Expressibility of the Logic SO^F on Classes of Structures of Bounded FO Types

Alejandro L. Grosso[1] and José M. Turull Torres[2]

[1] Dpto. de Informática, Universidad Nacional de San Luis, Argentina,
agrosso@unsl.edu.ar
[2] ICTIC, Universidad de la Cuenca del Plata, Corrientes, Argentina
and
Dpto. de Informática, Universidad Nacional de San Luis, Argentina
J.M.Turull@massey.ac.nz

Abstract. We introduce a new property for classes of structures (or relational database instances), that we call bounded FO types. Then we prove that on such classes the expressive power of SO^F collapses to first order logic FO. As a consequence of this we prove that SO^F is strictly included in SO.

Keywords: Finite Model Theory, Descriptive Complexity, Relational Machines

1. Introduction

Significant research has been done in the last few decades in regard to the relationship between finite model theory and computational complexity theory. There is a close relationship between computational complexity, the amount of resources we need to solve a problem over some Turing machine, and descriptive complexity, the logic we need to describe the positive instances of the problem. The most important result about this relationship was the result of Fagin [3]. This result establishes that the properties of finite relational structures (or relational database instances) which are defined by existential second order sentences coincide with the properties that belong to the complexity class NP. This result was extended by Stockmeyer [10] establishing a close relationship between second order logic and the polynomial hierarchy.

In [7] we introduced the logic SO^F and proved that the logic SO^ω defined by A. Dawar [2] and further studied by F. Ferrarotti and the second author [6], which is a fragment of the infinitary logic $\mathcal{L}^\omega_{\infty,\omega}$, is strictly included in SO^F. In SO^ω the second order quantifiers range over k'-ary relations closed by the equivalence relation $\equiv^k$ for $k' \leq k$, whereas in SO^F the second order quantifiers range over k-ary relations closed by the equivalence relation $\equiv^{FO}$ (see section 2 for definitions of $\equiv^k$ and $\equiv^{FO}$). We showed that in SO^F the rigidity query can be expressed. This property means that the structures only has one automorphism which is the identity function. Rigidity belongs to co-NP. In [7] we also characterize the existential fragment $\Sigma_1^{1,F}$ of SO^F with a modified version of the relational machine defined in [1].

In [8] we added oracles to our version of relational machine, and we introduced the polynomial time hierarchy PHF. We then showed the correspondence between the fragments $\Sigma_i^{1,F}$ of SO^F and the levels of the polynomial time hierarchy PHF. That result is analogous to the L. Stockmeyer's characterization of polynomial time hierarchy on Turing machine [10] and also to the F. Ferrarotti and the second author's characterization of the relational polynomial time hierarchy on relational machines [6].

Using the semantics of SO^F and fixing a σ-structure $\mathcal{A}$ it is possible to assign an FO formula equivalent to each SO^F formula. In this reduction lies the idea to define, for a fixed σ-structure $\mathcal{A}$, each possible k-ary relation closed under FO types for k-tuples using an FO formula. This match between sentences of SO^F and sentences of FO can be established for classes of structures which have a finite number of isolating formulae for FO types. This property holds for the class of structures with unary vocabulary.

As a consequence, the expressive power of SO^F collapses to FO over classes with a finite number of isolating formulae including the case of structures with unary vocabulary. It is well known that parity is expressible in SO but is not expressible in FO, then parity over sets is not expressible in SO^F. Therefore, SO^F is strictly included in SO.

2. Preliminaries

We only consider finite relational structure.

A vocabulary σ is a set of relational symbols $\{P_1, \ldots, P_s\}$ with associated arities, $r_1, \ldots, r_s \geq 1$. A σ-structure (also called model or relational database instance) $\mathcal{A} = \langle A, P_1^{\mathcal{A}}, \ldots, P_s^{\mathcal{A}} \rangle$ consists of a non empty set A called domain of $\mathcal{A}$ and a relation $P_i^{\mathcal{A}} \subseteq A^{r_i}$ for each relation symbol P_i in σ for $1 \leq i \leq s$. The domain of $\mathcal{A}$ is denoted with A or $dom(\mathcal{A})$.

An *m-ary query* q, for $m \geq 1$ is a function which maps structures of a fixed vocabulary σ to m-ary relations on the domain of the structures, and which preserves isomorphisms, i. e., when f is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ then $\bar{t} \in q(\mathcal{A})$ iff $f(\bar{t}) \in q(\mathcal{B})$. A 0-ary query, also called *Boolean query*, is a function from a class of σ-structures to {0,1} and can be identified with a class of σ-structures. By a class of structures in B_σ we means a class closed under isomorphism.

For the definitions of syntax and semantics of FO see [9] among others.

The truth value of a formula φ, $\mathcal{A} \models \varphi(x_1, \ldots, x_n)[v]$, with free variables $x_1, \ldots, x_n$, only depends on the values assigned by the valuation v to the free variables. Therefore, we denote with $\mathcal{A} \models \varphi[a_1, \ldots, a_n]$ the truth value of the φ in the structure $\mathcal{A}$ for a valuation that assigns to the free variable x_i the value a_i for $1 \leq i \leq n$. Then a formula $\varphi(x_1, \ldots, x_n)$ with n free variables defines an n-ary relation on $\mathcal{A}$, $\varphi^{\mathcal{A}} = \{(a_1, \ldots, a_n) \in A^n | \mathcal{A} \models \varphi(x_1, \ldots, x_n)[a_1, \ldots, a_n]\}$.

