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Singular value estimates of oblique projections

Jorge Antezana [a,c,1], Gustavo Corach [b,c,*,2]

[a] Departamento de Matemática, FCE-UNLP, La Plata, Argentina
[b] Departamento de Matemática, FI-UBA, Buenos Aires, Argentina
[c] IAM-CONICET, Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina

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Abstract

Let $\mathscr{W}$ and $\mathscr{M}$ be two finite dimensional subspaces of a Hilbert space $\mathscr{H}$ such that $\mathscr{H} = \mathscr{W} \oplus \mathscr{M}^{\perp}$, and let $P_{\mathscr{W} \| \mathscr{M}^{\perp}}$ denote the oblique projection with range $\mathscr{W}$ and nullspace $\mathscr{M}^{\perp}$. In this article we get the following formula for the singular values of $P_{\mathscr{W} \| \mathscr{M}^{\perp}}$:

$$2(s_k(P_{\mathscr{W} \| \mathscr{M}^{\perp}}) - 1) = \min_{(F,H)\in\mathfrak{X}(\mathscr{W},\mathscr{M})} s_k(F - H)^2,$$

where the minimum is taken over the set of all operator pairs (F, H) on $\mathscr{H}$ such that $R(F) = \mathscr{W}$, $R(H) = \mathscr{M}$ and $FH^* = P_{\mathscr{W} \| \mathscr{M}^{\perp}}$, and $k \in \{1, \ldots, \dim \mathscr{W}\}$. We also characterize all the pairs where the minimum is attained.

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1. Introduction

Given a Hilbert space $\mathscr{H}$, consider a decomposition of $\mathscr{H}$ as a direct sum of two subspaces $\mathscr{H} = \mathscr{W} \oplus \mathscr{M}^{\perp}$, and consider the oblique projection associated to this decomposition denoted

* Corresponding author. Address: Departamento de Matematica, FI-UBA, Buenos Aires, Argentina..
E-mail addresses: antezana@mate.unlp.edu.ar (J. Antezana), gcorach@fi.uba.ar (G. Corach).
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by $P_{\mathscr{W}\|\mathscr{M}^\perp}$. If $L(\mathscr{H})$ denote the algebra of bounded operators on $\mathscr{H}$, let $\mathfrak{X}(\mathscr{W}, \mathscr{M})$ be the subset of $L(\mathscr{H}) \times L(\mathscr{H})$ defined by

$$\mathfrak{X}(\mathscr{W}, \mathscr{M}) := \{(F, H) : R(F) = \mathscr{W}, R(H) = \mathscr{M} \text{ and } FH^* = P_{\mathscr{W}\|\mathscr{M}^\perp}\}.$$

In [2], it is proved that $\min \|F - H\|^2$ exists and it is equal to $2(\|P_{\mathscr{W}\|\mathscr{M}^\perp}\| - 1)$, where the minimum is taken over all pairs $(F, H) \in \mathfrak{X}(\mathscr{W}, \mathscr{M})$ (the notation used there for this set was $\mathfrak{X}_Q$, where $Q = P_{\mathscr{W}\|\mathscr{M}^\perp}$). There are many minimizing pairs, and some of them have been determined. The present paper is devoted to a similar problem, this time for singular values instead of the operator norm. More precisely, if $\mathscr{W}$ (and therefore $\mathscr{M}$) has a finite dimension, say n, then we prove that

$$\min s_k^2(F - H) = 2(s_k(P_{\mathscr{W}\|\mathscr{M}^\perp}) - 1) \tag{1}$$

for $k \in \{1, 2, \ldots, n\}$, and we find all minimizing pairs (F, H). These results, which are obvious if $\mathscr{W} = \mathscr{M}$ because in this case $P_{\mathscr{W}\|\mathscr{M}^\perp}$ is the orthogonal projection onto $\mathscr{W}$, $(P_{\mathscr{W}\|\mathscr{M}^\perp}, P_{\mathscr{W}\|\mathscr{M}^\perp}) \in \mathfrak{X}(\mathscr{W}, \mathscr{M})$ and therefore both members of (1) vanish, are not evident in the oblique case.

