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Some properties of frames of subspaces obtained by operator theory methods

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Abstract

We study the relationship among operators, orthonormal basis of subspaces and frames of subspaces (also called fusion frames) for a separable Hilbert space H. We get sufficient conditions on an orthonormal basis of subspaces $\mathcal{E} = \{E_i\}_{i\in I}$ of a Hilbert space K and a surjective $T \in L(K, H)$ in order that $\{T(E_i)\}_{i\in I}$ is a frame of subspaces with respect to a computable sequence of weights. We also obtain generalizations of results in [J.A. Antezana, G. Corach, M. Ruiz, D. Stojanoff, Oblique projections and frames, Proc. Amer. Math. Soc. 134 (2006) 1031–1037], which relate frames of subspaces (including the computation of their weights) and oblique projections. The notion of refinement of a fusion frame is defined and used to obtain results about the excess of such frames. We study the set of admissible weights for a generating sequence of subspaces. Several examples are given.

Keywords: Frames; Frames of subspaces; Fusion frames; Hilbert space operators; Oblique projections

1. Introduction

Let H be a real or complex (separable) Hilbert space. A sequence $\mathcal{F} = \{f_i\}_{i\in I}$ in H is a *frame* for H if there exist numbers $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{i\in I} \left|\langle f, f_i\rangle\right|^2 \leqslant B\|f\|^2, \quad \text{for every } f \in H. \tag{1}$$

This notion has been generalized to frames of subspaces by Casazza and Kutyniok [5] (see also [12] and [13]) in the following way: Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a sequence of closed subspaces of H, and let $w = \{w_i\}_{i\in I} \in \ell^\infty(I)$ such that $w_i > 0$ for every $i \in I$. The sequence $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ is a frame of subspaces (shortly: FS) for H if there exist $A_{\mathcal{W}_w}, B_{\mathcal{W}_w} > 0$ such that

$$A_{\mathcal{W}_w}\|f\|^2 \leqslant \sum_{i\in I} w_i^2 \|P_{W_i} f\|^2 \leqslant B_{\mathcal{W}_w}\|f\|^2, \quad \text{for every } f \in H,$$

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where each P_{W_i} denotes the orthogonal projection onto W_i. The relevance of this notion, as remarked in [5], is that it gives criteria for constructing a frame for H, by joining sequences of frames for subspaces of H (see Theorem 3.4 for details).

Recently, the frames of subspaces have been renamed as *fusion frames.* This notion has been intensely studied during the last years, and several new applications have been discovered. The reader is referred to the works by Casazza, Kutyniok, Li [7], Casazza and Kutyniok [6], Gavruta [14] and references therein.

If $\mathcal{W}_w$ is an FS, the synthesis, analysis and frame operators can be defined, and the properties of $\mathcal{W}_w$ can be studied using these operators, as well as for frames of vectors [3,5]. In [5], the synthesis operator $T_{\mathcal{W}_w}$ is defined as $T_{\mathcal{W}_w}: K_{\mathcal{W}} = \sum_{i\in I} \bigoplus W_i \to H$, given by $T_{\mathcal{W}_w}(g) = \sum_{i\in I} w_i g_i$, for every $g = (g_i)_{i\in I} \in K_{\mathcal{W}}$ (see also [3] where a different domain is used).

Although the synthesis operator of a frame of subspaces $\mathcal{W}_w$ is useful to study the properties of the frame, its definition is rigid, in the sense that it is difficult to find the synthesis operator of any perturbation of the FS. Observe that the action of $T_{\mathcal{W}_w}$ on each orthogonal summand of $K_{\mathcal{W}}$ is completely prescribed by its definition. One purpose of this work is to get more flexibility in the use of operator theory techniques for studying fusion frames. In this direction we get (sufficient) conditions on an orthonormal basis of subspaces $\mathcal{E} = \{E_i\}_{i\in I}$ of a Hilbert space K and a surjective operator $T \in L(K, H)$ in order to assure that the sequence $\mathcal{W} = \{T(E_i)\}_{i\in I}$ becomes an FS with respect to a computable sequence of weights (see Theorem 3.6). Then we use this result to describe properties of equivalent frames of subspaces, and to study the *excess* of such frames. We obtain generalizations of two results from [2], which relate FS (including the computation of their weights) and oblique projections (see also [3]). We define the notion of refinement of sequences of subspaces and frames of subspaces. This allows us to describe the excess of frames of subspaces, obtaining results which are similar to the known results in classical frame theory.

It is remarkable that several known results of frame theory are not valid in the FS setting. For example, we exhibit a frame of subspaces $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ for H such that, for every $G \in \mathrm{Gl}(H)$, the sequence $(v_i, G(W_i))_{i\in I}$ fails to be a Parseval FS for each $v \in \ell^\infty(I)$, including the case $G = S_{\mathcal{W}_w}^{-1/2}$, where $S_{\mathcal{W}_w}$ is the frame operator of $\mathcal{W}_w$ (see Examples 7.5 and 7.6). Several of these facts are exposed in the section of (counter)examples.

Finally we begin with the study of what is, in our opinion, the key problem of the theory of frames of subspaces: given a generating sequence $\mathcal{W} = \{W_i\}_{i\in I}$ of closed subspaces of H, to obtain a characterization of the set of its admissible weights,

$$\mathcal{P}(\mathcal{W}) = \{w \in \ell^\infty_+(I)\colon\ \mathcal{W}_w = (w_i, W_i)_{i\in I} \text{ is an FS for } H\}.$$

Particularly, we look for conditions on $\mathcal{W}$ which ensure that $\mathcal{P}(\mathcal{W}) \neq \emptyset$. We obtain some partial results about these problems, and we study an equivalence relation between weights, compatible with their admissibility with respect to a generating sequence. We give also several examples which illustrate the complexity of the problem.

The paper is organized as follows: Section 2 contains preliminary results about angles between closed subspaces, the reduced minimum modulus of operators, and frames of vectors. In Section 3 we introduce the frames of subspaces and we state the first results relating these frames and Hilbert space operators. In Section 4 the set of admissible weights of an FS is studied. Section 5 contains the results which relate oblique projections and frames of subspaces. Section 6 is devoted to refinement of sequences of subspaces and it contains several results about the excess of an FS. In Section 7 we present a large collection of examples.

2. Preliminaries and notations

Let H and K be separable Hilbert spaces and $L(H, K)$ the space of bounded linear transformations $A : H \to K$ (if $K = H$ we write $L(H)$). The group of invertible operators in $L(H)$ is denoted $\mathrm{Gl}(H)$, and $\mathrm{Gl}(H)^+$ is the set of positive definite invertible operators on H. For an operator $A \in L(H, K)$, $R(A)$ denotes the range of A, $N(A)$ the nullspace of A, $A^* \in L(K, H)$ the adjoint of A, and $\|A\|$ the operator norm of A.

By $M \sqsubseteq H$ we mean that M is a *closed subspace* of H. Given $M \sqsubseteq H$, P_M is the orthogonal (i.e., self-adjoint) projection onto M. If also $N \sqsubseteq H$, we write $M \ominus N := M \cap (M \cap N)^\perp$. Let I be a countable set. We denote by $\ell^\infty_+(I)$ the space of bounded sequences $\{a_i\}_{i\in I}$ of strictly positive numbers. We consider in $\ell^\infty_+(I)$ the usual product of $\ell^\infty(I)$ (i.e. coordinatewise product). With this product $\ell^\infty(I)$ is a von Neumann algebra. We denote by

$$G(I)^+ = \Big\{\{w_i\}_{i\in I} \in \ell^\infty_+(I)\colon\ \inf_{i\in I} w_i > 0\Big\} = \ell^\infty_+(I) \cap \mathrm{Gl}\big(\ell^\infty(I)\big). \tag{2}$$

In what follows we recall the definition and basic properties of angles between closed subspaces of H. We refer the reader to [1] for details and proofs. See also the survey by Deutsch [11] or the book by Kato [17].

Definition 2.1. Let $M, N \sqsubseteq H$. The cosine of the *angle* of M and N is

$$c[M,N] = \sup\{|\langle x, y\rangle|: x \in M \ominus N,\ y \in N \ominus M \text{ and } \|x\| = \|y\| = 1\}.$$

If $M \subseteq N$ or $N \subseteq M$, we define $c[M,N] = 0$, as if they were orthogonal. The *sine* of this angle is denoted by $s[M,N] = (1 - c[M,N]^2)^{1/2}$.

Proposition 2.2. *Let $M, N \sqsubseteq H$. Then*

1. $c[M,N] = c[N,M] = c[M \ominus N, N] = c[M, N \ominus M]$.
2. *If* $\dim M < \infty$, *then* $c[M,N] < 1$.
3. $c[M,N] < 1$ *if and only if* $M + N$ *is closed.*
4. $c[M,N] = c[M^\perp, N^\perp]$.
5. $c[M,N] = \|P_M P_{N \ominus M}\| = \|P_{M \ominus N} P_N\| = \|P_M P_N - P_{M\cap N}\|$.
6. $s[M,N] = \operatorname{dist}(N, \{x \in M \ominus N: \|x\| = 1\})$.

Definition 2.3. The *reduced minimum modulus* $\gamma(T)$ of $T \in L(H,K)$ is defined by $\gamma(T) = \inf\{\|Tx\|: \|x\| = 1,\ x \in N(T)^\perp\}$.