FO^k is the fragment of FO where we use up to k different variables.

In second order logic we add a set of second order variables which range over relations instead of elements of the structure domain.

Another way to extend the logic FO is by allowing conjunction and disjunction over sets of formulae with arbitrary cardinality, then we have the infinitary logic $\mathcal{L}_{\infty,\omega}$. The logic $\mathcal{L}^k_{\infty,\omega}$ is the fragment of $\mathcal{L}_{\infty,\omega}$ where we use up to k different variables. The logic $\mathcal{L}^\omega_{\infty,\omega}$ is the union of the fragments $\mathcal{L}^k_{\infty,\omega}$ for $k \geq 1$.

2.1. Element Types

Let $\mathcal{A}$ be a structure and $\bar{a}$ be an l-tuple of elements of A for $l \geq 1$, we define the *FO type* of $\bar{a}$ in $\mathcal{A}$, denoted by $type^{FO}_{\mathcal{A}}(\bar{a})$, as the set of FO formulae, φ, with free variables among $x_1, \ldots, x_l$ such that $\mathcal{A} \models \varphi[a_1, \ldots, a_l]$. A set τ of FO formulae is an FO type iff τ is the FO type for some tuple in some structure. If τ is an FO type, we say that the tuple $\bar{a}$ realize τ in $\mathcal{A}$ iff $\tau = type^{FO}_{\mathcal{A}}(\bar{a})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures and $\bar{a}$, $\bar{b}$ be two tuples of the same length in the structures $\mathcal{A}$ and $\mathcal{B}$ respectively. $(\mathcal{A}, \bar{a}) \equiv^{FO} (\mathcal{B}, \bar{b})$ iff $type^{FO}_{\mathcal{A}}(\bar{a}) = type^{FO}_{\mathcal{B}}(\bar{b})$. That is, two tuples of possibly different structures have the same FO type when they satisfy the same (maximally consistent) set of FO formulae. Let $k \geq 1$, and τ be an FO type, a formula $\varphi(\bar{x}) \in \tau$ is called an *isolating formula* of FO type τ for k-tuples when for all pairs of structures $\mathcal{A}, \mathcal{B}$ in B_σ and tuples $\bar{a} \in A^k$, $\bar{b} \in B^k$ it holds that $\mathcal{A} \models \varphi(x)[\bar{a}]$ and $\mathcal{B} \models \varphi(x)[\bar{b}]$ iff $\tau = type^{FO}_{\mathcal{A}}(\bar{a}) = type^{FO}_{\mathcal{B}}(\bar{b})$.

In a similar way we can define $type^k_{\mathcal{A}}(\bar{a})$ and $\equiv^k$ for the logic FO^k.

Fact 1 *Let $l \geq 1$, $\mathcal{A}$ a finite σ-structure and $\bar{a}$ and $\bar{b}$ be two l-tuples on $\mathcal{A}$. $(\mathcal{A}, \bar{a}) \equiv^{FO} (\mathcal{A}, \bar{b})$, if and only if, there is an automorphism f such that $f(a_i) = b_i$ for $1 \leq i \leq l$.*

3. Semantic Restrictions of SO

A. Dawar in [2] introduced a restriction on second order logic (SO$^\omega$)by restricting the class of relations that the quantified second order variables can be assigned to by valuations. In SO$^\omega$, the second order variables can only contain relations which are closed under the equivalence relation $\equiv^k$ for some $k \geq 1$. This implies that we cannot assign arbitrary relations to the variables. The relations have to be unions of FO^k types, i.e., they have to be unions of equivalence classes of $\equiv^k$.

In [7] we introduced the logic SO^F as a restriction of the second order logic where the second order quantifiers range over relations closed under the equivalence relation $\equiv^{FO}$, i.e., the quantifiers range over relations which are unions of FO types. These relations are redundant in the sense of [5].

Let $k \geq 1$. For a k-ary relation variable R, we define the second order quantifier $\exists^F R$ with the following semantics: $\mathcal{A} \models \exists^F R\varphi[v]$ if and only if there exists a relation $S \subseteq A^k$ such that S is closed under the equivalence relation $\equiv^{FO}$ in $\mathcal{A}$ for k-tuples, and $\mathcal{A} \models \varphi[v\frac{S}{R}]$. As usual $\forall^F R\varphi$ is an abbreviation of $\neg\exists^F R\neg\varphi$.

We add the following formation rules to the FO formation rules to obtain the formulae of SO^F: *1) If R is a k-ary second order variable, for $k \geq 1$, and*

$x_1, \ldots, x_k$ are first order variables, then $R(x_1, \ldots, x_k)$ is a formula (atomic) of SO^F. 2) If φ is an SO^F formula, and R is a k-ary second order variable, then $\exists^F R\varphi$ and $\forall^F R\varphi$ are formulae of SO^F. The fragment $\Sigma_i^{1,F}$ of SO^F consists of the formulae of SO^F which have a prefix of i alternated blocks of second order quantifiers followed by an FO formula. The prefix must begin with an existential quantifier block. Then, we can define: $\mathrm{SO}^F = \bigcup_i \Sigma_i^{1,F}$.

4. Collapse of SO^F to FO

Before we show the reduction to FO we will see that the FO types for r-tuples realized in a particular structure can be expressed by means of FO formulae with r free variables.