The paper is organized as follows: Section 2 contains preliminaries and a description of the tools needed for the proofs: an operator version of the arithmetic–geometric inequality, some 2×2 matrix computations and elementary facts about singular values. In Section 3, we state the main results of this paper. Section 4 is devoted to the proof of the results stated in the previous section.

1.1. Motivation of the problem

The results of this paper have a direct translation to frame theory and sampling formulae, and they have been motivated by practical problems that appear in those areas. Let PW be the subspace of all $f \in L^2(R)$ whose Fourier transform has support contained in the interval $[-\pi, \pi]$. Then, the classical Shannon (or Whittaker–Kotelnikov–Shannon, WKS) formula

$$f(x) = \sum f(n)\operatorname{sinc}(x - n), \quad f \in PW$$

is one of the first examples of sampling formulae, frequently used in sampling theory and signal processing. The facts that $s_n(x) = \operatorname{sinc}(x - n)$ form an orthonormal basis of PW and that $f(n) = \langle f, s_n\rangle$, first noticed by Hardy [14], show that

$$Pf = \sum \langle f, s_n\rangle s_n, \quad f \in \mathscr{H}$$

is the orthogonal projection onto PW, and is one of the obvious factorizations we mentioned above. In the survey by Unser [25] the reader can find historical notices and applications of the WKS formula, as well as a projection-based view of some sampling problems. Indeed, in modern sampling theory, factorizations of projections appear frequently. In fact, if $\mathscr{S}$ is a subspace of a space $\mathscr{H}$ of functions defined on a set X, a sampling formula is a collection of expansions like

$$f(x) = \sum f(t_n) f_n(x), \quad f \in \mathscr{S},$$

where $\{t_n\}_{n\in\mathbb{N}}$ is a sequence in X and $\{f_n\}$ is a sequence in $\mathscr{H}$ such that the expansions converge in a certain topology on $\mathscr{H}$. If $\mathscr{H}$ is a reproducing kernel Hilbert space, each evaluation functional, a fortiori the evaluations at t_n, is bounded and by Riesz representation theorem there exists a sequence $\{h_n\}_{n\in\mathbb{N}}$ in $\mathscr{H}$ such that the sampling formula above becomes

$$f = \sum \langle f, h_n\rangle f_n, \quad f \in \mathscr{H}.$$

It turns out that, under reasonable hypothesis on $\{f_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$, the expansion converges, not only for elements of $\mathscr{S}$ but also for every $f \in \mathscr{H}$, to an element of $\mathscr{S}$. Thus

$$Qf = \sum \langle f, h_n \rangle f_n, \quad f \in \mathscr{H}$$

defines a bounded linear projection on $\mathscr{H}$ with image $\mathscr{S}$. Moreover, if $\{e_n\}_{n\in\mathbb{N}}$ is the canonical basis of ℓ^2, then $Fe_n = f_n$ and $He_n = h_n$ define bounded operators $F, H : \ell^2 \to \mathscr{H}$ and $Q = FH^*$; $\{f_n\}_{n\in\mathbb{N}}$ is called the sequence of reconstruction vectors and $\{h_n\}_{n\in\mathbb{N}}$ that of sampling vectors.

The study of these type of factorizations as well as estimation for the norm of oblique projections are very useful to study different problems in modern harmonic analysis. For instance it has been used to study the biorthogonality of two multiresolution analyses , problems on perturbation of frames , and problems concerning sampling theory (see for example [16–18,8,19–21,26,7,3,9] and the references cited therein).

2. Preliminaries

Given a separable Hilbert space $\mathscr{H}$, $L(\mathscr{H})$ denotes the algebra of bounded linear operators on $\mathscr{H}$, and $L_f(\mathscr{H})$ the ideal of operators with finite dimensional range. Given $A \in L(\mathscr{H})$, $R(A)$ denotes the range or image of A, $N(A)$ the nullspace of A, $\sigma(A)$ the spectrum of A, A^* the adjoint of A, $|A| = (A^*A)^{1/2}$ the absolute value of A, $\|A\|$ the spectral norm of A.