Remark 2.4. The following properties are well known [1]. Let $T \in L(H,K)$.

1. $\gamma(T) = \gamma(T^*) = \gamma(T^*T)^{1/2}$.
2. $R(T) \sqsubseteq K$ if and only if $\gamma(T) > 0$.
3. If T is invertible, then $\gamma(T) = \|T^{-1}\|^{-1}$.
4. If $B \in \mathrm{Gl}(K)$, then
$$\|B^{-1}\|^{-1}\gamma(T) \leqslant \gamma(BT) \leqslant \|B\|\gamma(T). \tag{3}$$
5. Suppose that $R(T) \sqsubseteq K$ and take $M \sqsubseteq H$. Then
$$\gamma(T)s[N(T), M] \leqslant \gamma(TP_M) \leqslant \|T\|s[N(T), M]. \tag{4}$$
In particular, $T(M) \sqsubseteq K$ if and only if $c[N(T), M] < 1$.

We introduce now some basic facts about frames in Hilbert spaces. For a complete description of frame theory and its applications, the reader is referred to Daubechies, Grossmann and Meyer [10], the review by Heil and Walnut [15] or the books by Young [18] and Christensen [8].

Definition 2.5. Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a sequence in a Hilbert space H. $\mathcal{F}$ is called a *frame* for H if there exist numbers $A, B > 0$ such that Eq. (1) holds. The optimal constants $A_\mathcal{F}, B_\mathcal{F}$ for Eq. (1) are called the *frame bounds* for $\mathcal{F}$. The frame $\mathcal{F}$ is *tight* if $A_\mathcal{F} = B_\mathcal{F}$, and *Parseval if* $A_\mathcal{F} = B_\mathcal{F} = 1$.

Remark 2.6. Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame for H and let K be a separable Hilbert space with $\dim K = |I|$. Fix $\mathcal{B} = \{\varphi_i\}_{i\in I}$ an orthonormal basis (ONB) for K. Then there exists a surjective operator $T_{\mathcal{F},\mathcal{B}} \in L(K,H)$ such that $T_{\mathcal{F},\mathcal{B}}(\varphi_i) = f_i$ for every $i \in I$. We say that $T_{\mathcal{F},\mathcal{B}}$ is a *preframe operator* for $\mathcal{F}$. It holds that

$$A_\mathcal{F} = \gamma(T_{\mathcal{F},\mathcal{B}})^2 \quad \text{and} \quad B_\mathcal{F} = \|T_{\mathcal{F},\mathcal{B}}\|^2. \tag{5}$$

We have that $T^*_{\mathcal{F},\mathcal{B}} \in L(H,K)$ is given by $T^*_{\mathcal{F},\mathcal{B}}(x) = \sum_{i\in I}\langle x, f_i\rangle\varphi_i$, for $x \in H$. The operator $S_\mathcal{F} = T_{\mathcal{F},\mathcal{B}}T^*_{\mathcal{F},\mathcal{B}} \in L(H)^+$, called the *frame operator* of $\mathcal{F}$, satisfies $S_\mathcal{F} f = \sum_{i\in I}\langle f, f_i\rangle f_i$, for $f \in H$. In fact $S_\mathcal{F} \in \mathrm{Gl}(H)^+$. Moreover $A_\mathcal{F} I_H \leqslant S_\mathcal{F} \leqslant B_\mathcal{F} I_H$. Note that $S_\mathcal{F}$ does not depend on the preframe operator chosen. If one takes the canonical basis $\mathcal{E}$ for $\ell^2(I)$, then $T_\mathcal{F} = T_{\mathcal{F},\mathcal{E}}$ is called the *synthesis operator* of $\mathcal{F}$.

Remark 2.7. Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame for H. The cardinal number $e(\mathcal{F}) = \dim N(T_{\mathcal{F}})$ is called the *excess* of the frame. In [16] and [4], it is proved that $e(\mathcal{F}) = \sup\{|J|:\ J \subseteq I$ and $\{f_i\}_{i\notin J}$ is still a frame for $H\}$. For every preframe operator $T_{\mathcal{F},\mathcal{B}} \in L(K,H)$ of $\mathcal{F}$, it holds that $e(\mathcal{F}) = \dim N(T_{\mathcal{F},\mathcal{B}})$. The frame $\mathcal{F}$ is called a *Riesz basis* (or exact) if $e(\mathcal{F}) = 0$, i.e., if the preframe (synthesis) operators of $\mathcal{F}$ are invertible.

3. Frames of subspaces

Throughout this section, H will be a fixed separable Hilbert space, and $I \subseteq \mathbb{N}$ a fixed index set ($I = \mathbb{N}$ or $I = \mathbb{I}_n := \{1,\dots,n\}$ for $n \in \mathbb{N}$). Recall that $\ell^\infty_+(I)$ denotes the space of bounded sequences of (strictly) positive numbers, which will be considered as weights in the sequel. The element $e \in \ell^\infty_+(I)$ is the sequence with all its entries equal to 1. Following Casazza and Kutyniok [5], we define:

Definition 3.1. Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a sequence of closed subspaces of H, and let $w = \{w_i\}_{i\in I} \in \ell^\infty_+(I)$.

1. We say that $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ is a *Bessel sequence* of subspaces (BSS) if there exists $B > 0$ such that $\sum_{i\in I} w_i^2 \|P_{W_i} f\|^2 \leqslant B\|f\|^2$ for every $f \in H$, where $P_{W_i} \in L(H)$ is the orthogonal projection onto W_i.
2. We say that $\mathcal{W}_w$ is a *frame of subspaces* (*or a fusion frame*) *for* H, and write that $\mathcal{W}_w$ is an FS (respectively FS for $S \sqsubseteq H$) if there exist $A, B > 0$ such that
$$A\|f\|^2 \leqslant \sum_{i\in I} w_i^2 \|P_{W_i} f\|^2 \leqslant B\|f\|^2, \quad \text{for } f \in H \text{ (respectively } f \in S). \tag{6}$$
The optimal constants for (6) are denoted by $A_{\mathcal{W}_w}$ and $B_{\mathcal{W}_w}$.
3. $\mathcal{W}$ is a *minimal* sequence if $W_i \cap \overline{\operatorname{span}}\{W_j:\ j \neq i\} = \{0\}$, $i \in I$.

Suppose that $\mathcal{W}_w$ is a fusion frame for H. Then $\mathcal{W}_w$ is called:

4. *Tight* if $A_{\mathcal{W}_w} = B_{\mathcal{W}_w}$, and *Parseval* if $A_{\mathcal{W}_w} = B_{\mathcal{W}_w} = 1$.
5. *Riesz basis of subspaces* (shortly RBS) if $\mathcal{W}$ is a minimal sequence.
6. *Orthonormal basis of subspaces* (OBS) if $w = e$ and $W_i \perp W_j$ for $i \neq j$. Observe that, in this case, the sequence of projections $\{P_{W_i}\}_{i\in I}$ becomes a *resolution of the identity*, in the sense that every $f = \sum_{i\in I} P_{W_i} f$.

The synthesis, analysis and frame operators can be defined for a BSS:

Definition 3.2. Let $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ be a BSS for H. Consider the (external) product Hilbert space
$$K_{\mathcal{W}} = \sum_{i\in I} \bigoplus W_i := \left\{ g = (g_i)_{i\in I}:\ g_i \in W_i \text{ and } \|g\|^2 := \sum_{i\in I} \|g_i\|^2 < +\infty \right\}.$$
The operator $T_{\mathcal{W}_w} \in L(K_{\mathcal{W}}, H)$, given by $T_{\mathcal{W}_w}(g) = \sum_{i\in I} w_i g_i$, for every $g = (g_i)_{i\in I} \in K_{\mathcal{W}}$, is called the *synthesis operator* of $\mathcal{W}_w$. Its adjoint $T^*_{\mathcal{W}_w} \in L(H, K_{\mathcal{W}})$ is called the *analysis operator* of $\mathcal{W}_w$. It is easy to see that $T^*_{\mathcal{W}_w}(f) = \{w_i P_{W_i} f\}_{i\in I}$, for $f \in H$. The *frame operator* $S_{\mathcal{W}_w} = T_{\mathcal{W}_w} T^*_{\mathcal{W}_w} \in L(H)^+$ satisfies the formula $S_{\mathcal{W}_w} f = \sum_{i\in I} w_i^2 P_{W_i} f$, for $f \in H$.