When we use second order quantification we extend the structure with relations. In SO^F we extend the structure with redundant relations in the sense that has been studied in the work of F. Ferrarotti, A. Paoletti and the second author in [5]. In the extended structure $\langle \mathcal{A}, R\rangle$ where R is an r-ary relation closed under FO types for r-tuples for $r \geq 1$, the equivalence relation $\equiv^{FO}$ for r-tuples is the same for the original structure and the extended structure. That is, for all $\bar{a}, \bar{b} \in A^r$, $(\mathcal{A}, \bar{a}) \equiv^{FO} (\mathcal{A}, \bar{b})$ iff $(\langle \mathcal{A}, R\rangle, \bar{a}) \equiv^{FO} (\langle \mathcal{A}, R\rangle, \bar{b})$. This is not true in SO where the quantified relations can break the FO types. The following lemma is from [5].

Lemma 1. *Let $\mathcal{A}$ be a σ-structure. Let R be a r-ary relation closed by FO types for r-tuples in $\mathcal{A}$. Let $\bar{a} \in R$ and $\bar{b} \in A^r$. There is formula $\varphi_{\bar{a}}(x_1, \ldots, x_r)$ of $FO(\sigma)$ such that $\mathcal{A} \models \varphi_{\bar{a}}(x_1, \ldots, x_r)[\bar{b}]$ iff $tp_{\mathcal{A}}^{FO}(\bar{a}) = tp_{\mathcal{A}}^{FO}(\bar{b})$.*

The proof uses the diagram, $\Delta_{\mathcal{A}}$, of the structure $\mathcal{A}$. Let $|dom(\mathcal{A})| = n$ and $v\colon \{x_1, \ldots, x_n\} \longrightarrow dom(\mathcal{A})$ be an injective valuation such that $v(x_{i_1}) = a_1, \ldots, v(x_{i_r}) = a_r$ for $1 \leq i_1, \ldots, i_r \leq n$, then $\varphi_{\bar{a}}(y_1, \ldots, y_r) \equiv \exists x_1 \ldots \exists x_n\, (\delta_{\mathcal{A}} \wedge (\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j) \wedge \forall x_{n+1} (\bigvee_{1 \leq i \leq n} x_{n+1} = x_i) \wedge (x_{i_1} = y_1 \wedge \cdots \wedge x_{i_r} = y_r))$ where

$$\delta_{\mathcal{A}} = \bigwedge_{R \in \sigma} \{R(x_{i_1}, \ldots, x_{i_r}) : \mathcal{A} \models R(x_{i_1}, \ldots, x_{i_r})\, y\, 1 \leq i_1, \ldots, i_r \leq n\} \wedge$$
$$\bigwedge_{R \in \sigma} \{\neg R(x_{i_1}, \ldots, x_{i_r}) : \mathcal{A} \not\models R(x_{i_1}, \ldots, x_{i_r})\, y\, 1 \leq i_1, \ldots, i_r \leq n\}$$

that is, $\delta_{\mathcal{A}}$ is the conjunction of the atomic formulae and negated atomic formulae that hold in $\mathcal{A}$.

The formula above, without the conjunction $(x_{i_1} = y_1 \wedge \cdots \wedge x_{i_r} = y_r)$, is the diagram of $\mathcal{A}$. It is known that for all structure $\mathcal{B}$, $\mathcal{B} \models \Delta_{\mathcal{A}}$ iff $\mathcal{B}$ is isomorphic to $\mathcal{A}$.

The Lemma below is well known and holds for many logic used in Finite Model Theory. We include a proof for $\Sigma_k^{1,F}$ because we will make use of the construction later.

Lemma 2. *Let $\mathcal{A}$ be σ-structure and ψ be a sentence in $\Sigma_k^{1,F}$ for $k \geq 1$. There exists a sentence $\hat{\psi}_\mathcal{A}$ in FO(σ) such that $\mathcal{A} \models \psi$ iff $\mathcal{A} \models \hat{\psi}_\mathcal{A}$.*

Proof. Let $\psi \equiv \exists^F X_{11} \ldots \exists^F X_{1s_1} \forall^F X_{21} \ldots \forall^F X_{2s_2} \ldots Q\, X_{k1} \ldots Q\, X_{ks_k}$ $\varphi(X_{11}, \ldots, X_{1s_1}, X_{21}, \ldots, X_{2s_2}, \ldots, X_{k1}, \ldots, X_{ks_k})$ where $Q = \exists^F$ when k is odd or $Q = \forall^F$ when k is even.

Let $R_1^\mathcal{A}, \ldots, R_{n_r}^\mathcal{A}$ be all the r-ary relations closed by FO types for r-tuples on $\mathcal{A}$, then we can define each relation with an FO formula using the formulae defined in the Lemma 1.

$(\mathcal{A}, R_j^\mathcal{A}) \models R_j(y_1, \ldots, y_r)[b_1, \ldots, b_r]$ iff $\mathcal{A} \models \bigvee_{\bar{a} \in R_j^\mathcal{A}} \varphi_{\bar{a}}(y_1, \ldots . y_r)[b_1, \ldots, b_r]$, for $1 \leq j \leq n_r$. We define $\hat{\phi}_j \equiv \bigvee\limits_{\bar{a} \in R_j^\mathcal{A}} \varphi_{\bar{a}}(y_1, \ldots . y_r)$.

