{E}_I \longrightarrow \mathbf{E}_I$ and $B_h := \Lambda_h^{-1} A_h \Lambda_h : \mathbf{E}_I \longrightarrow \mathbf{E}_I$. For $\mathbf{x} \in \mathbf{E}_I$ we have

$$\|(A_{\mathbf{E}_I} - B_h)\mathbf{x}\| \leq \|A_{\mathbf{X}}(I - \Lambda_h)\mathbf{x}\| + \|(A_{\mathbf{X}} - A_h)\Lambda_h \mathbf{x}\| + \|(I - \Lambda_h^{-1})A_h \Lambda_h \mathbf{x}\|.$$

Now, for h small enough, by using (4.3) we have

$$\|A_{\mathbf{X}}(I - \Lambda_h)\mathbf{x}\| \leq \|A_{\mathbf{X}}\|\, Ch^\gamma \|\mathbf{x}\|,$$

by using **P2**,

$$\|(A_{\mathbf{X}} - A_h)\Lambda_h \mathbf{x}\| \le \|A_{\mathbf{X}} - A_h\|_h \, \|\Lambda_h \mathbf{x}\| \le Ch^\gamma \|\mathbf{x}\|$$

and, from (4.3) again and since $\|\Lambda_h^{-1}\| \le 2$,

$$\|(I - \Lambda_h^{-1}) A_h \Lambda_h \mathbf{x}\| = \|(I - \Lambda_h)\Lambda_h^{-1} A_h \Lambda_h \mathbf{x}\| \le Ch^\gamma \|A_h\| \|\mathbf{x}\|.$$

Then

$$\|A_{\mathbf{E}_I} - B_h\| = \sup_{\mathbf{x}\in\mathbf{E}_I \,:\, \|\mathbf{x}\|=1} \|(A_{\mathbf{E}_I} - B_h)\mathbf{x}\| \le Ch^\gamma.$$

Finally, taking into account that $A_{\mathbf{E}_I} = \lambda I$ and that $\lambda_1^h, \dots, \lambda_n^h$ are the eigenvalues of B_h, (i) follows from the continuity of the traces:

$$\left|\lambda - \frac{1}{n}\sum_{i=1}^{n} \lambda_i^h\right| = \frac{1}{n} \left|\operatorname{tr}(A_{\mathbf{E}_I}) - \operatorname{tr}(B_h)\right| \le C\|A_{\mathbf{E}_I} - B_h\| \le Ch^\gamma;$$

(ii) follows from the fact that for f an analytic function on I,

$$\|f(A_{\mathbf{E}_I}) - f(B_h)\| \le C\|A_{\mathbf{E}_I} - B_h\|$$

and (iii) can be deduced as in [14]. □

Claim (ii) in Theorem 4.4 shows that any eigenvector of the continuous problem, corresponding to the relevant eigenvalues $\lambda \in (0,1)$, can be approximated with an error of order h^γ. Claim (i) shows that all the eigenvectors of the discrete problem are approximants of those of the continuous one if h is small enough. Theorems 4.5 and 4.3 give analogous results for the eigenvalues. These theorems are valid for any discretization satisfying **P1** and **P2**. In the following section we prove these properties for our method.

5. Properties of the approximation

5.1 Property P1. To prove this property we introduce a $\mathbf{V}_h$–interpolant. Let $I_h : \left\{H^{\alpha,1}(\operatorname{div},\Omega_{\mathrm{F}}) \times \left[H^{1+\beta}(\Omega_{\mathrm{S}})\right]^2\right\} \cap \mathbf{V} \longrightarrow \mathbf{V}_h$ be defined in the following way:

$$I_h(\mathbf{u},\mathbf{v})|_T := \begin{cases} (\boldsymbol{L}\mathbf{v})|_T, & \text{if } T \subset \Omega_{\mathrm{S}} \\ (\boldsymbol{R}\mathbf{u})|_T, & \text{if } T \subset \Omega_{\mathrm{F}} \text{ and } \partial T \cap \Gamma_{\mathrm{I}} = \varnothing \\ (\widetilde{\boldsymbol{R}}\mathbf{u})\Big|_T, & \text{if } T \subset \Omega_{\mathrm{F}} \text{ and } \partial T \cap \Gamma_{\mathrm{I}} \neq \varnothing \end{cases}$$