Remark 3.3. Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a sequence of closed subspaces of H, and let $w \in \ell^\infty_+(I)$. In [5] the following results were proved: $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ is a BSS if and only if the synthesis operator $T_{\mathcal{W}_w}$ is well defined and bounded. In this case, $\mathcal{W}_w$ is an FS for H if and only if $T_{\mathcal{W}_w}$ is onto. If $\mathcal{W}_w$ is an FS for H,

1. $A_{\mathcal{W}_w} = \gamma(T_{\mathcal{W}_w})^2$ and $B_{\mathcal{W}_w} = \|T_{\mathcal{W}_w}\|^2$. So that $A_{\mathcal{W}_w} \cdot I_H \leqslant S_{\mathcal{W}_w} \leqslant B_{\mathcal{W}_w} \cdot I_H$.
2. $\mathcal{W}_w$ is an RBS if and only if $T_{\mathcal{W}_w}$ is invertible (i.e. injective) and $\mathcal{W}_w$ is an OBS if and only if $w = e$ and $T^*_{\mathcal{W}_w} T_{\mathcal{W}_w} = I_{K_{\mathcal{W}}}$.
3. $\mathcal{W}_w$ is tight if and only if $S_{\mathcal{W}_w} = A_{\mathcal{W}_w} \cdot I_H$, and $\mathcal{W}_w$ is Parseval if and only if $T_{\mathcal{W}_w}$ is a coisometry (i.e. $S_{\mathcal{W}_w} = I_H$). In this case, the sequence $\{w_i^2 P_{W_i}\}_{i\in I}$ is a resolution of the identity.

We state another useful result proved in [5].

Theorem 3.4. *Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a sequence of closed subspaces of H and let $w \in \ell_+^\infty(I)$. For each $i \in I$, let $\mathcal{G}_i = \{f_{ij}\}_{j\in J_i}$ be a frame for W_i. Suppose that $A = \inf_{i\in I} A_{\mathcal{G}_i} > 0$ and $B = \sup_{i\in I} B_{\mathcal{G}_i} < \infty$. Let $\mathcal{E}_i = \{e_{ik}\}_{k\in K_i}$ be an ONB for each W_i. Then the following conditions are equivalent*:

1. $\mathcal{F} = \{w_i f_{ij}\}_{i\in I,\, j\in J_i} = \{w_i \mathcal{G}_i\}_{i\in I}$ *is a frame for* H.
2. $\mathcal{E} = \{w_i e_{ik}\}_{i\in I,\, k\in K_i} = \{w_i \mathcal{E}_i\}_{i\in I}$ *is a frame for* H.
3. $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ *is an FS for* H.

In this case, the bounds of $\mathcal{W}_w$ satisfy the inequalities

$$\frac{A_{\mathcal{F}}}{B} \leqslant A_{\mathcal{W}_w} = A_{\mathcal{E}} \quad \text{and} \quad B_{\mathcal{E}} = B_{\mathcal{W}_w} \leqslant \frac{B_{\mathcal{F}}}{A}. \tag{7}$$

Also $T_{\mathcal{E},\mathcal{B}} = T_{\mathcal{W}_w}$, using the ONB $\mathcal{B} = \{e_{ik}\}_{i\in I,\, k\in K_i}$ of $K_{\mathcal{W}}$.

Operators and frames

Definition 3.5. Let $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ be a BSS for H, with synthesis operator $T_{\mathcal{W}_w}$. The excess of $\mathcal{W}_w$ is defined as $e(\mathcal{W}_w) = \dim N(T_{\mathcal{W}_w})$.

Theorem 3.6. *Let $\{E_i\}_{i\in I}$ be an OBS for K and let $T \in L(K,H)$ be surjective. Suppose that $0 < \inf_{i\in I} \frac{\gamma(TP_{E_i})}{\|TP_{E_i}\|}$. Let $0 < A, B < \infty$ be such that,*

$$\frac{A}{B} \leqslant \frac{\gamma(TP_{E_i})^2}{\|TP_{E_i}\|^2} \quad \text{i.e.,} \quad \frac{\|TP_{E_i}\|^2}{B} \leqslant \frac{\gamma(TP_{E_i})^2}{A}, \quad \forall i \in I. \tag{8}$$

Denote $W_i = T(E_i) \sqsubseteq H$, for $i \in I$. Let $w = \{w_i\}_{i\in I} \in \ell_+^\infty(I)$ such that

$$\frac{\|TP_{E_i}\|^2}{B} \leqslant w_i^2 \leqslant \frac{\gamma(TP_{E_i})^2}{A}, \quad \forall i \in I. \tag{9}$$

Then $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ is an FS for H. Moreover,

$$\frac{\gamma(T)^2}{B} \leqslant A_{\mathcal{W}_w} \quad \text{and} \quad B_{\mathcal{W}_w} \leqslant \frac{\|T\|^2}{A}.$$

If $N(T) \cap E_i = \{0\}$ for every $i \in I$, then $e(\mathcal{W}_w) = \dim N(T)$.

Proof. Suppose that (8) and (9) hold for every $i \in I$. Since $\gamma(TP_{E_i}) > 0$, Remark 2.4 assures that every $W_i = T(E_i) \sqsubseteq H$. Let $\{b_{ij}\}_{j\in J_i}$ be an ONB for each E_i. By Eqs. (5), (8) and (9), each sequence $\mathcal{G}_i = \{w_i^{-1} T b_{ij}\}_{j\in J_i}$ is a frame for W_i with

$$A_{\mathcal{G}_i} = w_i^{-2} \gamma(TP_{E_i})^2 \geqslant A \quad \text{and} \quad B_{\mathcal{G}_i} = w_i^{-2} \|TP_{E_i}\|^2 \leqslant B.$$

On the other hand, since $\{b_{ij}\}_{i\in I,\, j\in J_i}$ is an ONB for K and T is an epimorphism, the sequence $\mathcal{F} = \{Tb_{ij}\}_{i\in I,\, j\in J_i}$ is a frame for H. Finally, since $\mathcal{F} = \{w_i(w_i^{-1} T b_{ij})\}_{i\in I,\, j\in J_i} = \{w_i \mathcal{G}_i\}_{i\in I}$, Theorem 3.4 assures that $\mathcal{W}_w$ is an FS for H. The inequalities for the bounds for $\mathcal{W}_w$ follow from Eq. (7) and the fact that $A_{\mathcal{F}} = \gamma(T)^2$ and $B_{\mathcal{F}} = \|T\|^2$. Suppose that $N(T) \cap E_i = \{0\}$ for every $i \in I$. Then $N(TP_{E_i}) = E_i^\perp$ and $\gamma(TP_{E_i})\|z\| \leqslant \|TP_{E_i} z\|$ for every $z \in E_i$. By Eq. (9), for every $x \in K$ and $i \in I$,

$$A^{1/2} w_i \|P_{E_i} x\| \leqslant \gamma(TP_{E_i}) \|P_{E_i} x\| \leqslant \|TP_{E_i} x\| \leqslant B^{1/2} w_i \|P_{E_i} x\|,$$

and $\|x\|^2 = \sum_{i\in I} \|P_{E_i} x\|^2$. Let $K_{\mathcal{W}} = \sum_{i\in I} \bigoplus W_i$ (the domain of $T_{\mathcal{W}_w}$). Observe that $T(E_i) = W_i$ for every $i \in I$. Therefore the map $V : K \to K_{\mathcal{W}}$ given by $Vx = (w_i^{-1} TP_{E_i} x)_{i\in I}$, for $x \in K$, is well defined, bounded and invertible. By the definition of the synthesis operator $T_{\mathcal{W}_w}$, and using that $x = \sum_{i\in I} P_{E_i} x$ for every $x \in K$, we can deduce that $T_{\mathcal{W}_w} \circ V = T$. Therefore $\dim N(T) = \dim V^{-1}(N(T_{\mathcal{W}_w})) = \dim N(T_{\mathcal{W}_w}) = e(\mathcal{W}_w)$. □

Example 7.1 shows a surjective operator T and $\mathcal{E} = \{E_i\}_{i\in I}$, an OBS for a Hilbert space K, such that $\gamma(TP_{E_i}) > 0$ for every $i \in I$, but the sequence $T\mathcal{E} = \{T(E_i)\}_{i\in I}$ fails to be an FS for every $w \in \ell^\infty_+(I)$. Hence T and $\mathcal{E}$ do not satisfy Eq. (8). However, Eq. (8) is not a necessary condition to assure that $\mathcal{P}(T\mathcal{E}) \neq \emptyset$ (see Definition 4.1).

In Example 7.2 we show an FS which is the image of an OBS under an epimorphism that does not satisfy Eq. (8).

Remark 3.7. If $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ is an FS for H, then its synthesis operator $T_{\mathcal{W}_w}$, defined as in Definition 3.2 clearly satisfies Eq. (8). Moreover, it holds that $T_{\mathcal{W}_w} g = w_i g_i$ for every $g \in E_i$, the copy of W_i in $K_{\mathcal{W}}$. Hence $\gamma(T_{\mathcal{W}_w} P_{E_i}) = \|T_{\mathcal{W}_w} P_{E_i}\| = w_i$ for every $i \in I$. Then, if one takes $A = B = 1$, the unique weight which satisfies Eq. (9) for $T_{\mathcal{W}_w}$ is w itself.

Remark 3.8. Let $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ be an FS for H, and let $G \in \mathrm{Gl}(H)$. In [5], [7, Theorem 2.11] and [14, Theorem 2.4] it is proved that $G\mathcal{W}_w = (w_i, G(W_i))_{i\in I}$ must be also an FS for H.

We give a short proof of this fact, including extra information about the bounds and the excess of $G\mathcal{W}_w$, in order to illustrate the tools given by Theorem 3.6.