Then $\mathcal{A} \models \exists^F X_{11} \ldots \exists^F X_{1s_1} \; \forall^F X_{21} \ldots \forall^F X_{2s_2} \ldots Q\, X_{k1} \ldots Q\, X_{ks_k} \varphi(X_{11}, \ldots, X_{1s_1}, X_{21}, \ldots, X_{2s_2}, \ldots, X_{k1}, \ldots, X_{ks_k})$ iff $\mathcal{A} \models \hat{\psi}_\mathcal{A}$, with

$$\hat{\psi}_\mathcal{A} \equiv \bigvee_{1 \leq j_{11} \leq n_{r_{11}}} \cdots \bigvee_{1 \leq j_{1s_1} \leq n_{r_{1s_1}}} \bigwedge_{1 \leq j_{21} \leq n_{r_{21}}} \cdots \bigwedge_{1 \leq j_{2s_2} \leq n_{r_{2s_2}}} \cdots \Big(\bigvee / \bigwedge\Big)_{1 \leq j_{k1} \leq n_{r_{k1}}} \cdots \Big(\bigvee / \bigwedge\Big)_{1 \leq j_{ks_k} \leq n_{r_{ks_k}}} \varphi(\hat{\phi}_{j_{11}}/X_{11}, \ldots, \hat{\phi}_{j_{1s_1}}/X_{1s_1}, \ldots \hat{\phi}_{j_{21}}/X_{21}, \ldots, \hat{\phi}_{j_{2s_2}}/X_{2s_2}, \ldots, \hat{\phi}_{jk1}/X_{k1}, \ldots, \hat{\phi}_{j_{ks_k}}/X_{ks_k}),$$

where $n_{r_{tu}}$, for $1 \leq t \leq k$ and $1 \leq u \leq s_t$, is the number of different r_{tu}-ary relations closed by FO types for r_{tu}-tuples.

In this way the second order existential quantifiers are replaced by disjunctions over all r_{tu}-arity relations $R_{j_{tu}}^\mathcal{A}$ closed by FO types for r_{tu}-tuples. Theses relations are definible by the formulae $\hat{\phi}_{j_{tu}}$ belonging to FO(σ). Similarly, the second order universal quantifiers are replaced by conjunctions. □

Looking at the proof of Lemma 2, note that the formula $\hat{\psi}_\mathcal{A}$ depends on the σ-structure $\mathcal{A}$. That is, given the set of formulae $\hat{\Psi} = \{\hat{\psi}_\mathcal{A} \in \text{FO}(\sigma) | \mathcal{A} \in B_\sigma\}$, if we fix the σ-structure $\mathcal{A}$, then the formula $\psi \in \text{SO}^F(\sigma)$ is equivalent to the formula $\hat{\psi}_\mathcal{A} \in \hat{\Psi}$.

We will see now a property that we can define over a class $\mathcal{C}$ in order to reduce a formula $\psi \in \text{SO}^F(\sigma)$ to an equivalent formula $\hat{\psi}_\mathcal{C} \in \text{FO}(\sigma)$ that holds for every structure in $\mathcal{C}$, not only for a single structure $\mathcal{A} \in \mathcal{C}$ (module isomorphism).

Definition 1. *Let $\mathcal{C} \subseteq B_\sigma$ be a class of structures and $k \geq 1$. A finite set of formulae Φ^k is a* set of intra-isolating formula for FO types for k-tuples over $\mathcal{C}$ *when:*

i) For every structure $\mathcal{A} \in \mathcal{C}$, every $\bar{a}$, $\bar{b} \in A^k$ and $\varphi^k \in \Phi^k$ it holds: if $\mathcal{A} \models \varphi^k(x_1, \ldots, x_k)[\bar{a}]$ and $\mathcal{A} \models \varphi^k(x_1, \ldots, x_k)[\bar{b}]$ then $type_\mathcal{A}^{FO}(\bar{a}) = type_\mathcal{A}^{FO}(\bar{b})$

ii) For every structure $\mathcal{A} \in \mathcal{C}$, and every $\bar{a}$, $\bar{b} \in dom(\mathcal{A})^k$, if $type_\mathcal{A}^{FO}(\bar{a}) = type_\mathcal{A}^{FO}(\bar{b})$ then there exists $\varphi_i \in \Phi^k$ such that $\mathcal{A} \models \varphi_i(x_1, \ldots, x_k)[\bar{a}]$ and $\mathcal{A} \models \varphi_i(x_1, \ldots, x_k)[\bar{b}]$, and for all $\varphi_j \in \Phi^k$ with $j \neq i$ it holds $\mathcal{A} \not\models \varphi_j(x_1, \ldots, x_k)[\bar{a}]$ and $\mathcal{A} \not\models \varphi_j(x_1, \ldots, x_k)[\bar{b}]$.

iii) *For every structure* $\mathcal{A} \in \mathcal{C}$*, and every* $\bar{a} \in dom(\mathcal{A})^k$ *there exists* $\varphi^k \in \Phi^k$ *such that* $\mathcal{A} \models \varphi^k(x_1, \ldots, x_k)[\bar{a}]$.

A formula $\varphi^k \in \Phi^k$ can express that two k-tuples over a structure in $\mathcal{C}$ have the same FO type even if φ^k is not an isolating formula for FO type for k-tuples. Note that Def. 1 is based on tuples of the same structure and does not consider the case when the tuples belong to two different structures. Then, it may happen that the same intra-isolating formula is satisfied by two tuples from two distinct structures with different FO types. For example over the class of r-ary full trees with depth h (see example 1), for $r \geq 1$, the elements in the same level have the same FO type. One intra-isolating formula $\varphi^1(x_1)$ can just express that the element x_1 has depth d with $0 \leq d \leq h$. Given elements a and b with the same depth from full trees $\mathcal{T}_r$ (r-ary)and $\mathcal{T}_{r+1}$ (($r+1$)-ary) respectively, then a and b satisfy the intra-isolating formula $\varphi^1(x_1)$, but they don't have the same FO type, i.e., $type^{FO}_{\mathcal{T}_r}(a) \neq type^{FO}_{\mathcal{T}_{r+1}}(b)$. They have different FO type because, for example, they have a different number of siblings.