where $\boldsymbol{L}\mathbf{v}$ is the Lagrange interpolant of $\mathbf{v}$ in $[L_h(\Omega_{\mathrm{S}})]^2$, $\boldsymbol{R}\mathbf{u}$ is the standard Raviart–Thomas interpolant of $\mathbf{u}$ in $\mathbf{R}_h(\Omega_{\mathrm{F}})$ and $(\widetilde{\boldsymbol{R}}\mathbf{u})\Big|_T$ is the function in $\mathcal{R}_0(T)$ with degrees of freedom (normal components) given by

$$(\widetilde{\boldsymbol{R}}\mathbf{u})\Big|_\ell \cdot \boldsymbol{\nu} := \begin{cases} (\boldsymbol{R}\mathbf{u})|_\ell \cdot \boldsymbol{\nu}, & \text{if } \ell \not\subset \Gamma_{\mathrm{I}} \\ \frac{1}{|\ell|}\int_\ell (\boldsymbol{L}\mathbf{v})|_{T_\ell} \cdot \boldsymbol{\nu}, & \text{if } \ell \subset \Gamma_{\mathrm{I}} \end{cases}$$

with T_ℓ the triangle contained in Ω_{S} such that $\partial T \cap \partial T_\ell = \ell$. With this definition we ensure $I_h(\mathbf{u},\mathbf{v}) \in \mathbf{V}_h$.

In the following lemma we give a bound for $\|\boldsymbol{R}\mathbf{u} - \widetilde{\boldsymbol{R}}\mathbf{u}\|_{H(\operatorname{div},T)}$ for those triangles where $\widetilde{\boldsymbol{R}}$ is defined. For the sake of simplicity we assume that T has only one edge ℓ contained in Γ_{I}, but a similar result is true in general with obvious modifications.

Lemma 5.1. *There exists $C > 0$ (only depending on the regularity of T and T_ℓ) such that, for each* $(\mathbf{u}, \mathbf{v}) \in \left\{H^{\alpha,1}(\mathrm{div}, \Omega_{\mathrm{F}}) \times \left[H^{1+\beta}(\Omega_{\mathrm{S}})\right]^2\right\} \cap \mathbf{V}$,

$$\|\boldsymbol{R}\mathbf{u} - \widetilde{\boldsymbol{R}}\mathbf{u}\|_{H(\mathrm{div},T)} \leq Ch^\beta \|\mathbf{v}\|_{[H^{1+\beta}(T_\ell)]^2}.$$

Proof. Let φ_ℓ be the standard basis function of $\mathcal{R}_0(T)$ associated to ℓ.

$$(\boldsymbol{R}\mathbf{u} - \widetilde{\boldsymbol{R}}\mathbf{u})\Big|_T = \left[\frac{1}{|\ell|}\int_\ell (\mathbf{u} - \boldsymbol{L}\mathbf{v})\cdot\boldsymbol{\nu}\right]\varphi_\ell = \left[\frac{1}{|\ell|}\int_\ell (\mathbf{v} - \boldsymbol{L}\mathbf{v})\cdot\boldsymbol{\nu}\right]\varphi_\ell.$$

Hence,

$$\|\boldsymbol{R}\mathbf{u} - \widetilde{\boldsymbol{R}}\mathbf{u}\|_{H(\mathrm{div},T)} = \left|\frac{1}{|\ell|}\int_\ell (\mathbf{v} - \boldsymbol{L}\mathbf{v})\cdot\boldsymbol{\nu}\right| \|\varphi_\ell\|_{H(\mathrm{div},T)}.$$

