Bearings-Only Target Passive Location

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Abstract. This paper addresses a performance analysis of the principal algorithm for passive location target detection. This algorithm is based on maximum likelihood estimation. Finally a estimator accuracy comparison is done as a function of the number of sensors and measurements per sensor.

1 INTRODUCTION

The target position estimation has been boarded by different ways. One of then is emitting source position estimation from passive angle measurements that is widely investigated, such as estimation algorithms [9,11,12], location accuracy [5,6,8] and target observability [1,2,7].

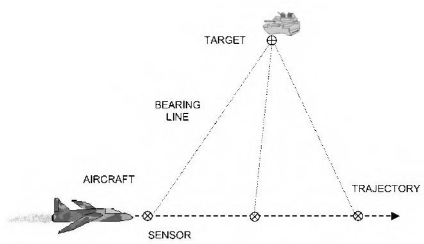

Fig. 1. Bearing lines from aircraft position.

We are interested on the study of the bearing measurements problem. The bearing measurements take two or more points along the moving observer trajectory, or collected by fixed direction finding sensors(typically electro-optical). This points can be intersected to determine the emitting target location, Fig. 1. Since the bearing measurements are noisy, statistical algorithms, sometimes called triangulation or fixing methods, are required in order to obtain optimal target position estimation. In this paper we used Maximum likelihood (ML) estimation algorithm because of its accuracy. We will analyze the properties of the estimator, based on the number of distributed sensors and its relation with the Cramer-Rao lower bound (CRLB).

The paper is organized as follows. Section 2 contains Estimation Methods. The performance analysis is presented in Section 3. In Section 4 and 5 the problem formulation for the two-dimensional case and simulation study is presented. Finally the conclusions are given in Section 6.

2 ESTIMATION METHODS

The components of an K-dimensional vector $\mathbf{x}$ that is to be estimated are the position coordinates in two or three dimensions. A set of N measurements r_i, $i = 1, 2, ..., N$ is collected at various positions. In the absence of random measurement errors, r_i is equal to a know function $f(\mathbf{x})$. In the presence of additive errors,

$$r_i = f_i(\mathbf{x}) + n_i, \qquad i = 1, 2, ..., N \tag{1}$$

These N equations can be written as a single equation for N-dimensional column vectors:

$$\mathbf{r} = \mathbf{f}\left(\mathbf{x}\right) + \mathbf{n} \tag{2}$$

The measurement error $\mathbf{n}$ is assumed to be a multivariate random vector with an $N \times N$ positive-definite covariance matrix,

$$M = E\left[\left(\mathbf{n} - E\left(\mathbf{n}\right)\right)\left(\mathbf{n} - E\left(\mathbf{n}\right)\right)^T\right] \tag{3}$$

where $E[\cdot]$ denotes the expected value and $(\cdot)^T$ denotes transposition. If $\mathbf{x}$ is regarded as an unknown but nonrandom vector and $\mathbf{n}$ is assumed to have a zero mean Gaussian distribution, then the conditional density function of $\mathbf{r}$ given $\mathbf{x}$ is:

$$p\left({}^{\mathbf{r}}/_{\mathbf{x}}\right) = \frac{1}{\left(2\pi\right)^{\frac{N}{2}}\left|M\right|^{\frac{1}{2}}} e^{-\frac{1}{2}\left[\mathbf{r}-\mathbf{f}(\mathbf{x})\right]^T M^{-1}\left[\mathbf{r}-\mathbf{f}(\mathbf{x})\right]} \tag{4}$$

where $|M|$ denotes the determinant of M and the superscript -1 denote the inverse. Because M is symmetric and positive definite, its inverse exists. The maximum likelihood estimator is that value of which maximizes (4) [3,10]. Thus the ML estimator minimizes the quadratic form

$$Q\left(\mathbf{x}\right) = \left[\mathbf{r} - \mathbf{f}\left(\mathbf{x}\right)\right]^T M^{-1}\left[\mathbf{r} - \mathbf{f}\left(\mathbf{x}\right)\right] \tag{5}$$