If $\mathscr{H} = \mathscr{W} \oplus \mathscr{M}^\perp$ then the projection onto $\mathscr{W}$ defined by this decomposition is denoted by $P_{\mathscr{W}\|\mathscr{M}^\perp}$. Observe that $P^*_{\mathscr{W}\|\mathscr{M}^\perp} = P_{\mathscr{M}\|\mathscr{W}^\perp}$. In the case of orthogonal projections, i.e., $\mathscr{W} = \mathscr{M}$, we write $P_{\mathscr{W}}$ instead of $P_{\mathscr{W}\|\mathscr{W}^\perp}$.

Given $A \in L_f(\mathscr{H})$, $s_1(A), s_2(A), \ldots$ denote the singular values of A arranged in non-increasing order, $\operatorname{tr}(A)$ the trace of A and $\|A\|_{\mathrm{F}}$ the Frobenius norm of A. Recall that $\|A\|_{\mathrm{F}}^2 = \operatorname{tr}(A^*A) = \sum_k s_k(A)^2$.

Remark 2.1. Throughout this paper we consider infinite and finite dimensional Hilbert spaces. In the first case, the sub-indexes of the singular values run over all the positive integers, while in the second case they belong to the set $\{1, \ldots, \dim \mathscr{H}\}$.

The following well-known operator version of the arithmetic-geometric inequality (see [5,1, 10]) is a key result in what follows:

Proposition 2.2. *Given $C, D \in L(\mathscr{H})$, then*

$$\|CD^*\| \leqslant \left\| \frac{|C|^2 + |D|^2}{2} \right\|.$$

If $C, D \in L_f(\mathscr{H})$, then

$$s_k(CD^*) \leqslant s_k\left(\frac{|C|^2 + |D|^2}{2}\right) \quad \forall k$$

and the equality for every k holds if and only if $|C|^2 = |D|^2$.

We end this preliminary section by recalling some basic facts on generalized inverses. The reader is referred to the books by Nashed [23], and Ben-Israel and Greville [6] for more information.

Definition 2.3. Let $A \in L(\mathcal{H})$. A *generalized inverse* of A is an operator $B \in L(\mathcal{H})$ such that $ABA = A$ and $BAB = B$.

It is a well-known fact that A has a (bounded) generalized inverse if and only if $R(A)$ is closed. In that case, the next proposition relates generalized inverses with oblique projections.

Proposition 2.4. *Let $A \in L(\mathcal{H})$ be a closed range operator*

(1) *If $B \in L(\mathcal{H})$ is a generalized inverse of A, then*:
- *AB is an oblique projection onto $R(A)$.*
- *BA is an oblique projection whose nullspace is $N(A)$.*

(2) *Given a pair of projections $Q, \widetilde{Q} \in L(\mathcal{H})$ such that $R(Q) = R(A)$ and $N(\widetilde{Q}) = N(A)$, there is a unique generalized inverse B of A such that $AB = Q$ and $BA = \widetilde{Q}$. In particular the unique one associated to the orthogonal projections $P_{R(A)}$ and $P_{N(A)^\perp}$ is called Moore–Penrose generalized inverse and it is denoted by $A^\dagger$. In terms of $A^\dagger$, the unique generalized inverse associated to the pair $(Q, \widetilde{Q})$ can be written in the following way*:

$$B = \widetilde{Q} A^\dagger Q.$$

3. Statements

In this section we state the main result of this paper, postponing its proof until the next section. Given two closed subspaces $\mathcal{W}$ and $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{M}^\perp$, recall that $\mathfrak{X}(\mathcal{W}, \mathcal{M})$ denotes the subset of $L(\mathcal{H}) \times L(\mathcal{H})$ defined by

$$\mathfrak{X}(\mathcal{W}, \mathcal{M}) := \{(F, H) : R(F) = \mathcal{W},\ R(H) = \mathcal{M} \text{ and } FH^* = P_{\mathcal{W}\|\mathcal{M}^\perp}\}.$$

Note that the pair $(P_{\mathcal{W}\|\mathcal{M}^\perp}, P^*_{\mathcal{W}\|\mathcal{M}^\perp}) = (P_{\mathcal{W}\|\mathcal{M}^\perp}, P_{\mathcal{M}\|\mathcal{W}^\perp})$ always belongs to this set, hence it is non-empty.