Now,

$$\int_T |\operatorname{div}\varphi_\ell|^2 = \frac{1}{|T|}\left|\int_T \operatorname{div}\varphi_\ell\right|^2 = \frac{1}{|T|}\left|\int_{\partial T}\varphi_\ell\cdot\boldsymbol{\nu}_T\right|^2 = \frac{|\ell|^2}{|T|}$$

and, by changing coordinates to a reference element, it can be seen that $\int_\ell \varphi_\ell^2 = \frac{|T|}{6}$. Therefore, $\|\varphi_\ell\|_{H(\mathrm{div},T)} \leq C$ for a constant C only depending on the regularity of T. On the other hand, by using a suitable trace theorem and standard interpolation results for Sobolev spaces of fractional order ([5]), we have

$$\begin{aligned}
\left|\frac{1}{|\ell|}\int_\ell (\mathbf{v} - \boldsymbol{L}\mathbf{v})\cdot\boldsymbol{\nu}\right| &\leq \frac{1}{|\ell|^{1/2}}\|\mathbf{v} - \boldsymbol{L}\mathbf{v}\|_{[L^2(\ell)]^2} \\
&\leq \frac{C}{|\ell|^{1/2}}\left\{h^{-1/2}|\mathbf{v} - \boldsymbol{L}\mathbf{v}|_{[L^2(T_\ell)]^2} + h^{1/2}|\mathbf{v} - \boldsymbol{L}\mathbf{v}|_{[H^1(T_\ell)]^2}\right\} \\
&\leq \frac{C}{|\ell|^{1/2}}h^{1/2+\beta}\|\mathbf{v}\|_{[H^{1+\beta}(T_\ell)]^2} \\
&\leq Ch^\beta\|\mathbf{v}\|_{[H^{1+\beta}(T_\ell)]^2},
\end{aligned}$$

for C a constant only depending on the regularity of T_ℓ. All together, we conclude the lemma. □

The following theorem gives an error estimate for the interpolant I_h.

Theorem 5.2. *There exists a strictly positive constant C such that, for all* $(\mathbf{u}, \mathbf{v}) \in H^{\alpha,1}(\mathrm{div}, \Omega_{\mathrm{F}}) \times \left[H^{1+\beta}(\Omega_{\mathrm{S}})\right]^2$,

$$\|(\mathbf{u}, \mathbf{v}) - I_h(\mathbf{u}, \mathbf{v})\| \leq Ch^\gamma\left\{\|\mathbf{u}\|_{H^{\alpha,1}(\mathrm{div},\Omega_{\mathrm{F}})} + \|\mathbf{v}\|_{[H^{1+\beta}(\Omega_{\mathrm{S}})]^2}\right\}. \tag{5.1}$$

Proof. By using the interpolation results in [5] we obtain

$$\|\mathbf{v} - \boldsymbol{L}\mathbf{v}\|_{[H^1(T)]^2} \leq Ch^\beta\|\mathbf{v}\|_{[H^{1+\beta}(T)]^2}, \qquad \text{for } T \subset \Omega_{\mathrm{S}}, \tag{5.2}$$

and

$$\|\mathbf{u}-\boldsymbol{R}\mathbf{u}\|_{H(\mathrm{div},T)} \leq \boldsymbol{C} h^{\alpha}\|\mathbf{u}\|_{H^{\alpha,1}(\mathrm{div},T)}, \qquad \text{for } T \subset \Omega_{\mathrm{F}}. \tag{5.3}$$

From (5.3) and Lemma 5.1 we have, for $T \subset \Omega_{\mathrm{F}}$ such that $\partial T \cap \Gamma_{\mathrm{I}} \neq \varnothing$,

$$\begin{aligned}\|\mathbf{u}-\widetilde{\boldsymbol{R}}\mathbf{u}\|_{H(\mathrm{div},T)} &\leq \|\mathbf{u}-\boldsymbol{R}\mathbf{u}\|_{H(\mathrm{div},T)} + \|\boldsymbol{R}\mathbf{u}-\widetilde{\boldsymbol{R}}\mathbf{u}\|_{H(\mathrm{div},T)} \\ &\leq \boldsymbol{C}\left\{ h^{\alpha}\|\mathbf{u}\|_{H^{\alpha,1}(\mathrm{div},T)} + \sum_{\substack{T'\subset\Omega_{\mathrm{S}}\\ T'\cap T\neq\emptyset}} h^{\beta}\|\mathbf{v}\|_{[H^{1+\beta}(T')]^2}\right\}.\end{aligned} \tag{5.4}$$

So, by using (5.2), (5.3) and (5.4), we conclude the theorem. □

Now, **P1** is a simple corollary of Theorem 5.2.