Corollary 3.9. *Let $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ be an FS for H, and let $G \in L(H, H_1)$ be* invertible. *Then $G\mathcal{W}_w = (w_i, G(W_i))_{i\in I}$ is an FS for H_1, with $e(\mathcal{W}_w) = e(G\mathcal{W}_w)$. Also*

$$\left(\|G\|\|G^{-1}\|\right)^{-2} A_{\mathcal{W}_w} \leqslant A_{G\mathcal{W}_w} \quad \text{and} \quad B_{G\mathcal{W}_w} \leqslant \left(\|G\|\|G^{-1}\|\right)^2 B_{\mathcal{W}_w}.$$

Proof. Denote by E_i the copy of each W_i in $K_{\mathcal{W}}$, i.e., $K_{\mathcal{W}} = \bigoplus_{i\in I} E_i$. Define $T = GT_{\mathcal{W}_w} \in L(K_{\mathcal{W}}, H_1)$, which is clearly surjective (since $T_{\mathcal{W}_w}$ is). By Eq. (3) and Remark 3.7, $\gamma(TP_{E_i}) \geqslant \gamma(G)\cdot\gamma(T_{\mathcal{W}_w} P_{E_i}) = \gamma(G) w_i$, and

$$\|TP_{E_i}\| \leqslant \|G\|\|T_{\mathcal{W}_w} P_{E_i}\| = \|G\| w_i, \quad \text{for every } i \in I.$$

In particular, $T(E_i) \sqsubseteq H_1$. Then, we can apply Theorem 3.6 for T, $A = \gamma(G)^2$ and $B = \|G\|^2$. Indeed, for every $i \in I$, we have that

$$\frac{\|TP_{E_i}\|^2}{\|G\|^2} \leqslant w_i^2 \leqslant \frac{\gamma(TP_{E_i})^2}{\gamma(G)^2} \quad \text{so that} \quad \frac{\gamma(G)^2}{\|G\|^2} \leqslant \frac{\gamma(TP_{E_i})^2}{\|TP_{E_i}\|^2}.$$

Therefore, $G\mathcal{W}_w = (w_i, G(W_i))_{i\in I}$ is an FS for H_1 by Theorem 3.6. In order to prove the inequalities for the bounds, by Eq. (3) and item 2 of Remark 3.3 we have that $\gamma(GT_{\mathcal{W}_w}) \geqslant \gamma(G)\gamma(T_{\mathcal{W}_w}) = \|G^{-1}\|^{-1} A^{1/2}_{\mathcal{W}_w}$ and

$$\|GT_{\mathcal{W}_w}\| \leqslant \|G\|\|T_{\mathcal{W}_w}\| = \|G\| B^{1/2}_{\mathcal{W}_w}.$$

Now apply Theorem 3.6 with $A = \|G^{-1}\|^{-2}$ and $B = \|G\|^2$. It is easy to see that $N(T) = N(T_{\mathcal{W}_w})$. Then $N(T) \cap E_i = \{0\}$ $(i \in I)$. By Theorem 3.6, we deduce that $e(\mathcal{W}_w) = \dim N(T_{\mathcal{W}_w}) = \dim N(T) = e(G\mathcal{W}_w)$. □

4. Admissible weights

Definition 4.1. We say that $\mathcal{W} = \{W_i\}_{i\in I}$ is a *generating* sequence of H, if $W_i \sqsubseteq H$ for every $i \in I$, and $\overline{\mathrm{span}}\{W_i\colon i \in I\} = H$. In this case, we define

$$\mathcal{P}(\mathcal{W}) = \left\{w \in \ell^\infty_+(I)\colon \mathcal{W}_w = (w_i, W_i)_{i\in I} \text{ is an FS for } H\right\} \subseteq \ell^\infty_+(I),$$

the set of *admissible* sequences of weights for $\mathcal{W}$.

It is clear that, if $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ is an FS for H, then $\mathcal{W} = \{W_i\}_{i\in I}$ is a generating sequence. Nevertheless, in Examples 7.1 and 7.3 we shall see that there exist generating sequences $\mathcal{W} = \{W_i\}_{i\in I}$ for H such that $\mathcal{P}(\mathcal{W}) = \emptyset$.

Proposition 4.2. *Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a generating sequence of H.*

1. *If $w \in \mathcal{P}(\mathcal{W})$, then $a \cdot w \in \mathcal{P}(\mathcal{W})$ and $e(\mathcal{W}_w) = e(\mathcal{W}_{a\cdot w})$, for every $a \in G(I)^+$ (the set of sequences defined in Eq.* (2)).

2. *If $\mathcal{W}_w = (w, \mathcal{W})$ is an RBS, for some $w \in \ell^\infty_+(I)$, then $\mathcal{P}(\mathcal{W}) = G(I)^+$, and $(a, \mathcal{W})$ is still an RBS for every $a \in G(I)^+$.*
3. *Let $G \in \mathrm{Gl}(H)$. Then $\mathcal{P}(\mathcal{W}) = \mathcal{P}(\{G(W_i)\}_{i\in I})$. In other words, a sequence $w \in \ell^\infty_+(I)$ is admissible for $\mathcal{W}$ if and only if it is admissible for $G\mathcal{W} = \{G(W_i)\}_{i\in I}$.*

Proof. Let $K_\mathcal{W} = \sum_{i\in I} \bigoplus W_i$, and denote by $E_i \sqsubseteq K_\mathcal{W}$ the copy of each W_i in $K_\mathcal{W}$. For every $a \in G(I)^+$, consider the limit $D_a = \sum_{i\in I} a_i P_{E_i}$ (with respect to the strong operator topology). Then $D_a \in \mathrm{Gl}(K_\mathcal{W})^+$. Therefore, if $T_{\mathcal{W}_w} \in L(K_\mathcal{W}, H)$ is the synthesis operator of $\mathcal{W}_w$, then $T_{\mathcal{W}_w} D_a$ is, by definition, the synthesis operator of $(a \cdot w, \mathcal{W})$. Since $T_{\mathcal{W}_w} D_a$ is bounded and surjective, then $(a \cdot w, \mathcal{W})$ is also an FS. Note that $N(T_{\mathcal{W}_{a\cdot w}}) = N(T_{\mathcal{W}_w} D_a) = D_a^{-1}(N(T_{\mathcal{W}_w}))$, proving item 1.

If $\mathcal{W}_w$ is an RBS for H, then $T_{\mathcal{W}_w}$ is invertible. Since $T_{\mathcal{W}_w} x = w_i x_i$ and $\|x\| = \|x_i\|$ for every$x \in E_i$, then $w_i \geqslant \gamma(T_{\mathcal{W}_w}) = A^{1/2}_{\mathcal{W}_w}$ for every $i \in I$. This implies that $w \in G(I)^+$. Note that $w \cdot G(I)^+ = G(I)^+$ (because $w^{-1} \in G(I)^+$). Then $G(I)^+ \subseteq \mathcal{P}(\mathcal{W})$ by item 1. But, for every $a \in \mathcal{P}(\mathcal{W})$, we have that $\mathcal{W}_a$ is an RBS, because $\mathcal{W}$ is still minimal.

To prove 3, apply Corollary 3.9 for G and G^{-1}. □

Definition 4.3. Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a generating sequence of H. Given $v, w \in \mathcal{P}(\mathcal{W})$, we say that v and w are *equivalent* if there exists $a \in G(I)^+$ such that $v = a \cdot w$.

Remark 4.4. Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a generating sequence of H. By Proposition 4.2, if $w \in \mathcal{P}(\mathcal{W})$, then its equivalence class $w \cdot G(I)^+ \subseteq \mathcal{P}(\mathcal{W})$. On the other hand, in Example 7.5 below we shall see that there exist generating sequences $\mathcal{W}$ of H with infinitely many non equivalent sequences $w \in \mathcal{P}(\mathcal{W})$. If $\mathcal{W}_w$ is an RBS for H, then by Proposition 4.2 all admissible sequences for $\mathcal{W}$ are equivalent to w, since $\mathcal{P}(\mathcal{W}) = G(I)^+$. Since $\mathcal{W}_v = (v, \mathcal{W})$ is an RBS for H for every $v \in G(I)^+$, from now on we will not mention the weights. We just say that the sequence of subspaces $\mathcal{W}$ is a Riesz basis of subspaces. In Example 7.3, we shall see that there exist minimal sequences which are generating for H, but with $\mathcal{P}(\mathcal{W}) = \emptyset$.

Proposition 4.5. *Let $\mathcal{E} = \{E_i\}_{i\in I}$ be an OBS for H. Let $G \in L(H, H_1)$ be an invertible operator. Then the sequence $\mathcal{W} = \{G(E_i)\}_{i\in I}$ is an RBS for H_1.*

Proof. It is a consequence of Corollary 3.9. □

Remark 4.6. If $\mathcal{F} = \{f_i\}_{i\in I}$ is a frame for H with frame operator $S_\mathcal{F}$ (see Remark 2.6), it is well known that the sequence $\{S_\mathcal{F}^{-1/2} f_i\}_{i\in I}$ is a Parseval frame. Nevertheless, if $\mathcal{W}_w$ is an FS, then $S_{\mathcal{W}_w}^{-1/2}\mathcal{W}_w$ may be a non Parseval frame of subspaces (see Example 7.5 below), not even changing the sequence of weights. Moreover, there exist frames of subspaces $\mathcal{W}_w = (w, \mathcal{W})$ for H such that the sequence $(v, G\mathcal{W})$ fails to be a Parseval FS for H for every $G \in \mathrm{Gl}(H)$ and $v \in \ell^\infty_+(I)$ (see Example 7.6). In the next proposition we show that the situation is different for an RBS.