Minimization of $Q(\mathbf{x})$ is a reasonable criterion for the determination of an estimator even when then additive error cannot be assumed to be Gaussian. In this case, the resulting estimate is called the least squares estimate and M^{-1} is regarded as a matrix weighting coefficient. In general, $\mathbf{f}(\mathbf{x})$ is nonlinear vector function. To determine a reasonably simple estimator, $\mathbf{f}(\mathbf{x})$ can be linearized by expanding it in a Taylor series about a reference point specified by the vector $\mathbf{x}_0$ and retaining the first two terms; that is,

$$\mathbf{f}(\mathbf{x}) \simeq \mathbf{f}(\mathbf{x}_0) + G(\mathbf{x} - \mathbf{x}_0) \tag{6}$$

where $\mathbf{x}$ and $\mathbf{x}_0$ are $K \times 1$ column vectors and G is the $N \times K$ matrix of derivatives evaluated at $\mathbf{x}_0$:

$$G = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{\mathbf{x}=\mathbf{x_o}} & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_{\mathbf{x}=\mathbf{x_o}} \\ \vdots & & \vdots \\ \left.\frac{\partial f_n}{\partial x_1}\right|_{\mathbf{x}=\mathbf{x_o}} & \cdots & \left.\frac{\partial f_n}{\partial x_n}\right|_{\mathbf{x}=\mathbf{x_o}} \end{bmatrix} \tag{7}$$

Each row of this matrix is the gradient vector of one of the components of $\mathbf{f}(\mathbf{x})$. The vector $\mathbf{x}_0$ could be an estimate of $\mathbf{x}$ determined from a previous iteration of the estimation procedure or based upon a priori information. It is assumed in the subsequent analysis that $\mathbf{x}_0$ is sufficiently close to $\mathbf{x}$ so that (6) is an accurate approximation. Combining (5) and (6) gives

$$Q(\mathbf{x}) = [\mathbf{r}_1 - G\mathbf{x}]^T M^{-1} [\mathbf{r}_1 - G\mathbf{x}] \tag{8}$$

where

$$\mathbf{r}_1 = \mathbf{r} - \mathbf{f}(\mathbf{x}) - G\mathbf{x_0} \tag{9}$$

To determine the necessary condition for the estimator $\hat{\mathbf{x}}$ that minimizes, we calculate the gradient of $Q(\mathbf{x})$, defined by

$$\nabla_{\mathbf{x}} Q\left(\mathbf{x}\right) = \left[\, \frac{\partial Q}{\partial x_1} \cdots \frac{\partial Q}{\partial x_n} \,\right]^T \tag{10}$$

and then solve for the $\mathbf{x}$ such that $\nabla_{\mathbf{x}} Q\left(\mathbf{x}\right) = \mathbf{0}$. From its definition, M is a symmetric matrix, that is, $M^T = M$. Since $(M^{-1})^T = (M^T)^{-1}$, it follows that $(M^{-1})^T = M^{-1}$, which implies that M^{-1} is a symmetric matrix. Therefore,

$$\left.\nabla_{\mathbf{x}} Q\left(\mathbf{x}\right)\right|_{\mathbf{x}=\hat{\mathbf{x}}} = 2G^T M^{-1} G\hat{\mathbf{x}} - 2G^T M^{-1} \mathbf{r_1} = \mathbf{0} \tag{11}$$

We assume that the matrix $G^T M G$ is nonsingular. Thus the solution of (11) is

$$\hat{\mathbf{x}} = \mathbf{x}_o + \left(G^T M^{-1} G\right)^{-1} G^T M^{-1} \left[\mathbf{r} - \mathbf{f}\left(\mathbf{x}_o\right)\right] \tag{12}$$

Therefore, $\hat{\mathbf{x}} = \arg\min Q(\mathbf{x})$ is called the linearized least square estimator. This estimator can be implemented using the Newton-Gauss iteration, with an initial estate, $\hat{\mathbf{x}}_0$, close enough to the true minimum of the cost function [5,9]. Such initial estate may be available from prior information, or can be obtained using a simple (but suboptimal) procedure.

3 PERFORMANCE ANALYSIS

Substituting (2) into (12) and rearranging terms, the expression for (12) can be written as

$$\hat{\mathbf{x}} = \mathbf{x} + \left(G^T M^{-1} G\right)^{-1} G^T M^{-1} \left[\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_o) - G(\mathbf{x} - \mathbf{x}_o) + \mathbf{n}\right] \tag{13}$$

Which shows how the estimate error is affected by the linearization error and the noise [4]. The bias of the estimator $\hat{\mathbf{x}}$ is defined as $\mathbf{f}(\mathbf{x})$. Using (13), we obtain:

$$\mathbf{b} = \left(G^T M^{-1} G\right)^{-1} G^T M^{-1} \left\{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_o) - G(\mathbf{x} - \mathbf{x}_o) + E[\mathbf{n}]\right\} \tag{14}$$