Theorem 3.1. *Let $\mathcal{W}$ and $\mathcal{M}$ be finite dimensional subspaces of a Hilbert space $\mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{M}^\perp$. Then for $(F, H) \in \mathfrak{X}(\mathcal{W}, \mathcal{M})$*

$$s_k(F - H)^2 \geqslant \begin{cases} 2(s_k(P_{\mathcal{W}\|\mathcal{M}^\perp}) - 1) & if\ k \in \{1, \dots, n\}, \\ 0 & if\ k > n, \end{cases} \tag{2}$$

where $n = \dim \mathcal{W}$ $(= \dim \mathcal{M})$ and $k \leqslant \dim \mathcal{H}$ or $k \in \mathbb{N}$ if $\dim \mathcal{H} = \infty$. Moreover, given F_0 with $R(F_0) = \mathcal{W}$, if $H_0 = (F_0^\dagger P_{\mathcal{W}\|\mathcal{M}^\perp})^*$ then $(F_0, H_0) \in \mathfrak{X}(\mathcal{W}, \mathcal{M})$, and the equality for every $k \in \mathbb{N}$ is attained precisely at those pairs (F_0, H_0) that also satisfy $F_0 F_0^* = |P^*_{\mathcal{W}\|\mathcal{M}^\perp}| = |P_{\mathcal{M}\|\mathcal{W}^\perp}|$.

Remark 3.2. Note that, one of the consequences of Theorem 3.1 is the following identity:

$$2(\|P_{\mathcal{W}\|\mathcal{M}^\perp}\| - 1) = \min_{(F,H) \in \mathfrak{X}(\mathcal{W}, \mathcal{M})} \|F - H\|^2. \tag{3}$$

As we mentioned in the introduction, this identity has been proved in [2], not only for finite dimensional spaces but also for for infinite dimensional closed subspaces. However, a complete characterization of the pairs $(F, H) \in \mathfrak{X}(\mathcal{W}, \mathcal{M})$ where the minimum is attained in (3) is still unknown. If we only look for minimizers for the spectral norm, besides the pairs (F_0, H_0) such that $F_0F_0^* = |P^*_{\mathscr{W}\|\mathscr{M}^\perp}| = |P_{\mathscr{M}\|\mathscr{W}^\perp}|$ and $H_0 = (F_0^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*$ there may be more.

Remark 3.3. Theorem 3.1 can be restated in terms of the so-called principal angles between subspaces. Recall that, given two (non-trivial) finite dimensional subspaces $\mathscr{W}$ and $\mathscr{M}$ of a Hilbert space the principal angles between $\mathscr{W}$ and $\mathscr{M}$ are defined as the values θ_k in $[0, \pi/2]$ whose cosines are the nonzero singular values of $P_{\mathscr{M}}P_{\mathscr{W}}$ (see [22,11,12,27]). If in addition $\mathscr{H} = \mathscr{W} \oplus \mathscr{M}^\perp$, as in Theorem 3.1, then $P_{\mathscr{W}\|\mathscr{M}^\perp} = (P_{\mathscr{M}}P_{\mathscr{W}})^\dagger$. Indeed, as $\mathscr{H} = \mathscr{W} \oplus \mathscr{M}^\perp$, we get

$$R(P_{\mathscr{M}}P_{\mathscr{W}}) = \mathscr{M} \quad \text{and} \quad R(P_{\mathscr{W}}P_{\mathscr{M}}) = \mathscr{W}.$$