Proposition 4.7. *Let $\mathcal{W} = \{W_i\}_{i\in I}$ be an RBS for H. Then, for every $w \in G(I)^+$, the sequence $\{S_{\mathcal{W}_w}^{-1/2}(W_i)\}_{i\in I}$ is an OBS.*

Proof. Let $\{e_{ik}\}_{k\in K_i}$ be an ONB of each W_i. According to Theorem 3.4, the sequence $\mathcal{E} = \{w_i e_{ik}\}_{i\in I, k\in K_i}$ satisfies that $T_\mathcal{E} = T_{\mathcal{W}_w}$, which is invertible. So that $\mathcal{E}$ is a Riesz basis for H. Hence the sequence $\{w_i S_\mathcal{E}^{-1/2} e_{ik}\}_{i\in I, k\in J_i}$ is an ONB for H. Since $S_{\mathcal{W}_w} = S_\mathcal{E}$, then $\{w_i S_{\mathcal{W}_w}^{-1/2} e_{ik}\}_{k\in K_i}$ is an ONB of each subspace $S_{\mathcal{W}_w}^{-1/2}(W_i)$. Therefore $\{S_{\mathcal{W}_w}^{-1/2}(W_i)\}_{i\in I}$ is an OBS for H. □

5. Projections and frames

The Naimark's Theorem can be formulated in a frame version: A sequence $\{f_n\}_{n\in\mathbb{N}}$ in H is a Parseval frame for H if and only if there exist a Hilbert space K containing H and an ONB $\{e_n\}_{n\in\mathbb{N}}$ for K such that $f_n = P_H e_n$, for every $n \in \mathbb{N}$.

There are generalizations of this theorem obtained by replacing the orthonormal basis by a Riesz basis and also by considering oblique projections instead of orthonormal projections (see [2]). In this section we obtain a generalization of these results, relating FS (including the computation of their weights) and oblique projections (see also [3]). Unlike the case of vector frames, all the results are in "one direction." The converses fail in general (see Example 7.4 and Remarks 5.3 and 5.5).

Theorem 5.1. *Let $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ be an FS for H. Then there exists a Hilbert space $L \supseteq H$ and an RBS $\{B_i\}_{i\in I}$ for L such that*

$$P_H(B_i) = W_i \quad \text{and} \quad A_{\mathcal{W}_w}^{1/2}\|P_H P_{B_i}\| \leqslant w_i \leqslant B_{\mathcal{W}_w}^{1/2}\|P_H P_{B_i}\|, \quad \forall i \in I.$$

Also, we have that $e(\mathcal{W}_w) = \dim L \ominus H$.

Proof. Denote by E_i the copy of each W_i in $K_{\mathcal{W}} = \sum_{i\in I} \bigoplus W_i$. Let $T_{\mathcal{W}_w} \in L(K_{\mathcal{W}}, H)$ be the synthesis operator for $\mathcal{W}_w$. Denote by $M = N(T_{\mathcal{W}_w})$ and $L = H \oplus M$. We can identify H with $H \oplus \{0\} \sqsubseteq L$. Let $U : K_{\mathcal{W}} \to L$ given by $U(x) = T_{\mathcal{W}_w}x \oplus \gamma(T_{\mathcal{W}_w})P_M x$, $x \in K_{\mathcal{W}}$. Since $K_{\mathcal{W}} = M^\perp \perp M$ and $T_{\mathcal{W}_w}|_{M^\perp} : M^\perp \to H$ is invertible, we can deduce that U is bounded and invertible. Moreover, it is easy to see that

$$\|U^{-1}\|^{-1} = \gamma(U) = \gamma(T_{\mathcal{W}_w}) = A_{\mathcal{W}_w}^{1/2} \quad \text{and} \quad \|U\| = \|T_{\mathcal{W}_w}\| = B_{\mathcal{W}_w}^{1/2}. \tag{10}$$

By Proposition 4.5, the sequence $\{B_i\}_{i\in I} = \{U(E_i)\}_{i\in I}$ is an RBS for L. Observe that $P_H(B_i) = P_H U(E_i) = T_{\mathcal{W}_w}(E_i) \oplus \{0\} = W_i \oplus \{0\} \sim W_i$, for every $i \in I$. Let y be a unit vector of $B_i = U(E_i)$. Then $y = Ux$ with $x \in E_i$, and $\gamma(U)\|x\| \leqslant \|Ux\| = \|y\| = 1 \leqslant \|U\|\|x\|$. If $x \in E_i$, we denote by x_i its component in W_i (the others are zero). Using that $\|P_H y\| = \|T_{\mathcal{W}_w}x\| = w_i\|x_i\| = w_i\|x\|$ and Eq. (10), we can conclude that for every $y \in B_i$,

$$A_{\mathcal{W}_w}^{1/2}\|P_H y\| = \gamma(T_{\mathcal{W}_w})\|P_H y\| = w_i\gamma(U)\|x\| \leqslant w_i \quad \Rightarrow \quad A_{\mathcal{W}_w}^{1/2}\|P_H P_{B_i}\| \leqslant w_i.$$

Similarly, $w_i \leqslant w_i\|U\|\|x\| = B_{\mathcal{W}_w}^{1/2}\|P_H y\| \leqslant B_{\mathcal{W}_w}^{1/2}\|P_H P_{B_i}\|$. □

As a particular case of Theorem 5.1, we get a result proved by Asgari and Khosravi [3], with some extra information concerning the weights:

Corollary 5.2. *Let $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ be a Parseval FS for H. Then there exists a Hilbert space $L \supseteq H$ and an* orthonormal basis *of subspaces $\{F_i\}_{i\in I}$ for L such that $P_H(F_i) = W_i$ and $w_i = c[H, F_i] = \|P_H P_{F_i}\|$ for every $i \in I$.*

Proof. We use the notations of the proof of Theorem 5.1. If $\mathcal{W}_w$ is Parseval, then $A_{\mathcal{W}_w} = B_{\mathcal{W}_w} = 1$. By Eq. (10), this implies that the operator $U \in L(K_{\mathcal{W}}, L)$ becomes unitary. Hence, in this case, the sequence $\{F_i\}_{i\in I} = \{U(E_i)\}_{i\in I}$ is an OBS for L. Also, by Theorem 5.1, we have that $w_i = \|P_H P_{F_i}\|$ for every $i \in I$. It is easy to see that $F_i \cap (H \oplus \{0\}) \neq \{0\}$ implies that $w_i = 1$ and $F_i \subseteq (H \oplus \{0\})$ (because U is unitary). So $\|P_H P_{F_i}\| = c[H, F_i]$ for every $i \in I$. □

Remark 5.3. Although the converse of Corollary 5.2 fails in general, it holds with some additional assumptions, based on Theorem 3.6: If $\mathcal{E} = \{E_i\}_{i\in I}$ is an OBS for $L \supseteq H$ such that $0 < \inf_{i\in I} \frac{\gamma(P_H P_{E_i})}{\|P_H P_{E_i}\|}$, then $\mathcal{P}(\mathcal{W}) \neq \emptyset$, where $W_i = P_H(E_i)$, $i \in I$. Moreover, as in Theorem 3.6, it can be found a concrete $w \in \mathcal{P}(\mathcal{W})$. Nevertheless, we cannot assure that $\mathcal{W}_w$ is a Parseval FS.

Theorem 5.4. *Let $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ be an FS for H such that $1 \leqslant A_{\mathcal{W}_w}$. Denote $H_1 = H \oplus K_{\mathcal{W}}$. Then there exist an* oblique *projection $Q \in L(H_1)$ with $R(Q) = H \oplus \{0\}$ and an* orthonormal system *of subspaces $\{B_i\}_{i\in I}$ in H_1, such that*

$$W_i \oplus 0 = Q(B_i) \quad \text{and} \quad w_i = \|QP_{B_i}\| = \gamma(QP_{B_i}), \quad \text{for every } i \in I.$$

Moreover, if $e(\mathcal{W}_w) = \infty$, then the sequence $\{B_i\}_{i\in I}$ can be assumed to be an OBS *for H_1.*

Proof. Write $T_{\mathcal{W}_w} = T$. Then $TT^* \geqslant A_{\mathcal{W}_w} I_H \geqslant I_H$. Denote by $X = (TT^* - I_H)^{1/2} \in L(H)^+$. Consider the (right) polar decomposition $T = |T^*|V$, where $V \in L(K_{\mathcal{W}}, H)$ is a partial isometry with initial space $N(T)^{\perp}$ and final space H, so that $VV^* = I_H$. Consider the "ampliation" $\tilde{T} \in L(K_{\mathcal{W}}, H_1)$ given by $\tilde{T}x = Tx \oplus 0$. Then

$$\tilde{T}\tilde{T}^* = \begin{bmatrix} TT^* & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} H \\ K_{\mathcal{W}} \end{matrix} \in L(H_1).$$

Define

$$Q = \begin{bmatrix} I_H & XV \\ 0 & 0 \end{bmatrix} \begin{matrix} H \\ K_{\mathcal{W}} \end{matrix} \in L(H_1).$$