If $\mathbf{f}(\mathbf{x})$ is lineal, and $E[\mathbf{x}] = \mathbf{0}$, then the least squares estimator is unbiased. When $\mathbf{f}(\mathbf{x})$ is nonlinear the bias can be estimated by expanding it in a Taylor series about $\mathbf{x}_0$ and retaining the second-order term. Let P denote the covariance matrix of $\hat{\mathbf{x}}$. Equation (13) yields

$$P = E\left[(\hat{\mathbf{x}} - E[\hat{\mathbf{x}}])(\hat{\mathbf{x}} - E[\hat{\mathbf{x}}])^T\right] = \left(G^T M^{-1} G\right)^{-1} \tag{15}$$

The diagonal elements of P are the variances of each component of $\hat{\mathbf{x}}$. If $\mathbf{n}$ is zero-mean white Gaussian noise, the maximum likelihood estimator for the linearized model has minimum variance. Therefore P is equal to the Fisher information matrix [5].

If $\mathbf{r}$ is a Gaussian random vector, then (12) indicates that for $\hat{\mathbf{x}}$ is also a Gaussian random vector. Then, the probability density function of $\hat{\mathbf{x}}$ is

$$p_{\hat{\mathbf{x}}}(\xi) = \frac{1}{(2\pi)^{\frac{K}{2}} |P|^{\frac{1}{2}}} e^{-\frac{1}{2}[\xi - \mathbf{m}]^T P^{-1} [\xi - \mathbf{m}]} \tag{16}$$

where $\mathbf{m} = E[\hat{\mathbf{x}}]$ is the mean vector, and P is the covariance matrix expressed on (15). By definition, P is symmetric and positive definite. The loci of constant density function values are described by equations of the form

$$(\xi - \mathbf{m})^T P^{-1} (\xi - \mathbf{m}) = \kappa \tag{17}$$

where κ is a constant that determines the size of the K-dimensional region enclosed by the surface. In two dimensions, the surface is an ellipse; in three dimensions, it is an ellipsoid; in the general case of K dimension, is may be considered a hyperellipsoid [10,12]. Then the probability that $\hat{\mathbf{x}}$ lies inside the hyperellipsoid of (17) is

$$P(\kappa) = \int_R \int \dots \int p_{\hat{\mathbf{x}}}(\xi)\; d\xi_1\; d\xi_2 \dots d\xi_n \tag{18}$$

where the region of integration is

$$R = \left\{\xi : (\xi - \mathbf{m})^{\mathbf{T}} P^{-1} (\xi - \mathbf{m}) \le \kappa\right\} \tag{19}$$

The size of the semiaxes of this hyperellipsoids is defined by the eigenvalues of P, and the orientation of these semiaxes is defined by the eigenvectors of P. The area of the ellipse corresponding to the two-dimensional case can be expressed as

$$A = \pi\sqrt{|P|}\kappa^2 \tag{20}$$

Thus, target observability is based on the area minimization give by (20). This area only depends on the estimation error covariance matrix. Thereby by minimizing the area of uncertainty we will enhance the estimation accuracy, because the probability density function (16) is tuned and stretched up.

4 PROBLEM FORMULATION FOR TWO-DIMENSIONAL CASE

The two-dimensional bearings-only location problem can be formulated as follows. Let $\mathbf{x} = (x_T, y_T)^T$ be the target coordinate vector to be estimated from the bearing measurements $\mathbf{r} = (r_1, r_2, ..., r_N)^T$. The target bearings are measured from an own-ship with known trajectory or from fixed sensors at known locations. Denote by $\mathbf{x} = (x_{sn}, y_{sn})^T$ the sensor coordinates associated with the measurement r_n. The problem geometry is depicted in Fig. (2).

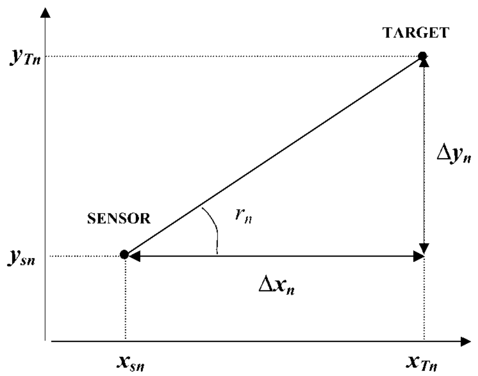

Fig. 2. Problem geometry.