Theorem 5.3. *(**P1**) For each eigenfunction $(\mathbf{u},\mathbf{v})$ of $A_{\mathbf{X}}$ associated to $\lambda \in (0,1)$ with $\|(\mathbf{u},\mathbf{v})\| = 1$, there exist strictly positive constants $\boldsymbol{C}$ and h_0 such that, if $h \leq h_0$, then*

$$\mathrm{dist}\left((\mathbf{u},\mathbf{v}),\mathbf{V}_h\right) \leq \boldsymbol{C} h^{\gamma}.$$

Proof. Since $A_{\mathbf{X}}(\mathbf{u},\mathbf{v}) = \lambda(\mathbf{u},\mathbf{v})$, by applying Theorems 2.6 and 2.5, we obtain

$$\left\{\|\mathbf{u}\|_{H^{\alpha,1}(\mathrm{div},\Omega_{\mathrm{F}})} + \|\mathbf{v}\|_{[H^{1+\beta}(\Omega_{\mathrm{S}})]^2}\right\} \leq \frac{\boldsymbol{C}}{\lambda}|(\mathbf{u},\mathbf{v})| \leq \frac{\boldsymbol{C}}{\lambda}.$$

Hence, the theorem follows from (5.1). □

5.2 Property P2. It only remains to prove the following theorem.

Theorem 5.4. *(**P2**) There exist strictly positive constants $\boldsymbol{C}$ and h_0 such that, if $h \leq h_0$, then*

$$\|A_{\mathbf{X}} - A_h\|_h \leq \boldsymbol{C} h^{\gamma}.$$

Proof. Let $\mathbf{S}_h$ be the orthogonal complement in $\mathbf{V}_h$ of $\mathbf{K}\cap\mathbf{X}_h = \mathbf{K}\cap\mathbf{V}_h$. Because of (3.1), we know that

$$\mathbf{S}_h = \left\{(\mathbf{u},\mathbf{v}) \in \mathbf{V}_h : \int_{\Omega_{\mathrm{F}}} \mathbf{u}\cdot\mathbf{curl}\,\xi = 0,\ \forall\xi \in L_h(\Omega_{\mathrm{F}}),\ \xi|_{\Gamma_{\mathrm{I}}} = 0\right\}.$$

Since $A_{\mathbf{X}}|_{\mathbf{K}\cap\mathbf{X}_h} = A_h|_{\mathbf{K}\cap\mathbf{X}_h}$ (in fact, both are the identity on $\mathbf{K}\cap\mathbf{X}_h$), we have

$$\|A_{\mathbf{X}} - A_h\|_h = \sup_{\substack{(\mathbf{u},\mathbf{v})\in\mathbf{S}_h\\ \|(\mathbf{u},\mathbf{v})\|=1}} \|(A_{\mathbf{X}} - A_h)(\mathbf{u},\mathbf{v})\|$$

Now, let $(\mathbf{u},\mathbf{v}) \in \mathbf{S}_h$ with $\|(\mathbf{u},\mathbf{v})\| = 1$. We can write the orthogonal decomposition

$$(\mathbf{u},\mathbf{v}) = (\,\mathbf{curl}\,\zeta, 0) + (\nabla q, \mathbf{v}) \tag{5.5}$$

with $\zeta \in H^1(\Omega_{\mathrm{F}})$ and $q \in H^{1+\alpha}(\Omega_{\mathrm{F}})$. In fact, we can take q as a solution of the Neumann problem

$$\begin{cases} \Delta q = \operatorname{div} \mathbf{u}, & \text{in } \Omega_{\mathrm{F}} \\ \frac{\partial q}{\partial \nu} = \mathbf{v} \cdot \boldsymbol{\nu}, & \text{on } \Gamma_{\mathrm{I}} \end{cases}$$