Then it is clear that Q is an oblique projection with $R(Q) = H \oplus 0$. Moreover,

$$QQ^* = \begin{bmatrix} I_H + XX^* & 0 \\ 0 & 0 \end{bmatrix} = \tilde{T}\tilde{T}^* \quad \Rightarrow \quad |Q^*| = |\tilde{T}^*|.$$

Define $U \in L(K_{\mathcal{W}}, H_1)$ by $Ux = VP_{N(T)^{\perp}}x \oplus P_{N(T)}x$, for $x \in K_{\mathcal{W}}$. Then U is an *isometry*, because the initial space of V is $N(T)^{\perp}$. Note that also $\tilde{T} = |\tilde{T}^*|U$. The partial isometry of the right polar decomposition of Q extends to a unitary operator W on H_1, because $\dim N(Q) = \dim R(Q)^{\perp}$. Moreover, $Q = |Q^*|W$. Then $\tilde{T} = |\tilde{T}^*|U = |Q^*|U = QW^*U$. Therefore, if we consider the OBS $\{E_i\}_{i\in I}$ for $K_{\mathcal{W}}$,

$$W_i = T(E_i) \sim T(E_i) \oplus 0 = \tilde{T}(E_i) = QW^*U(E_i) = Q(B_i), \quad i \in I,$$

where $\{B_i\}_{i\in I} = \{W^*U(E_i)\}_{i\in I}$, which is clearly an orthonormal system in H_1. If $y \in B_i$ is a unit vector, then $y = W^*Ux$ for $x \in E_i$ with $\|x\| = 1$, and $w_i = \|Tx\| = \|QW^*Ux\| = \|Qy\|$. Hence $w_i = \|QP_{B_i}\| = \gamma(QP_{B_i})$. Suppose now that $\dim N(T) = \infty$. Then the isometry U can be changed to a unitary operator from $K_{\mathcal{W}}$ onto H_1, still satisfying that $\tilde{T} = |\tilde{T}^*|U$. Indeed, take $U'x = VP_{N(T)^{\perp}}x \oplus YP_{N(T)}x$, for $x \in H$, where $Y \in L(K_{\mathcal{W}})$ is a partial isometry with initial space $N(T)$ and final space $K_{\mathcal{W}}$. It is easy to see that U' is unitary. Then the sequence $\{B_i'\}_{i\in I} = \{W^*U'(E_i)\}_{i\in I}$ is an OBS for H_1. □

Remark 5.5. As in Remark 5.3, a converse to Theorem 5.4 holds under the assumption $\inf_{i\in I} \frac{\gamma(QP_{B_i})}{\|QP_{B_i}\|} > 0$.

6. Refinements of frames of subspaces

In [5] it is shown by an example that an FS with $e(\mathcal{W}_w) > 0$ can be exact, i.e. $(w_i, W_i)_{i\in J}$ is not an FS for H, for every proper $J \subset I$. In this section, we introduce the notion of refinements of subspace sequences, which is a natural way to recover the connection between excess and erasures.

Definition 6.1. Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a sequence of closed subspaces. A *refinement* of $\mathcal{W}$ is a sequence $\mathcal{V} = \{V_i\}_{i\in J}$ of closed subspaces such that $J \subseteq I$ and $\{0\} \neq V_i \subset W_i$ for every $i \in J$. In this case, the excess of $\mathcal{W}$ over $\mathcal{V}$ is the cardinal number $e(\mathcal{W}, \mathcal{V}) = \sum_{i\in J} \dim(W_i \ominus V_i) + \sum_{i\notin J} \dim W_i$. If $w \in \mathcal{P}(\mathcal{W})$, $\mathcal{V}_w = (w_i, V_i)_{i\in J}$ is an *FS refinement* (FSR) of $\mathcal{W}_w$ if $\mathcal{V}_w$ is an FS for H.

Remark 6.2. It is easy to see that if $\mathcal{V}$ is a refinement of $\mathcal{W}$ and $\mathcal{V}'$ is a refinement of $\mathcal{V}$, then $\mathcal{V}'$ is a refinement of $\mathcal{W}$ and $e(\mathcal{W}, \mathcal{V}') = e(\mathcal{W}, \mathcal{V}) + e(\mathcal{V}, \mathcal{V}')$. The next result uses basic Fredholm theory. We refer to J.B. Conway's book [9, Chapter XI].

Lemma 6.3. *Let $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ be an FS for H and let $\mathcal{V} = \{V_i\}_{i\in J}$ be a refinement of $\mathcal{W}$. We consider $K_{\mathcal{V}} = \sum_{i\in J} \bigoplus V_i$ as a subspace of $\sum_{i\in I} \bigoplus W_i = K_{\mathcal{W}}$. Then we have that $\mathcal{V}_w = (w_i, V_i)_{i\in J}$ an FSR of $\mathcal{W}_w$ if and only if $T_{\mathcal{W}_w} P_{K_{\mathcal{V}}}$ is surjective. In this case,*

1. $\dim K_{\mathcal{V}}^{\perp} = e(\mathcal{W}, \mathcal{V}) \leqslant e(\mathcal{W}_w)$.
2. *If $e(\mathcal{W}, \mathcal{V}) < \infty$, then $e(\mathcal{V}_w) = e(\mathcal{W}_w) - e(\mathcal{W}, \mathcal{V})$.*

Proof. For each $i \in I$, denote by E_i (respectively F_i) the copy of W_i (respectively V_i, or $F_i = \{0\}$ if $i \notin J$) in $K_{\mathcal{W}}$. Then $K_{\mathcal{V}}^{\perp} = \bigoplus_{i \in I}(E_i \ominus F_i)$, which implies that $e(\mathcal{W}, \mathcal{V}) = \dim K_{\mathcal{V}}^{\perp}$. Let $P = P_{K_{\mathcal{V}}}$. By construction, $T_{\mathcal{V}_w} = T_{\mathcal{W}_w}|_{K_{\mathcal{V}}} \in L(K_{\mathcal{V}}, H)$. Then $R(T_{\mathcal{W}_w} P) = R(T_{\mathcal{V}_w}) = H$ if and only if $\mathcal{V}_w$ an FSR of $\mathcal{W}_w$. In this case, $N(PT_{\mathcal{W}_w}^*) = \{0\}$. Since $R(T_{\mathcal{W}_w}^*) = N(T_{\mathcal{W}_w})^{\perp}$, then $N(T_{\mathcal{W}_w})^{\perp} \cap N(P) = \{0\}$. This implies that

$$e(\mathcal{W}, \mathcal{V}) = \dim N(P) \leqslant \dim N(T_{\mathcal{W}_w}) = e(\mathcal{W}_w).$$

Note that $T_{\mathcal{W}_w}$ is a semi-Fredholm operator, with

$$\operatorname{Ind}(T_{\mathcal{W}_w}) = \dim N(T_{\mathcal{W}_w}) - 0 = e(\mathcal{W}_w).$$

If $e(\mathcal{W}, \mathcal{V}) < \infty$, then P is Fredholm, with $\operatorname{Ind}(P) = 0$. Hence,

$$e(\mathcal{W}_w) = \operatorname{Ind}(T_{\mathcal{W}_w}) + \operatorname{Ind}(P) = \operatorname{Ind}(T_{\mathcal{W}_w} P) = \dim N(T_{\mathcal{W}_w} P).$$

Finally, since $T_{\mathcal{V}_w} = T_{\mathcal{W}_w}|_{K_{\mathcal{V}}}$, then

$$e(\mathcal{V}_w) = \dim N(T_{\mathcal{V}_w}) = \dim N(T_{\mathcal{W}_w} P) - \dim N(P) = e(\mathcal{W}_w) - e(\mathcal{W}, \mathcal{V}),$$

which completes the proof. □

Lemma 6.4. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be an FS for H with $e(\mathcal{W}_w) > 0$. Then there exists an FSR $\mathcal{V}_w = (w_i, V_i)_{i \in J}$ of $\mathcal{W}_w$ with $e(\mathcal{W}, \mathcal{V}) = 1$.*

Proof. For each $i \in I$, denote by E_i the copy of W_i in $K_{\mathcal{W}}$. Suppose that there is no FSR $\mathcal{V}_w$ of $\mathcal{W}_w$ with $e(\mathcal{W}, \mathcal{V}) = 1$. Then, by Lemma 6.3, for every $i \in I$ and every unit vector $e \in E_i$, it holds that $R(T_{\mathcal{W}_w} P_{\{e\}^{\perp}}) \neq H$. By 2 and 4 of Proposition 2.2, Eq. (3), $c[N(T_{\mathcal{W}_w}), \{e\}^{\perp}] = c[N(T_{\mathcal{W}_w})^{\perp}, \operatorname{span}\{e\}] < 1$.