On the other hand,

$$(P_{\mathscr{M}}P_{\mathscr{W}})P_{\mathscr{W}\|\mathscr{M}^\perp} = P_{\mathscr{M}}P_{\mathscr{W}\|\mathscr{M}^\perp} = P_{\mathscr{M}},$$
$$P_{\mathscr{W}\|\mathscr{M}^\perp}(P_{\mathscr{M}}P_{\mathscr{W}}) = P_{\mathscr{W}\|\mathscr{M}^\perp}P_{\mathscr{W}} = P_{\mathscr{W}}$$

and therefore $P_{\mathscr{W}\|\mathscr{M}^\perp} = (P_{\mathscr{M}}P_{\mathscr{W}})^\dagger$ as we claimed (see also [13]). This implies that the non zero singular values of $P_{\mathscr{W}\|\mathscr{M}^\perp}$ are the secant of the principal angles between $\mathscr{W}$ and $\mathscr{M}$. Therefore, formulae (2) can be rewritten in terms of principal angles as follows: for every $(F, H) \in \mathfrak{X}(\mathscr{M}, \mathscr{N})$ and every $k \in \{1, \ldots, \dim \mathscr{W}\}$:

$$\cos(\theta_k) \geqslant \frac{2}{2 + s_{n-k+1}(F-H)^2}.$$

The following estimate of the trace norm of an oblique projection can be also obtained as a consequence of Theorem 3.1:

Corollary 3.4. *Let $\mathscr{W}$ and $\mathscr{M}$ be finite dimensional subspaces of a Hilbert space $\mathscr{H}$ such that $\mathscr{H} = \mathscr{W} \oplus \mathscr{M}^\perp$. Then, for every pair $(F, H) \in \mathfrak{X}(\mathscr{W}, \mathscr{M})$*

$$\|P_{\mathscr{W}\|\mathscr{M}^\perp}\|_1 \leqslant \frac{2n + \|F - H\|_2}{2},$$

where $n = \dim \mathscr{W} = \dim \mathscr{M}$.

4. Proof of the main result

Let $f : [0, +\infty) \to [0, +\infty)$ be the function defined by $f(x) = x + \frac{b}{x}$, where $b > 0$. A simple analysis of this function shows that it attains a global minimum at $x = \sqrt{b}$ and $f(\sqrt{b}) = 2\sqrt{b}$. The first step towards a proof of Theorem 3.1 is an extension of this result to operators on Hilbert spaces. The proof of this generalization is a simple consequence of the arithmetic-geometric inequality stated in Proposition 2.2:

Proposition 4.1. *Let $B \in L(\mathscr{H})$ be a positive and invertible operator. Then, for every positive invertible operator $A \in L(\mathscr{H})$ it holds that*

$$2\|B^{1/2}\| \leqslant \|A + A^{-1/2}BA^{-1/2}\|. \tag{4}$$

If $\dim \mathscr{H} = n < \infty$, then

$$2s_k(B^{1/2}) \leqslant s_k(A + A^{-1/2}BA^{-1/2}) \quad \forall k \in \{1, \ldots, n\}. \tag{5}$$

Moreover, the equality for all $k \in \{1, \ldots, n\}$ holds if and only if $A = B^{1/2}$.

Proof. Use the arithmetic–geometric inequality (Proposition 2.2) with $C = A^{1/2}$ and $D = B^{1/2}A^{-1/2}$. $\square$

In order to prove Theorem 3.1, we also need the following lemmas:

Lemma 4.2. *Let $\mathscr{W}$ and $\mathscr{M}$ be two closed subspaces of a Hilbert space $\mathscr{H}$ such that $\mathscr{H} = \mathscr{W} \oplus \mathscr{M}^{\perp}$, and let $(F, H) \in \mathfrak{X}(\mathscr{W}, \mathscr{M})$. Then FH^* and H^*F are projections with $R(FH^*) = R(F)$ and $N(H^*F) = N(F)$ such that*

$$H^* = H^*FF^{\dagger}FH^*. \tag{6}$$

Proof. Since by assumption $FH^* = P_{\mathscr{W}\|\mathscr{M}^{\perp}}$ and $R(F) = \mathscr{W}$, FH^* is a projection and $R(FH^*) = R(F)$.