This problem is compatible since for $(\mathbf{u}, \mathbf{v}) \in \mathbf{V}_h$, $\int_\ell \mathbf{u} \cdot \boldsymbol{\nu} = \int_\ell \mathbf{v} \cdot \boldsymbol{\nu}$ for any edge $\ell \subset \Gamma_{\mathrm{I}}$. The a–priori estimate for this Neumann problem ([7]) shows that $q \in H^{1+\alpha}(\Omega_{\mathrm{F}})$ and

$$\|q\|_{H^{1+\alpha}(\Omega_{\mathrm{F}})} \leq C \left[\|\mathbf{v} \cdot \boldsymbol{\nu}\|_{H^{1/2}(\Gamma_{\mathrm{I}})} + \|\operatorname{div} \mathbf{u}\|_{L^2(\Omega_{\mathrm{F}})} \right] \leq C \|(\mathbf{u}, \mathbf{v})\| = C. \qquad (5.6)$$

Now, let $\Pi_h : \mathbf{X} \longrightarrow \mathbf{V}_h$ be the orthogonal projection in $a^*(\cdot, \cdot)$. We have

$$\begin{aligned} \|(A_{\mathbf{X}} - A_h)(\mathbf{u}, \mathbf{v})\| \leq \|(I - \Pi_h) A_{\mathbf{X}}(\operatorname{\mathbf{curl}} \zeta, \mathbf{0})\| + \|(I - \Pi_h) A_{\mathbf{X}}(\nabla q, \mathbf{v})\| \\ + \|(\Pi_h A_{\mathbf{X}} - A_h)(\mathbf{u}, \mathbf{v})\|. \end{aligned}$$

In the following lemmas we prove that each one of the three terms in the r.h.s. is bounded by $C h^\gamma$, concluding therefore the Theorem. □

Lemma 5.5. *Under the assumptions and with the notation of Theorem 5.4,*

$$\|(I - \Pi_h) A_{\mathbf{X}}(\operatorname{\mathbf{curl}} \zeta, \mathbf{0})\| \leq C h^\alpha.$$

Proof. Since Π_h is the projection in the norm $a^*(\cdot, \cdot)^{1/2}$, which is equivalent to $\|\cdot\|$, we have

$$\begin{aligned} \|(I - \Pi_h) A_{\mathbf{X}}(\operatorname{\mathbf{curl}} \zeta, \mathbf{0})\| &\leq C\, a^*(A_{\mathbf{X}}(\operatorname{\mathbf{curl}} \zeta, \mathbf{0}), A_{\mathbf{X}}(\operatorname{\mathbf{curl}} \zeta, \mathbf{0}))^{1/2} \\ &\leq C \sup_{\substack{(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{V} \\ \|(\boldsymbol{\phi}, \boldsymbol{\psi})\| = 1}} a^*(A_{\mathbf{X}}(\operatorname{\mathbf{curl}} \zeta, \mathbf{0}), (\boldsymbol{\phi}, \boldsymbol{\psi})) \\ &= C \sup_{\substack{(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{V} \\ \|(\boldsymbol{\phi}, \boldsymbol{\psi})\| = 1}} b((\operatorname{\mathbf{curl}} \zeta, \mathbf{0}), (\boldsymbol{\phi}, \boldsymbol{\psi})) \\ &\leq C \|\operatorname{\mathbf{curl}} \zeta\|_{[L^2(\Omega_{\mathrm{F}})]^2} \end{aligned} \qquad (5.7)$$

Now, from (5.5) we can write

$$\|\operatorname{\mathbf{curl}} \zeta\|^2_{[L^2(\Omega_{\mathrm{F}})]^2} = \int_{\Omega_{\mathrm{F}}} (\nabla q - \mathbf{u}) \cdot (\nabla q - \mathbf{R}(\nabla q)) + \int_{\Omega_{\mathrm{F}}} (\nabla q - \mathbf{u}) \cdot (\mathbf{R}(\nabla q) - \mathbf{u}), \quad (5.8)$$

where $\mathbf{R}$ is the Raviart–Thomas interpolant as above. Therefore, $(\mathbf{R}(\nabla q) - \mathbf{u}) \in \mathbf{R}_h(\Omega_{\mathrm{F}})$ and so, for every $T \subset \Omega_{\mathrm{F}}$,