The angle measurements consist of a true angle corrupted by additive noise $\mathbf{n}$, that it is assumed to be zero-mean white Gaussian noise with $N \times N$ covariance matrix,

$$M = diag\left(\sigma_1^2, \sigma_2^2, ..., \sigma_N^2\right) \tag{21}$$

Thus the problem is described by a nonlinear equation

$$\mathbf{r} = \mathbf{f}(\mathbf{x}) + \mathbf{n} \tag{22}$$

where $\mathbf{f}(\mathbf{x}) = [\mathbf{f}_1(\mathbf{x}), \mathbf{f}_2(\mathbf{x}), \ldots \mathbf{f}_N(\mathbf{x})]$

$$f_i(\mathbf{x}) = \arctan\left(\frac{\mathbf{\Delta}_{yn}}{\mathbf{\Delta}_{xn}}\right) \tag{23}$$

$$\begin{aligned} \Delta_{xn} &= x_T - x_{sn} \\ \Delta_{yn} &= y_T - y_{sn} \\ d_n^2 &= \Delta_{yn}^2 + \Delta_{xn}^2 \end{aligned} \tag{24}$$

and the index n indicates the sensor, where $n \in [1, N]$. From (7) the gradient vector is

$$G = \begin{bmatrix} -\frac{\Delta_{y1}}{d_1^2} & \frac{\Delta_{x1}}{d_1^2} \\ \vdots & \vdots \\ -\frac{\Delta_{yN}}{d_N^2} & \frac{\Delta_{xN}}{d_N^2} \end{bmatrix} \tag{25}$$

The least squares minimization performed by the Newton-Gauss iteration, with an initial estate $\hat{\mathbf{x}}_0$ is

$$\hat{\mathbf{x}}_{n+1} = \hat{\mathbf{x}}_n + \left(G^T N^{-1} G\right)^{-1} G^T N^{-1} \left[\mathbf{r} - \mathbf{f}(\hat{\mathbf{x}}_n)\right], \quad i = 0, 1, 2, ... \tag{26}$$

However, in the present problem the covariance matrix P defined in (15) can be expressed as

$$P = \begin{bmatrix} \sum\limits_{n=1}^{N} \frac{(\Delta_{yn})^2}{d_n^4 \sigma_n^2} & -\sum\limits_{n=1}^{N} \frac{\Delta_{xn}\Delta_{yn}}{d_n^4 \sigma_n^2} \\ -\sum\limits_{n=1}^{N} \frac{\Delta_{xn}\Delta_{yn}}{d_n^4 \sigma_n^2} & \sum\limits_{n=1}^{N} \frac{(\Delta_{xn})^2}{d_n^4 \sigma_n^2} \end{bmatrix}^{-1} \tag{27}$$

As it was mentioned, the area of the ellipse (20) can be approximately expressed as

$$A \simeq \pi\sqrt{|P|} \tag{28}$$

It is easy to prove that when N is increased (more sensors), the area (28) decreases. Therefore, this implies that the estimator accuracy is improved.

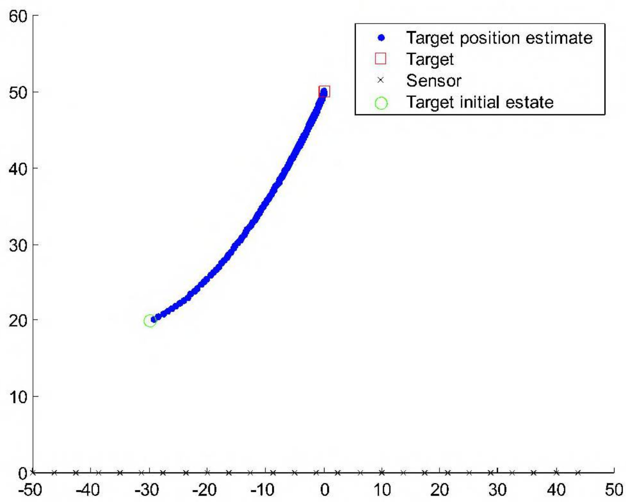

Fig. 3. Estimated target trajectory using Newton-Gauss iteration, with an initial position $\hat{\mathbf{x}}_0$.