Then, if a class $\mathcal{C}$ has a set Φ^k, we can define each k-ary relation closed by FO types for k-tuples over all structures in $\mathcal{C}$. We recall that a relation closed by FO types over $\mathcal{A}$ is the union of FO types realized in $\mathcal{A}$. And by making the disjunction of different intra-isolating formulae of Φ^k we have unions of FO types.

Lemma 3. *Let* $\mathcal{C}$ *be a class of structures,* $k \geq 1$ *and* Φ^k *be a set of intra-isolating formulae for FO types for* k*-tuples over* $\mathcal{C}$*. Then every* k*-ary relation closed by FO types for* k*-tuples* $R^{\mathcal{A}} \subseteq dom(\mathcal{A})^k$ *with* $\mathcal{A} \in \mathcal{C}$ *is definable from* Φ^k*.*

Proof. By definition of Φ^k, for each $\mathcal{A}$, and each $\bar{a} \in dom(\mathcal{A})^k$ there exist a formula $\varphi^k_{\bar{a}} \in \Phi^k$ such that $\mathcal{A} \models \varphi^k_{\bar{a}}(x_1, \ldots, x_k)[\bar{a}]$. By Def. 1, $\varphi^k_{\bar{a}}(x_1, \ldots, x_k)$ is satisfied for k-tuplas on $\mathcal{A}$ that have the same FO type as $\bar{a}$. Then every relation closed by FO types $R^{\mathcal{A}} \subseteq dom(\mathcal{A})^k$ is definable with the following formula: $\varphi(x_1, \ldots, x_k) \equiv \bigvee_{\bar{a} \in R^{\mathcal{A}}} \varphi^k_{\bar{a}}(x_1, \ldots, x_k)$. □

Lemma 4. *Let* $\mathcal{C}$ *be a class of structures,* $k \geq 1$ *and* $\Phi^k = \{\varphi_1, \varphi_2, \ldots, \varphi_{n_k}\}$*, with* $n_k \geq 1$*, be a set of intra-isolating formulae for FO types for* k*-tuples over* $\mathcal{C}$*. Then, the set* $\{R^{\mathcal{A}} | \mathcal{A} \in \mathcal{C},\ R^{\mathcal{A}} \subseteq dom(\mathcal{A})^k$ *and* $R^{\mathcal{A}}$ *is closed by FO types* $\}$ *is equal to the set* $\{\varphi^{\mathcal{A}} | \varphi(x_1, \ldots, x_k) \equiv \bigvee_{i \in D} \varphi_i(x_1, \ldots, x_k),\ D \neq \emptyset,\ D \subseteq \{1, 2, \ldots, n_k\}$ *and* $\mathcal{A} \in \mathcal{C}\} \cup \{\emptyset\}$*.*

Proof. (⇒)Let $R^{\mathcal{A}}$ be a non empty k-ary relation closed by FO types for some $\mathcal{A} \in \mathcal{C}$. Then, by the Lemma 3, $R^{\mathcal{A}} = \varphi^{\mathcal{A}}$ such that $\varphi(x_1, \ldots, x_k) \equiv \bigvee_{\bar{a} \in R^{\mathcal{A}}} \varphi_{\bar{a}}(x_1, \ldots, x_k)$ and $\varphi_{\bar{a}}(x_1, \ldots, x_k) \equiv \varphi_i(x_1, \ldots, x_k)$ for some $i \in \{1, \ldots, n_k\}$.

(⇐) Every formula $\varphi(x_1, \ldots, x_k) \equiv \bigvee_{i \in D} \varphi_i(x_1, \ldots, x_k)$, for some $D \subseteq \{1, 2, \ldots, n_k\}$ and $D \neq \emptyset$, define a relation closed by FO types since the disjunction implies the union of the FO types which are isolated by the formulae $\varphi_i(x_1, \ldots, x_k)$. If $R^{\mathcal{A}} = \emptyset$, then $\varphi(x_1, \ldots, x_k) \equiv x_1 \neq x_1 \wedge \cdots \wedge x_k \neq x_k$ □

Definition 2. *Let $k \geq 1$, the class of structures $\mathcal{C}$ is a* class with bounded FO types for k-tuples *if $\mathcal{C}$ has a set Φ^k of intra-isolating formulae for k-tuples. $\mathcal{C}$ is a* class with bounded FO types *when for each $k \geq 1$ it holds that $\mathcal{C}$ is a class with bounded FO types for k-tuples.*

Theorem 1. *Let $\mathcal{C}$ be a class with bounded FO types, then SO^F is equivalent to FO on $\mathcal{C}$.*

Proof. Let $\mathcal{A}$ be σ-structure in $\mathcal{C}$, and $\psi \equiv \exists^F X_{11} \ldots \exists^F X_{1s_1} \forall^F X_{21} \ldots \forall^F X_{2s_2} \ldots Q\, X_{k1} \ldots Q\, X_{ks_k} \varphi(X_{11}, \ldots, X_{1s_1}, X_{21}, \ldots, X_{2s_2}, \ldots, X_{k1}, \ldots, X_{ks_k})$ where $Q = \exists^F$ when k is odd and $Q = \forall^F$ when k is even. Let $\Phi^{r_{tu}} = \{\varphi_1, \ldots, \varphi_{n_{r_{tu}}}\}$ be a set of intra-isolating formulae for FO types for r_{tu}-tuples for $\mathcal{C}$ with $n_{r_{tu}}$ the cardinality of such set.