$$\begin{aligned} \operatorname{div} (\mathbf{R}(\nabla q) - \mathbf{u})|_T &= \frac{1}{|T|} \int_T \operatorname{div} (\mathbf{R}(\nabla q) - \mathbf{u}) = \frac{1}{|T|} \int_{\partial T} (\mathbf{R}(\nabla q) - \mathbf{u}) \cdot \boldsymbol{\nu}_T \\ &= \frac{1}{|T|} \int_{\partial T} (\nabla q - \mathbf{u}) \cdot \boldsymbol{\nu}_T = \frac{1}{|T|} \int_T \operatorname{div} (\nabla q - \mathbf{u}) = 0 \end{aligned}$$

and, for every edge $\ell \subset \Gamma_{\mathrm{I}}$,

$$(\mathbf{R}(\nabla q) - \mathbf{u})|_\ell \cdot \boldsymbol{\nu} = \frac{1}{|\ell|}\int_\ell (\mathbf{R}(\nabla q) - \mathbf{u}) \cdot \boldsymbol{\nu} = \frac{1}{|\ell|}\int_\ell (\nabla q \cdot \boldsymbol{\nu} - \mathbf{v} \cdot \boldsymbol{\nu}) = 0.$$

So, $((\mathbf{R}(\nabla q) - \mathbf{u}), \mathbf{0}) \in \mathbf{K} \cap \mathbf{X}_h$ and hence, from (3.1), there exists $\xi \in L_h(\Omega_{\mathrm{F}})$ with $\xi|_{\Gamma_{\mathrm{I}}} = 0$ such that $(\mathbf{R}(\nabla q) - \mathbf{u}) = \mathbf{curl}\,\xi$. Therefore, since $(\mathbf{u}, \mathbf{v}) \in \mathbf{S}_h$,

$$\int_{\Omega_{\mathrm{F}}} (\nabla q - \mathbf{u}) \cdot (\mathbf{R}(\nabla q) - \mathbf{u}) = \int_{\Omega_{\mathrm{F}}} (\nabla q - \mathbf{u}) \cdot \mathbf{curl}\,\xi = \int_{\Gamma_{\mathrm{I}}} q \frac{\partial \xi}{\partial \boldsymbol{\tau}} = 0.$$

Now, by using this equality in (5.8), the error estimate for the $\mathbf{R}_h$–interpolation, (5,5) and the a–priori estimate (5.6), we obtain

$$\begin{aligned}
\|\,\mathbf{curl}\,\zeta\|^2_{[L^2(\Omega_{\mathrm{F}})]^2} &\le \|\nabla q - \mathbf{u}\|_{[L^2(\Omega_{\mathrm{F}})]^2}\, \|\nabla q - \mathbf{R}(\nabla q)\|_{[L^2(\Omega_{\mathrm{F}})]^2} \\
&\le \|\,\mathbf{curl}\,\zeta\|_{[L^2(\Omega_{\mathrm{F}})]^2}\, C h^\alpha\, \|\nabla q\|_{[H^\alpha(\Omega_{\mathrm{F}})]^2} \\
&\le C h^\alpha\, \|\,\mathbf{curl}\,\zeta\|_{[L^2(\Omega_{\mathrm{F}})]^2}.
\end{aligned}$$

Finally, from this inequality and (5.7) we conclude the lemma. □

Lemma 5.6. *Under the assumptions and with the notation of Theorem 5.4,*

$$\|(I - \Pi_h) A_{\mathbf{X}}(\nabla q, \mathbf{v})\| \le C h^\gamma.$$

Proof. Let $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) := A_{\mathbf{X}}(\nabla q, \mathbf{v})$. Since $(\nabla q, \mathbf{v}) \in \mathbf{G}$, then, applying Theorem 2.5, it turns out that $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \in \mathbf{V} \cap \left\{ H^{\alpha,1}(\mathrm{div}, \Omega_{\mathrm{F}}) \times \left[H^{1+\beta}(\Omega_{\mathrm{S}})\right]^2 \right\}$ and $\|\tilde{\mathbf{u}}\|_{H^{\alpha,1}(\mathrm{div},\Omega_{\mathrm{F}})} + \|\tilde{\mathbf{v}}\|_{[H^{1+\beta}(\Omega_{\mathrm{S}})]^2} \le C|(\nabla q, \mathbf{v})| \le C\|(\mathbf{u}, \mathbf{v})\| = C$, the last inequality being true because of (5.6). Hence, by using Theorem 5.2 and the equivalence between $a^*(\cdot,\cdot)^{1/2}$ and $\|\cdot\|$, we obtain