5 SIMULATION STUDY

In this section we are interested in the ML performance verification via Monte Carlo simulations. Consider a target located at $x_T = 0$, $y_T = 50$. Own-ship moves with constant velocity along the x axis and its trajectory is defined by $x_S = -50 + 0.15t$, $y_S = 0$. Own-ship collects target bearings measurements every 25 time units. The distance, velocity, and time units should be read in consistent units, kilometer, kilometer per second, and seconds, respectively. The measurement noise is identically distributed white Gaussian process with standard deviation $\sigma_n = 3°, \forall n$. Let us look at observation times $0 \leq t \leq \mathrm{T}$, where T is between 75 and 475 time units. Fig. 3 shows the convergence of the target position estimator trough the true position with 20 sensors and starting from an initial position. Fig. 4 shows the error bias and standard deviation for the x and y component using ML estimator as a function of the number of independent sensors. The circles represent simulation results, where each point has been obtained from 200 Monte Carlo experiments.
We note that as the number of sensor increases, the bias and standard deviation tend to zero. These results verify the theory. Finally, Fig. 5 shows the error bias and standard deviation with 10 sensors as a function of the Monte Carlo experiments.

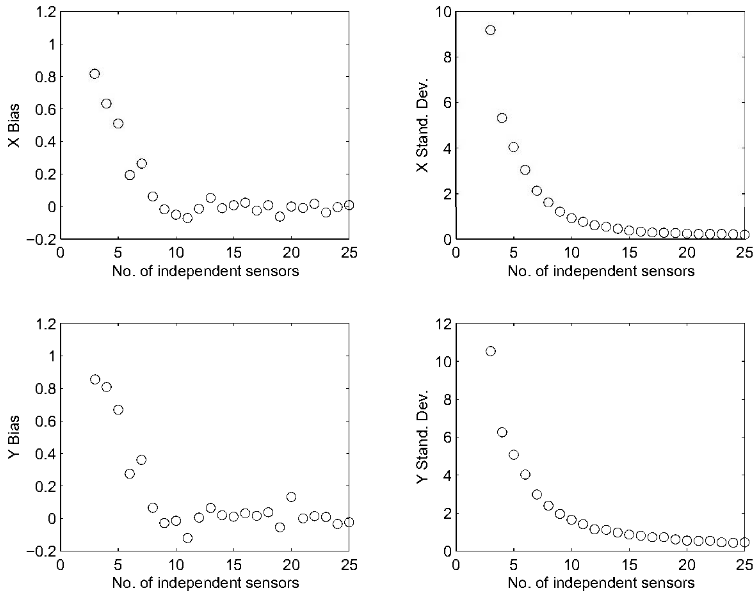

Fig. 4. Performance of ML estimator versus number of independent sensors.

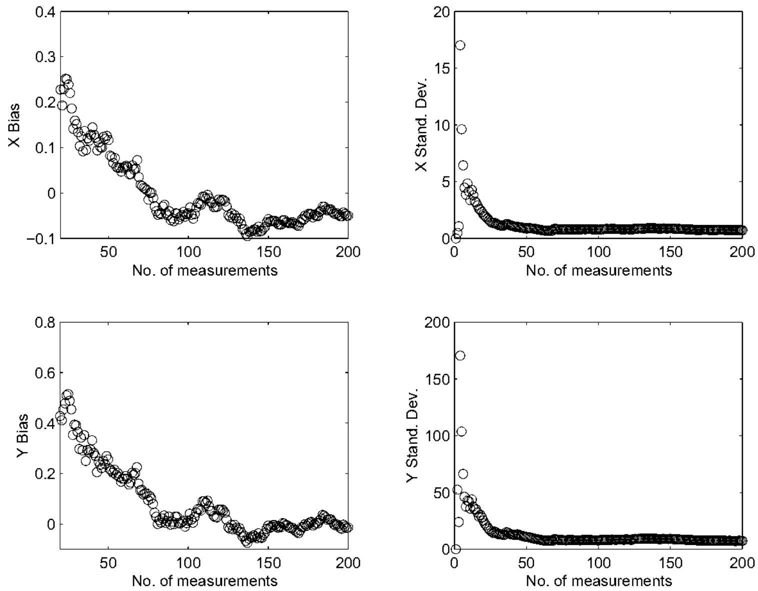

Fig. 5. Performance of ML estimator versus number of measurements for 10 independents sensors.

6 CONCLUSIONS

The results given in the previous section show that the bias of the ML estimator vanishes when the number of measurements per sensor increases. This estimator property is well-known as asymptotically unbiased. On the other hand, this ML estimator is minimum variance. Therefore, the estimator covariance matrix realizes the Cramer-Rao lower bound [10]. However, if we increase the number of sensors, this lower bound decreases. This implies that the estimator accuracy improves as an incremental function of the number of independent sensors. Finally, the ML target position estimation is a simple method which has nice important statistical properties without prior information about the target position.

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