We define the set $\hat{\Gamma}_{r_{tu}} = \{\hat{\gamma}_{r_{ij}} | \hat{\gamma}_{r_{tu}} = \bigvee_{s \in D} \varphi_s(x_1, \ldots, x_{r_{tu}})$ for $D \subseteq \{1, \ldots, n_{r_{tu}}\}$, $D \neq \emptyset$ and $\varphi_s \in \Phi^{r_{tu}}\} \cup \{x_1 \neq x_1 \wedge \cdots \wedge x_{r_{tu}} \neq x_{r_{tu}}\}$. Note that $\hat{\Gamma}_{r_{tu}}$ is the set of FO(σ) formulas that define every possible r_{tu}-ary relation closed by FO types for r_{tu}-tuples on $\mathcal{C}$.

Then

$$\mathcal{A} \models \exists^F X_{11} \ldots \exists^F X_{1s_1} \forall^F X_{21} \ldots \forall^F X_{2s_2} \ldots Q\, X_{k1} \ldots Q\, X_{ks_k} \varphi(X_{11}, \ldots, X_{1s_1}, X_{21}, \ldots, X_{2s_2}, \ldots, X_{k1}, \ldots, X_{ks_k}) \text{ iff } \mathcal{A} \models \hat{\psi}_{\mathcal{C}}.$$

where

$$\hat{\psi}_{\mathcal{C}} \equiv \bigvee_{\hat{\gamma}_{r_{11}} \in \hat{\Gamma}_{r_{11}}} \cdots \bigvee_{\hat{\gamma}_{r_{1s_1}} \in \hat{\Gamma}_{r_{1s_1}}} \bigwedge_{\hat{\gamma}_{r_{21}} \in \hat{\Gamma}_{r_{21}}} \cdots \bigwedge_{\hat{\gamma}_{r_{2s_2}} \in \hat{\Gamma}_{r_{2s_2}}} \cdots \left(\bigvee / \bigwedge\right)_{\hat{\gamma}_{r_{k1}} \in \hat{\Gamma}_{r_{k1}}} \cdots \left(\bigvee / \bigwedge\right)_{\hat{\gamma}_{r_{ks_k}} \in \hat{\Gamma}_{r_{ks_k}}} \varphi(\hat{\gamma}_{r_{11}}/X_{11}, \ldots, \hat{\gamma}_{r_{1s_1}}/X_{1s_1}, \ldots \hat{\gamma}_{r_{21}}/X_{21}, \ldots, \hat{\gamma}_{r_{2s_2}}/X_{2s_2}, \hat{\gamma}_{r_{k1}}/X_{k1}, \ldots, \hat{\gamma}_{r_{ks_k}}/X_{ks_k}).$$

We note that all the sets $\hat{\Gamma}_{r_{tu}}$, for $1 \leq t \leq k$ and $1 \leq u \leq s_t$, are finite by Lemma 4. Then, since all disjunctions and conjunctions are finite, $\hat{\psi}_{\mathcal{C}} \in$ FO. $\square$

Example 1. Fixing $h \geq 1$, the class of full trees with arbitrary arity r and depth h is an example of a class with bounded FO types. This class has an infinite number of FO types but has a finite set of intra-isolating formulae. Given a tuple $\bar{a}$ in an r-ary full tree $\mathcal{T}_r$ and a tuple $\bar{b}$ in an $(r+1)$-ary full tree $\mathcal{T}_{r+1}$, the FO type of $\bar{a}$ is different to the FO type of $\bar{b}$ but they can satisfy the same intra-isolaiting formula. Nevertheless, for a particular full tree $\mathcal{T}_r$ if two tuples on $\mathcal{T}_r$ satisfy the same intra-isolating formula then they have the same FO type. In every full tree there are $h+1$ different FO types for elements. Since the elements at the same level have the same FO type. As an example, let $h = 2$ and $r = 3$, then we have the following FO types for pairs of elements:

We have an equivalence class C_1 for the pairs of elements which are siblings at level 2. This class is symmetrical. The equivalence class C_2 consists of pairs which are "cousins" at level 2. This class is symmetrical. The equivalence class C_3 consists of pairs of elements which are siblings at level 1. C_3 is symmetrical. The equivalence class C_4 consists of pairs where the first component is an element in level 1 and the second component is a child in level two. C_4 has an associated class C_5 with the symmetrical pairs. The class C_6 consists of pairs where the first component is the root node and the second component is an element of level 2. C_6 has an associated class C_7 with the symmetrical pairs. The class C_8 consists of the pairs where the first component is an element of level 1 and the second component is an element of level two which is not a child of the element of level 1. C_8 has an associated class C_9 with the symmetrical pairs. The class C_{10} consists of pairs where the first component is the root node and the second component is a child of root node. C_{10} has an associated class C_{11} with the symmetrical pairs. The class C_{12} consists of a reflexive pair with the root node. The class C_{13} consists of reflexive pairs with elements of level 1. The class C_{14} consists of reflexive pairs with elements of level 2. All of these classes of pairs can be expressed with FO formulae with two free variables and all of them satisfy an unique intra-isolating formula.

When we consider 3-tuples instead of 2-tuples the combinatorics grow but they remain finite independently of the arity of the tree. Then all queries in SO^F on the class of full tree with fixed depth can be translated to equivalent queries in FO.