$$\begin{aligned}
\|(I - \Pi_h) A_{\mathbf{X}}(\nabla q, \mathbf{v})\| &\le C\|(I - I_h)(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})\| \\
&\le C h^\gamma \left\{ \|\tilde{\mathbf{u}}\|_{H^{\alpha,1}(\mathrm{div},\Omega_{\mathrm{F}})} + \|\tilde{\mathbf{v}}\|_{[H^{1+\beta}(\Omega_{\mathrm{S}})]^2} \right\} \le C h^\gamma.
\end{aligned}$$

□

Lemma 5.7. *Under the assumptions and with the notation of Theorem 5.4,*

$$\|(\Pi_h A_{\mathbf{X}} - A_h)(\mathbf{u}, \mathbf{v})\| \le C h.$$

Proof. Since $(\Pi_h A_{\mathbf{X}} - A_h)(\mathbf{u}, \mathbf{v}) \in \mathbf{V}_h$ and $a^*(\cdot,\cdot)^{1/2}$ is equivalent to $\|\cdot\|$, we have

$$\begin{aligned}
\|(\Pi_h A_{\mathbf{X}} - A_h)(\mathbf{u}, \mathbf{v})\| &\le C \sup_{\substack{(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathbf{V}_h \\ \|(\boldsymbol{\phi},\boldsymbol{\psi})\| = 1}} a^*((\Pi_h A_{\mathbf{X}} - A_h)(\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) \\
&= C \sup_{\substack{(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathbf{V}_h \\ \|(\boldsymbol{\phi},\boldsymbol{\psi})\| = 1}} \left[a^*(\Pi_h A_{\mathbf{X}}(\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) - b((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) \right].
\end{aligned} \tag{5.9}$$

Let $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = A_{\mathbf{X}}(\mathbf{u}, \mathbf{v})$ as in the previous lemma. From Theorem 2.8 we know that $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \in H^{0,1}(\mathrm{div}, \Omega_{\mathrm{F}}) \times \left[H^1(\Omega_{\mathrm{S}})\right]^2$ and

$$\left\{ \|\tilde{\mathbf{u}}\|_{H^{0,1}(\mathrm{div},\Omega_{\mathrm{F}})} + \|\tilde{\mathbf{v}}\|_{[H^1(\Omega_{\mathrm{S}})]^2} \right\} \leq C \|(\mathbf{u}, \mathbf{v})\| = C. \tag{5.10}$$

Arguing as in Theorem 2.1, it can be shown that $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ is the solution of the strong problem

$$\begin{aligned}
&- c^2 \nabla(\,\mathrm{div}\,\tilde{\mathbf{u}}) + \tilde{\mathbf{u}} = \mathbf{u}, && \text{in } \Omega_{\mathrm{F}} \\
&- \mathcal{L}(\tilde{\mathbf{v}}) + \rho_{\mathrm{S}} \tilde{\mathbf{v}} = \rho_{\mathrm{S}} \mathbf{v}, && \text{in } \Omega_{\mathrm{F}} \\
&\boldsymbol{\sigma}(\tilde{\mathbf{v}})\boldsymbol{\nu} - (\rho_{\mathrm{F}} c^2 \,\mathrm{div}\,\tilde{\mathbf{u}})\boldsymbol{\nu} = \mathbf{0}, && \text{on } \Gamma_{\mathrm{I}} \\
&(\tilde{\mathbf{u}} - \tilde{\mathbf{v}}) \cdot \boldsymbol{\nu} = 0, && \text{on } \Gamma_{\mathrm{I}} \\
&\boldsymbol{\sigma}(\tilde{\mathbf{v}})\boldsymbol{\eta} = \mathbf{0}, && \text{on } \Gamma_{\mathrm{N}} \\
&\tilde{\mathbf{v}} = \mathbf{0}, && \text{on } \Gamma_{\mathrm{D}}
\end{aligned}$$