As $R(H) = \mathscr{M} = N(P_{\mathscr{W}\|\mathscr{M}^{\perp}})^{\perp}$, then $N(H^*) = \mathscr{M}^{\perp}$. On the other hand,

$$R(I - FH^*) = N(P_{\mathscr{W}\|\mathscr{M}^{\perp}}) = \mathscr{M}^{\perp}.$$

So, we can conclude that $H^*(I - FH^*) = 0$, that is, $H^* = H^*FH^*$. In particular this proves that H^*F is a projection because

$$(H^*F)^2 = H^*FH^*F = H^*F.$$

Moreover, since $R(F) = \mathscr{W}$ and $N(H^*) = \mathscr{M}^{\perp}$, by assumption $R(F) \cap N(H) = \{0\}$. This implies that $N(H^*F) = N(F)$.

Finally, as $FF^{\dagger}F = F$ (Proposition 2.4) we obtain

$$H^*FF^{\dagger}FH^* = H^*FH^* = H^*.$$

which concludes the proof. $\square$

Lemma 4.3. *Let $\mathscr{W}$ and $\mathscr{M}$ be two closed subspaces of a Hilbert space $\mathscr{H}$ such that $\mathscr{H} = \mathscr{W} \oplus \mathscr{M}^{\perp}$, and let $(F, H) \in \mathfrak{X}(\mathscr{W}, \mathscr{M})$. Then*

$$|(F - (F^{\dagger}P_{\mathscr{W}\|\mathscr{M}^{\perp}})^*)^*|^2 \leqslant |(F - H)^*|^2 \quad \forall k \in \mathbb{N}. \tag{7}$$

Proof. By Lemma 4.2, $H^* = QF^{\dagger}P_{\mathscr{W}\|\mathscr{M}^{\perp}}$ where $Q = H^*F$ is an oblique projection such that $N(Q) = N(F)$. So, we obtain that

$$\begin{aligned} |(F - H)^*|^2 &= FF^* + HH^* - (P_{\mathscr{W}\|\mathscr{M}^{\perp}} + P^*_{\mathscr{W}\|\mathscr{M}^{\perp}}) \\ &= FF^* + P^*_{\mathscr{W}\|\mathscr{M}^{\perp}}(F^{\dagger})^*Q^*QF^{\dagger}P_{\mathscr{W}\|\mathscr{M}^{\perp}} - (P_{\mathscr{W}\|\mathscr{M}^{\perp}} + P^*_{\mathscr{W}\|\mathscr{M}^{\perp}}). \end{aligned}$$

Consider the matrix representation of Q with respect to the decomposition $\mathscr{H} = N(F)^{\perp} \oplus N(F)$

$$Q = \begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}.$$

In this representation, the (1,2)- and (2,2)-entries are zero because $N(Q) = N(F)$. On the other side, since $FH^* = P_{\mathscr{W}\|\mathscr{M}^{\perp}}$ and $R(F) = \mathscr{W}$, it holds that $FH^*FF^* = FF^*$, or equivalently

$$\langle H^*FF^*x, F^*y\rangle = \langle F^*x, F^*y\rangle$$

for every $x, y \in \mathscr{H}$. This shows that the (1,1)-entry is I. Using the above matrix representation of Q we obtain that

$$Q^*Q = \begin{pmatrix} 1 & x^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix} = \begin{pmatrix} 1+x^*x & 0 \\ 0 & 0 \end{pmatrix} \geqslant \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_{N(F)^\perp}. \tag{8}$$

Thus, as $R(F^\dagger) = N(F)^\perp$, we have

$$\begin{aligned}|(F-H)^*|^2 &\geqslant FF^* + P^*_{\mathscr{W}\|\mathscr{M}^\perp}(F^\dagger)^*F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp} - (P_{\mathscr{W}\|\mathscr{M}^\perp} + P^*_{\mathscr{W}\|\mathscr{M}^\perp}) \\ &= |(F-(F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*)^*|^2,\end{aligned}$$

which proves the lemma. □

Corollary 4.4. *Let F, H and $P_{\mathscr{W}\|\mathscr{M}^\perp}$ as in Theorem 3.1. Then*

$$s_k(F-(F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*) \leqslant s_k(F-H) \quad \forall k \tag{9}$$

and the equality for every k holds if and only if $H = (F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*$.