5. SO^F over Sets

A special case of structures with bounded FO types is any recursive class of structures with a unary vocabulary. Note that in these structures the relations are sets. We can build formulae which isolate the FO types for elements in a given structure considering how each element participates in the different sets. Using this formulae we can build formulae which isolate the FO type for k-tuples in a given structure. It is interesting to see that, fixing a unary vocabulary, there is a finite set of intra-isolating formulae for FO types for k-tuples.

Definition 3. *Let $\sigma = \{S_1, \ldots, S_t\}$ be a unary vocabulary.* A binary conjunction *is a formula $\varphi_{i_1 \ldots i_t}(x) = l_{i_1} \wedge \cdots \wedge l_{i_t}$ with $l_{i_j} = S_j(x)$ when $i_j = 1$, and $l_{i_j} = \neg S_j(x)$ when $i_j = 0$ for $1 \leq j \leq t$ and $i_j \in \{0, 1\}$.*

For example, for $\sigma = \{S_1, S_2, S_3\}$ we have the following binary conjunctions:

$\varphi_{000}(x) = \neg S_1(x) \wedge \neg S_2(x) \wedge \neg S_3(x)$

$\varphi_{001}(x) = \neg S_1(x) \wedge \neg S_2(x) \wedge S_3(x)$

$\vdots$

$\varphi_{111}(x) = S_1(x) \wedge S_2(x) \wedge S_3(x)$

Lemma 5. *Let $\sigma = \{S_1, \ldots, S_t\}$ be a unary vocabulary with $t \geq 1$. Let $\mathcal{C} \subseteq B_\sigma$ and $\Phi^1 = \{\varphi_i(x) | i \in \{0, 1\}^t$ and φ_i is a binary conjunction$\}$. Then, Φ^1 is a set of intra-isolating formulae for FO types for elements on $\mathcal{C}$.*

Proof. We must prove that Φ^1 satisfies Def. 1.

Let $\mathcal{A} \in \mathcal{C}$. First we will see that condition *i)* holds. Let a_1 and a_2 be elements in $dom(\mathcal{A})$ such that $\mathcal{A} \models \varphi_i(x)[a_1]$ and $\mathcal{A} \models \varphi_i(x)[a_2]$ for $i \in \{0,1\}^t$, then the bijective function f that exchanges a_1 with a_2 and fixes the other elements in $dom(\mathcal{A})$ is an automorphism on $\mathcal{A}$. For $S_j \in \sigma$, with $1 \leq j \leq t$ it holds that $\mathcal{A} \models S_j(x)[a_1] \Leftrightarrow \mathcal{A} \models S_j(x)[a_2]$ since S_j is in $\varphi_i(x)$ either positively or negatively. All elements $a_l \notin \{a_1, a_2\}$ trivially satisfy $\mathcal{A} \models S_j(x)[a_l] \Leftrightarrow \mathcal{A} \models S_j(x)[a_l]$. Then for all $a_l \in dom(\mathcal{A})$ and for all $S_j \in \sigma$ it holds $\mathcal{A} \models S_j(x)[a_l] \Leftrightarrow \mathcal{A} \models S_j(x)[f(a_l)]$. Therefore $type^{FO}_{\mathcal{A}}(a_1) = type^{FO}_{\mathcal{A}}(a_2)$.

For condition *ii)* Let a and b satisfy $type^{FO}_{\mathcal{A}}(a) = type^{FO}_{\mathcal{A}}(b)$, let $i = i_1 \dots i_t$ with $i \in \{0,1\}^t$ and $i_j = 0$ when $\mathcal{A} \not\models S_j(x)[a]$ and $i_j = 1$ when $\mathcal{A} \models S_j(x)[a]$ for $1 \leq j \leq t$, then $\mathcal{A} \models \varphi_{i_1 \dots i_t}(x)[a]$ and $\mathcal{A} \models \varphi_{i_1 \dots i_t}(x)[b]$ but $\mathcal{A} \not\models \varphi_r(x)[a]$ and $\mathcal{A} \not\models \varphi_r(x)[b]$ for $r \in \{0,1\}^t$ and $r \neq i$.

We prove condition *iii)*.For i and j as above, let a be an element in $dom(\mathcal{A})$. Let $i_j = 0$ when $\mathcal{A} \models \neg S_j(x)[a]$ and $i_j = 1$ when $\mathcal{A} \models S_j(x)[a]$. Then $\mathcal{A} \models \varphi_{i_1,\dots,i_t}(x)[a]$ and $\varphi_{i_1,\dots,i_t}(x) \in \Phi^1$. □

Lemma 6. *Let $k \geq 1$, $t \geq 1$, $\sigma = \{S_1, \dots, S_t\}$ a unary vocabulary and $\mathcal{C} \subseteq B_\sigma$. Let $\Phi^k = \{\varphi_{u_1}(x_1) \wedge \dots \wedge \varphi_{u_k}(x_k) |\ \varphi_{u_l}$ is a binary conjunction, $u_l \in \{0,1\}^t$ and $1 \leq l \leq k\}$. Φ^k is a set of intra-isolating formulae for FO types for k-tuples on $\mathcal{C}$.*