Hence, integrating by parts, we obtain, for $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbf{V}_h$,

$$a^*(\Pi_h A_{\mathbf{X}}(\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) - b((\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi})) = \rho_{\mathrm{F}} c^2 \int_{\Gamma_{\mathrm{I}}} \mathrm{div}\,\tilde{\mathbf{u}}\,(\boldsymbol{\phi} - \boldsymbol{\psi}) \cdot \boldsymbol{\nu}. \tag{5.11}$$

For $\ell \subset \Gamma_{\mathrm{I}}$ let $T \subset \Omega_{\mathrm{F}}$ and $T_\ell \subset \Omega_{\mathrm{S}}$ be the triangles such that $\partial T \cap \partial T_\ell = \ell$. Let P_ℓ denote the $L^2(\ell)$–projection of $H^{1/2}(\ell)$ onto the constants. Since $\boldsymbol{\phi}\cdot\boldsymbol{\nu} = P_\ell(\boldsymbol{\psi}\cdot\boldsymbol{\nu})$, we have

$$\begin{aligned}
\left| \int_\ell \mathrm{div}\,\tilde{\mathbf{u}}\,(\boldsymbol{\phi} - \boldsymbol{\psi}) \cdot \boldsymbol{\nu} \right| &= \left| \int_\ell [\,\mathrm{div}\,\tilde{\mathbf{u}} - P_\ell(\,\mathrm{div}\,\tilde{\mathbf{u}})]\,[P_\ell(\boldsymbol{\psi} \cdot \boldsymbol{\nu}) - \boldsymbol{\psi} \cdot \boldsymbol{\nu}] \right| \\
&\leq \|\,\mathrm{div}\,\tilde{\mathbf{u}} - P_\ell(\,\mathrm{div}\,\tilde{\mathbf{u}})\|_{L^2(\ell)}\, \|P_\ell(\boldsymbol{\psi} \cdot \boldsymbol{\nu}) - \boldsymbol{\psi} \cdot \boldsymbol{\nu}\|_{L^2(\ell)}\,.
\end{aligned}$$

Now, if P_T denotes the $L^2(T)$–projection of $H^1(T)$ onto the constants, by using a trace theorem and the standard error estimate for the L^2–projection, we have

$$\begin{aligned}
\|\,\mathrm{div}\,\tilde{\mathbf{u}} - P_\ell(\,\mathrm{div}\,\tilde{\mathbf{u}})\|_{L^2(\ell)} &\leq \|\,\mathrm{div}\,\tilde{\mathbf{u}} - P_T(\,\mathrm{div}\,\tilde{\mathbf{u}})\|_{L^2(\ell)} \\
&\leq C \Big[h^{-1/2}\, \|\,\mathrm{div}\,\tilde{\mathbf{u}} - P_T(\,\mathrm{div}\,\tilde{\mathbf{u}})\|_{L^2(T)} \\
&\qquad + h^{1/2}\, |\,\mathrm{div}\,\tilde{\mathbf{u}} - P_T(\,\mathrm{div}\,\tilde{\mathbf{u}})|_{H^1(T)} \Big] \\
&\leq C h^{1/2}\, |\,\mathrm{div}\,\tilde{\mathbf{u}}|_{H^1(T)}\,.
\end{aligned}$$

Analogously, $\|P_\ell(\boldsymbol{\psi} \cdot \boldsymbol{\nu}) - \boldsymbol{\psi} \cdot \boldsymbol{\nu}\|_{L^2(\ell)} \leq C h^{1/2}\, |\boldsymbol{\psi}|_{H^1(T_\ell)}$. Hence

$$\left| \int_{\Gamma_{\mathrm{I}}} \mathrm{div}\,\tilde{\mathbf{u}}\,(\boldsymbol{\phi} - \boldsymbol{\psi}) \cdot \boldsymbol{\nu} \right| \leq C h\, |\,\mathrm{div}\,\tilde{\mathbf{u}}|_{H^1(\Omega_{\mathrm{F}})}\, |\boldsymbol{\psi}|_{H^1(\Omega_{\mathrm{S}})}\,. \tag{5.12}$$

So, from (5.9), (5.11), (5.12) and (5.10) we conclude the lemma. □

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