Proof. Using the so-called minimax principle for singular values (see [24,4, p. 75]) and Lemma 4.3, we get for every $k \in \mathbb{N}$

$$\begin{aligned}s_k((F-(F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*)^*)^2 &= \max_{\mathscr{S}\subseteq\mathscr{H},\dim\mathscr{S}=k}\ \min_{x\in\mathscr{S},\|x\|=1} \langle |(F-(F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*)^*|^2x, x\rangle \\ &\leqslant \max_{\mathscr{S}\subseteq\mathscr{H},\dim\mathscr{S}=k}\ \min_{x\in\mathscr{S},\|x\|=1} \langle |(F-H)^*|^2x, x\rangle \\ &= s_k((F-H)^*)^2\end{aligned}$$

and inequality (9) follows by taking square roots and using that $s_k((F-H)^*) = s_k(F-H)$ and $s_k((F-(F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*)^*) = s_k(F-(F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*)$ for every $k \in \mathbb{N}$. In order to prove the uniqueness part, suppose that the equality in (9) holds for every k. Then

$$\begin{aligned}\operatorname{tr}(|(F-H)^*|^2) &= \sum_{k=1}^{\infty} s_k(F-H)^2 \\ &= \sum_{k=1}^{\infty} s_k(F-(F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*)^2 \\ &= \operatorname{tr}(|(F-(F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*)^*|^2).\end{aligned}$$

Expanding the absolute values inside both traces and using the linearity of the trace we obtain

$$\operatorname{tr}(P^*_{\mathscr{W}\|\mathscr{M}^\perp}(F^\dagger)^*Q^*QF^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp}) = \operatorname{tr}(P^*_{\mathscr{W}\|\mathscr{M}^\perp}(F^\dagger)^*F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp}).$$

Since $R(F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp}) = N(F)^\perp$ and $Q^*Q \geqslant P_{N(F)^\perp}$, this equality implies that $Q^*Q = P_{N(F)^\perp}$, which holds if and only if $Q = P_{N(F)^\perp}$. □

Proof (*Proof of Theorem* 3.1). Let $F \in L(\mathscr{H})$ such that $R(F) = \mathscr{W}$ and let $H := (F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*$. To show that $(F, H) \in \mathfrak{X}(\mathscr{W}, \mathscr{M})$, we have to prove the relations

$$R(H) = \mathscr{M} \quad \text{and} \quad FH^* = P_{\mathscr{W}\|\mathscr{M}^\perp}.$$

Since by definition

$$R(H) = N(H^*)^\perp = N(F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^\perp$$

and $F^\dagger$ is injective on $R(F) = R(P_{\mathscr{W}\|\mathscr{M}^\perp})$, we can conclude

$$R(H) = N(P_{\mathscr{W}\|\mathscr{M}^\perp})^\perp = (\mathscr{M}^\perp)^\perp = \mathscr{M}.$$

Next, as $FF^\dagger = P_{R(F)}$ and $R(P_{\mathscr{W}\|\mathscr{M}^\perp}) = R(F) = \mathscr{W}$, one has

$$FH^* = FF^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp} = P_{\mathscr{W}\|\mathscr{M}^\perp}$$

proving the relations. Therefore $(F, H) \in \mathfrak{X}(\mathscr{W}, \mathscr{M})$. Moreover, by Corollary 4.4 it is enough to prove the theorem for the pairs (F, H) so that $R(F) = \mathscr{W}$ and $H = (F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*$. Thus, let (F, H) be one of such pairs. The decomposition $\mathscr{H} = \mathscr{W} \oplus \mathscr{W}^\perp$ induces the following 2×2 matrix representation of $P_{\mathscr{W}\|\mathscr{M}^\perp}$ and FF^*:

$$P_{\mathscr{W}\|\mathscr{M}^\perp} = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \quad FF^* = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},$$

where $a : \mathscr{W} \to \mathscr{W}$ is invertible because $R(F) = \mathscr{W}$. Note that, as the projection $P_{\mathscr{W}\|\mathscr{M}^\perp}$ is fixed, the operator x is also fixed.