Proof. Let $\mathcal{A} \in \mathcal{C}$. We prove the condition *i)* of Def. 1. Let $\bar{a}, \bar{b} \in dom(\mathcal{A})^k$ and $\varphi_{u_1}(x_1) \wedge \dots \wedge \varphi_{u_k}(x_k) \in \Phi^k$ such that $\mathcal{A} \models \varphi_{u_1}(x_1) \wedge \dots \wedge \varphi_{u_k}(x_k)[\bar{a}]$ and $\mathcal{A} \models \varphi_{u_1}(x_1) \wedge \dots \wedge \varphi_{u_k}(x_k)[\bar{b}]$. Let f_l be the bijective function that exchange a_l with b_l for $1 \leq l \leq k$ and fixes the other elements in $dom(\mathcal{A})$, f_l is an automorphism as we show in Lemma 5. Then the composition $f_1 \circ \dots \circ f_k$ is an automorphism that exchanges the k-tuple $\bar{a}$ with the k-tuple $\bar{b}$, and fixes the other elements in $dom(\mathcal{A})$. Therefore $type^{FO}_{\mathcal{A}}(\bar{a}) = type^{FO}_{\mathcal{A}}(\bar{b})$.

For condition *ii)* we suppose that for k-tuples $\bar{a}$ and $\bar{b}$ it holds that$type^{FO}_{\mathcal{A}}(\bar{a}) = type^{FO}_{\mathcal{A}}(\bar{b})$. Let $\varphi_{u_1}(x_1) \wedge \dots \wedge \varphi_{u_k}(x_k)$ with $u_l \in \{0,1\}^t$ and $u_l = i_1 \dots i_t$ such that $\varphi_{i_1 \dots, i_t}(x)$ is as the proof of Lemma 5 and $1 \leq l \leq k$. Then $\mathcal{A} \models \varphi_{u_l}(x_l)[a_l]$ and $\mathcal{A} \models \varphi_{u_l}(x_l)[b_l]$ and for all $r \in \{0,1\}^t$ such that $u_l \neq r$, $\mathcal{A} \not\models \varphi_r(x_l)[a_l]$ and $\mathcal{A} \not\models \varphi_r(x_l)[b_l]$ for $1 \leq l \leq k$. Therefore $\mathcal{A} \models \varphi_{u_1}(x_1) \wedge \dots \wedge \varphi_{u_k}(x_k)[\bar{a}]$ and $\mathcal{A} \models \varphi_{u_1}(x_1) \wedge \dots \wedge \varphi_{u_k}(x_k)[\bar{b}]$,with $\varphi_{u_1}(x_1) \wedge \dots \wedge \varphi_{u_k}(x_k) \in \Phi^k$ and for all $\varphi_{v_1}(x_1) \wedge \dots \wedge \varphi_{v_k}(x_k) \in \Phi^k$ with $v_1 \dots v_k \in \{0,1\}^t$ and $u_1 \dots u_k \neq v_1 \dots v_k$ it holds that $\mathcal{A} \not\models \varphi_{v_1}(x_1) \wedge \dots \wedge \varphi_{v_k}(x_k)[\bar{a}]$ and $\mathcal{A} \not\models \varphi_{v_1}(x_1) \wedge \dots \wedge \varphi_{v_k}(x_k)[\bar{b}]$

We prove conowndition *iii)*. Let $\bar{a} = (a_1, \dots, a_k) \in dom(\mathcal{A})^k$. Let $u_{l_j} = 0$ when $\mathcal{A} \models \neg S_j(x_l)[a_l]$ and $u_{lj} = 1$ when $\mathcal{A} \models S_j(x_l)[a_l]$ for $1 \leq l \leq k$ and $1 \leq j \leq t$, then $\mathcal{A} \models \varphi_{u_l}(x_l)[a_l]$ for $u_l = u_{l_1} \dots u_{l_t}$. Therefore $\mathcal{A} \models \varphi_{u_1}(x_1) \wedge \dots \wedge \varphi_{u_k}(x_k)[\bar{a}]$ and $\varphi_{u_1}(x_1) \wedge \dots \wedge \varphi_{u_k}(x_k) \in \Phi^k$. □

Theorem 2. *Let $\mathcal{C}$ be a class of structures with unary vocabulary, then SO^F is equivalent to FO on $\mathcal{C}$.*

Proof. Let $t \geq 1$, and $\sigma = \{S_1, \dots, S_t\}$, and $\mathcal{C} \in B_\sigma$ then, for all $k \geq 1$, by Lemma 6, $\Phi^k = \{\varphi_{u_1}(x_1) \wedge \dots \wedge \varphi_{u_k}(x_k) | \varphi_{u_l}$ is a binary conjuntion, $u_l \in$

$\{0,1\}^t$ and $1 \leq l \leq k\}$, is a set of intra-isolating formulae for FO types for k-tuples over $\mathcal{C}$. Therefore, by Theorem 1, SO^F is equivalent to FO on $\mathcal{C}$. □

Among other queries, the parity query[3] over sets is not expressible in SO^F since as it is well known parity is not expressible in FO. However this query is expressible in SO for structures of arbitrary vocabulary, in particular for unary vocabulary. Then, the following Corollary is immediate.

Corollary 1. *SO^F is strictly included in SO.*

6. Conclusion

In the existential fragment $\Sigma_1^{1,F}$ of SO^F we can express co-NP problems . In [4] we proved that there are NP complete problems that can be expressed in $\Sigma_1^{1,F}$. However we cannot express in full SO^F the parity query which is in P. Then we can conclude that different logics allow orthogonal classifications of the problems with respect to the classic classification of computational complexity. These orthogonal classifications can be used to refine the classic computational complexity classes, providing us more information about certain problems.

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[3] class of σ-structures with an even domain