Then, by Eq. (4) of Remark 2.4, $R(T_{\mathcal{W}_w} P_{\{e\}^{\perp}}) \sqsubseteq H$. Take a unit vector $x_e \in R(T_{\mathcal{W}_w} P_{\{e\}^{\perp}})^{\perp} = N(P_{\{e\}^{\perp}} T_{\mathcal{W}_w}^*)$. Then $0 \neq T_{\mathcal{W}_w}^* x_e \in \operatorname{span}\{e\}$, i.e., $e \in R(T_{\mathcal{W}_w}^*)$. Hence $\bigcup_{i \in I} E_i \subseteq R(T_{\mathcal{W}_w}^*)$ (which is closed), so that $T_{\mathcal{W}_w}^*$ is surjective and $e(\mathcal{W}_w) = 0$. □

Theorem 6.5. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be an FS for H. Then*

$$e(\mathcal{W}_w) = \sup\bigl\{e(\mathcal{W}, \mathcal{V}) \colon \mathcal{V}_w = (w_i, V_i)_{i \in J} \text{ is an FSR of } \mathcal{W}_w\bigr\}. \tag{11}$$

If $e(\mathcal{W}_w) = \infty$, then for every $n \in \mathbb{N}$ there exists an FSR $\mathcal{V}_w = (w_i, V_i)_{i \in J}$ of $\mathcal{W}_w$ such that $e(\mathcal{W}, \mathcal{V}) = n$.

Proof. Denote by α the supremum of Eq. (11). Then $\alpha \leqslant e(\mathcal{W}_w)$ by Lemma 6.3. If $e(\mathcal{W}_w) < \infty$, using Remark 6.2 and Lemmas 6.4 and 6.3 one obtains an inductive argument which shows that $\alpha \geqslant e(\mathcal{W}_w)$. If $e(\mathcal{W}_w) = \infty$, a similar argument shows that, for every $n \in \mathbb{N}$, there exists an FSR $\mathcal{V}_w$ of $\mathcal{W}_w$ such that $e(\mathcal{W}, \mathcal{V}) = n$. □

Corollary 6.6. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be an FS of H such that $e(\mathcal{W}_w) < \infty$. Then $w \in G(I)^+$ and there exists $\mathcal{V}_w = (w_i, V_i)_{i \in J}$, a FSR of $\mathcal{W}_w$, such that $\mathcal{V}$ is an RBS for H and $e(\mathcal{W}, \mathcal{V}) = e(\mathcal{W}_w)$.*

Proof. By Theorem 6.5, there exists $\mathcal{V}_w = (w_i, V_i)_{i \in J}$, an FSR of $\mathcal{W}_w$, such that $e(\mathcal{W}, \mathcal{V}) = e(\mathcal{W}_w)$. By Lemma 6.3, $e(\mathcal{V}_w) = 0$. This means that $\mathcal{V}_w$ is an RBS for H. Then, by Proposition 4.2, the sequence $\{w_i\}_{i \in J} \in G(J)^+$. Since $e(\mathcal{W}, \mathcal{V}) < \infty$, then $I \setminus J$ is finite and we get that $w \in G(I)^+$. □

Corollary 6.7. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be an FS for H such that $e(\mathcal{W}_w) < \infty$. Then $\mathcal{P}(\mathcal{W}) = G(I)^+$ and $e(\mathcal{W}_v) = e(\mathcal{W}_w)$ for every other $v \in \mathcal{P}(\mathcal{W})$.*

Proof. By Corollary 6.6, we know that $w \in G(I)^+$. By Proposition 4.2, we deduce that $G(I)^+ \subseteq \mathcal{P}(\mathcal{W})$. Let $\mathcal{V}_w = (w_i, V_i)_{i \in J}$ be an FSR of $\mathcal{W}_w$ such that $\mathcal{V}$ is an RBS (which exists by Corollary 6.6). Let $v \in \mathcal{P}(\mathcal{W})$. Then, the sequence $\mathcal{V}_v = (v_i, V_i)_{i \in J}$ is also an FSR of $\mathcal{W}_v$. Indeed, consider $T_{\mathcal{V}_v} = T_{\mathcal{W}_v}|_{K_{\mathcal{V}}} \in L(K_{\mathcal{V}}, H)$. By Lemma 6.3, $\dim K_{\mathcal{V}}^{\perp} = e(\mathcal{W}, \mathcal{V}) < \infty$. As in the proof of Lemma 6.4, this implies that $R(T_{\mathcal{V}_v}) = R(T_{\mathcal{W}_v} P_{K_{\mathcal{V}}}) \sqsubseteq H$. On the other hand, $\operatorname{span}\{\bigcup_{i \in J} V_i\} \subseteq R(T_{\mathcal{V}_v})$. But $\operatorname{span}\{\bigcup_{i \in J} V_i\}$ is dense in H, because $T_{\mathcal{V}_w}$ is surjective (recall that $\mathcal{V}_w$ is an FS for H). This shows that also $T_{\mathcal{V}_v}$ is surjective, i.e. $\mathcal{V}_v$ is an FS for H, as claimed. Summarizing, we have that $\mathcal{V}$ is an RBS, and $v_J = \{v_i\}_{i\in J} \in \mathcal{P}(\mathcal{V})$. By Proposition 4.2, $v_J \in G(J)^+$. As before, this implies that $v \in G(I)^+$. Again, by Proposition 4.2, we conclude that $e(\mathcal{W}_v) = e(\mathcal{W}_w)$, because $v = (vw^{-1})w$ and $vw^{-1} \in G(I)^+$. □

Theorem 6.8. *Let $\mathcal{W}_w = (w_i, W_i)_{i\in I}$ be an FS for H. Then $e(\mathcal{W}_v) = e(\mathcal{W}_w)$ for every other $v \in \mathcal{P}(\mathcal{W})$.*

Proof. If $e(\mathcal{W}_w) < \infty$, apply Corollary 6.7. If $e(\mathcal{W}_w) = \infty$ and $v \in \mathcal{P}(\mathcal{W})$, then $e(\mathcal{W}_v) = \infty$, since otherwise we could apply Corollary 6.7 to $\mathcal{W}_v$. □

7. Examples

Observe that, if $\{E_i\}_{i\in I}$ is an OBS for K and $T \in L(K, H)$ is a surjective operator such that $T(E_i) \sqsubseteq H$ for every $i \in I$, then $\mathcal{W} = \{T(E_i)\}_{i\in I}$ is a generating sequence for H. Nevertheless, our first example shows that $\mathcal{W}$ may have $\mathcal{P}(\mathcal{W}) = \emptyset$.

Example 7.1. Let $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ be an ONB of H. Take the sequence $\mathcal{E} = \{E_k\}_{k\in\mathbb{N}}$ given by $E_k = \operatorname{span}\{e_{2k-1}, e_{2k}\}$, $k \in \mathbb{N}$. Consider the (densely defined) operator $T : H \to H$ given by

$$Te_n = \begin{cases} 2^{-k}e_1 & \text{if } n = 2k-1, \\ e_{k+1} & \text{if } n = 2k. \end{cases}$$

Then, T can be extended to $L(H)$ as a surjective operator also denoted by T. Let $\mathcal{W} = \{W_k\}_{k\in\mathbb{N}}$ be given by $W_k = T(E_k) = \operatorname{span}\{e_1, e_{k+1}\}$, $k \in \mathbb{N}$. Then $\mathcal{P}(\mathcal{W}) = \emptyset$. Indeed, suppose that $w \in \mathcal{P}(\mathcal{W})$. Then by Eq. (6) applied to $f = e_1 \in \bigcap_{k\in\mathbb{N}} W_k$, we would have that $w \in \ell^2(\mathbb{N})$. But

$$A_{\mathcal{W}_w} = A_{\mathcal{W}_w}\|e_{k+1}\|^2 \leqslant \sum_{j\in\mathbb{N}} w_j^2 \|P_{W_j}e_{k+1}\|^2 = w_k^2 \xrightarrow[k\to\infty]{} 0,$$

which is a contradiction. Note that $\frac{\gamma(TP_{E_k})}{\|TP_{E_k}\|} = 2^{-k} \xrightarrow[k\to\infty]{} 0$.

The following example shows that, if $\{E_i\}_{i\in I}$ is an OBS for K and $T \in L(K, H)$ is a surjective operator, then Eq. (8) in Theorem 3.6 is not a necessary condition to assure that $\mathcal{P}(\mathcal{W}) \neq \emptyset$, where $\mathcal{W} = T\mathcal{E}$.

Example 7.2. Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis for H and consider the frame (of vectors) $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ given by

$$f_n = \begin{cases} e_k & \text{if } n = 2k-1, \\ \frac{e_{k+1}}{\sqrt{k+1}} & \text{if } n = 2k. \end{cases}$$

Let $T = T_{\mathcal{F}} \in L(\ell^2(\mathbb{N}), H)$ be its synthesis operator (which is surjective). If $\{b_n\}_{n\in\mathbb{N}}$ is the canonical basis of $\ell^2(\mathbb{N})$, then $Tb_n = f_n$. For each $k \in \mathbb{N}$ we set $E_k = \operatorname{span}\{b_{2k-1}, b_{2k}\}$. Then, by construction, $\{E_k\}_{k\in\mathbb{N}}$ is an OBS for $\ell^2(\mathbb{N})$. Take the sequences $w = e \in \ell^\infty_+(\mathbb{N})$ and $\mathcal{W} = \{W_k\}_{k\in\mathbb{N}}$ given by $W_k = T(E_k) = \operatorname{span}\{e_k, e_{k+1}\}$, $k \in \mathbb{N}$. By Theorem 3.4, $\mathcal{W}_w = (w_k, W_k)_{k\in\mathbb{N}}$ is an FS for H. Nevertheless, T does not satisfy Eq. (8), since $\gamma(TP_{E_k}) = \frac{1}{\sqrt{k+1}}$, while $\|TP_{E_k}\| = 1$, for every $k \in \mathbb{N}$.