Since $FF^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp} = P_{\mathscr{W}\|\mathscr{M}^\perp}$

$$FF^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp} = P_{\mathscr{W}\|\mathscr{M}^\perp} = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (F^\dagger)^* F^\dagger = (FF^*)^\dagger = \begin{pmatrix} a^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$\begin{aligned}
&(F - (F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*)(F^* - F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp}) \\
&= FF^* - (FF^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^* - FF^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp} + P^*_{\mathscr{W}\|\mathscr{M}^\perp}(F^\dagger)^* F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp} \\
&= FF^* - P^*_{\mathscr{W}\|\mathscr{M}^\perp} - P_{\mathscr{W}\|\mathscr{M}^\perp} + P^*_{\mathscr{W}\|\mathscr{M}^\perp}(FF^*)^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp} \\
&= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2 & x \\ x^* & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ x^* & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} a + a^{-1} - 2 & (a^{-1} - 1)x \\ x^*(a^{-1} - 1) & x^* a^{-1} x \end{pmatrix} \\
&= \begin{pmatrix} a^{-1/2} - a^{1/2} & 0 \\ x^* a^{-1/2} & 0 \end{pmatrix} \cdot \begin{pmatrix} a^{-1/2} - a^{1/2} & 0 \\ x^* a^{-1/2} & 0 \end{pmatrix}^*.
\end{aligned}$$

This implies

$$\begin{aligned}
s_k(F - (F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*)^2 &= s_k\left(\begin{pmatrix} a^{-1/2} - a^{1/2} & 0 \\ x^* a^{-1/2} & 0 \end{pmatrix}\right)^2 \\
&= s_k\left(\begin{pmatrix} a^{-1/2} - a^{1/2} & a^{-1/2}x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^{-1/2} - a^{1/2} & 0 \\ x^* a^{-1/2} & 0 \end{pmatrix}\right) \\
&= s_k\left(\begin{pmatrix} a^{-1} + a - 2 + a^{-1/2} x x^* a^{-1/2} & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&= s_k\left(\begin{pmatrix} a + a^{-1/2}(1 + xx^*) a^{-1/2} - 2 & 0 \\ 0 & 0 \end{pmatrix}\right).
\end{aligned}$$

Therefore, it holds that

$$s_k(F - (F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*)^2 = \begin{cases} s_k(a + a^{-1/2}(1 + xx^*)a^{-1/2}) - 2 & \text{if } 1 \leqslant k \leqslant n, \\ 0 & \text{if } k > n. \end{cases} \tag{10}$$

Since $\dim \mathscr{W} = n < \infty$, we can use Proposition 4.1 and get for every $k \in \{1, \ldots, n\}$

$$\begin{aligned} s_k(F - (F^\dagger P_{\mathscr{W}\|\mathscr{M}^\perp})^*)^2 &= s_k(a + a^{-1/2}(1 + xx^*)a^{-1/2}) - 2 \\ &\geqslant 2s_k((1 + xx^*)^{1/2}) - 2 = 2s_k((P_{\mathscr{W}\|\mathscr{M}^\perp} P^*_{\mathscr{W}\|\mathscr{M}^\perp})^{1/2}) - 2 \\ &= 2(s_k(P_{\mathscr{W}\|\mathscr{M}^\perp}) - 1), \end{aligned}$$

which concludes the proof of (2). On the other side, the equality holds for every $k \in \{1, \ldots, n\}$ if and only if $s_k(a + a^{-1/2}(1 + xx^*)a^{-1/2}) = 2s_k((1 + xx^*)^{1/2})$. So, by Proposition 4.1, it holds if and only if $a = (1 + xx^*)^{1/2}$, which is equivalent to $FF^* = |P^*_{\mathscr{W}\|\mathscr{M}^\perp}| = |P_{\mathscr{M}\|\mathscr{W}^\perp}|$. The equality for $k > n$ follows from (10). This completes the proof. □

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