The key argument in Example 7.1 was that $\bigcap_{i\in I} W_i \neq \{0\}$. This fact is a sufficient condition for the emptiness of $\mathcal{P}(\mathcal{W})$ if $\operatorname{span}\{W_i\colon 1 \leqslant i \leqslant n\} \neq H$ for every $n \in \mathbb{N}$. Nevertheless, the next example shows a *minimal* and generating sequence $\mathcal{W}$ of finite dimensional subspaces such that $\mathcal{P}(\mathcal{W}) = \emptyset$.

Example 7.3. Let $\mathcal{B} = \{e_i\}_{i\in\mathbb{N}}$ be an ONB for H. Consider the unit vector $g = \sum_{k=1}^\infty \frac{e_{2k}}{2^{k/2}} \in H$. For every $n \in \mathbb{N}$, denote by $P_n \in L(H)$ the orthogonal projection onto $H_n = \operatorname{span}\{e_1, \ldots, e_n\}$. Let $\mathcal{W} = \{W_k\}_{k\in\mathbb{N}}$ be the generating sequence given by

$$W_k = \operatorname{span}\{P_{2k}g, e_{2k-1}\} = \operatorname{span}\left\{\sum_{j=1}^k \frac{e_{2j}}{2^{j/2}}, e_{2k-1}\right\}, \quad k \in \mathbb{N}.$$

It is not difficult to prove that $\mathcal{W}$ is a minimal sequence. The problem is that $c[W_i, W_j] \xrightarrow[i,j\to\infty]{} 0$ exponentially, and for this reason $\mathcal{P}(\mathcal{W}) = \emptyset$. Indeed, suppose that $w \in \mathcal{P}(\mathcal{W})$, so that $\mathcal{W}_w = (w, \mathcal{W})$ is an FS. Then

$$B_{\mathcal{W}_w}\|g\|^2 \geqslant \sum_{k\in\mathbb{N}} w_k^2\|P_{W_k}g\|^2 = \sum_{k\in\mathbb{N}} w_k^2\|P_{2k}g\|^2 = \sum_{k\in\mathbb{N}} w_k^2\big(1-2^{-k}\big). \tag{12}$$

Then $w_k \xrightarrow[k\to\infty]{} 0$. But $A_{\mathcal{W}_w}\|e_{2k-1}\|^2 \leqslant \sum_{i\in\mathbb{N}} w_i^2\|P_{W_i}e_{2k-1}\|^2 = w_k^2$, which is a contradiction. Therefore $\mathcal{P}(\mathcal{W}) = \emptyset$.

Example 7.4. Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis for H. Consider the unit vector $g = \sum_{k\in\mathbb{N}} \frac{e_{2k-1}}{2^{k/2}}$ and the subspace $M = \overline{\operatorname{span}}\{\{g\}\cup\{e_{2k}: k\in\mathbb{N}\}\}$. On the other hand, take $\mathcal{E} = \{E_k\}_{k\in\mathbb{N}}$ given by $E_k = \operatorname{span}\{e_{2k-1}, e_{2k}\}$ ($k\in\mathbb{N}$), which is an OBS for H. Consider the sequence $\mathcal{W} = \{W_k\}_{k\in\mathbb{N}}$ given by $W_k = P_M(E_k) = \operatorname{span}\{g, e_{2k}\}$, $k\in\mathbb{N}$. Then $\mathcal{P}(\mathcal{W}) = \emptyset$ (as an FS for M) by the same reason as in Example 7.1, because $g \in \bigcap_{k\in\mathbb{N}} W_k \neq \{0\}$. Compare with Corollary 5.2.

Example 7.5. Let $\mathcal{E} = \{e_k\}_{k\in\mathbb{N}}$ be an ONB for H. Take the sequence $\mathcal{W} = \{W_k\}_{k\in\mathbb{N}}$ given by $W_1 = \overline{\operatorname{span}}\{e_k: k\geqslant 2\} = \{e_1\}^\perp$ and $W_k = \operatorname{span}\{e_1, e_k\}$ for $k\geqslant 2$. Observe that $\mathcal{P}(\mathcal{W}) = \{w\in\ell^2(\mathbb{N}):$ every $w_i > 0\}$. Indeed, one inclusion is clear, and

$$w\in\mathcal{P}(\mathcal{W}) \quad\Rightarrow\quad \sum_{k=2}^{\infty} w_k^2 = \sum_{k=2}^{\infty} w_k^2\|P_{W_k}e_1\|^2 \leqslant B_{\mathcal{W}_w} \quad\Rightarrow\quad w\in\ell^2(\mathbb{N}).$$

Now we shall see that $\mathcal{W}_w$ cannot be a *Parseval* FS for any $w\in\mathcal{P}(\mathcal{W})$. Indeed, in this case, $1 = \|e_k\|^2 = \sum_{i\in\mathbb{N}} w_i^2\|P_{W_i}e_k\|^2 = w_1^2 + w_k^2$, for every $k\geqslant 2$. Then $w\notin\ell^2(\mathbb{N})$, a contradiction. Our next step is to show that the frame operator $S_{\mathcal{W}_w}\in L(H)$ is diagonal with respect to $\mathcal{E}$, for every $w\in\mathcal{P}(\mathcal{W})$. Indeed, $T^*_{\mathcal{W}_w}e_1 = \{w_kP_{W_k}e_1\}_{k\in\mathbb{N}} = 0\oplus\{w_ke_1\}_{k\geqslant 2}$, so that

$$S_{\mathcal{W}_w}e_1 = T_{\mathcal{W}_w}T^*_{\mathcal{W}_w}e_1 = \left(\sum_{k=2}^{\infty} w_k^2\right)e_1.$$

Straightforward computations show that $S_{\mathcal{W}_w}e_j = (w_1^2 + w_j^2)e_j$, for every $j\geqslant 2$. In particular, $S_{\mathcal{W}_w}^{-1/2}$ is also diagonal. This implies that $S_{\mathcal{W}_w}^{-1/2}\mathcal{W} = \mathcal{W}$, which cannot be Parseval for any sequence of weights.

This example shows that there exists an FS such that, if one deletes one coordinate, the remaining sequence of subspaces is still generating, but it does not form an FS for any sequence of weights. Indeed, if $w = \{2^{-n}\}_{n\in\mathbb{N}}$, then $\mathcal{W}_w$ is an FS for H, but $\mathcal{P}(\{W_k\}_{k>1}) = \emptyset$, because $\bigcap_{k>1} W_k \neq \{0\}$. The sequence $\{W_k\}_{k>1}$ is generating since $\bigcup_{k>1} W_k$ contains an ONB for H.

Example 7.6. Let $\mathcal{B}_4 = \{e_n\}_{n\leqslant 4}$ be an ONB for $\mathbb{C}^4$. Take the sequence

$$W_1 = \operatorname{span}\{e_1, e_2\}, \qquad W_2 = \operatorname{span}\{e_1, e_3\} \quad\text{and}\quad W_3 = \operatorname{span}\{e_4\}.$$

We claim that, for every $G\in \mathrm{Gl}(4,\mathbb{C})$ and every $w\in\mathbb{R}_+^3$, the sequence $G\mathcal{W}_w = (w_k, G(W_k))_{k\in\mathbb{I}_3}$ fails to be a Parseval FS. Indeed, consider the unit vectors $g_1 = \|Ge_1\|^{-1}Ge_1$, $g_4 = \|Ge_4\|^{-1}Ge_4$ and choose g_2 and g_3 in such a way that $\{g_1, g_2\}$ is an ONB for $G(W_1)$ and $\{g_1, g_3\}$ is an ONB for $G(W_2)$.

Let $b_1 = g_1$, $b_2 = g_2$, $b_3 = g_1$, $b_4 = g_3$, $b_5 = g_4$, be the ONB for $K_{G\mathcal{W}_w}$. If $G\mathcal{W}_w$ were a Parseval FS, by Theorem 3.4 the frame $\mathcal{E} = \{T_{G\mathcal{W}_w}b_k\}_{k\in\mathbb{I}_5}$ would be also Parseval. Its rearrangement, $\mathcal{E}' = \{w_1g_1, w_2g_1, w_1g_2, w_2g_3, w_3g_4\}$, is also Parseval. Consider the matrix $T\in\mathcal{M}_{4,5}(\mathbb{C})$ with the vectors of $\mathcal{E}'$ as columns. After a unitary change of coordinates in $\mathbb{C}^4$, T has the form

$$T = \begin{bmatrix} w_1 & w_2 & \vec{v} \\ 0 & 0 & V \end{bmatrix} \begin{matrix}\mathbb{C} \\ \mathbb{C}^3\end{matrix} \quad \text{with } \vec{v} = (0,0,a)\in\mathbb{C}^3 \text{ and } V\in\mathcal{M}_3(\mathbb{C}).$$

Since $TT^* = I_4$, it is easy to see that V is unitary. But this is impossible because the first two columns of V have norms $\|w_1g_2\| = w_1$ and $\|w_2g_3\| = w_2$, while $1 = w_1^2 + w_2^2 + |a|^2$.

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